Haïm Brezis: an exceptional master between pure mathematics and applied mathematics

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On July 7, Haïm Brezis, passed away in Jerusalem. He was one of the most eminent mathematicians of the 20th century, he combined an extraordinary capacity for mathematical analysis with the study of the differential equations that govern the processes that model modern science. He was born in Riom-ès-Montagnes, a small town in Auvergne, France, on June 1, 1944, just 80 years ago. The work Haïm Brezis left behind is an excellent proof of how pure mathematics can shed light on sophisticated problems posed by applied mathematics. This is a small tribute of two of his students to celebrate his life and legacy.

Haïm was initially trained in the environment of pure mathematics (his first thesis advisor was Gustave Choquet) but his second advisor, Jacques-Louis Lions, directed him towards more applied topics (Note: in France, in those years, the Thèse d'État had to deal with two disjoint topics having different advisors). Leaving behind the reigning Bourbaki style, Haïm soon oriented himself towards a world where Functional Analysis is applied as a tool for the study of "nonlinear" Partial Differential Equations (PDEs). Indeed, in the 19sixties of the last century, much was a mystery on these issues, and there was a lack of systematic "methods" to address a plethora of very diverse problems that were apparently unconnected. For example, the results on the regularity of solutions on "unilateral problems", collected in his doctoral thesis, were entirely published in the prestigious journal *J. Math. Pures Appliquées*, see citation [B72], and opened the door for the efficient application of numerical methods; these would be developed later on by R. Glowinski, J.-L. Lions himself, and many others.

The emergence of an exceptional figure

The work of great mathematical figures, such as E. De Giorgi, L. Hörmander, P. Lax, J. L. Lions, L. Nirenberg, J. Serrin, G. Stampacchia, and others had prepared the way, and H. Brezis would take brilliant relay in the 70 collaborating with most of them and also with F.E. Browder, M.G. Crandall, T. Kato, A. Friedman, E. Lieb, D. Kinderlehrer, L. Peletier, J. Bourgain, L.C. Evans and many others, among whom we must count the long list of his students. According to the *Mathematics Genealogy Project*, Haïm supervised 58 doctoral theses and had 1,161 scientific descendants. They include Fields Medal winners such as P.L. Lions (recent Fields medalist A. Figalli is among his collaborators), and also distinguished specialists from more than 16 countries.

At the beginning of the 1970s, the young Brezis caused a sensation among the afore-mentioned experts for his mastery of functional analysis applied to non-linear EDPS and his interest in the applications that then occupied the minds of the great figures he was meeting. He had a prodigious mind that knew how to quickly see the solution to a problem where others could not, and he expressed his solution with passion and elegance. His vision was often based on finding subtle connections between the multiple concepts that were needed to describe apparently very diverse phenomena, and on linking the obtained results to the "classical theory of Analysis." He combined his visionary creativity with a sharp instinct to detect what was important and

what was mere technical. The name Brezis is associated with many concepts used in the analysis of nonlinear equations, such as maximal monotone operators, nonlinear semigroups of contractions, variational inequalities, the appropriate concept of weak solution, solutions with compact support, singularities of different types, vortices, Ginzburg-Landau equations, and so on.

In 1973 Haïm wrote an important book, [BOMM], explaining in it the theory of maximal monotone operators and their relationship with the generation of semigroups, as a technique for solving differential equations of evolution by functional methods. It is a well-known theory today, but it was required reading for many of us when we began our research lives because at that time it was a great novelty. He had the collaboration of the first of his students, Philippe Bénilan (1940-2001), whose early death has so deeply affected those of us who treated him assiduously. The way the book came about is curious. Brezis gave his doctoral course in Paris VI, while Bénilan took notes, and between the two of them they edited the text as it emerged.

Let us see another example of Brezis in action. After establishing his connection with the US, he cultivated the friendship of the great mathematician-physicist Elliott Lieb. Both delved into the mathematical difficulties of the Thomas-Fermi atomic model [BL79] with notable results for the theory of existence of solutions of nonlinear PDEs. Changing register, in 1983 they published the famous "Brezis-Lieb lemma" about convergence of L^p functions, [BL83]. Returning to the other extreme of applied problems, one can cite his pioneering work in the modeling of nuclear fusion plasmas in Tokamaks (developed together with H. Berestycki [BB80] and later analyzed numerically by R. Glowinski and his Spanish student E. Fernández Cara), the study of liquid crystals [BCL86], superconductivity, and so on. Brezis cultivated all his life the "art of inequalities", a fundamental tool of applied functional analysis, among the many examples we can cite the Brezis-Gallouët inequality [BG80]. Futher below we will cite the "Hardy inequality".

Let us examine one more example of how Haïm's approach of using Functional Analysis techniques opened new perspectives in classical applied problems. We are talking about the problems of heat propagation and filtration in porous media that use non-linear parabolic equations as models. This topic would occupy the "young Spanish school" ever since, and it saw great contributions from Ph. Bénilan (who left such a pleasant memory in Spain). These models are non-linear PDEs in which it is crucial (even for a good numerical treatment of the problem) to choose the functional space in which the model is naturally well posed (that is, the space where there are good estimates and where solutions must be sought, both theoretical solutions or numerical ones): the crucial clue that Brezis gave to Bénilan, at the beginning of the latter's thesis, is that this space would not be the Hilbert space of the L^2 functions, usual for many other problems in Mathematical Physics, but the non-reflexive L^1 space of merely integrable functions.

At that time, Brezis was already preparing what would be one of his most innovative articles, Brezis-Strauss [BS74]. The change of space is not trivial, since it forces us to deal with a space with less pleasant analytical properties, a space more typical of "pure mathematicians and probabilists." Such a space turns out to be essential, and the " Brezis school" was ready to use it and benefit from it. We cannot fail to mention here the figure of M. Crandall and his distinguished student L.C. Evans in the USA, who had such close relation with Brezis and his students. The article [BBC75] is worth quoting for its classic style. Although it is not well known, Brezis addressed the numerical analysis of this type of problems in his beautiful article [BBR]. The article mentions the convergence of the implicit semi-discretized scheme and other numerical algorithms that are convergent thanks to an abstract result: the so-called "Chernoff formula".

Maturity and universal recognition

As time went by, Brezis developed an increasing activity, he was known throughout the world to this day as one of the most cited mathematical authors for his articles and research books, and also famous for the clarity and amenity of his lectures. Due to the brevity of this story we will leave out the study of so many fascinating topics for future authors. Among which we could his pioneering work on the mathematical structure of the Ginzburg-Landau vortices collected in the book with F. Bethuel and F. Hélein [BBH94]. The Ginzburg-Landau model has been a surprising source of new problems and new ideas in analysis, geometry, and topology.

Brezis was a universal figure: He was a professor at the University of Paris VI (officially, Univ. Pierre and Marie Curie, now Paris-Sorbonne), from 1972 until his retirement in 2007. But his extraordinary energy and ability allowed him to double his duties for a semester to devote the other one to visiting other countries, especially the United States (he was "visiting distinguished professor" at Rutgers University since 1987) and Israel (Technion Institute, Haifa); his family currently resides in Jerusalem.

His influence among mathematicians of his generation, and especially among later ones, was exceptional in a field like mathematics, so different in its ways from other experimental sciences. His "Spanish school" appears in a prominent place. His personal contribution to the progress and international recognition of young Spanish mathematicians was very special and had a unique strategic importance since it acted as a detonator for new ideas in Spain and then expanded to other areas.

Notes on his relationship with Spain

This is just a brief story of a relationship that began unexpectedly and ended up changing the professional lives of many Spanish mathematicians. Haïm's first visit to Madrid took place in 1974 at the invitation of A. Dou and M. de Guzmán, who had such a keen vision in those years of where the future was headed. Let us also mention that In those years the French Embassy in Madrid played a patronage role of capital importance, given the still precarious situation of research in our country**.**

The research relationship began in subsequent years in the Department of Functional Equations of the Universidad Complutense, and crystallized with the doctoral theses of the authors of this Obituary (in 1976 and 1979, respectively), which were joined by those of J Hernandez (1977), M.A. Herrero (1979), J. Carrillo (1981) and M. Escobedo (1988). His contribution to the training of many other Spaniards was also outstanding (G. Díaz (1980), F. Bernis (1982), X. Cabré (1998), D. Gómez-Castro (2015), and so on). For example, Haïm visited Seville in 1978, at the invitation of A. Valle, a great supporter of relations with France, and we also know of visits to Granada and other Spanish universities.

Brezis's great contribution was the extreme care he gave to the research careers of his students, inserting them into an international community that fostered the research collaboration that It barely existed in our country. This did not happen in similar way and amount in the case of other foreign figures who also directed in those times solid theses to Spaniards outside of our borders.

Some samples of the collaboration with Brezis

Haïm collaborated very actively in launching new mathematical journals in our country. For example, Haïm published, together with the second author of this obituary, an important article in the newly born Revista Matemática de la Universidad Complutense de Madrid, cited as [BV97]. That article dealt with a classic problem, already addressed by Lord Kelvin. It dealt with

the model an isothermal gas ball in gravitational equilibrium: a model that was later used in the description of certain equilibrium configurations in combustion by numerous authors, among others I.M. Gelfand (1959). The equation is simple, $\Delta u + \lambda f(u) = 0,$ in a bounded domain Ω in *n*-dimensional space and *f* is an increasing function. It is shown that there exists a un $\lambda^* > 0$, called the extreme parameter, such that the equation has energy solutions only up to λ^*. In the work, a nonlinearity *f* of exponential type is taken in particular, and it is shown that for dimension *n>9* the extremal solution cannot be bounded (something that numerically is quite difficult to detect). A key tool was the authors' refinement of the famous "Hardy inequality with best constant and remainder term." Once again, "abstract" mathematics clarifies the understanding of a topic in applied to science, in this case concerning fluid mechanics. The many citations that this article received benefited the international recognition of the UCM magazine. Authors such as Adimurthi, X. Cabré, S. Filippas, N. Goussoub, M. Marcus, I. Peral, X. Ros-Oton, A. Tertikas and E. Zuazua contributed to this line of work with remarkable results.

Brezis also provided similar patronage work with the launch of the journal RACSAM by the Royal Academy of Sciences of Spain, whereby he sent an important article, [BM01]. The Main Editor of RACSAM was the first author of this writing. The topic of this work is further developed to his latest book by H. Brezis and P. Mironescu, [BM21].

Haïm had a decisive influence in encouraging his students to write their main results in a clear style with convincing arguments, without refusing to spend the necessary time. At this point we would like to show our debt to his teaching by remembering two of our books, [D85] and [V07], which we believe have traces of his style and interests, since they deal with problems that are part of the legacy he left us.

Honors and distinctions

Brezis was a member of the editorial board of numerous prestigious journals, his dedication was incessant, and in our opinion exhausting. We recall that this task is as important as it is poorly appreciated. Thanks to it, the journals keep their level and the researchers see their work read and their efforts rewarded. Brezis was quick in his replies and his judgment was usually impartial and accurate. His work at the helm of *JEMS, the Journal of the European Mathematical Society* at the beginning of this century was impressive. On top of this work he combined intense activity as head of book collections for Pitman and Birkhäuser.

For all the activity in so many fronts, his services were recognized with a multitude of honors and awards evident in his resume (accessible on Wikipedia). They began with the Prix Peccot in 1974 and his entry into the Académie des Sciences de Paris in 1988. He was also a member of the Academia Europaea (1988), of the United States National Academy of Sciences (2003), of the American Academy of Arts and Sciences. Also, Distinguished Fellow of the American Mathematical Society since 2012, among many other distinctions, including the Steele Prize for Lifetime Achievement, awarded by the AMS in 2024. Haïm was also for a time Vice President of the American Mathematical Society.

His work in Spain was recognized with his appointment as Foreign Member of RAC, the Royal Academy of Sciences, in 1999, and with Dr. Honoris Causa by Universidad Autónoma de Madrid in 2002. It gives an idea of his commitment to Spanish mathematics that he was the main French driving force of the agreement between the French and Spanish Academies of Sciences signed in Paris in 2002 (the first agreement in the history of our country) and the celebration in Paris, in 2003, of a joint Congress between both academies, the minutes of which were published in

the newly created RACSAM magazine and edited by him and the first signatory of this obituary in [BD03].

Some personal details

Haïm was very committed to his religious convictions. The book "*Haïm Brezis: Un mathématicien juif",* [LBr99], gives an idea of his ideas and his great intellectual stature, as also possessed by his widow, Michal Govrin (poet, writer and theater director), and his two daughters , Rachel and Miriam.

Haïm (which means "Life" in Hebrew) was a frugal person; when he came to Madrid he liked to stay in small hotels in bustling central neighborhoods to experience the life of the old city, like the "Hotel Inglés" on Echegaray Street.

His books, written in a style of great elegance and clarity, remain textbooks in numerous universities around the world. His book *"Analyse Fonctionnelle Théorie et Applications",* [B83], was required reading for so many of us and shaped a whole new philosophy for addressing problems in Applied Mathematics. The master's mark will remain forever and will serve as an example for future generations.

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BIBLIOGRAPHY

[BBC75] Ph. Bénilan, H. Brezis, M.G. Crandall. *A semilinear equation in L^1(R^N).* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 4, 523–555

[BB80] H. Berestycki, H. Brezis. On a free boundary problem arising in plasma physics. Nonlinear Anal. 4 (1980), no. 3, 415–436.

[BBR] A. Berger, H. Brezis, J. Rogers. *A numerical method for solving the problem* $u_t - \Delta f(u) = 0$, RAIRO Anal. Num. 13 (1979), p. 297-312.

[BBH94] F. Bethuel, H. Brezis, F. Hélein. *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, Inc., Boston, MA, 1994.

[B72] H. Brezis. *Problèmes unilatéraux*. J. Math. Pures Appl. (9) 51 (1972), 1–168.

[BOMM] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Mathematics Studies, No. 5. Notas de Matemática, No. 50.

North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[B83] H. Brezis. *Analyse fonctionnelle. Théorie et applications.* [Análisis Funcional, Teoría y aplicaciones] Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983. Versión renovada y extendida*: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.*

[LBr99] *"Haïm Brezis: Un mathématicien juif",* Libro-entrevista con J. Vauthier. Ed. Beauchesne, Paris, 1999.

[BCL86] H. Brezis, J.-M. Coron, E. H. Lieb. *Harmonic maps with defects*. Comm. Math. Phys. 107 (1986), no. 4, 649–705.

[BD03] H. Brézis, J.I. Díaz, edts. *Matemáticas y Medio Ambiente, Volumen especial de RACSAM, (Rev. R. Acad. Cien., Serie A Matem.), 96, nº 3, 2003*.

[BG80] H. Brezis, T. Gallouët. *Nonlinear Schrödinger evolution equations.* Nonlinear Anal. 4 (1980), no. 4, 677–681.

[BL83] H. Brézis, E. Lieb. *A relation between pointwise convergence of functions and convergence of functionals.* Proc. Amer. Math. Soc. 88 (1983), no. 3, 486–490. Tiene 1608 citas en Mathscinet.

[BL79] H. Brezis, E. Lieb. *Long range atomic potentials in Thomas-Fermi theory*. Comm. Math. Phys. 65 (1979), no. 3, 231–246.

[BM01] H. Brezis, P. Mironescu. *On some questions of topology for S1-valued fractional Sobolev spaces*. RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 95 (2001), no. 1, 121–143.

[BM21] H. Brezis, P. Mironescu. *Sobolev Maps to the Circle- From the Perspective of Analysis, Geometry and Topology*, Birkhäuser, New York (2021).

[BS74] H. Brezis, W. A. Strauss. *Semi-linear second-order elliptic equations in L^1.* J. Math. Soc. Japan 25 (1973), 565–590.

[BV97] H. Brezis, J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 443–469.

[D85] J.I. Díaz. *Nonlinear partial differential equations and free boundaries.* Research Notes in Mathematics, 106. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[V07] J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Haïm Brezis, in 2014, in Paris : Photo by J.I. Diaz