IMRN International Mathematics Research Notices
2005, No. 12

On the Brauer Class of Modular Endomorphism Algebras

Eknath Ghate, Enrique González-Jiménez, and Jordi Quer

1 Introduction

In this paper we study the Brauer class of the endomorphism algebra X_f of the motive attached to a primitive elliptic modular cusp form f without complex multiplication (CM). Our study includes the case of forms of weight 2, where the associated motive is an abelian variety.

It is a fundamental fact that X_f has a natural crossed product structure. This was proved by Ribet [8] and Momose [6] in the case of weight 2, and extended to forms of higher weight in [2] subject to an injectivity constraint, which we remove here. It follows that X_f is a central simple algebra over a subfield F of the Hecke field of f. Moreover X_f is 2-torsion when considered as an element of the Brauer group of F. Thus X_f is isomorphic to a matrix algebra over F, or a matrix algebra over a quaternion division algebra over F. Ribet has remarked in [8] that it seems difficult to distinguish these cases by pure thought. His remark pertains to the case of weight 2, but is equally relevant in higher weight. The chief motivation of this paper (and to a large extent [2]) is to give as complete a picture as possible of the Brauer class of X_f.

In recent years the notion of slope has played an important role in the theory of elliptic modular forms. For instance this notion is fundamental in parameterizing families of elliptic modular cusp forms, as in the work of Hida (slope 0) and Gouvêa, Mazur,

Received 27 April 2004. Revision received 17 December 2004.
Communicated by Peter Sarnak.

and Coleman (finite slope). Remarkably, the notion of slope turns out to be useful in studying the Brauer class of modular endomorphism algebras as well. In fact our main result is that at a finite place of F not dividing the level of f the ramification of X_f is essentially completely determined by the *parity* of the (normalized) slope of f when this slope is finite.

At places of F dividing the level our knowledge of the ramification of X_f is less complete. However we show that it is still governed to some extent by the slopes of f, at least at certain places where the underlying local representation is in the principal series. On the other hand at places for which the local representation is of Steinberg type one knows quite a bit about the ramification of X_f (see [2]). Predicting the ramification of X_f at the remaining bad places (the supercuspidal places) is still an open problem.

We end this paper with tables of the Brauer class of X_f for all forms f of small weight and level with $F = \mathbb{Q}$.

2 Statement of results

We now give more precise statements of our results. Let $f = \sum a_n q^n$ be a primitive cusp form of weight $k \geq 2$, level $N \geq 1$, and nebentypus ϵ. Here *primitive* means that f is a normalized newform that is a common eigenform of all the Hecke operators. Let M_f denote the abelian variety associated to f as constructed by Shimura when $k = 2$, and let M_f denote the Grothendieck motive attached to f constructed by Scholl in [11] when $k > 2$. Let $\mathrm{End}(M_f)$ be the ring of endomorphisms of M_f defined over $\overline{\mathbb{Q}}$. When $k > 2$, we work modulo cohomological equivalence. Set $X_f := \mathrm{End}(M_f) \otimes \mathbb{Q}$.

The first result is that X_f has a natural structure of a crossed product algebra. To state this result more precisely we need some notation. Let $E = \mathbb{Q}(a_n)$ denote the Hecke field of f. Then E is either a totally real or a CM number field. Assume from now on that f does not have CM. A pair (γ, χ_γ) where $\gamma \in \mathrm{Aut}(E)$ and χ_γ is an E-valued Dirichlet character is said to be an extra twist for f if $a_p^\gamma = a_p \cdot \chi_\gamma(p)$ for all but finitely many primes p. Let Γ denote the set of $\gamma \in \mathrm{Aut}(E)$ such that f has a twist by (γ, χ_γ) for some E-valued Dirichlet character χ_γ. In turns out that Γ is an abelian subgroup of $\mathrm{Aut}(E)$. For $\gamma, \delta \in \Gamma$ set

$$c(\gamma, \delta) = \frac{G\left(\chi_\gamma^{-1}\right) G\left(\chi_\delta^{-\gamma}\right)}{G\left(\chi_{\gamma\delta}^{-1}\right)}, \tag{2.1}$$

where $G(\chi)$ is the Gauss sum of the primitive Dirichlet character associated to χ. Then $c \in Z^2(\Gamma, E^\times)$ is a 2-cocycle which turns out to be $E^\times$-valued. Let X denote the crossed product algebra associated to c. For the reader's convenience we recall the definition of

X. For each $\gamma \in \Gamma$ let x_γ denote a formal symbol. Then as an E-vector space X is finite-dimensional with basis given by the symbols x_γ:

$$X = \bigoplus_{\gamma \in \Gamma} E x_\gamma, \tag{2.2}$$

and as an algebra X has structure given by the relations

$$\begin{aligned} x_\gamma \cdot e &= \gamma(e) x_\gamma, \\ x_\gamma \cdot x_\delta &= c(\gamma, \delta) x_{\gamma\delta}, \end{aligned} \tag{2.3}$$

where $e \in E$ and $\gamma, \delta \in \Gamma$.

If $k = 2$, then it is a result of Ribet [8, Theorem 5.1] and Momose [6, Theorem 4.1] that X_f is isomorphic to X. On the other hand if $k > 2$, then X_f contains a subalgebra isomorphic to X (see [6] and [2, Theorem 1.0.1]). Here, building on the above results, we prove the following.

Theorem 2.1. Let $k \geq 2$. Then X_f is isomorphic to X. $\square$

Let F be the number field contained in E which is the fixed field of Γ. Then X is isomorphic to a central simple algebra over F which is easily seen to be 2-torsion when considered as an element of the Brauer group of F. As a result, X is either a matrix algebra over F or a matrix algebra over a quaternion division algebra over F. We wish to distinguish these cases.

Recall that by global class field theory there is an injection

$$\mathrm{Br}(F) \hookrightarrow \oplus_v \mathrm{Br}\left(F_v\right), \tag{2.4}$$

where v runs through the places of F and F_v is the completion of F at v. Thus to study the Brauer class of X it suffices to study its image $X_v = X \otimes_F F_v$ for each place v of F under the above map. Since X is 2-torsion in the Brauer group of F, the algebra X_v is *a fortiori* either a matrix algebra over F_v or a matrix algebra over a quaternion division algebra over F_v.

As far as the infinite places are concerned, the field F is easily seen to be totally real, and it follows from a result of Momose [6, Theorem 3.1(ii)] (see also [2, Theorem 3.1.1]) that X is totally indefinite if k is even or totally definite if k is odd.

Now suppose that v is a finite place of F of residue characteristic p with p coprime to N. Then $a_p^2 \epsilon(p)^{-1} \in F$. Set

$$m_v := \left[F_v : \mathbb{Q}_p\right] \cdot v\left(a_p^2 \epsilon(p)^{-1}\right) \in \mathbb{Z} \cup \{\infty\}, \tag{2.5}$$

where v is normalized so that $v(p) = 1$. We will show that the structure of X_v is essentially determined by the *parity* of m_v when it is finite. If w is a place of E lying over v, then $w(a_p) = (1/2)v(a_p^2\epsilon(p)^{-1})$ is called the *slope* of f at p (with respect to w). Thus we show that there is a close connection between the ramification of X_v away from the level and the parity of (normalized) slopes. More precisely we have the following theorem.

Theorem 2.2. Let p be a prime such that

(i) p does not divide N,

(ii) $p \neq 2$ if $F \neq \mathbb{Q}$.

Also assume $a_p \neq 0$. Let v be a place of F lying over p. Then X_v *is a matrix algebra over* F_v *if and only if* $m_v \in \mathbb{Z}$ *is even*, except possibly in the exceptional case that p splits in all the quadratic fields cut out by the extra twists of f, in which case X_v is necessarily a matrix algebra over F_v. □

A result of this kind was proved in [2, Theorem 1.0.4] for cusp forms having quadratic extra twists, or equivalently, real nebentypus character. That such a result might be true in general became clear after extensive numerical computations were made by the second author. The proof of the general case combines ideas from [2, 7].

If f is ordinary at v (i.e., $v(a_p^2\epsilon(p)^{-1}) = 0$), then $m_v = 0$, and it follows from the theorem that X_v is a matrix algebra over F_v. This was already known even if $p = 2$ (see [9, Theorem 6] for $k = 2$ and [2, Theorem 3.3.1] for $k > 2$). Thus the theorem above may be considered as a generalization of these results.

For a more detailed explanation of the exceptional case mentioned in the statement of the theorem the reader is referred to Theorem 4.3 below. It is also possible to deal with the case $a_p = 0$ (and p still coprime to N). In this case m_v blows up but it may easily be substituted for by a closely related (finite) integer (see Proposition 4.5). We point out here that the above-mentioned results imply that X can only be ramified at the primes dividing

$$2 \cdot N \cdot \mathrm{disc}(E) \cdot \infty, \tag{2.6}$$

where $\mathrm{disc}(E)$ is the discriminant of E and ∞ is the unique infinite place of $\mathbb{Q}$ (Corollary 4.7).

The proof of the theorem above is based on an explicit computation of symbols appearing in a formula for the Brauer class of X. A more conceptual approach, based on a study of the filtered (ϕ, N)-modules of Fontaine attached to the local Galois representations associated to f, is available. This approach has so far been more successful for studying the ramification of X only when the slope is small compared to the weight (see, e.g., [2, Theorem 1.0.3]). However, it can be applied to study the ramification of X at some bad places $v \mid N$ (see [2, Theorems 1.0.5 and 1.0.6]). Here we push this approach further and prove the following result.

Theorem 2.3. Suppose that $p \mid N$ and that the power of p dividing N is the same as the power of p dividing the conductor of ϵ. Let v be a place of F lying over p. Let $\alpha \in \mathbb{Q}$ be such that $0 \leq \alpha < (k-1)/2$ and α has odd denominator. If for each place w of E lying over v either

$$w(a_p) = \alpha \qquad \text{or} \qquad \bar{w}(a_p) = \alpha, \tag{2.7}$$

then X_v is a matrix algebra over F_v. □

If the power of p dividing N is larger than the power of p dividing the conductor of ϵ, our knowledge of X_v is less complete, except in the very special case when $p \| N$ and the conductor of ϵ is prime to p (the Steinberg case). In this last case it turns out that the ramification of X_v is related to the parity of the weight k of f. For more precise statements see the end of [2, Section 3].

3 Crossed product structure

In this section we prove that X_f is isomorphic to the crossed product algebra X (Theorem 2.1). In view of the work of Ribet and Momose we will assume in this section that $k > 2$.

Let ℓ be a prime and let M_ℓ denote the ℓ-adic realization of M_f. Recall that M_ℓ is a $\mathbb{Q}_\ell$-vector space with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a subgroup H of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ let $\mathrm{End}_H(M_\ell)$ denote the endomorphisms of M_ℓ which commute with H. An endomorphism of M_f gives rise to an endomorphism of each of its realizations. One therefore obtains a map

$$\alpha : \mathrm{End}\left(M_f\right) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{End}_H\left(M_\ell\right), \tag{3.1}$$

where H is a sufficiently deep finite index subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In [2] it was shown that X_f contains a subalgebra generated by certain twisting operators which is isomorphic to the crossed product algebra X. Moreover it was shown that if α is injective, then X_f is isomorphic to X because of dimension considerations. Indeed, by [8, Theorem 4.4] (whose proof carries over to the case $k > 2$), the $\mathbb{Q}_\ell$-dimension of $\mathrm{End}_H(M_\ell)$ is $[E : F][E : \mathbb{Q}]$, which is also the $\mathbb{Q}_\ell$-dimension of $X \otimes \mathbb{Q}_\ell$.

So it suffices to prove that α is injective. To do this we work slightly more generally. Let X be a smooth irreducible projective variety over $\mathbb{Q}$ of dimension d. Let $Z(X \times X)$ be the rational vector space generated by the irreducible subvarieties of $X \times X$ over $\overline{\mathbb{Q}}$ of codimension d. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $H_B^{2d}(X \times X_{/\mathbb{C}})(d)$ denote Betti cohomology (with coefficients $(2\pi i)^d\mathbb{Q}$) and let $c_B : Z(X \times X) \to H_B^{2d}(X \times X_{/\mathbb{C}})(d)$ be the cycle class map. Let $Z_h(X \times X)$ be the quotient

$$Z_h(X \times X) := Z(X \times X)/\ker\left(c_B\right) = Z(X \times X)/\sim, \tag{3.2}$$

where $\sim$ is the cohomological equivalence relation. Thus for $Z \in Z(X \times X)$ one has $Z \sim 0$ if and only if the image of Z in $H_B^{2d}(X \times X_{/\mathbb{C}})(d)$ under c_B is zero. Recall that $Z_h(X \times X)$ has a natural ring structure where multiplication is induced by the composition product of correspondences. Let $p \in Z_h(X \times X)$ be a projector. Let $M = (X, p)$ be a motive, where X and p are as above. Recall that by definition

$$\operatorname{End}(M) := \frac{\left\{Z \in Z_h(X \times X) : Z \circ p = p \circ Z\right\}}{\left\{Z \in Z_h(X \times X) : Z \circ p = p \circ Z = 0\right\}}. \tag{3.3}$$

We show that in this setting the natural map

$$\alpha : \operatorname{End}(M) \otimes \mathbb{Q}_\ell \longrightarrow \operatorname{End}\left(M_\ell\right) \tag{3.4}$$

is injective. Note that α is equivariant for the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on both sides.

Consider the cycle class map $c_\ell : Z(X \times X) \to H_\ell^{2d}(X \times X)(d)$ to ℓ-adic cohomology. There is a comparison isomorphism

$$I_\ell : H_B^{2d}\left(X \times X_{/\mathbb{C}}\right)(d) \otimes \mathbb{Q}_\ell \cong H_\ell^{2d}\left(X \times X_{/\mathbb{C}}\right)(d) \cong H_\ell^{2d}(X \times X)(d), \tag{3.5}$$

where the first isomorphism is (a twist of) the canonical comparison isomorphism between Betti and ℓ-adic cohomology for smooth projective varieties over $\mathbb{C}$ (see [5, Theorem 3.12]), and the second isomorphism is (again a twist of the one) induced by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ via the proper base change theorem. The two cycle class maps c_B and c_ℓ are related via I_ℓ, that is, $c_\ell = I_\ell \circ (c_B \otimes 1)$ (see [3, page 21] or [4, page 58]). It follows that the ℓ-adic cycle class map factors through the cycles (Betti-) cohomologically equivalent to zero, inducing a map $Z_h(X \times X) \hookrightarrow H_\ell^{2d}(X \times X)(d)$. (It also follows that $Z_h(X \times X)$ and hence $\operatorname{End}(M)$ is defined independently of the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ fixed above.) Since Betti cohomology gives a rational structure on ℓ-adic cohomology, the induced map

$$Z_h(X \times X) \otimes \mathbb{Q}_\ell \hookrightarrow H_\ell^{2d}(X \times X)(d) \tag{3.6}$$

continues to be injective. This is the key observation that was missed in [2]. To deduce from this that the map α in (3.4) is also injective is purely formal. We recall the argument here for the sake of completeness. By the Künneth formula,

$$H_\ell^{2d}(X \times X)(d) = \bigoplus_{q=0}^{2d} H_\ell^q(X) \otimes H_\ell^{2d-q}(X)(d) = \bigoplus_{q=0}^{2d} \mathrm{End}\left(H_\ell^q(X)\right), \tag{3.7}$$

where the last equality follows since $H_\ell^{2d-q}(X)(d)$ is dual to $H_\ell^q(X)$. Let

$$H_\ell^q(X)(p) = \mathrm{Im}\left(p : H_\ell^q(X) \longrightarrow H_\ell^q(X)\right). \tag{3.8}$$

By definition $\mathrm{End}(M_\ell) = \bigoplus_{q=0}^{2d} \mathrm{End}(H_\ell^q(X)(p))$. Now, the map in (3.6) induces a map

$$\left\{Z \in Z_h(X \times X) : Z \circ p = p \circ Z\right\} \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{End}\left(M_l\right). \tag{3.9}$$

Let $Z = Z \otimes 1 \in Z_h(X \times X) \otimes \mathbb{Q}_\ell$ belong to the kernel of (3.9). Thus we have $Zp = pZ$ and $Z = 0$ in $\mathrm{End}(M_\ell)$. Write $Z = Zp + Z(1-p)$ and note that $Z(1-p) \circ p = 0$, so $Z(1-p)$ acts as 0 on each $H_\ell^q(X)(p)$. Thus $Zp = Z - Z(1-p)$ acts as 0 on each $H_\ell^q(X)(p)$. Since Zp clearly acts as 0 on the spaces

$$\ker\left(p : H_\ell^q(X) \longrightarrow H_\ell^q(X)\right), \tag{3.10}$$

we see that $Zp = 0$ in $H_\ell^{2d}(X \times X)(d)$. By the injectivity of (3.6) we see that $Zp = pZ = 0$ in $Z_h(X\times X)$. It follows that the kernel of the map (3.9) is $\{Z \in Z_h(X\times X) : Z\circ p = p\circ Z = 0\}\otimes\mathbb{Q}_\ell$. Thus (3.9) induces an injective map $\mathrm{End}(M)\otimes\mathbb{Q}_\ell \hookrightarrow \mathrm{End}(M_\ell)$, which is precisely the map α in (3.4). This proves Theorem 2.1.

4 Ramification and slopes

Let $X \cong X_f$ denote the endomorphism algebra of M_f. We now study the relation between the ramification of X and the slopes of f.

Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. We will sometimes consider Dirichlet characters as characters of G. In particular ϵ is a character of G. For each $g \in G$, let $\sqrt{\epsilon(g)}$ be a fixed square root of $\epsilon(g)$. Now, for every extra twist (γ, χ_γ) of f and every preimage of γ in G_F, which we again denote by γ, there is a unique primitive Dirichlet character ψ_γ

of order 1 or 2 such that $\chi_\gamma(g) = \psi_\gamma(g) \cdot \sqrt{\epsilon(g)}^{\gamma-1}$ for all $g \in G$ (cf. [6, Lemma 1.5] and [7, Lemme 2]). Let $\mathbb{Q}(\sqrt{t_\gamma})$ be the quadratic field corresponding to ψ_γ (in the case that it has order 2). We assume that t_γ is also the discriminant of this field. The characters $\{\psi_\gamma \mid \gamma \in G_F\}$ form an elementary 2-group. Fix once and for all a subset $\Gamma_0 \subset G_F$ such that $\{\psi_\gamma \mid \gamma \in \Gamma_0\}$ is a basis for this group. For each $\gamma \in \Gamma_0$ choose square-free positive integers n_γ prime to N such that $a_{n_\gamma} \neq 0$, and such that for all $\gamma' \in \Gamma_0$,

$$\psi_{\gamma'}(n_\gamma) = \begin{cases} -1 & \text{if } \gamma' = \gamma, \\ 1 & \text{if not.} \end{cases} \tag{4.1}$$

Also, for a square-free integer n which is prime to N, set $z_n = a_n^2 \epsilon(n)^{-1} \in F$.

Let $[c_\epsilon]$ denote the class of the cocycle $c_\epsilon \in Z^2(G_F, \pm 1)$ defined by

$$c_\epsilon(g, h) = \sqrt{\epsilon(g)}\sqrt{\epsilon(h)}\sqrt{\epsilon(gh)}^{-1}, \tag{4.2}$$

for $g, h \in G_F$ (see [7, Section 2]).

The following result expressing the Brauer class of X in terms of symbols was proved in [7, Théorème 3] in the case $k = 2$.

Theorem 4.1. Let $k \geq 2$. Then

$$X = [c_\epsilon] \otimes \bigotimes_{\gamma \in \Gamma_0} (z_{n_\gamma}, t_\gamma) \tag{4.3}$$

up to Brauer equivalence. □

Proof. We make some brief remarks which show that the proof given in [7] for the case $k = 2$ continues to hold if $k > 2$.

Each $g \in G$ acts naturally as an automorphism on X and this automorphism fixes E since the elements of E are basically Hecke operators and so are defined over $\mathbb{Q}$. By the Skolem-Noether theorem the action of g on X must be given by inner conjugation by some element $e \in X$, which is well defined modulo $F^\times$. Since E is its own commutant in X, we see that $e \in E$. The association $g \mapsto e$ defines a continuous character $\alpha : G \to E^\times/F^\times$, where the target has the discrete topology. Write $\tilde{\alpha} : G \to E^\times$ for any lift of α.

The first point to note is that the result [10, Theorem 5.6], which says that the class of X in $\mathrm{Br}(F)$ is cut out by the 2-cocycle

$$(g, h) \longmapsto \frac{\tilde{\alpha}(g)\tilde{\alpha}(h)}{\tilde{\alpha}(gh)} \tag{4.4}$$

for $g, h \in G_F$, continues to hold if $k > 2$. Indeed, the class of X in $\mathrm{Br}(F)$ is the same as the class of the cocycle $c(g, h) \in Z^2(G_F, \bar{F}^\times)$, naturally obtained from the Jacobi sum cocycle $c(\gamma, \delta) \in Z^2(\Gamma, E^\times)$ in (2.1) by inflation. On the other hand by [9, Proposition 1], which is easily seen to hold for $k > 2$, this last cocycle cuts out the same class in $\mathrm{Br}(F)$ as the cocycle

$$(g, h) \longmapsto \chi_g(h), \tag{4.5}$$

where $\chi_g = \chi_\gamma$ for the image $\gamma \in \Gamma$ of $g \in G_F$, and χ_g is thought of as a character of G. Now using [9, Theorem 4], which says that $\tilde{\alpha}(h)^{g-1} = \chi_g(h)$, we see that the cocycle (4.5) defines the same class in $\mathrm{Br}(F)$ as the cocycle $(g, h) \mapsto \tilde{\alpha}(h)^{g-1}$. The proof of [9, Theorem 4] uses the Tate conjecture for the abelian variety attached to f. This is proved in [2, Corollary 1.0.2] for the motive M_f when $k > 2$ subject to an injectivity hypothesis which we have removed in Section 3. Finally, this last cocycle differs from the cocycle (4.4) by the map

$$(g, h) \longmapsto \frac{\tilde{\alpha}(h)^g \tilde{\alpha}(g)}{\tilde{\alpha}(gh)}, \tag{4.6}$$

which is a coboundary.

The second point to note is that the cocycle (4.4) is equal to $c_\epsilon \cdot c_d$ up to a coboundary, where c_d is the cocycle defined by

$$c_d(g, h) = \left(\frac{\tilde{\alpha}(h)}{\sqrt{\epsilon(h)}}\right)^{1-g}, \tag{4.7}$$

for $g, h \in G_F$ (see the beginning of [7, Section 2]). The rest of the proof of the theorem, which involves writing the class of c_d as a product of symbols, proceeds exactly as in the case $k = 2$ without any change. This proves the theorem. ∎

Remark 4.2. A formula similar to that appearing in the theorem above was proved in [2, Theorem 4.1.3] in the case that all the χ_γ are quadratic characters. This formula was proved by directly computing quadratic Gauss sums.

We wish to evaluate the symbols that appear in the expression for X in Theorem 4.1. To do this we recall some general facts about symbols from [12]. Let F be an arbitrary number field. Let v denote a place of F which is either finite or infinite. Let F_v denote the completion of F at v. It is well known that

$$\mathrm{Br}\left(F_v\right) \cong \mathbb{Q}/\mathbb{Z} \tag{4.8}$$

if v is finite (and is $\mathbb{Z}/2$ if v is infinite and real, and is trivial if v is infinite and complex). Now let a and b be nonzero elements of F. Then the symbol (a, b) determines an element in $\mathrm{Br}(F)[2]$. For each finite place v of F, let $(a, b)_v$ denote the induced element of $\mathrm{Br}(F_v)[2]$. By (4.8) the symbol $(a, b)_v$ is completely specified by a sign $+1$ or -1. This sign can be computed in terms of the v-adic valuations of a and b. There are two cases: the tame case, $v \nmid 2$, and the wild case, $v \mid 2$.

First assume that v is prime to 2. Fix a uniformizer π_v of the ring of integers of F_v. Write

$$\begin{aligned} a &= \pi_v^{v(a)} \cdot a', \\ b &= \pi_v^{v(b)} \cdot b', \end{aligned} \tag{4.9}$$

where we consider v here to be normalized such that $v(\pi_v) = 1$. In this section v will refer to a valuation which is normalized in this way unless explicitly stated otherwise. Then one has

$$(a, b)_v = (-1)^{(Nv-1)/2 \, v(a) v(b)} \cdot \left(\frac{b'}{v}\right)^{v(a)} \cdot \left(\frac{a'}{v}\right)^{v(b)}. \tag{4.10}$$

Here the symbol $(\frac{c}{v})$ takes the values ± 1 and is 1 exactly when the image of c is a square in the residue field at v.

Now assume that $v \mid 2$. We will only treat the case $F = \mathbb{Q}$ so that $v = 2$. For a unit $u \in \mathbb{Q}_2^\times$ let $\varepsilon(u)$ denote the residue of $(u-1)/2$ in $\mathbb{Z}/2$ and let $\omega(u)$ denote the residue of $(u^2-1)/8$ in $\mathbb{Z}/2$. Then for units u, v in $\mathbb{Q}_2^\times$ we have

$$(u, v)_2 = (-1)^{\varepsilon(u)\varepsilon(v)}, \tag{4.11}$$

$$(2, u)_2 = (-1)^{\omega(u)}. \tag{4.12}$$

Note that these formulas completely determine $(a, b)_2$ for $a, b \in \mathbb{Q}_2^\times$.

Now we return to our situation. Thus F is the center of X and contains $a_p^2 \epsilon(p)^{-1}$ for p prime to N. The usual local-global exact sequence for the Brauer group of F shows that the Brauer class of X is completely determined by the Brauer classes of the X_v, which are in turn completely determined by specifying a sign, one for each v. For notational convenience we write $X_v \sim a$ for an integer a if the sign of the Brauer class of X_v is the same as $(-1)^a$.

Recall that by a result of Momose [6, Theorem 3.1] one knows $X_v \sim k$ if v is infinite. On the other hand if $v \mid p$ is a finite place of F with $p \nmid N$, we have $X_v \sim 0$ if v is ordinary for f, that is, if $\nu(a_p^2\epsilon(p)^{-1}) = 0$ (see [9, Theorem 6] for the case $k = 2$ and [2, Theorem 3.3.1] for $k > 2$). The following theorem generalizes this result. To state it we introduce a positive integer m_ν for each place ν of F of residue characteristic $p \nmid N$ with $a_p \neq 0$:

$$m_\nu := \left[F_\nu : \mathbb{Q}_p\right] \cdot \nu\left(a_p^2 \epsilon(p)^{-1}\right). \tag{4.13}$$

In the definition of m_ν we take the valuation ν which is normalized such that $\nu(p) = 1$. Then we have the following theorem (it is a more precise version of Theorem 2.2).

Theorem 4.3. Let p be a prime such that $p \nmid N$ and $a_p \neq 0$. Let ν be a place of F lying over p. If $p \neq 2$, then

$$X_\nu \sim \begin{cases} 0 & \text{if } \psi_\gamma(p) = 1 \ \forall \gamma \in \Gamma_0, \\ m_\nu & \text{otherwise.} \end{cases} \tag{4.14}$$

If $p = 2$, then the same conclusion holds if $F = \mathbb{Q}$. □

Proof. The proof is similar to the proof of [2, Theorem 4.1.11], the main difference being that we use Theorem 4.1 instead of [2, Theorem 4.1.3] to compute X locally.

Note $[c_\epsilon]_\nu = 1$ if and only if the local component ϵ_p of ϵ is even. Since p is prime to N, ϵ_p is in fact trivial, so that $[c_\epsilon]_\nu = 1$. It follows from Theorem 4.1 that

$$X_\nu = \bigotimes_{\gamma \in \Gamma_0} \left(z_{n_\gamma}, t_\gamma\right)_\nu. \tag{4.15}$$

Since ν is prime to N, we have $\nu(t_\gamma) = 0$. First assume that $p \neq 2$. Then ν is prime to 2 so that by (4.10) we have $(z_{n_\gamma}, t_\gamma)_\nu = (\frac{t_\gamma}{\nu})^{\nu(z_{n_\gamma})}$. But $(\frac{t_\gamma}{\nu}) = (\frac{t_\gamma}{p})^{f_\nu}$ since every element of $\mathbb{F}_p$ has a square root over a quadratic extension of $\mathbb{F}_p$. We conclude that

$$\left(z_{n_\gamma}, t_\gamma\right)_\nu = \left(\frac{t_\gamma}{p}\right)^{f_\nu \cdot \nu(z_{n_\gamma})}. \tag{4.16}$$

Thus if $\psi_\gamma(p) = (\frac{t_\gamma}{p}) = 1$ for all $\gamma \in \Gamma_0$, then $X_\nu = 1$ as desired.

Suppose on the other hand that the subset S^- of the set $\{t_\gamma \mid \gamma \in \Gamma_0\}$ consisting of those t_γ for which $(\frac{t_\gamma}{p}) = -1$ is nonempty. Write the elements of S^- as $t_1, t_2, \ldots, t_m$ with $m \geq 1$. Define distinct primes r_j for $j = 0, 1, \ldots, r_{m-1}$ as follows: set $r_0 = p$ and define r_j

for $j = 1, \ldots, m-1$ recursively by

$$\left(\frac{t_i}{r_j}\right) = (-1)^{\delta_{ij}} \cdot \left(\frac{t_i}{r_{j-1}}\right) \quad \forall i = 1, \ldots, m, \tag{4.17}$$

$$\left(\frac{t_\gamma}{r_j}\right) = 1 \quad \text{if } t_\gamma \notin S^-. \tag{4.18}$$

We may and do assume that each $a_{r_j} \neq 0$. This can be done for $j = 0$ since $a_p \neq 0$ by hypothesis. For the other r_j's we simply note that if $a_{r_j} = 0$ for all r_j defined by the congruence conditions (4.17) and (4.18), then the set of primes p for which $a_p = 0$ would have a positive density contradicting Serre [13, Theorem 15].

Corresponding to $t_i \in S^-$ set

$$n_i = \begin{cases} r_{i-1} \cdot r_i & \text{if } 1 \leq i \leq m-1, \\ r_{m-1} & \text{if } i = m. \end{cases} \tag{4.19}$$

Clearly the n_i are square-free positive integers prime to the level satisfying $a_{n_i} \neq 0$ since Fourier coefficients are multiplicative on distinct primes and the r_j were chosen so that each $a_{r_j} \neq 0$. Furthermore the n_i satisfy the congruence conditions (4.1). Indeed suppose that t_i corresponds to $\gamma \in \Gamma_0$. Assume first that $i < m$. Then

$$\psi_\gamma(n_i) = \psi_\gamma(r_{i-1})\psi_\gamma(r_i) = -1 \tag{4.20}$$

since, by (4.17), $\psi_\gamma(r_{i-1})$ and $\psi_\gamma(r_i)$ differ by a sign. Similarly if γ' corresponds to t_j for $j \neq i$, then $\psi_{\gamma'}(r_{i-1}) = \psi_{\gamma'}(r_i)$ by (4.17) again so that $\psi_{\gamma'}(n_i) = 1$. Finally if γ' corresponds to some $t_{\gamma'} \notin S^-$, then by (4.18) $\psi_{\gamma'}(n_i) = 1$. Now assume that $i = m$. Then for any $\gamma' \in \Gamma_0$,

$$\psi_{\gamma'}(n_m) = \psi_{\gamma'}(r_{m-1}) = \left(\frac{t_{\gamma'}}{r_{m-1}}\right). \tag{4.21}$$

But (4.17) shows that

$$\left(\frac{t_i}{r_{m-1}}\right) = \begin{cases} -\left(\dfrac{t_i}{p}\right) = 1 & \text{if } i \leq m-1, \\ \left(\dfrac{t_i}{p}\right) = -1 & \text{if } i = m. \end{cases} \tag{4.22}$$

So this along with (4.18) shows that $\psi_{\gamma'}(n_m) = -1$ if and only if $\gamma' = \gamma$ as desired.

We are now ready to begin computing our symbols. Only those γ for which $t_i \in S^-$ contribute to the sign of X_ν in (4.15) since if $t_\gamma \notin S^-$, then $(\frac{t_\gamma}{\nu}) = (\frac{t_\gamma}{p}) = 1$ and the corresponding local symbol is trivial by (4.16). Now for $t_i \in S^-$ we have $z_{n_i} = a_{n_i}^2 \epsilon(n_i)^{-1}$ and $(\frac{t_i}{p}) = -1$ so that by (4.16) we have

$$\left(z_{n_i}, t_i\right)_\nu \sim f_\nu \cdot \nu\left(a_{n_i}^2 \epsilon\left(n_i\right)^{-1}\right). \tag{4.23}$$

Substituting for n_i from (4.19) above and multiplying over all i in $\{1, \ldots, m\}$, there is a mod 2 telescoping effect, the result of which is

$$X_\nu \sim f_\nu \cdot \nu\left(a_p^2 \epsilon(p)^{-1}\right). \tag{4.24}$$

If we take the $\nu(p) = 1$ normalization for ν, then the right-hand side becomes m_ν, proving the theorem in the case $p \neq 2$.

Now assume that $p = 2$ and that $F = \mathbb{Q}$. Write $\nu_2(z_{n_\gamma})$ for the power of 2 that divides z_{n_γ} and define z'_{n_γ} by $z_{n_\gamma} = 2^{\nu_2(z_{n_\gamma})} \cdot z'_{n_\gamma}$. We have

$$\left(z_{n_\gamma}, t_\gamma\right)_2 = \left(2, t_\gamma\right)_2^{\nu_2(z_{n_\gamma})} \cdot \left(z'_{n_\gamma}, t_\gamma\right)_2. \tag{4.25}$$

Since N is prime to $p = 2$, by hypothesis t_γ must be odd. One can easily check that $(2, t_\gamma)_2$ is equal to $(-1)^{\omega(t_\gamma)}$ by (4.12) which may again be easily checked to be the same as $(\frac{t_\gamma}{2})$ using the fact 2 splits in $\mathbb{Q}(\sqrt{t_\gamma})$ if and only if $t_\gamma \equiv 1 \bmod 8$. On the other hand $(z'_{n_\gamma}, t_\gamma)_2 = 1$ by (4.11) since $\varepsilon(t_\gamma) \equiv 0 \bmod 2$. Thus

$$\left(z_{n_\gamma}, t_\gamma\right)_2 = \left(\frac{t_\gamma}{2}\right)^{\nu_2(z_{n_\gamma})}. \tag{4.26}$$

Now the argument proceeds as in the case $p \neq 2$ proving the theorem in this case as well. ■

Remark 4.4. The assumption that $F = \mathbb{Q}$ when $p = 2$ could probably be removed if one had formulas for wild symbols other than in the case $F = \mathbb{Q}$.

As in [2] it is possible to treat the case $a_p = 0$ (and p still prime to N) with minimal effort. The structure of X_ν is not determined by the parity of m_ν since $m_\nu = \infty$. Thus the notion of slope is not useful in measuring the ramification in this case. However as we now show the structure of X_ν is still determined by the ν-adic valuation of a Fourier coefficient at a prime $p^\dagger$, closely related to p.

In fact we take $p^\dagger$ to be any prime such that $pp^\dagger \equiv 1 \bmod N$ and such that $a_{p^\dagger} \neq 0$. Serre's result, quoted above, guarantees that one can always find such a $p^\dagger$. Set

$$m_v^\dagger := [F_v : \mathbb{Q}_p] \cdot v\big(a_{p^\dagger}^2 \epsilon(p^\dagger)\big). \tag{4.27}$$

Theorem 4.3 now has the following avatar when $a_p = 0$.

Proposition 4.5. Let v be a place of F of residue characteristic p prime to N and assume $a_p = 0$. Let $m_v^\dagger$ be as above. If $p \neq 2$, then

$$X_v \sim \begin{cases} 0 & \text{if } \psi_\gamma(p) = 1 \ \forall \gamma \in \Gamma_0, \\ m_v^\dagger & \text{otherwise.} \end{cases} \tag{4.28}$$

If $p = 2$, then the same conclusion holds if $F = \mathbb{Q}$. □

Proof. Since $pp^\dagger \equiv 1 \bmod N$, we have

$$\left(\frac{t_\gamma}{p}\right) = \left(\frac{t_\gamma}{p^\dagger}\right) \tag{4.29}$$

so that the proof of Theorem 4.3 goes through replacing p with $p^\dagger$. ■

We record the following easy consequences of the above results.

Corollary 4.6. Let v be a place of F of residue characteristic p with $p \nmid 2N$. If v is unramified in E, then X_v is a matrix algebra over F_v. □

Proof. This is immediate from Theorem 4.3 and Proposition 4.5 since in this case the integer m_v or $m_v^\dagger$ is necessarily even. It may also be proved directly by studying the formula in Theorem 4.1. Indeed if $v \nmid 2N$ is a finite prime of F for which X_v is ramified, then the normalized v-adic valuation of at least one of the entries in the symbols appearing in that theorem must be odd. Since $t_\gamma \mid N$ for all $\gamma \in \Gamma$, the only possibility is that the normalized valuation $v(z_q)$ must be odd for some prime q. One checks easily then that v ramifies in $F(a_q)$ so that v ramifies in E. ■

Corollary 4.7. If X is ramified at v, then v must divide either the discriminant of the field E, or $2N$, or ∞. □

Proof. This is immediate from the previous corollary. ■

5 Bad places

In the previous section we showed how the ramification of X at the good places ($v \mid p \nmid N$) is essentially determined by the normalized slopes of f at p. In this section we continue our investigation of the ramification of X_v at the bad places ($v \mid p \mid N$) that was begun in [2]. We recall some notation introduced in that paper.

Assume now that $p \mid N$. Let N_p be the exponent of the exact power of p that divides N. Let C denote the conductor of ϵ and let C_p denote the exponent of the exact power of p that divides C_p. Note $N_p \geq C_p$ and $N_p \geq 1$. We consider three cases:

(1) $N_p = C_p$, in which case $|a_p| = p^{(k-1)/2}$ (ramified principal series),

(2) $N_p = 1$ and $C_p = 0$, in which case $a_p^2 = \epsilon(p)p^{k-2}$ (Steinberg),

(3) $N_p \neq C_p$ and $N_p \geq 2$, in which case $a_p = 0$ (other).

In the second case the local factor at p in the automorphic representation corresponding to f is the Steinberg representation or a twist of it by an unramified character. This case is treated in [9, Theorem 3] in the case $k = 2$, and in [2] for higher weight forms (see in particular [2, Theorems 3.4.6 and 3.4.8]). Almost nothing is known in the third case, where $a_p = 0$. This case includes twists of previous cases and also cases where the local automorphic representation is supercuspidal. Here we will be concerned with the first case, in which the local automorphic representation is in the ramified principal series. The following theorem contains [2, Theorems 3.4.1 and 3.4.2] as special cases, and was stated without proof as [2, Theorem 3.4.4].

Theorem 5.1. Suppose $p \mid N$ with $N_p = C_p$ and let v be a place of F lying over p. Let $\alpha \in \mathbb{Q}$ be such that

$$0 \leq \alpha < \frac{k-1}{2}, \tag{5.1}$$

and α has odd denominator. If for each place w of E lying over v either $w(a_p) = \alpha$ or $\bar{w}(a_p) = \alpha$, then X_v is a matrix algebra over F_v. □

Proof. Let $M_{\mathrm{crys},v}$ denote the crystal attached to f and v. We recall the definition. Fix a place $w \mid v$ of E. The local Galois representation $\rho_f|_{G_p} : G_p \to GL_2(E_w)$ is potentially crystalline. In fact if $K = \mathbb{Q}(\mu_{p^r})$ where $r = N_p = C_p$, then $\rho_f|_{G_K}$ is unramified. Let $D_w = D_{\mathrm{st}}(\rho_f|_{G_K})$ be the associated filtered module. It is a free module of rank 2 over E_w. Set $M_{\mathrm{crys},v} = \oplus_{w|v} D_w$. This has dimension $2[E : F][F_v : \mathbb{Q}_p]$ over $\mathbb{Q}_p$. A study of the crystals D_w now show that the crystalline Frobenius $\phi : M_{\mathrm{crys},v} \to M_{\mathrm{crys},v}$ has characteristic polynomial

$$H(x) = \prod_{w|v} \mathrm{Norm}_{E_w/\mathbb{Q}_p}\left((x - a_p)(x - \epsilon'(p)\bar{a}_p)\right), \tag{5.2}$$

where ϵ' is the prime-to-p part of ϵ. By hypothesis the Newton polygon of $H(x)$ has two distinct slopes, namely α and $k-1-\alpha$, each occurring with equal multiplicity, say n. Let $\bar{M}_{\mathrm{crys},v} = M_{\mathrm{crys},v} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{un}}$. It follows that $\bar{M}_{\mathrm{crys},v} \cong C_\alpha^n \times C_{k-1-\alpha}^n$ where C_α and $C_{k-1-\alpha}$ are the simple crystals over $\mathbb{Q}_p^{\mathrm{un}}$ of slopes α and $k-1-\alpha$. Now $\dim_{\mathbb{Q}_p^{\mathrm{un}}} C_\alpha = \dim_{\mathbb{Q}_p^{\mathrm{un}}} C_{k-1-\alpha} = s$ where $\alpha = r/s$ as a fraction in lowest terms. It follows that

$$2[E:F]\left[F_v:\mathbb{Q}_p\right] = \dim_{\mathbb{Q}_p^{\mathrm{un}}} \bar{M}_{\mathrm{crys},v} = 2sn. \tag{5.3}$$

Now let $V = \mathrm{Hom}(C_\alpha, \bar{M}_{\mathrm{crys},v}) = \mathrm{Hom}(C_\alpha, C_\alpha)^n$. This is a left X_v-module of dimension

$$\dim_{F_v} V = \frac{s^2 n}{\left[F_v:\mathbb{Q}_p\right]} = s[E:F]. \tag{5.4}$$

Since s is odd, it follows from the representation theory of the algebra X_v that X_v must split. ■

6 Tables of QM-modular motives

In this section we give complete tables of the endomorphism algebras of all modular motives of small weight ($2 \le k \le 4$) and small level ($1 \le N \le 100$) with $F = \mathbb{Q}$. (For $k = 5$ see the version of this paper on the first author's web page.) An entry appears in Tables 6.1, 6.2, and 6.3 only if the corresponding motive has quaternionic multiplication (QM), that is, only if the class of X is nonzero in the Brauer group of $\mathbb{Q}$.

Recall that twisting a form by a Dirichlet character does not change the Brauer class of X (see [9, Proposition 3] for the weight 2 case; the proof there works in all weights). In the interest of conserving space we do not list those entries that are obtained from forms of smaller level by twisting.

That X_f can have nontrivial Brauer class was discovered by Shimura: the example in [14, page 166] appears as the fifth entry in Table 6.1.

We describe how the tables are labeled. The format is similar to that used in the tables in the appendix of [1]. The first column contains Galois conjugacy classes of primitive forms of given level, weight, and nebentypus. The ordering we use to list these forms is described by a function which maps a primitive form f to a label of the form NX_ϵ^k

Table 6.1 Modular QM-abelian varieties of level ≤ 100.

Label	ord(ϵ)	E	Extra twists	Ramification	Slope
$28A_{[1,1]}$	6	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[1,0],[1,5]$	2 3	RPS 1
$35A_{[1,3]}$	4	$\mathbb{Q}(\sqrt{10},\sqrt{-1})$	$[0,3],[3,3]$	2 5	1 RPS
$44A_{[1,5]}$	2	$\mathbb{Q}(\sqrt{2},\sqrt{-3})$	$[1,5],[1,0]$	2 3	RPS 1
$56B_{[1,1,3]}$	2	$\mathbb{Q}(\sqrt{-1},\sqrt{6})$	$[1,1,0],[1,1,3]$	2 3	RPS 1
$57A_{[1,9]}$	2	$\mathbb{Q}(\sqrt{2},\sqrt{-5})$	$[1,9],[0,9]$	2 5	1 1
$60A_{[0,1,1]}$	4	$\mathbb{Q}(\sqrt{5},\sqrt{-1})$	$[0,0,3],[0,1,0]$	2 5	$a_p = 0$ RPS
$63A_{[3,1]}$	6	$\mathbb{Q}(\sqrt{-2},\sqrt{6})$	$[3,5],[0,5]$	2 3	1 $a_p = 0$
$77B_{[3,5]}$	2	$\mathbb{Q}(\sqrt{10},\sqrt{-2})$	$[0,5],[3,5]$	2 5	1 1
$80B_{[1,0,1]}$	4	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[0,0,3],[1,0,3]$	2 3	$a_p = 0$ 1
$92A_{[1,11]}$	2	$\mathbb{Q}(\sqrt{-1},\sqrt{14})$	$[0,11],[1,11]$	2 7	RPS 1
$93D_{[1,5]}$	6	$\mathbb{Q}(\sqrt{2},\sqrt{-3})$	$[1,0],[1,25]$	2 3	1 RPS
$95A_{[1,3]}$	12	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[3,15],[3,0]$	2 3	1 1
$95B_{[1,3]}$	12	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[3,15],[3,0]$	2 3	∞ 1

(e.g., $19B^3_{[9]}$), where N is the level of f, X is a letter or string of letters in $\{A, B, \ldots, Z, AA, BB, \ldots\}$, ϵ is an encoding of the nebentypus of f (described in more detail below), and k is the weight of f. When $k = 2$, we omit the superscript 2. To construct X assume that k, N, and ϵ are fixed. To $f = \sum a_n q^n$ associate the infinite sequence of integers $\mathbf{t}_f = (\mathrm{Tr}_{E/\mathbb{Q}}\, a_1, \mathrm{Tr}_{E/\mathbb{Q}}\, a_2, \ldots)$. Choose $X \in \{A, B, \ldots, Z, AA, BB, \ldots\}$ according to the position of $\mathbf{t}_f$ in the set $\{\mathbf{t}_g : g \text{ primitive of weight } k,\ \text{level } N,\ \text{and nebentypus } \epsilon\}$ sorted in increasing dictionary order. Notice that $\mathbf{t}_f$ determines the Galois conjugacy class of f. The above ordering was introduced by J. Cremona in the case of trivial nebentypus and weight 2 and by W. Stein in the general situation.

Table 6.2 Modular QM-motives of weight 3 and level ≤ 100.

Label	ord(ϵ)	E	Extra twists	Ramification	Slope
$9A^3_{[1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[5]	3	RPS
$10A^3_{[0,1]}$	4	$\mathbb{Q}(\sqrt{-1})$	[3]	2	St
$12A^3_{[1,0]}$	2	$\mathbb{Q}(\sqrt{-3})$	[1, 0]	3	St
$15A^3_{[1,0]}$	2	$\mathbb{Q}(\sqrt{-5})$	[1, 0]	5	St
$18A^3_{[0,3]}$	2	$\mathbb{Q}(\sqrt{-2})$	[3]	2	St
$19B^3_{[9]}$	2	$\mathbb{Q}(\sqrt{-13})$	[9]	13	1
$20A^3_{[0,1]}$	4	$\mathbb{Q}(\sqrt{-1})$	[0, 3]	2	$a_p = 0$
$21A^3_{[0,1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[0, 5]	3	St
$21B^3_{[0,1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[0, 5]	3	St
$21C^3_{[0,1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[0, 5]	3	St
$21A^3_{[0,3]}$	2	$\mathbb{Q}(\sqrt{-3})$	[0, 3]	3	St
$21B^3_{[1,2]}$	6	$\mathbb{Q}(\sqrt{-3}, \sqrt{15})$	[0, 4], [1, 4]	5	1
$22A^3_{[0,5]}$	2	$\mathbb{Q}(\sqrt{-2})$	[5]	2	St
$24A^3_{[0,0,1]}$	2	$\mathbb{Q}(\sqrt{-2})$	[0, 0, 1]	2	$a_p = 0$
$24C^3_{[0,1,1]}$	2	$\mathbb{Q}(\sqrt{2}, \sqrt{-7})$	[0, 0, 1], [0, 1, 1]	2	RPS
$25A^3_{[5]}$	4	$\mathbb{Q}(\sqrt{-1}, \sqrt{6})$	[15], [5]	3	1
$26A^3_{[0,3]}$	4	$\mathbb{Q}(\sqrt{-1})$	[9]	2	St
$28A^3_{[0,1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[0, 5]	3	1
$28A^3_{[0,3]}$	2	$\mathbb{Q}(\sqrt{-6})$	[0, 3]	3	1
$30A^3_{[0,1,2]}$	2	$\mathbb{Q}(\sqrt{2}, \sqrt{-17})$	[0, 2], [1, 0]	2	St
$31A^3_{[5]}$	6	$\mathbb{Q}(\sqrt{-3})$	[25]	3	1
$31A^3_{[15]}$	2	$\mathbb{Q}(\sqrt{-26})$	[15]	13	1
$33A^3_{[1,0]}$	2	$\mathbb{Q}(\sqrt{-11})$	[1, 0]	11	St
$35A^3_{[0,3]}$	2	$\mathbb{Q}(\sqrt{-5})$	[0, 3]	5	St
$35B^3_{[0,3]}$	2	$\mathbb{Q}(\sqrt{-5})$	[0, 3]	5	St
$35C^3_{[2,3]}$	2	$\mathbb{Q}(\sqrt{10}, \sqrt{-1})$	[0, 3], [2, 3]	5	RPS
$36A^3_{[1,2]}$	6	$\mathbb{Q}(\sqrt{-3})$	[1, 4]	3	RPS
$38A^3_{[0,9]}$	2	$\mathbb{Q}(\sqrt{-2})$	[9]	2	St
$39C^3_{[1,6]}$	2	$\mathbb{Q}(\sqrt{-35}, \sqrt{3})$	[1, 6], [1, 0]	7	1

Table 6.2 Continued.

Label	ord(ϵ)	E	Extra twists	Ramification	Slope
$40A^3_{[0,0,1]}$	4	$\mathbb{Q}(\sqrt{-1})$	$[0,0,3]$	2	$a_p=0$
$42A^3_{[0,1,2]}$	6	$\mathbb{Q}(\sqrt{-2},\sqrt{6})$	$[0,4],[1,4]$	2	St
$45A^3_{[0,1]}$	4	$\mathbb{Q}(\sqrt{-1},\sqrt{10})$	$[3,0],[3,3]$	5	RPS
$45A^3_{[3,2]}$	2	$\mathbb{Q}(\sqrt{7},\sqrt{-2})$	$[0,2],[3,2]$	7	1
$47A^3_{[23]}$	2	$\mathbb{Q}(\sqrt{-78})$	$[23]$	13	1
$48A^3_{[1,0,0]}$	2	$\mathbb{Q}(\sqrt{-3})$	$[1,0,0]$	3	St
$50A^3_{[0,5]}$	4	$\mathbb{Q}(\sqrt{-1})$	$[15]$	2	St
$54A^3_{[0,9]}$	2	$\mathbb{Q}(\sqrt{-2})$	$[9]$	2	St
$55D^3_{[2,5]}$	2	$\mathbb{Q}(\sqrt{-21},\sqrt{5})$	$[2,0],[2,5]$	3	1
$56A^3_{[1,1,2]}$	6	$\mathbb{Q}(\sqrt{-3})$	$[1,1,4]$	2	RPS
$57A^3_{[0,9]}$	2	$\mathbb{Q}(\sqrt{-3})$	$[0,9]$	3	St
$60A^3_{[0,1,0]}$	2	$\mathbb{Q}(\sqrt{-5})$	$[0,1,0]$	5	St
$60A^3_{[0,1,2]}$	2	$\mathbb{Q}(\sqrt{-1},\sqrt{5})$	$[0,0,2],[0,1,0]$	5	RPS
$60A^3_{[1,0,2]}$	2	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[1,0,2],[1,0,0]$	3	St
$63E^3_{[0,1]}$	6	$\mathbb{Q}(\sqrt{-3},\sqrt{13})$	$[3,5],[0,5]$	3	$a_p=0$
$64A^3_{[1,8]}$	2	$\mathbb{Q}(\sqrt{-1},\sqrt{3})$	$[1,8],[1,0]$	3	1
$72A^3_{[0,0,3]}$	2	$\mathbb{Q}(\sqrt{-2})$	$[0,0,3]$	2	$a_p=0$
$72C^3_{[1,1,0]}$	2	$\mathbb{Q}(\sqrt{10},\sqrt{-6})$	$[0,0,3],[1,1,0]$	2	RPS
$74A^3_{[0,9]}$	4	$\mathbb{Q}(\sqrt{-1})$	$[27]$	2	St
$74B^3_{[0,9]}$	4	$\mathbb{Q}(\sqrt{-1})$	$[27]$	2	St
$74C^3_{[0,9]}$	4	$\mathbb{Q}(\sqrt{-1})$	$[27]$	2	St
$75A^3_{[0,5]}$	4	$\mathbb{Q}(\sqrt{-1},\sqrt{6})$	$[0,15],[0,5]$	3	St
$75B^3_{[0,5]}$	4	$\mathbb{Q}(\sqrt{-1},\sqrt{6})$	$[0,15],[0,5]$	3	St
$75C^3_{[1,0]}$	2	$\mathbb{Q}(\sqrt{-11})$	$[1,0]$	11	1
$76A^3_{[0,9]}$	2	$\mathbb{Q}(\sqrt{-29})$	$[0,9]$	29	1
$77A^3_{[0,5]}$	2	$\mathbb{Q}(\sqrt{-7})$	$[0,5]$	7	St
$77A^3_{[2,5]}$	6	$\mathbb{Q}(\sqrt{-3},\sqrt{21})$	$[4,5],[2,0]$	7	RPS
$78A^3_{[0,1,6]}$	2	$\mathbb{Q}(\sqrt{2},\sqrt{-5})$	$[0,6],[1,0]$	2	St
$78B^3_{[0,1,6]}$	2	$\mathbb{Q}(\sqrt{2},\sqrt{-5})$	$[0,6],[1,0]$	2	St

Table 6.2 Continued.

Label	ord(ϵ)	E	Extra twists	Ramification	Slope
$80B^3_{[1,0,2]}$	2	$\mathbb{Q}(\sqrt{2}, \sqrt{-3})$	[0, 0, 2], [1, 0, 0]	2	$a_p = 0$
$84A^3_{[0,0,1]}$	6	$\mathbb{Q}(\sqrt{-3})$	[0, 0, 5]	3	St
$84A^3_{[0,0,3]}$	2	$\mathbb{Q}(\sqrt{-3})$	[0, 0, 3]	3	St
$84B^3_{[0,1,2]}$	6	$\mathbb{Q}(\sqrt{-3}, \sqrt{15})$	[0, 0, 2], [0, 1, 4]	5	1
$84C^3_{[0,1,2]}$	6	$\mathbb{Q}(\sqrt{-3}, \sqrt{105})$	[0, 0, 4], [0, 1, 4]	5	1
$84A^3_{[1,0,2]}$	6	$\mathbb{Q}(\sqrt{-3})$	[1, 0, 4]	3	St
$86A^3_{[0,21]}$	2	$\mathbb{Q}(\sqrt{-2})$	[21]	2	St
$90A^3_{[0,0,1]}$	4	$\mathbb{Q}(\sqrt{-1})$	[0, 3]	2	St
$90B^3_{[0,0,1]}$	4	$\mathbb{Q}(\sqrt{-1})$	[0, 3]	2	St
$90A^3_{[0,3,2]}$	2	$\mathbb{Q}(\sqrt{2}, \sqrt{-1})$	[0, 2], [3, 2]	2	St
$91A^3_{[3,4]}$	6	$\mathbb{Q}(\sqrt{-3}, \sqrt{39})$	[0, 4], [3, 0]	13	RPS
$91C^3_{[3,6]}$	2	$\mathbb{Q}(\sqrt{26}, \sqrt{-1})$	[3, 0], [3, 6]	13	RPS
$93A^3_{[0,15]}$	2	$\mathbb{Q}(\sqrt{-3})$	[0, 15]	3	St
$96A^3_{[0,0,1]}$	2	$\mathbb{Q}(\sqrt{-2}, \sqrt{3})$	[0, 0, 1], [1, 0, 0]	3	RPS
$96B^3_{[0,0,1]}$	2	$\mathbb{Q}(\sqrt{7}, \sqrt{-2})$	[0, 0, 1], [1, 0, 1]	7	1
$99C^3_{[0,5]}$	2	$\mathbb{Q}(\sqrt{-138}, \sqrt{-3})$	[3, 0], [0, 5]	23	1
$100B^3_{[0,5]}$	4	$\mathbb{Q}(\sqrt{-1}, \sqrt{6})$	[0, 15], [0, 5]	3	3

The encoding of the nebentypus $\epsilon : (\mathbb{Z}/N)^* \to \mathbb{C}^*$ is done as follows. Let $N = \prod p_n^{\alpha_n}$ be the prime-ordered factorization of N. Then for each p_n, there exists a unique Dirichlet character $\epsilon_{p_n} : (\mathbb{Z}/p_n^{\alpha_n})^\times \to \mathbb{C}^*$ such that $\epsilon = \prod \epsilon_{p_n}$. Fix $p = p_n$ momentarily and write ϵ_p for ϵ_{p_n}. If p is odd, let g_p be the smallest positive integer that generates $(\mathbb{Z}/p^\alpha)^\times$, and if $p = 2$ and $\alpha \leq 2$, let $g_p = -1$. In the above cases ϵ_p is determined by the integer $e_p \in [0, \varphi(p^\alpha))$ such that $\epsilon_p(g_p) = e^{2\pi i e_p/\varphi(p^\alpha)}$. If $p = 2$ and $\alpha > 2$, then $(\mathbb{Z}/2^\alpha)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{\alpha-2}$ where the first factor is generated by -1 and the second factor by 5. Thus in this case ϵ_2 is determined by a pair of integers $e_2' \in [0, 2)$, $e_2'' \in [0, 2^{\alpha-2})$ such that $\epsilon_2(-1) = e^{2\pi i e_2'/2}$ and $\epsilon_2(5) = e^{2\pi i e_2''/2^{\alpha-2}}$. We denote the pair e_2', e_2'' by e_2. Finally we denote ϵ by $[e_{p_n} : p_n \mid N]$.

The middle columns are as follows: column 2 contains the order of ϵ, column 3 contains the Hecke field E (recall that $F = \mathbb{Q}$), column 4 lists a generating set for the extra

Table 6.3 Modular QM-motives of weight 4 and level $\leq$ 100.

Label	ord(ϵ)	E	Extra twists	Ramification	Slope
$12A^4_{[1,1]}$	2	$\mathbb{Q}(\sqrt{-5},\sqrt{3})$	[0, 1], [1, 0]	3 5	RPS 1
$21B^4_{[1,3]}$	2	$\mathbb{Q}(\sqrt{-6},\sqrt{102})$	[1, 3], [0, 3]	3 17	RPS 1
$27C^4_{[0]}$	1	$\mathbb{Q}(\sqrt{2})$	[9]	2 3	1 $a_p = 0$
$35A^4_{[1,3]}$	4	$\mathbb{Q}(\sqrt{5},\sqrt{-1})$	[0, 3], [3, 0]	2 5	3 RPS
$36B^4_{[1,3]}$	2	$\mathbb{Q}(\sqrt{30},\sqrt{-2})$	[0, 3], [1, 0]	2 3	RPS $a_p = 0$
$48B^4_{[1,0,1]}$	2	$\mathbb{Q}(\sqrt{-2},\sqrt{6})$	[1, 0, 0], [1, 0, 1]	2 3	$a_p = 0$ RPS
$56B^4_{[1,1,3]}$	2	$\mathbb{Q}(\sqrt{-3},\sqrt{21})$	[0, 0, 3], [1, 1, 3]	3 7	1 RPS
$57B^4_{[1,9]}$	2	$\mathbb{Q}(\sqrt{17},\sqrt{-10})$	[1, 9], [0, 9]	5 17	1 1
$63B^4_{[3,3]}$	2	$\mathbb{Q}(\sqrt{-222},\sqrt{-2})$	[3, 0], [0, 3]	2 3	3 $a_p = 0$
$72C^4_{[0,1,0]}$	2	$\mathbb{Q}(\sqrt{22},\sqrt{-10})$	[0, 0, 3], [0, 1, 0]	2 5	RPS 1
$80B^4_{[1,0,1]}$	4	$\mathbb{Q}(\sqrt{-1},\sqrt{35})$	[1, 0, 0], [1, 0, 3]	2 7	$a_p = 0$ 1
$100D^4_{[1,5]}$	4	$\mathbb{Q}(\sqrt{11},\sqrt{-1},\sqrt{5})$	[1, 10], [0, 5], [1, 5]	11 2	1 RPS

twists (encoded in a similar manner as described above for ϵ), and column 5 lists the primes where the endomorphism algebra X ramifies.

The last column lists the numbers m_ν (normalized slope) if p is prime to N and $a_p \neq 0$. If a (finite) integer occurs in this column, it is always *odd* as predicted by the main result of this paper (Theorem 2.2). If $a_p = 0$ and p is still prime to N, then $m_\nu = \infty$ and the ramification is controlled by Proposition 4.5. On the other hand if some $p \mid N$ is a prime of ramification, then we give some further information as follows. Recall that C

denotes the conductor of ϵ, and N_p and C_p denote the exponent of p of the exact power of p diving N and C, respectively. If $N_p = C_p$, we write RPS for ramified principal series. If $N_p = 1$ and $C_p = 0$, then we write St for Steinberg. Finally a_p vanishes in the remaining cases $N_p \neq C_p$. These include the cases which are twists of previous cases, in which case the ramification can be sometimes explained, but also includes the cases where the local representation is supercuspidal. In either case we simply write $a_p = 0$. It is worth mentioning that while a_p can vanish both when $p \nmid N$ and $p \mid N$, in the former case it only occurs once (see $95B_{[1,3]}$ in Table 6.1) within the scope of the tables.

Corrections to [2]. We take this opportunity to correct some errors in [2]. The last two errors do not occur in the electronic version of the paper.

(i) In page 1655, line 8, $N_p \geq C_p \geq 1$ should be $N_p \geq C_p \geq 0$.

(ii) In page 1669, line 12, $(1/2)(u^2 - 1)$ should be $(u^2 - 1)/8$.

(iii) In the last few lines of page 1670, "supersingular primes" should read "primes p for which $a_p = 0$," and the last phrase should read "for non-CM forms the density of primes p for which $a_p = 0$ is 0".

(iv) In page 1672, line 9 from the bottom, "ordinary primes" should read "primes p for which $a_p = 0$".

Acknowledgments

The first author would like to thank N. Fakhruddin, M. Nori, and V. Srinivas for useful discussions concerning the material in Section 3. He would also like to thank the third author for arranging visits to Barcelona in 2002 and 2004. The second author held the position of *Contratado Doctor C.A.M. (02/0294/2002)* during the preparation of this work. The third author is supported by research projects BFM-2003-0678-C02-01 and 2002SGR 00148. The tables in this paper were prepared using W. Stein's Magma modular forms package.

References

[1] M. Baker, E. González-Jiménez, J. González, and B. Poonen, *Finiteness results for modular curves of genus at least* 2, to appear in Amer. J. Math.

[2] A. Brown and E. Ghate, *Endomorphism algebras of motives attached to elliptic modular forms*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 6, 1615–1676.

[3] P. Deligne, *Hodge cycles on abelian varieties*, Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math., vol. 900, Springer-Verlag, New York, 1982, pp. 9–100.

[4] U. Jannsen, *Mixed Motives and Algebraic* K*-Theory*, Lecture Notes in Mathematics, vol. 1400, Springer-Verlag, Berlin, 1990.

[5] J. Milne, *Étale Cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, New Jersey, 1980.

[6] F. Momose, *On the* ℓ*-adic representations attached to modular forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 1, 89–109.

[7] J. Quer, *La classe de Brauer de l'algèbre d'endomorphismes d'une variété abélienne modulaire* [*The Brauer class of the endomorphism algebra of a modular abelian variety*], C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 3, 227–230 (French).

[8] K. Ribet, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. **253** (1980), no. 1, 43–62.

[9] ———, *Endomorphism algebras of abelian varieties attached to newforms of weight* 2, Seminar on Number Theory (Paris, 1979–1980), Progr. Math., vol. 12, Birkhäuser Boston, Massachusetts, 1981, pp. 263–276.

[10] ———, *Abelian varieties over* $\mathbf{Q}$ *and modular forms*, Algebra and Topology (Taejŏn, 1992), Korea Advanced Institute of Science and Technology, Taejŏn, 1992, pp. 53–79, reprinted in Modular Curves and Abelian Varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 241–261.

[11] A. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), no. 2, 419–430.

[12] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979.

[13] ———, *Quelques applications du théorème de densité de Chebotarev* [*Some applications of the Chebotarev density theorem*], Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401 (French).

[14] G. Shimura, *Class fields over real quadratic fields and Hecke operators*, Ann. of Math. (2) **95** (1972), 130–190.

Eknath Ghate: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
E-mail address: eghate@math.tifr.res.in

Enrique González-Jiménez: Departamento de Matemáticas, Facultad de Ciencias, Universidad de Alcalá, 28871 Madrid, Spain
E-mail address: enrique.gonzalezj@uah.es

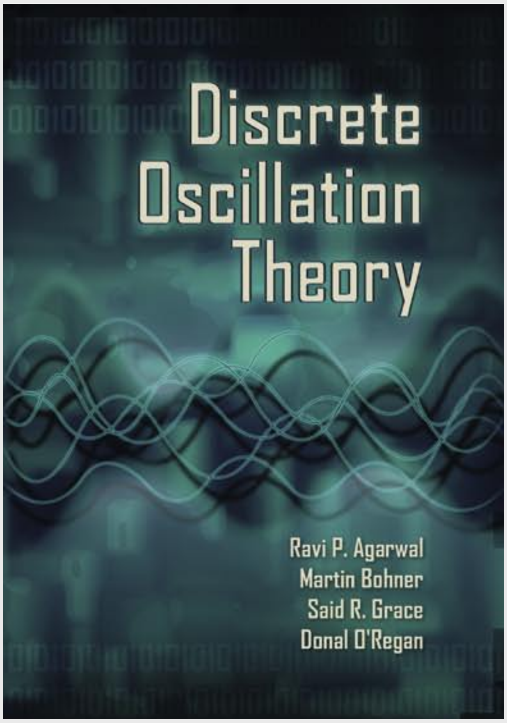
Jordi Quer: Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Jordi Girona 1-3, Campus Nord, Edifici Omega, Despatx 438, 08034 Barcelona, Spain
E-mail address: jordi.quer@upc.es

Contemporary Mathematics and Its Applications

CMIA Book Series, Volume 1, ISBN: 977-5945-19-4

DISCRETE OSCILLATION THEORY

Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan



This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.

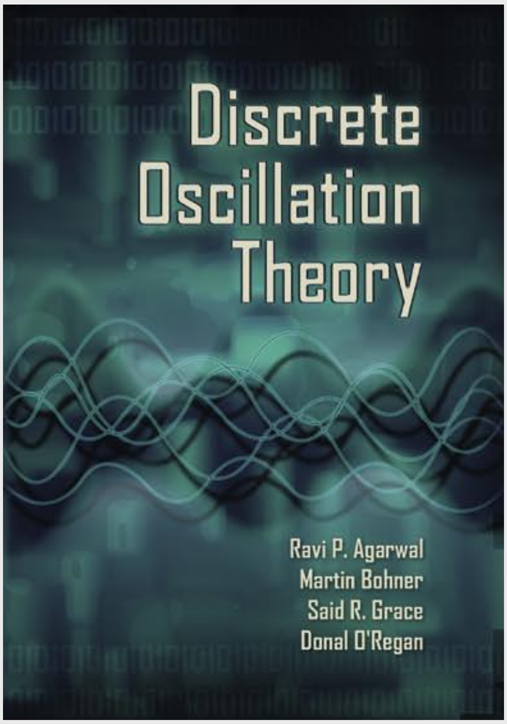
For more information and online orders please visit http://www.hindawi.com/books/cmia/volume-1
For any inquires on how to order this title please contact books.orders@hindawi.com

"Contemporary Mathematics and Its Applications" is a book series of monographs, textbooks, and edited volumes in all areas of pure and applied mathematics. Authors and/or editors should send their proposals to the Series Editors directly. For general information about the series, please contact cmia.ed@hindawi.com.

CMIA Book Series, Volume 1, ISBN: 977-5945-19-4

DISCRETE OSCILLATION THEORY

Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan



This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.

For more information and online orders please visit http://www.hindawi.com/books/cmia/volume-1
For any inquires on how to order this title please contact books.orders@hindawi.com

"Contemporary Mathematics and Its Applications" is a book series of monographs, textbooks, and edited volumes in all areas of pure and applied mathematics. Authors and/or editors should send their proposals to the Series Editors directly. For general information about the series, please contact cmia.ed@hindawi.com.