Boundary behavior of optimal approximants

Catherine Bénéteau, Myrto Manolaki and Daniel Seco

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Spaces over the disc

Definition

Dirichlet-type space, D_α, is:

$$\{f \in Hol(\mathbb{D}) : f(z) = \sum_{k\in\mathbb{N}} a_k z^k, ||f||_\alpha^2 = \sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha < \infty\}$$

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$\alpha = 1$, $\mathcal{D} = Hol(\mathbb{D}) \cap \{A(f(\mathbb{D})) < \infty\}$

Cyclicity and invariant subspaces

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In other spaces, much known but still to be understood.

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- BFKSS: When f not cyclic, $p_n f \to I\overline{I(0)}$, I "*inner part of f*".

Solution

We solved these optimization problems:

Theorem (BCLSS, JdAM'15; FMS, CMFT'14)

$p_n^*(z) = \sum_{j=0}^{n} c_j z^j$ *only solution to* $Mc = b$ *where*

$$c = (c_j)_{j=0}^{n}, \quad M_{j,k} =< z^j f, z^k f >_\alpha, \quad b_k =< 1, z^k f >_\alpha .$$

Applications to OPs

Later we discovered a relation with OPs: Let ϕ_j of degree j defined by:

$$\left\langle \phi_j f, \phi_k f \right\rangle_\omega = \delta_{j,k},$$

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Then we can obtain ϕ_j from p_j and p_{j-1} since:

Theorem (BKLSS, JLMS'16)

$$p_n(z) = \overline{f(0)} \sum_{k=0}^{n} \overline{\phi_k(0)} \phi_k(z)$$

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YES, if f polynomial.

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- Additional restrictions:

$$(1 - p_n f)(1) = (1 - p_n f)(2) = 1$$

A general closed formula

Theorem

$\exists A_n = (A_{1,n}, ..., A_{d,n})^*$ (ind. of k): for $k = 0, ..., n + d$,

$$d_{k,n} = \frac{1}{\omega_k} \sum_{i=1}^{d} A_{i,n} \overline{z_i^k}. \tag{1}$$

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So inverting a $d \times d$ matrix we can obtain a closed formula for all n.
Also, for p_n and hence for ϕ_k.

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$$dist^2(1, \mathcal{P}_n f) = -\sum_{i=1}^{d} A_{i,n} = v_0 E_{Z,n}^{-1} v_0^*.$$

In particular,

$$\sum_{i=1}^{d} A_{i,n} \in [-1, 0].$$

Also, if $Z(f) \subset \mathbb{D}$*, then*

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Notice $E_{Z,\infty,l,m} = k_{z_m}(z_l)$.

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Perhaps, true if $f \in A(\mathbb{T})$?

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To be continued...
Coming up work BMS and Ivrii


END OF TRAIL