CONTROLLABILITY OF THE LINEAR SYSTEM OF THERMOELASTICITY

by

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Abstract. We prove that the linear system of thermoelasticity is controllable in the following sense: If the control time is large enough and we act in the equations of displacement by means of a control supported in a neighborhood of the boundary of the thermoelastic body, then we may control exactly the displacement and simultaneously the temperature in an approximate way. The method of proof combines: (i) A decoupling result for the system of thermoelasticity due to D. Henry, O. Lopes and A. Perissinitto, (ii) the variational approach to controllability developed recently by C. Fabre, J. P. Puel and the author and (iii) some new observability inequalities for the system of thermoelasticity.

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1. INTRODUCTION AND MAIN RESULTS

Let us consider an isotropic and homogeneous thermoelastic body occupying an open and bounded set Ω of $\mathbb{R}^n (n \geq 1)$ with boundary $\Gamma = \partial\Omega$ of class C^2.

We denote by $x = (x_1, \ldots, x_n)$ a point of Ω while t stands for the time variable. The displacement-vector is denoted by $u = (u_1, \ldots, u_n)(u_i = u_i(x,t), i = 1, .., n)$ and the temperature by $\theta = \theta(x,t)$.

In the absence of exterior forces and heat sources the linear system of thermoelasticity is as follows:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \text{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega \times (0,\infty) \\ \theta_t - \Delta\theta + \beta \text{div} u_t = 0 & \text{in } \Omega \times (0,\infty) \\ u = \theta = 0 & \text{on } \Gamma \times (0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\mu, \lambda > 0$ are Lamé's constants and $\alpha, \beta > 0$ the coupling parameters.

It is well known that system (1.1) is well-posed in

$$H = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega).$$

More precisely, for every initial data $(u^0, u^1, \theta^0) \in H$ there exists a unique solution

$$(u, u_t, \theta) \in C([0, \infty); H).$$

This solution is given by

$$(u(t), u_t(t), \theta(t)) = S(t)(u^0, u^1, \theta^0), t > 0$$

where $S(t) : H \to H, t > 0$, is the strongly continuous semigroup generated by system (1.1). We will denote by $S_i(t), i = 1, 2, 3$ the three components of $S(t)$.

On the other hand, the energy

$$E(t) = \frac{1}{2}\int_\Omega \left[\mid u_t \mid^2 + \mu \mid \nabla u \mid^2 + (\lambda + \mu) \mid \mathrm{div} u \mid^2 + \frac{\alpha}{\beta}\theta^2 \right] dx \tag{1.2}$$

satisfies

$$\frac{dE(t)}{dt} = -\frac{\alpha}{\beta}\int_\Omega \mid \nabla\theta \mid^2 dx$$

We fix a control time $T > 0$ and a control region ω; an open and non-empty subset of Ω. We are allowed to act on the system through the equations of displacement by means of a control function

$$f = f(x, t) \in (L^2(\Omega \times (0, T)))^n$$

that represents an exterior force. The support of the control is restricted to the control region ω. In the sequel, by χ_ω we denote the characteristic function of the set ω.

The thermoelastic system in the presence of the control f reads as follows

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \alpha\nabla\theta = f\chi_\omega & \text{in } \Omega \times (0, T) \\ \theta_t - \Delta\theta + \beta \mathrm{div} u_t = 0 & \text{in } \Omega \times (0, T) \\ u = \theta = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u^0,\ u_t(0) = u^1, \theta(0) = \theta^0. \end{cases} \tag{1.3}$$

It is well known that system (1.3) possesses an unique solution $(u, u_t, \theta) \in C([0, T]; H)$ given in terms of the semigroup S by the variation of constants formula.

The first controllability problem one may consider is the exact controllability one: Find sufficient conditions on (ω, T) (control region and control time) such that for every initial and final data

$$(u^0, u^1, \theta^0), (v^0, v^1, \eta^0) \in H$$

there exists a control

$$f \in (L^2(\Omega \times (0, T)))^n,$$

such that the solution of (1.3) satisfies

$$u(T) = v^0, u_t(T) = v^1, \theta(T) = \eta^0.$$

Due to the irreversibility of the system of thermoelasticity and the regularizing effect of the heat equation that the temperature satisfies this exact controllability property may not hold. Therefore, it is natural to relax the problem to the following exact-approximate controllability one:Given $(u^0, u^1, \theta^0), (v^0, v^1, \eta^0) \in H$ and $\varepsilon > 0$, to find a control f such that the solution of (1.3) satisfies

$$\begin{cases} u(T) = v^0, u_t(T) = v^1 \\ \|\theta(T) - \eta^0\|_{L^2(\Omega)} \leq \varepsilon. \end{cases} \tag{1.4}$$

In other words, we request the exact controllability of the displacement and the approximate controllability of the temperature.

In the sequel, if this property holds we will say that system (1.3) is exact-approximately controllable.

We have the following result.

Theorem 1. *Let ω be a neighborhood of the boundary Γ in Ω, i. e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of Γ in $I\!R^n$. Suppose that $T > diam(\Omega \setminus \omega)/\sqrt{\mu}$. Then, system (1.3) is exact-approximately controllable in time T.*

One of the main ingredients of the proof of Theorem 1 is the following observability inequality for the adjoint system of thermoelasticity:

$$\begin{cases} \varphi_{tt} - \mu\Delta\varphi - (\lambda + \mu)\nabla \text{div } \varphi + \beta\nabla\psi_t = 0 & \text{in } \Omega \times (0, T) \\ -\psi_t - \Delta\psi - \alpha \text{div}\varphi = 0 & \text{in } \Omega \times (0, T) \\ \varphi = 0, \psi = 0 & \text{on } \partial\Omega \times (0, T) \\ \varphi(x, T) = \varphi^0(x), \varphi_t(x, T) = \varphi^1(x) & \text{in } \Omega \\ \psi(x, T) = \psi^0(x) & \text{in } \Omega \end{cases} \tag{1.5}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is the adjoint displacement variable and ψ the temperature.

Proposition 1. *Under the assumptions of Theorem 1, for every bounded set B of $L^2(\Omega)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_\omega |\varphi|^2 dxdt \tag{1.6}$$

holds for every solution of (1.5) with initial data such that

$$\|(\varphi^0, \varphi^1 + \beta\nabla\psi^0)\|_{(L^2(\Omega))^n \times (H^{-1}(\Omega))^n} \geq 1, \psi^0 \in B.$$

By suitably adapting the methods developed by J. L. Lions [Li3] and C. Fabre, J. P. Puel and the author in [FPuZ1,2,3,4] we will show that Theorem 1 is a consequence of Proposition 1 and Hölmgren's Uniqueness Theorem. To prove Proposition 1 we will combine multiplier techniques, compactness arguments, Hölmgren's Uniqueness Theorem and the following deep result due to D. Henry, O. Lopes and A. Perissinitto [HeLP]:

Theorem A [HeLP]. *Let P be the orthogonal projection from $(L^2(\Omega))^n$ into $F = \{\nabla\varphi : \varphi \in H_0^1(\Omega)\}$ and let us denote by $\{S^0(t)\}_{t\geq 0}$ the strongly continuous semigroup in H associated to the following decoupled system*

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\beta P u_t = 0 & \text{in } \Omega\times(0,\infty) \\ \theta_t - \Delta\theta + \beta \mathrm{div} u_t = 0 & \text{in } \Omega\times(0,\infty) \\ u = \theta = 0 & \text{on } \Gamma\times(0,\infty) \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega \end{cases} \tag{1.7}$$

Then, $S(t) - S^0(t) : H \to C([0,T]; H)$ is continuous and compact.

We will denote by $S_i^0(t), i = 1,2,3$ the three components of $S^0(t)$.

Let us consider now the one-dimensional problem. Suppose that $\Omega = (0,L) \subset \mathbb{R}$ and $\omega = (\ell_1, \ell_2) \subset (0,L)$. The system of thermoelasticity in the presence of a control f reeds now as follows:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = f\chi_\omega, \quad 0<x<L, \quad 0<t<T \\ \theta_t - \theta_{xx} + \beta u_{xt} = 0, \quad 0<x<L, \quad 0<t<T \\ u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad 0<t<T \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), \theta(x,0) = \theta^0(x), 0<x<L \end{cases} \tag{1.10}$$

Notice that u is now a scalar-valued function.

In this one-simensional case we have the following result:

Theorem 2. *Suppose that $T > 2\max(\ell_1, L-\ell_2)$, then system* (1.10) *is exact-approximately controllable at time T.*

The proof of Theorem 2 is simpler since the decoupling operator P of Theorem A is now:

$$P\varphi = \varphi - \frac{1}{L}\int_0^L \varphi dx, \qquad \forall\varphi \in L^2(0,L).$$

Let us compare our results with those existing in the literature. To our knowledge, the first results on controllability of thermoelastic systems are due to K. Narukawa [*N*]. But in [*N*] as well as in the more recent works by J. Lagnese [La], J. Lagnese and J. L. Lions [LaLi] and J. L. Lions [Li2], only "partial controllability" results are established. More precisely, in these works various models of thermoelasticity are considered and it is proved that, when the coupling parameters are small enough, one may control exactly the displacement by means of one control acting in the equations of displacement but nothing is said about the controllability of the temperature. The results of the present paper prove that, without any restriction on the size of the coupling parameters, in addition to the exact controllability of the displacement, one may achieve the approximate controllability of the temperature by means of one sole control. More recently, S. Hansen [*H*] has proved the null controllability of the system of the thermoelasticity in one space dimension by means of one sole boundary-control, for various boundary conditions and without restrictions in the size of the coupling parameters. In [*H*] the controllability of both displacement and temperature is proved. The methods of $[H]$ are based on moment problems and nonharmonic Fourier series and they do not seem to extend to several space dimensions. The extension of Hansen's results to several space dimensions (i. e. boundary control problem) remains to be done.

The rest of the paper is organized as follows. In Section 2 we will give some consequences of Hölmgren's Uniqueness Theorem that we will used in the proof of our results. In Section 3 we prove the observability inequality of Proposition 1. In Section 4 we proof the controllability result (Theorem 1). In Section 5 we address the one-dimensional problem. In Section 6 we comment some possible extensions of the results of this paper and some open problems. Finally in an Appendix, for the sake of completeness, we present a brief sketch of the proof of Theorem A of [HeLP]

2. UNIQUENESS RESULTS

We have the following preliminary lemma.

Lemma 2.1. *Let (u,θ) be a solution of the system of thermoelasticity in the set*

$$\mathcal{A}=\bigcup_{0<s<T/2}\{B(0,1+\sqrt{\mu}s)\times(s,T-s)\}.$$

Suppose that there exists a vector $e\in\mathbb{R}^n$ and a real number $c\in\mathbb{R}$ such that $(u,\theta)=(e,c)$ in $B(0,1)\times(0,T)$. Then $(u,\theta)=(e,c)$ in $\mathcal{A}$.

Remark. By $B(x_0,r)$ we denote the ball of radious r centered at $x_0\in\mathbb{R}^n$. This lemma shows that the "level sets" of solutions of the system of thermoelasticity expand as fast as in the scalar wave equation $v_{tt}-\mu\Delta v=0$.

By translation invariance (in space and time) and by scaling we deduce that if $(u,\theta)=(e,c)$ in $B(x_0,\rho)\times(t_1,t_2)$ then $(u,\theta)=(e,c)$ in

$$\bigcup_{0<s<(t_2-t_1)/2}\{B(x_0,\rho+\sqrt{\mu}s)\times(t_1+s,t_2-s)\}. \quad \blacksquare$$

Proof of Lemma 2.1. Without loss of generality we may assume that $e=0$ and $c=0$.

We observe that v=curl u satisfies

$$v_{tt}-\mu\Delta v=0.$$

Then, by Hölmgren's Uniqueness Theorem (see, for instance, J. L. Lions [Li1]) we deduce that $v=0$ in $\mathcal{A}$. Therefore, there exists a scalar function $\Phi=\Phi(x,t)$ such that

$$u=\nabla\Phi \text{ in } \mathcal{A}. \tag{2.1}$$

Now we observe that $w=\text{div}u$ and θ satisfy

$$\begin{cases} w_{tt}-(\lambda+2\mu)\Delta w+\alpha\Delta\theta=0 & \text{in } \mathcal{A}\\ \theta_t-\Delta\theta+\beta w_t=0 & \text{in } \mathcal{A}.\end{cases} \tag{2.2}$$

We apply to system (2.2) an argument due to A. Haraux [Ha].

Let us denote by φ_k and ρ_k the eigenfunctions and eigenvalues of the Laplace-Beltrami operator on the sphere S^n of $\mathbb{R}^n$. Multiplying in (2.2) by φ_k and integrating on S^n we deduce that $w_k = w_k(r,t) = \int_{S^n} w\varphi_k d\sigma$ and $\theta_k = \theta_k(r,t) = \int_{S^n} \theta\varphi_k d\sigma$ satisfy

$$\begin{cases} w_{k,tt} - (\lambda+2\mu)\left[w_{k,rr} + \dfrac{n-1}{r}w_{k,r} - \dfrac{\rho_k}{r^2}w_k\right] + \alpha\left[\theta_{k,rr} + \dfrac{n-1}{r}\theta_{k,r} - \dfrac{\rho_k}{r^2}\theta_k\right] = 0 \\ \theta_{k,t} - \left[\theta_{k,rr} + \dfrac{n-1}{r}\theta_{k,r} - \dfrac{\rho_k}{r^2}\theta_k\right] + \beta w_{k,t} = 0. \end{cases} \tag{2.3}$$

in

$$\mathcal{B} = \bigcup_{0<s<T/2} \{(0, 1+\sqrt{\mu}s) \times (s, T-s)\}.$$

Since $(w_k, \theta_k) = (0,0)$ for $0 \le r < 1$ and $0 < t < T$ it is sufficient to consider the system (2.3) in the region

$$\tilde{\mathcal{B}} = \bigcup_{0<s<T/2} \{(1/2, 1+\sqrt{\mu}s) \times (s, T-s)\}$$

where its coefficients are analytic.

The characteristic lines of system (2.3) are as follows

$$\begin{cases} t = \text{constant}, \\ t = \pm\dfrac{r+r_0}{\sqrt{\lambda+2\mu}}. \end{cases}$$

They correspond to the underlying heat and wave equations of system (2.3), respectively.

By Hölmgren's Uniqueness Theorem (see F. John [J]) we deduce that $(w_k, \theta_k) = (0,0)$ in $\tilde{\mathcal{B}}$ (and, thus, in $\mathcal{B}$) for all $k \in \mathbb{N}$. Thus

$$(\text{div} u, \theta) = (w, \theta) = (0,0) \text{ in } \mathcal{A}. \tag{2.4}$$

Therefore, we already have $\theta = 0$ in $\mathcal{A}$. But, on the other hand, by (2.1) and (2.4):

$$w = \text{div} u = \Delta\Phi = 0 \text{ in } \mathcal{A}. \tag{2.5}$$

Since $u = \nabla\Phi$ and $u = 0$ in $B(0,1) \times (0,T)$ we have that $\Phi = f(t)$ in $B(0,1) \times (0,T)$. In virtue of (2.5) we deduce that $\Phi = f(t)$ in $\mathcal{A}$, but then $u = \nabla\Phi = 0$ in $\mathcal{A}$. ∎

As an immediate consequence of Lemma 2.1 we have the following result.

Corollary 2.2. *Suppose that $u = 0$ in $B(0,1) \times (0,T)$. Then, there exists some $c \in \mathbb{R}$ such that $(u, \theta) = (0, c)$ in $\mathcal{A}$.*

Proof: Note that since $u = 0$ in $B(0,1) \times (0,T)$ then, by the equations of displacement and temperature:

$$\nabla\theta = \theta_t - \Delta\theta = 0 \text{ in } B(0,1) \times (0,T).$$

This implies the existence of $c \in \mathrm{I\!R}$ such that $\theta = c$ in $B(0,1) \times (0,T)$. The corollary is now a direct consequence of Lemma 2.1. ∎

Now we state the main uniqueness result for the system of thermoelasticity:

Proposition 2.3. *Suppose that* $T > \mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}$. *Let* (u, θ) *be a solution of the Dirichlet problem for the system of thermoelasticity such that* $u = 0$ *in* $\omega \times (0,T)$. *Then* $u \equiv \theta \equiv 0$ *in* $\Omega \times (0,T)$.

Proof. In view of Corollary 2.2 (see J. L. Lions [Li1], Chapter 1) we deduce the existence of $c \in \mathrm{I\!R}$ such that

$$(u, \theta) = (0, c) \text{ in } \Omega \times (\mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}, T - \mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}).$$

Since $\theta = 0$ on $\Gamma \times (0,T)$ we deduce that $c = 0$.

By forward uniqueness we obtain that

$$(u, \theta) = (0, 0) \text{ in } \Omega \times (\mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}, T).$$

By backward uniqueness (see J. L. Lions and B. Malgrange [LiM]) we deduce that

$$(u, \theta) = (0, 0) \text{ in } \Omega \times (0, \mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}).$$

This concludes the proof of the proposition. ∎

In the one-dimensional frame of Theorem 2 we have the following uniqueness result:

Proposition 2.4. *Suppose that* $T > 2\max(\ell_1, L - \ell_2)$. *Let* (u, θ) *be a solution of* (1.10) *with* $f \equiv 0$ *such that* $u = 0$ *in* $(\ell_1, \ell_2) \times (0,T)$. *Then* $u \equiv \theta \equiv 0$ *in* $(0, L) \times (0,T)$.

Let us now consider the adjoint system:

$$\begin{cases} \varphi_{tt} - \mu \Delta \varphi - (\lambda + \mu) \nabla \mathrm{div} \varphi + \beta \nabla \sigma_t = 0 & \text{in } \Omega \times (0,T) \\ -\sigma_t - \Delta \sigma - \alpha \mathrm{div} \varphi = 0 & \text{in } \Omega \times (0,T) \\ \varphi = \sigma = 0 & \text{on } \Gamma \times (0,T). \end{cases} \tag{2.6}$$

The following uniqueness result will be one of the main ingredients of the proof of the controllability result:

Proposition 2.5. *Suppose that* $T > \mathrm{diam}(\Omega \setminus \omega)/\sqrt{\mu}$. *Let* (φ, σ) *be a solution of system* (2.6) *such that* $\varphi = 0$ *in* $\omega \times (0,T)$. *Then* $\varphi \equiv \sigma \equiv 0$ *in* $\Omega \times (0,T)$.

Proof: Reversing the time variable we get

$$\begin{cases} \tilde{\varphi}_{tt} - \mu \Delta \tilde{\varphi} - (\lambda + \mu) \nabla \mathrm{div} \tilde{\varphi} - \beta \nabla \tilde{\sigma}_t = 0 & \text{in } \Omega \times (0,T) \\ \tilde{\sigma}_t - \Delta \tilde{\sigma} - \alpha \mathrm{div} \tilde{\varphi} = 0 & \text{in } \Omega \times (0,T) \\ \tilde{\varphi} = \tilde{\sigma} = 0 & \text{on } \Gamma \times (0,T) \end{cases}$$

where $\tilde{\varphi}(x,t) = \varphi(x, T-t)$ and $\tilde{\sigma}(x,t) = \sigma(x, T-t)$.

Note that $(\tilde{\varphi}, \tilde{\sigma}_t)$ are solutions of the system of thermoelasticity (1.3) where α (resp. β) has been replaced by $-\beta$ (resp.$-\alpha$). Thus, as a consequence of Proposition 2.3 we get $\tilde{\varphi} = 0$ and $\tilde{\sigma}_t = 0$. On the other hand, it is easy to see that if $\tilde{\varphi} = 0$ and $\tilde{\sigma} = \tilde{\sigma}(x)$ then, necessarily, $\tilde{\sigma} = 0$. This concludes the proof of this proposition. ∎

Let us finally consider the adjoint system in one space dimension:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \beta\sigma_{xt} = 0 & \text{in } (0,L) \times (0,T) \\ -\sigma_t - \sigma_{xx} - \alpha\varphi_x = 0 & \text{in } (0,L) \times (0,T) \\ \varphi(0,t) = \varphi(L,t) = \sigma(0,t) = \sigma(L,t) = 0 & \text{for } t \in (0,T) \end{cases} \tag{2.7}$$

The corresponding uniqueness result is as follows:

Proposition 2.6. *Suppose that $T > 2\max(\ell_1, L - \ell_2)$. Then, if (φ, σ) solves (2.7) and $\varphi = 0$ in $(\ell_1, \ell_2) \times (0,T)$ we have $(\varphi, \sigma) = (0,0)$ in $(0,L) \times (0,T)$.*

Remark. In the statements above we have not made any regularity assumption on the solutions. These results apply to any weak solutions in the sense of distributions (of course, the boundary conditions, if any, have to be incorporated in the weak formulation).

In the sequel we will apply these uniqueness results to solutions of the adjoint system of thermoelasticity with data (at time $t = T$) in $(L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times L^2(\Omega)$. ∎

3. THE OBSERVABILITY INEQUALITY

This section is devoted to the proof of Proposition 1. We will use the notation $\tilde{H} = (L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times L^2(\Omega)$

First we observe that if (φ, ψ) solves (1.5), then (ϕ, ψ), where

$$\phi(x,t) = -\int_t^T \varphi(x,s)\,ds + \chi(x)$$

with

$$\begin{cases} -\mu\Delta\chi - (\lambda+\mu)\nabla \text{div}\chi = -\varphi^1 - \beta\nabla\psi^0 & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega, \end{cases}$$

solves

$$\begin{cases} \phi_{tt} - \mu\Delta\phi - (\lambda+\mu)\nabla\text{div}\phi + \beta\nabla\psi = 0 & \text{in } \Omega \times (0,T) \\ -\psi_t - \Delta\psi - \alpha\text{div}\phi_t = 0 & \text{in } \Omega \times (0,T) \\ \phi = 0, \psi = 0 & \text{on } \partial\Omega \times (0,T) \\ \phi(x,T) = \chi(x), \phi_t(x,T) = \varphi^0(x), \psi(x,T) = \psi^0(x) & \text{in } \Omega \end{cases} \tag{3.1}$$

Taking into account that $\|\chi\|_{(H_0^1(\Omega))^n}$ and $\|\varphi^1 + \beta\nabla\psi^0\|_{(H^{-1}(\Omega))^n}$ are equivalent norms, we see that Proposition 1 is equivalent to the following one:

Proposition 3.1. *Under the assumptions of Theorem 1, for every bounded set B of $L^2(\Omega)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_\omega |\phi_t|^2 dxdt \tag{3.2}$$

holds for every solution of (3.1) *with initial data such that*

$$\|(\chi, \varphi^0)\|_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \geq 1, \psi^0 \in B. \tag{3.3}$$

Thus, it is sufficient to prove Proposition 3.1.

To prove Proposition 3.1, first we introduce the decoupled system associated to (3.1):

$$\begin{cases} \tilde{\phi}_{tt} - \mu\Delta\tilde{\phi} - (\lambda+\mu)\nabla \text{div}\tilde{\phi} - \alpha\beta P\tilde{\phi}_t = 0 & \text{in } \Omega \times (0,T) \\ -\tilde{\psi}_t - \Delta\tilde{\psi} - \alpha \text{div}\tilde{\phi}_t = 0 & \text{in } \Omega \times (0,T) \\ \tilde{\phi} = 0, \tilde{\psi} = 0 & \text{on } \partial\Omega \times (0,T) \\ \tilde{\phi}(x,T) = \chi(x), \tilde{\phi}_t(x,T) = \varphi^0(x), \tilde{\psi}(x,T) = \psi^0(x) & \text{in } \Omega \end{cases} \tag{3.4}$$

and consider the subsystem that $\tilde{\phi}$ satisfies:

$$\begin{cases} \tilde{\phi}_{tt} - \mu\Delta\tilde{\phi} - (\lambda+\mu)\nabla \text{div}\tilde{\phi} - \alpha\beta P\tilde{\phi}_t = 0 & \text{in } \Omega \times (0,T) \\ \tilde{\phi} = 0 & \text{on } \partial\Omega \times (0,T) \\ \tilde{\phi}(x,T) = \chi(x), \tilde{\phi}_t(x,T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{3.5}$$

We have the following observability inequality for system (3.5).

Proposition 3.2. *Suppose that* $T > \text{diam}(\Omega \setminus \omega)/\sqrt{\mu}$. *Then, there exists a constant* $C > 0$ *and a semi-norm* $X : (H_0^1(\Omega))^n \times (L^2(\Omega))^n \to I\!R^+$ *such that*

$$\|\chi\|^2_{(H_0^1(\Omega))^n} + \|\varphi^0\|^2_{(L^2(\Omega))^n} \leq C\Big[\int_0^T \int_\omega |\tilde{\phi}_t|^2 dxdt + X^2(\chi, \varphi^0)\Big] \tag{3.6}$$

holds for every solution of (3.5), $X : (H_0^1(\Omega))^n \times (L^2(\Omega))^n \to I\!R^+$ *being continuous and compact.*

The proof of this proposition will be given at the end of this section. Let us now conclude the proof of Proposition 3.1 by assuming that Proposition 3.2 holds.

We decompose the solution of (3.1) as $(\phi, \psi) = (\tilde{\phi}, \tilde{\psi}) + (\xi, \eta)$ where $(\tilde{\phi}, \tilde{\psi})$ solves (3.4) and (ξ, η) satisfies

$$\begin{cases} \xi_{tt} - \mu\Delta\xi - (\lambda+\mu)\nabla \text{div}\xi = -\alpha\beta P\tilde{\phi}_t - \beta\nabla\psi & \text{in } \Omega \times (0,T) \\ -\eta_t - \Delta\eta - \alpha \text{div}\xi_t = 0 & \text{in } \Omega \times (0,T) \\ \xi = 0, \eta = 0 & \text{on } \partial\Omega \times (0,T) \\ \xi(x,T) = \xi_t(x,T) = 0, \eta(x,T) = 0 & \text{in } \Omega \end{cases} \tag{3.7}$$

As a consequence of Proposition 3.2 we have

$$\|\chi\|^2_{(H_0^1(\Omega))^n} + \|\varphi^0\|^2_{(L^2(\Omega))^n} \leq C\Big[\int_0^T \int_\omega (|\phi_t|^2 + |\xi_t|^2) dxdt + X^2(\chi, \varphi^0)\Big]. \tag{3.8}$$

We argue by contradiction. Suppose that Proposition 3.1 does not hold. Then, there exists a bounded set B of $L^2(\Omega)$ and a sequence of initial data $(\chi_j, \varphi_j^0, \psi_j^0)$ with $\psi_j^0 \in B$ satisfying (3.3) such that

$$\int_0^T \int_\omega |\phi_{j,t}|^2 \mathrm{dxdt} \to 0 \text{ as } j \to \infty. \tag{3.9}$$

In view of (3.8) and taking into account (3.9) and that $\|(\chi_j, \varphi_j^0)\|_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \geq 1$ holds, we deduce that

$$\lim_{j\to\infty} \inf [\int_0^T \int_\omega |\xi_{j,t}|^2 \mathrm{dxdt} + X^2(\chi_j, \varphi_j^0)] > 0. \tag{3.10}$$

We introduce the normalized data

$$(\hat{\chi}_j^0, \hat{\varphi}_j^0, \hat{\psi}_j^0) = (\chi_j^0, \varphi_j^0, \psi_j^0)/[\|\xi_{j,t}\|_{(L^2(\omega\times(0,T)))^n} + X^2(\chi_j, \varphi_j^0)]^{1/2}$$

and the corresponding solutions $(\hat{\phi}_j, \hat{\psi}_j)$ and $(\hat{\xi}_j, \hat{\eta}_j)$ of (3.1) and (3.7). We have then

$$\int_0^T \int_\omega |\hat{\xi}_{j,t}|^2 \mathrm{dxdt} + X^2(\hat{\chi}_j, \hat{\varphi}_j^0) = 1, \forall j \geq 1; \quad \int_0^T \int_\omega |\hat{\phi}_{j,t}|^2 \mathrm{dxdt} \to 0 \text{ as } j \to \infty. \tag{3.11}$$

In view of (3.8) we deduce that

$$\|(\hat{\chi}_j^0, \hat{\varphi}_j^0)\|_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \leq C.$$

On the other hand, by (3.10) and taking into account that $\psi_j^0 \in B$ we know that $(\hat{\psi}_j^0)$ remains in a bounded set $\hat{B}$ of $L^2(\Omega)$. By extracting subsequences we deduce that

$$\begin{cases} (\hat{\chi}_j, \hat{\varphi}_j^0) \longrightarrow (\hat{\chi}, \hat{\varphi}^0) & \text{weakly in } (H_0^1(\Omega))^n \times (L^2(\Omega))^n \\ \hat{\psi}_j^0 \longrightarrow \hat{\psi}^0 & \text{weakly in } L^2(\Omega) \end{cases}$$

and

$$\begin{cases} \hat{\phi}_{j,t} \longrightarrow \hat{\phi}_t & \text{weakly in } (L^2(\Omega \times (0,T)))^n \\ \hat{\xi}_{j,t} \longrightarrow \hat{\xi}_t & \text{weakly in } (L^2(\Omega \times (0,T)))^n \end{cases} \tag{3.12}$$

as $j \to \infty$, where $(\hat{\varphi}, \hat{\psi})$, $(\hat{\phi}, \hat{\psi})$ and $(\hat{\xi}, \hat{\eta})$ are the solutions of (1.5), (3.1) and (3.7) corresponding to the limit initial data.

On the other hand, in virtue of Theorem A we know that $(\hat{\xi}_{j,t})$ is relatively compact in $C([0,T]; (L^2(\Omega))^n)$ and therefore

$$\hat{\xi}_{j,t} \longrightarrow \hat{\xi}_t \quad \text{strongly in } (L^2(\Omega \times (0,T)))^n. \tag{3.13}$$

As a consequence of (3.11) and (3.12) we deduce that

$$\hat{\varphi} = \hat{\phi}_t = 0 \text{ in } \omega \times (0, T). \tag{3.14}$$

In view of (3.14) and applying Proposition 2.5 we obtain that $(\hat{\varphi}, \hat{\psi}) \equiv 0$ in $\Omega \times (0, T)$ and therefore

$$(\hat{\varphi}^0, \hat{\varphi}^1, \hat{\psi}^0) \equiv 0. \tag{3.15}$$

This implies that

$$\hat{\xi} \equiv 0. \tag{3.16}$$

However, combining (3.11), (3.13) and the fact that $X : (H_0^1(\Omega))^n \times (L^2(\Omega))^n \to \mathbb{R}^+$ is compact we deduce that

$$\|\hat{\xi}_t\|^2_{(L^2(\omega\times(0,T)))^n} + X^2(\hat{\chi}, \hat{\varphi}^0) = 1. \tag{3.17}$$

and this contradicts (3.15)-(3.16). ∎

Proof of Proposition 3.2

To simplify the notation we shall denote by ϕ the solution of system (3.5) and by (ϕ^0, ϕ^1) its initial data. In the sequel, by $X(\phi^0, \phi^1)$ we denote a generic term in our estimates that is continuous and compact from $(H_0^1(\Omega)^n \times (L^2(\Omega))^n$ into $\mathbb{R}^+$ and that may change from line to line.

We proced in several steps.

Step 1. *Estimates near the boundary.*

Let $\rho = \rho(x)$ be a non-negative smooth function such that

$$\rho > 0 \text{ in } \omega, \quad \rho = 1 \text{ on } \Gamma, \quad \rho = 0 \text{ in } \Omega \setminus \omega \tag{3.18}$$

$$\nabla\rho/\rho^{1/2} \in L^\infty(\Omega). \tag{3.19}$$

This function is easy to construct. It is sufficient to take $\rho = \tilde{\rho}^2$ where $\tilde{\rho}$ is any function satisfying (3.18).

Let us also consider a non-negative function $h : [0, T] \to \mathbb{R}^+$ such that

$$h = 1 \text{ in } [\varepsilon, T - \varepsilon], \quad h(0) = h(T) = 0$$

with $\varepsilon > 0$ such that $T > diam(\Omega \setminus \omega)/\sqrt{\mu} + 2\varepsilon$.

Multiplying in (3.5) by $\rho(x)h(t)\phi(x,t)$ and integrating by parts in $\Omega \times (0, T)$ we obtain:

$$\int_0^T \int_\Omega \rho(x)h(t)(\mu \mid \nabla\phi \mid^2 + (\lambda + \mu) \mid \text{div}\phi \mid^2) dxdt \le \int_0^T \int_\Omega \rho \mid \phi_t \mid^2 dxdt + X^2(\phi^0, \phi^1).$$

This implies that for any open subset ω_0 of ω such that $cl(\omega_0) \cap \Omega \subset \omega$ and for any $\varepsilon_0 > \varepsilon$ we have

$$\int_{\varepsilon_0}^{T-\varepsilon_0} \int_{\omega_0} (\mu \mid \nabla\phi \mid^2 + (\lambda + \mu) \mid \text{div}\phi \mid^2) dxdt \le \int_0^T \int_\omega \mid \phi_t \mid^2 dxdt + Y^2(\phi^0, \phi^1).$$

Since $T > diam(\Omega \setminus \omega)/\sqrt{\mu} + 2\varepsilon$ we can find ω_0 and ε_0 verifying the conditions above and such that $T > diam(\Omega \setminus \omega_0)/\sqrt{\mu} + 2\varepsilon_0$. Denoting ω_0 by ω for the sake of simplicity and taking into account that the system is translation invariant with respect to time, we see that it suffices to obtain the following estimate:

$$\|(\phi^0, \phi^1)\|^2_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \leq C \int_0^T \int_\omega (|\nabla\phi|^2 + |\phi_t|^2) dxdt + X^2(\phi^0, \phi^1) \tag{3.20}$$

Let us consider now a function $\tilde{\phi} = \rho\phi$ with ρ as above.Then, $\tilde{\phi}$ verifies:

$$\begin{cases} \tilde{\phi}_{tt} - \mu\Delta\tilde{\phi} - (\lambda+\mu)\nabla \mathrm{div}\tilde{\phi} - \alpha\beta P\tilde{\phi}_t = -Z_1 + \alpha\beta Z_2 & \text{in } \Omega \times (0,T) \\ \tilde{\phi} = 0 \quad \text{on } \Gamma \times (0,T) \\ \tilde{\phi}(T) = \rho\phi^0, \tilde{\phi}_t(T) = \rho\phi^1 \quad \text{in } \Omega \end{cases} \tag{3.21}$$

with

$$Z_1 = \mu[2\nabla\rho \cdot \nabla\phi + \Delta\rho\phi] + (\lambda+\mu)[\nabla\rho \mathrm{div}\phi + \nabla(\nabla\rho \cdot \phi)]$$

and

$$Z_2 = -\nabla(-\Delta)^{-1}[\nabla\rho \cdot \phi_t + 2\nabla\rho \cdot \nabla((-\Delta)^{-1}(\mathrm{div}\phi_t)) + \Delta\rho(-\Delta)^{-1}(\mathrm{div}\phi_t)]$$
$$-\nabla\rho((-\Delta)^{-1}(\mathrm{div}\phi_t)).$$

Since system (3.21) is well posed in $(H_0^1(\Omega))^n \times (L^2(\Omega))^n$ and

$$\| - Z_1 + \alpha\beta Z_2\|^2_{L^1(0,T;(L^2(\Omega))^n)} \leq C \int_0^T \int_\omega (|\nabla\phi|^2 + |\phi_t|^2) dxdt + X^2(\phi^0, \phi^1) \tag{3.22}$$

we obtain

$$\|(\rho\phi^0, \rho\phi^1)\|^2_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \leq C \int_0^T \int_\omega (|\nabla\phi|^2 + |\phi_t|^2) dxdt + X^2(\phi^0, \phi^1). \tag{3.23}$$

Indeed, let us introduce the energy

$$E(\rho\phi^0, \rho\phi^1; t) = \frac{1}{2} \int_\Omega [| \phi_t(x,t) |^2 + \mu | \nabla\phi(x,t) |^2 + (\lambda+\mu) | div\phi(x,t) |^2] dx$$

We have

$$\frac{dE(\rho\phi^0, \rho\phi^1; t)}{dt} = \int_\Omega (-Z_1 + \alpha\beta Z_2) \cdot \phi_t dx.$$

Therefore, by Gronwall's inequality and using (3.22) we deduce that

$$E(\rho\phi^0, \rho\phi^1; T) \leq CE(\rho\phi^0, \rho\phi^1; t) + X^2(\phi^0, \phi^1)$$

for any $t \in [0,T]$. Integrating this inequality for $t \in [0,T]$ we obtain (3.23).

Step 2. *Estimates in the interior.*

Let ϕ be a solution of (3.5). We set $v = \text{div}\phi$ and $w = \text{curl}\phi$. We have

$$\begin{cases} v_{tt} - (\lambda + 2\mu)\Delta v - \alpha\beta v_t = 0 & \text{in } \Omega \times (0,T) \\ v(x,T) = v^0(x) = \text{div}\phi^0, v_t(x,T) = v^1(x) = \text{div}\phi^1(x) & \text{in } \Omega \end{cases} \tag{3.24}$$

and

$$\begin{cases} w_{tt} - \mu\Delta w = 0 & \text{in } \Omega \times (0,T) \\ w(x,T) = w^0(x) = \text{curl}\phi^0, w_t(x,T) = w^1(x) = \text{curl}\phi^1(x) & \text{in } \Omega \end{cases} \tag{3.25}$$

since $\text{curl}P\cdot = 0$ and $\text{div}P\cdot = \text{div}\cdot$.

Let $\rho = \rho(x)$ be a non-negative and smooth function such that $\rho = 1$ in $\Omega \setminus \omega$ and $\rho = 0$ on Γ. We set $\tilde{v} = e^{-\alpha\beta(t-T)/2}\rho v$ and $\tilde{w} = \rho w$. We have:

$$\begin{cases} \tilde{v}_{tt} - (\lambda + 2\mu)\Delta\tilde{v} - \dfrac{(\alpha\beta)^2}{4}\tilde{v} = \\ \qquad\qquad = -(\lambda + 2\mu)e^{-\alpha\beta(t-T)/2}(2\nabla\rho \cdot \nabla v + \Delta\rho v) \quad \text{in } \Omega \times (0,T) \\ \tilde{v}(x,T) = \tilde{v}^0(x) = \rho v^0, \tilde{v}_t(x,T) = \tilde{v}^1 = \rho v^1 - \dfrac{\alpha\beta}{2}\rho v^0 \quad \text{in } \Omega \\ \tilde{v} = 0 \quad \text{on } \Gamma \times (0,T) \end{cases} \tag{3.26}$$

and

$$\begin{cases} \tilde{w}_{tt} - \mu\Delta\tilde{w} = -\mu(2\nabla\rho \cdot \nabla w + \Delta\rho w) & \text{in } \Omega \times (0,T) \\ \tilde{w}(x,T) = \tilde{w}^0(x) = \rho w^0(x), \tilde{w}_t(x,T) = \tilde{w}^1(x) = \rho w^1(x) & \text{in } \Omega \\ \tilde{w} = 0 & \text{on } \Gamma \times (0,T). \end{cases} \tag{3.27}$$

Let us consider first system (3.27).

We note that

$$\| -2\nabla\rho \cdot \nabla w - \mu\Delta\rho w\|^2_{L^1(0,T;(H^{-1}(\Omega))^n)} \leq C\int_0^T \int_\omega |\nabla\phi|^2 \, dxdt. \tag{3.28}$$

On the other hand, $\tilde{w}$ can be decomposed as $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$ where $\tilde{w}_1$ solves

$$\begin{cases} \tilde{w}_{1,tt} - \mu\Delta\tilde{w}_1 = 0 & \text{in } \Omega \times (0,T) \\ \tilde{w}_1(x,T) = \tilde{w}^0(x) = \rho w^0(x), \tilde{w}_{1,t}(x,T) = \tilde{w}^1(x) = \rho w^1(x) & \text{in } \Omega \\ \tilde{w}_1 = 0 & \text{on } \Gamma \times (0,T) \end{cases} \tag{3.29}$$

and $\tilde{w}_2$ satisfies

$$\begin{cases} \tilde{w}_{2,tt} - \mu\Delta\tilde{w}_2 = -\mu(2\nabla\rho \cdot \nabla w + \Delta\rho w) & \text{in } \Omega \times (0,T) \\ \tilde{w}_2(x,T) = \tilde{w}_{2,t}(x,T) = 0 & \text{in } \Omega \\ \tilde{w}_2 = 0 & \text{on } \Gamma \times (0,T). \end{cases} \tag{3.30}$$

In view of (3.28) and taking into account that system (3.30) is well-posed in $(L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ we obtain, in particular, that

$$\|\tilde{w}_2\|^2_{L^2(\Omega\times(0,T))} \leq C\int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt. \tag{3.31}$$

On the other hand, proceeding as in [Li1] we obtain that, if $T > diam(\Omega)/\sqrt{\mu}$ then

$$\|(\rho \text{curl}(\phi^0), \rho\text{curl}(\phi^1))\|^2_{(L^2(\Omega))^n\times(H^{-1}(\Omega))^n} \leq C\int_0^T\int_\omega \mid \tilde{w}_1 \mid^2 dxdt$$
$$\leq C\int_0^T\int_\omega [\mid \tilde{w} \mid^2 + \mid \tilde{w}_2 \mid^2] dxdt$$

which, combined with (3.31) yields

$$\|(\rho \text{curl}(\phi^0), \rho\text{curl}(\phi^1))\|^2_{(L^2(\Omega))^n\times(H^{-1}(\Omega))^n} \leq C\int_0^T\int_\omega \mid \tilde{w} \mid^2 dxdt$$
$$+\int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt \leq C\int_0^T\int_\omega \mid \rho\text{curl}(\phi) \mid^2 dxdt + \int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt$$
$$\leq C\int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt. \tag{3.32}$$

Let us observe that, in the argument above, the function ρ may be chosen such that it vanishes in ω_0 with ω_0 any neighborhood of Γ such that $cl(\omega_0) \subset \omega$. Applying the same argument to system (3.30) in the domain $\Omega \setminus \omega_0$, the estimate (3.32) is obtained for $T > diam(\Omega\setminus\omega_0)/\sqrt{\mu}$ and therefore, for any $T > diam(\Omega\setminus\omega)/\sqrt{\mu}$.

Let us now consider system (3.29).The same arguments allows to get a similar estimate. However, because of the presence of a lower order term in this equation we obtain:

$$\|(\rho\text{div}(\phi^0), \rho\text{div}(\phi^1))\|^2_{(L^2(\Omega))^n\times(H^{-1}(\Omega))^n} \leq C\int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt + X^2(\phi^0,\phi^1) \tag{3.33}$$

for any $T > diam(\Omega\setminus\omega)/\sqrt{(\lambda+2\mu)}$.

Combining (3.32) and (3.33) we get

$$\|(\text{div}(\rho\phi^0), \text{div}(\rho\phi^1))\|^2_{(L^2(\Omega))^n\times(H^{-1}(\Omega))^n} + \|(\text{curl}(\rho\phi^0), \text{curl}(\rho\phi^1))\|^2_{(L^2(\Omega))^n\times(H^{-1}(\Omega))^n}$$
$$\leq C\int_0^T\int_\omega \mid \nabla\phi \mid^2 dxdt + X^2(\phi^0,\phi^1) \tag{3.34}$$

for any $T > diam(\Omega\setminus\omega)/\sqrt{\mu}$.

Combining (3.34) with the basic inequality

$$\|\nabla\varphi\|^2_{(H_0^1(\Omega))^n} \leq C\{\|\text{div}(\varphi)\|^2_{L^2(\Omega)} + \|\text{curl}(\varphi)\|^2_{(L^2(\Omega))^n}\},\ \forall\varphi\in(H_0^1(\Omega))^n \tag{3.35}$$

we obtain

$$\|(\rho\phi^0, \rho\phi^1)\|^2_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \leq C \int_0^T \int_\omega |\nabla\phi|^2 \, dxdt + X^2(\phi^0, \phi^1). \tag{3.36}$$

Finally, combining (3.23) and (3.36) we deduce (3.6).

4. PROOF OF THEOREM 1

First we observe that it is sufficient to consider the case where $u^0 \equiv u^1 \equiv 0$ and $\theta^0 \equiv 0$. Indeed, given any initial and final data $(u^0, u^1, \theta^0), (v^0, v^1, \eta^0) \in H$ and any $\varepsilon > 0$, let (w, σ) be the solution of (1.3) with initial data (u^0, u^1, θ^0) and right hand side $f = 0$. Then, set $(\hat{u}, \hat{\theta}) = (u, \theta) - (w, \sigma)$. It is easy to check that finding $f \epsilon (L^2(\Omega \times (0, T))^n$ such that (1.4) holds is equivalent to finding $f \epsilon (L^2(\Omega \times (0, T))^n$ so that the solution $(\hat{u}, \hat{\theta})$ of (1.3) with this control and zero initial data satisfies

$$\begin{cases} \hat{u}(T) = v^0 - w(T), \hat{u}_t(T) = v^1 - w_t(T) \\ \|\hat{\theta}(T) - \eta^0 + \sigma(T)\|_{L^2(\Omega)} \leq \varepsilon \end{cases}$$

Therefore, in the sequel we will assume that $u^0 \equiv u^1 \equiv 0$ and $\theta^0 \equiv 0$.

Given any $(v^0, v^1, \eta^0) \epsilon H$ and $\varepsilon > 0$ we introduce the functional $J : \tilde{H} \to \mathbb{R}$ defined as follows

$$J(\varphi^0, \varphi^1, \psi^0) = \frac{1}{2} \int_0^T \int_\omega |\varphi|^2 dxdt - \int_\Omega v^1 . \varphi^0 \mathrm{dx} + < v^0, \varphi^1 > \tag{4.1}$$

$$+\varepsilon \|\psi^0\|_{L^2(\Omega)} - \int_\Omega (\eta^0 + \beta \text{ div } v^0) \psi^0 dx$$

where $< .,. >$ denotes the duality product between $(H_0^1(\Omega))^n$ and $(H^{-1}(\Omega))^n$ and (φ, ψ) the solution of (1.5).

The functional J is coercive in $\tilde{H}$. More precisely, we have the following result:

Lemma 4.1. *Under the assumptions of Theorem 1,*

$$\liminf_{\|(\varphi^0, \varphi^1 + \beta\nabla\psi^0, \psi^0)\|_{\tilde{H}} \to \infty} \frac{J(\varphi^0, \varphi^1, \psi^0)}{\|(\varphi^0, \varphi^1 + \beta\nabla\psi^0, \psi^0)\|_{\tilde{H}}} \geq \varepsilon. \tag{4.2}$$

Proof: Let us consider a sequence $(\varphi_j^0, \varphi_j^1, \psi_j^0)$ in $\tilde{H}$ such that

$$N_j = \|(\varphi_j^0, \varphi_j^1 + \beta\nabla\psi_j^0, \psi_j^0)\|_{\tilde{H}} \to \infty, \text{ as } j \to \infty.$$

We introduce the normalized initial data

$$(\hat{\varphi}_j^0, \hat{\varphi}_j^1, \hat{\psi}_j^0) = (\varphi_j^0, \varphi_j^1, \psi_j^0)/N_j$$

and the corresponding solutions of (1.5):

$$(\hat{\varphi}_j, \hat{\psi}_j) = (\varphi_j, \psi_j)/N_j.$$

We have

$$I_j/N_j = J(\varphi_j^0,\varphi_j^1,\psi_j^0)/N_j = \frac{N_j}{2}\int_\omega |\hat\varphi_j|^2 \mathrm{dxdt} - \int_\Omega v^1.\hat\varphi_j^0 \mathrm{dx}$$

$$+ < v^0, \hat\varphi_j^1 > + \varepsilon \|\hat\psi_j^0\|_{L^2(\Omega)} - \int_\Omega (\eta^0 + \beta \text{ div } v^0)\hat\psi_j^0 dx.$$

We distinguish the following two cases:
(i)

$$\liminf_{j\to\infty} \int_0^T \int_\omega |\hat\varphi_j|^2 dxdt > 0.$$

(ii) there exists a subsequence (denoted by the index j to simplify the notation) such that

$$\int_0^T \int_\omega |\hat\varphi_j|^2 dxdt \to 0, \text{ as } j \to \infty. \tag{4.3}$$

In the first case we have clearly

$$\liminf_{j\to\infty} I_j/N_j = \infty.$$

Let us consider the second one. Since $(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0)$ is bounded in $\tilde H$ we can extract a subsequence (still denoted by the index j) such that $(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0) \longrightarrow (\hat\varphi^0, \hat\varphi^1, \hat\psi^0)$ weakly in $\tilde H$ as $j \to \infty$. Let us denote by $(\hat\varphi, \hat\psi)$ the corresponding solutions of (1.5). In view of (4.3) we have

$$\hat\varphi \equiv 0 \text{ in } \omega \times (0,T)$$

and therefore, as a consequence of Proposition 2.5, $(\hat\varphi^0, \hat\varphi^1, \hat\psi^0) \equiv 0$. Thus

$$(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0) \longrightarrow (0,0,0) \text{ weakly in } \tilde H. \tag{4.4}$$

From (4.4) we deduce that

$$\liminf_{j\to\infty} I_j/N_j = \liminf_{j\to\infty} \frac{N_j}{2}\int_0^T \int_\omega |\hat\varphi_j|^2 \mathrm{dxdt} + \varepsilon \|\hat\psi_j^0\|_{L^2(\Omega)}. \tag{4.5}$$

Clearly, (4.2) holds if

$$\liminf_{j\to\infty} \|\hat\psi_j^0\|_{L^2(\Omega)} = 1. \tag{4.6}$$

Let us suppose that

$$\liminf_{j\to\infty} \|\hat\psi_j^0\|_{L^2(\Omega)} < 1.$$

Then, since $\|(\hat\varphi_j^0, \hat\varphi_j^1 + \beta\nabla\hat\psi_j^0, \hat\psi_j^0)\|_{\tilde H} = 1$ for all j,

$$\liminf_{j\to\infty} \|(\hat\varphi_j^0, \hat\varphi_j^1 + \beta\nabla\hat\psi_j^0)\|_{(L^2(\omega))^n \times (H^{-1}(\Omega))^n} > 0 \tag{4.7}$$

From (4.7) and the fact that $\hat{\psi}_j^0$ is bounded in $L^2(\Omega)$, as a consequence of Proposition 1, we deduce that

$$\liminf_{j\to\infty} \int_0^T \int_\omega |\hat{\varphi}_j|^2 \mathrm{dxdt} > 0 \tag{4.8}$$

but this contradicts (4.3). Then, necessarilly (4.6) holds and as a consequence of this, (4.2). ∎

As a consequence of the coercivity property (4.2) it is easy to check that the infimum of J over $\tilde{H}$ is achieved at some $(\hat{\varphi}^0, \hat{\varphi}^1, \hat{\psi}^0) \in \tilde{H}$. At this minimimizer $(\hat{\varphi}^0, \hat{\varphi}^1, \hat{\psi}^0)$ we have the following optimality conditions:

$$|\int_0^T \int_\omega \hat{\varphi} \cdot \rho \; \mathrm{dxdt} - \int_\Omega v^1 \cdot \rho^0 \mathrm{dx} + < v^0, \rho^1 > - \int_\Omega (\eta^0 + \beta \mathrm{div} v^0) \xi^0 dx| \leq \varepsilon \|\xi^0\|_{L^2(\Omega)} \tag{4.9}$$

for all $(\rho^0, \rho^1, \xi^0) \in \tilde{H}$ where $(\hat{\varphi}, \hat{\psi})$ denotes the solution of (1.5) corresponding to the minimizer $(\hat{\varphi}^0, \hat{\varphi}^1, \hat{\psi}^0)$ and (ρ, ξ) the solution of (1.5) with data (ρ^0, ρ^1, ξ^0).

Observe that if (u, θ) solve (1.3) with $u^0 \equiv u^1 \equiv 0$ and $\theta^0 = 0$ and $f = \hat{\varphi}$ then

$$\int_0^T \int_\omega \hat{\varphi} \cdot \rho dx dt = \int_\Omega u_t(T) \cdot \rho^0 dx - < u(T), \rho^1 > + \int_\Omega (\theta(T) + \beta \mathrm{div} u(T)) \xi^0 dx. \tag{4.10}$$

Combining (4.9) and (4.10) and taking $\xi^0 = 0$ we immediately see that

$$u(T) = v^0, u_t(T) = v^1. \tag{4.11}$$

From (4.9) – (4.11) we obtain

$$|\int_\Omega (\theta(T) - \eta^0) \xi^0 dx| \leq \varepsilon \|\xi^0\|_{L^2(\Omega)}$$

and this is equivalent to

$$\|\theta(T) - \eta^0\|_{L^2(\Omega)} \leq \varepsilon.$$

This concludes the proof of Theorem 1.

5.THE ONE-DIMENSIONAL CASE

The method of proof of Theorem 2 is the same as in Theorem 1. In fact, it is sufficient to prove the following observability inequality:

Proposition 5.1. *Under the assumptions of Theorem 2, for every bouded set B of $L^2(0, L)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_{l_1}^{l_2} |\varphi|^2 dx dt \tag{5.1}$$

holds for every solution of

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \beta\sigma_{xt} = 0 & \text{in } (0,L)\times(0,T) \\ -\sigma_t - \sigma_{xx} - \alpha\varphi_x = 0 & \text{in } (0,L)\times(0,T) \\ \varphi(0,t) = \varphi(L,t) = \sigma(0,t) = \sigma(L,t) = 0 & \text{for } t\in(0,T) \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x); \sigma(x,T) = \sigma^0(x) & \text{in } (0,L) \end{cases} \tag{5.2}$$

with initial data such that

$$\|(\varphi^0, \varphi^1 + \beta\sigma_x^0)\|_{L^2(0,L)\times H^{-1}(0,L)} \geq 1, \sigma^0 \in B.$$

Proceeding as in section 3, it is easy to see that the proof of this proposition may be reduced to prove the following one:

Proposition 5.2. *Under the assumptions of Theorem 2, for every bouded set B of $L^2(0,L)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_{l_1}^{l_2} |\phi_t|^2 dxdt \tag{5.3}$$

holds for every solution of

$$\begin{cases} \phi_{tt} - \phi_{xx} + \beta\sigma_x = 0 & \text{in } (0,L)\times(0,T) \\ -\sigma_t - \sigma_{xx} - \alpha\phi_{x,t} = 0 & \text{in } (0,L)\times(0,T) \\ \phi(0,t) = \phi(L,t) = \sigma(0,t) = \sigma(L,t) = 0 & \text{for } t\in(0,T) \\ \phi(x,T) = \phi^0(x), \phi_t(x,T) = \phi^1(x); \sigma(x,T) = \sigma^0(x) & \text{in } (0,L) \end{cases} \tag{5.4}$$

with initial data such that

$$\|(\phi^0, \phi^1)\|_{H_0^1(0,L)\times L^2(0,L)} \geq 1, \sigma^0 \in B.$$

To prove Proposition 5.2, we may proceed as in the proof of Proposition 3.1. Then, it is easily seen that it is sufficient to consider the wave equation satisfied by the displacement in the decoupled system associated with (5.4):

$$\begin{gathered} \phi_{tt} - \phi_{xx} - \alpha\beta P\phi_t = 0 \quad \text{in } (0,L)\times(0,T) \\ \phi(0,T) = \phi(L,t) = 0 \quad \text{for } t\in(0,T) \\ \phi(x,T) = \phi^0(x), \phi_t(x,T) = \phi^1(x) \quad \text{in } (0,L) \end{gathered} \tag{5.5}$$

where

$$P\phi = \phi - \frac{1}{L}\int_0^L \phi(x)dx$$

and to prove the following observability inequality:

Proposition 5.3. *Suppose that* $T > 2max(l_1, L - l_2)$. *Then, there exists a positive constant* $C > 0$ *such that*

$$\|\phi^0\|^2_{H_0^1(0,L)} + \|\phi^1\|^2_{L^2(0,L)} \leq C \int_0^T \int_{l_1}^{l_2} |\phi_t|^2 \mathrm{dxdt} \tag{5.6}$$

holds for every solution of (5.5).

Proof of Proposition 5.3: We decompose ϕ as $\phi = \phi_1 + \phi_2$ where ϕ_1 solves

$$\begin{aligned} \phi_{1,tt} - \phi_{1,xx} - \alpha\beta\phi_{1,t} = 0 \quad & \text{in } (0,L)\times(0,T) \\ \phi_1(0,t) = \phi_1(L,t) = 0 \quad & \text{for } t \in (0,T) \\ \phi_1(x,T) = \phi^0(x), \phi_{1,t}(x,T) = \phi^1(x) \quad & \text{in } (0,L). \end{aligned} \tag{5.7}$$

and ϕ_2 satisfies

$$\begin{cases} \phi_{2,tt} - \phi_{2,xx} = -\dfrac{\alpha\beta}{L}\displaystyle\int_0^L \phi_t(x,t)dx & \text{in } (0,L)\times(0,T) \\ \phi_2(0,t) = \phi_2(L,t) = 0 & \text{for } t \in (0,T) \\ \phi_2(x,T) = \phi_{2,t}(x,T) = 0 & \text{in } (0,L) \end{cases} \tag{5.8}$$

It is easy to check that, since $T > 2max(1_1, L - l_2)$, there exists $C > 0$ such that

$$\|\phi^0\|^2_{H^1(0,L)} + \|\phi^1\|^2_{L^2(0,L)} \leq C \int_0^T \int_{l_1}^{l_2} |\phi_{1,t}|^2 \mathrm{dxdt}. \tag{5.9}$$

Indeed, if ϕ_1 solves (5.7), then

$$\psi = e^{\alpha\beta(t-T)/2}\phi_1$$

satisfies

$$\begin{cases} \psi_{tt} - \psi_{xx} - \dfrac{(\alpha\beta)^2}{4}\psi = 0 & \text{in } (0,L)\times(0,T) \\ \psi(0,t) = \psi(L,t) = 0 & \text{for } t \in (0,T) \\ \psi(x,T) = \phi^0(x), \psi_t(x,T) = \phi^1(x) - \dfrac{\alpha\beta}{2}\phi^0(x) & \text{in } (0,L) \end{cases}$$

In view of [Z3] we have

$$\|\phi^0\|^2_{H^1(0,L)} + \|\phi^1\|^2_{L^2(0,L)} \leq C[\int_0^T \int_{l_1}^{l_2} |\psi_t|^2 \mathrm{dxdt} + \|\phi^0\|^2_{L^2(0,L)}]$$

and this implies that

$$\|\phi^0\|^2_{H^1(0,L)} + \|\phi^1\|^2_{L^2(0,L)} \leq C[\int_0^T \int_{l_1}^{l_2} |\phi_{1,t}|^2 \mathrm{dxdt} + \int_0^T \int_{l_1}^{l_2} |\phi_1|^2 \mathrm{dxdt} + \|\phi^0\|^2_{L^2(0,L)}].$$

A classical compactness-uniqueness argument shows then that

$$\int_0^T \int_{l_1}^{l_2} |\phi_1|^2 dxdt + \|\phi^0\|^2_{L^2(0,L)} \leq C \int_0^T \int_{l_1}^{l_2} |\phi_{1,t}|^2 dxdt$$

and therefore (5.9) holds.

From (5.9) we deduce that

$$\|\phi^0\|^2_{H_0^1(0,L)} + \|\phi^1\|^2_{L^2(0,L)} \leq C[\int_0^T \int_{l_1}^{l_2} |\phi_t|^2 dxdt + \int_0^T \int_{l_1}^{l_2} |\phi_{2,t}|^2 dxdt] \tag{5.10}$$

Thus, it is sufficient to show that

$$\int_0^T \int_{l_1}^{l_2} |\phi_{2,t}|^2 dxdt \leq C \int_0^T \int_{l_1}^{l_2} |\phi_t|^2 dxdt. \tag{5.11}$$

We argue by contradiction. If (5.11) does not hold, there exists a sequence of initial data (ϕ_k^0, ϕ_k^1) in $H_0^1(0,L) \times L^2(0,L)$ such that the corresponding solutions of (5.2) and (5.8) satisfy

$$\int_0^T \int_{l_1}^{l_2} |\phi_{k,t}|^2 dxdt \to 0 \text{ as } k \to \infty \tag{5.12}$$

$$\int_0^T \int_{l_1}^{l_2} |\phi_{2,k,t}|^2 dxdt = 1 \quad \text{ for all } k. \tag{5.13}$$

Combining (5.10) with (5.12) and (5.13) we deduce that (ϕ_k^0, ϕ_k^1) is bounded in $H_0^1(0,L) \times L^2(0,L)$. By extracting subsequences (that we still denote by the index k) we have

$$\begin{cases} (\phi^0, \phi_k^1) \to (\phi^0, \phi^1) & \text{weakly in } H_0^1(0,L) \times L^2(0,L) \\ \phi_{k,t} \to \phi_t & \text{weakly in } L^2((0,L) \times (0,T)) \\ \phi_{2,k,t} \to \phi_{2,t} & \text{strongly in } L^2((0,L) \times (0,T)) \end{cases} \tag{5.14}$$

as $k \to \infty$ where ϕ and ϕ_2 are the solutions of (5.2) and (5.8) corresponding to the limit initial data (ϕ^0, ϕ^1). The fact that $\phi_{2,k,t}$ converges strongly in $L^2((0,L) \times (0,T))$ is a consequence of the fact that the right hand side of (5.8) is independent of x and bounded in $C^1([0,T])$.

In view of (5.12) we have $\phi_t \equiv 0$ in $(l_1, l_2) \times (0,T)$. This implies that

$$(\phi^0, \phi^1) \equiv 0 \text{ in } (0,L) \tag{5.15}$$

Indeed, $\psi = \phi_x$ solves the wave equation

$$\psi_{tt} - \psi_{xx} - \alpha\beta\psi_t = 0$$

and $\psi \equiv 0$ in $(l_1, l_2) \times (0, T)$. By unique continuation we deduce that

$$\psi \equiv 0 \text{ in } Q = (0, L) \times (\max(l_1, L - L_2), T - \max(l_1, L - l_2))$$

and therefore

$$\phi = \phi(t) \text{ in } Q.$$

But then, since ϕ takes homegeneous Dirichlet boundary data we condude that

$$\phi \equiv 0 \text{ in } Q$$

which implies (5.15).

Combining (5.12) and (5.15) we deduce that

$$\int_0^T \int_{l_1}^{l_2} |\phi_{2,t}|^2 dxdt = 1. \tag{5.16}$$

On the other hand, in view of (5.15) we have

$$\phi_2 \equiv 0. \tag{5.17}$$

But clearly (5.16) and (5.17) are in contradiction. This concludes the proof of proposition 5.3.

6. COMMENTS

6.1 The techniques of this paper allow us to consider the case where the control f acts on a neighborhood of an open subset of the boundary $\Gamma_0 \subset \partial\Omega$. Indeed, let ω be an open subset of Ω such that the following condition holds:

(C) If $T > T_0$ there exists a constant $C > 0$ such that (3.6)

holds for every finite energy solution of (3.5).

Then the conclusions of Theorem 1 hold.

The proof of this result is a straightforward application of the proof of Theorem 1.

Condition (C) holds for $\alpha\beta$ sufficiently small if it holds for the solutions of the system of elasticity (with $\alpha\beta = 0$) (see [Z1]). Concerning the system of elasticity we know that (3.6) holds if ω is a neighborhood of a subset of the boundary of the form

$$\Gamma(x^0) = \{x \in \partial\Omega : (x - x^0) \cdot n(x) > 0\}$$

for any $x^0 \in R^n$ where $n(x)$ denotes the outward unit normal to Ω at $x \in \partial\Omega$ (see [Li1]).

It would be interesting to have a charaterization of those subsets ω such that condition (C) holds, in terms of the geometric control conditions introduced by C. Bardos, G. Lebeau and J. Rauch in [BLR1,2].

6.2 *Boundary Control*. The methods of this paper allow also to treat boundary control problems. Let us consider for instance the system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,T) \\ \theta_t - \Delta\theta + \beta \mathrm{div} u_t = 0 & \text{in } \Omega\times(0,T) \\ u = v; \theta = 0 & \text{on } \Gamma\times(0,T) \\ u(0) = u^0,\ u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{6.1}$$

The natural functional setting is now $v \in L^2(\partial\Omega\times(0,T))$ and $(u^0, u^1, \theta^0) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times L^2(\Omega)$. However we can not conclude the exact-approximate controllability of this sytem since we do not know whether the following uniqueness result holds:

$$\text{If } T > \mathrm{diam}(\Omega)/\sqrt{\mu}, \text{ and } (\varphi,\sigma) \text{ is a solution of (2.6)}$$

$$\text{such that } \partial\varphi/\partial n = 0 \text{ on } \partial\Omega\times(0,T), \text{ then } \varphi\equiv\sigma\equiv 0.$$

6.3 *Control in the temperature*. Let us consider now the system of thermoelasticity with one control acting in the equation of the temperature:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,T) \\ \theta_t - \Delta\theta + \beta \mathrm{div} u_t = g & \text{in } \Omega\times(0,T) \\ u = \theta = 0 & \text{on } \Gamma\times(0,T) \\ u(0) = u^0,\ u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{6.2}$$

In the one-dimensional case, it is easy to prove that the exact-approximate controllability holds as in Theorem 2, with controls $g \in L^2(0,T;H^{-1}(\Omega)) + H^{-1}(0,T;H_0^1(\Omega))$ supported in $\omega = (l_1, l_2)$. Indeed, as an immediate consequence of Proposition 5.1, we have that

$$\delta \le \int_0^T \|(\sigma_t + \sigma_{xx})\|^2_{H^{-1}(l_1,l_2)} dt$$

for every solution of (5.2) with initial data such that

$$\|(\varphi^0, \varphi^1 + \beta\sigma_x^0)\|_{L^2(0,L)\times H^{-1}(0,L)} \ge 1, \sigma^0 \in B.$$

This type of result can not be expected in several space dimensions since an estimate of ψ (in any norm) over the set ω in system (1.5), only provides an estimate on divφ but does not give any extra information about curlφ.

APPENDIX

For the sake of completeness, in this section we describe the main steps of the proof of theorem A of D. Henry, O. Lopes and A. Perissinitto [HeLP].

Let B be a bounded set of H. We set

$$(u(t), u_t(t), \theta(t)) = [S(t)](u^0, u^1, \theta^0),$$

$$(\tilde{u}(t), \tilde{u}_t(t), \tilde{\theta}(t)) = [S^0(t)](u^0, u^1, \theta^0)$$

and

$$(v(t), v_t(t), \eta(t)) = [S(t) - S^0(t)](u^0, u^1, \theta^0)$$

for any $(u^0, u^1, \theta^0) \in B$.

We have:

$$\begin{cases} v_{tt} - \mu\Delta v - (\lambda+\mu)\nabla \mathrm{div} v + \alpha\nabla\eta = \alpha[\beta P\tilde{u}_t - \nabla\tilde{\theta}] & \text{in } \Omega\times(0,T) \\ \eta_t - \Delta\eta + \beta \mathrm{div} v_t = 0 & \text{in } \Omega\times(0,T) \\ v = \eta = 0 & \text{on } \Gamma\times(0,T) \\ v(0) = v_t(0) = 0, \eta(0) = 0. & \text{in } \Omega \end{cases} \qquad (A.1)$$

It is sufficient to check that $\beta P\tilde{u}_t - \nabla\tilde{\theta}$ is bounded in $L^1(0,T;(H^s(\Omega))^n)$ for some $s > 0$ when (u^0, u^1, θ^0) varies in B.

Let us decompose $\beta P\tilde{u}_t - \nabla\tilde{\theta}$ as follows:

$$\beta P\tilde{u}_t - \nabla\tilde{\theta} = \nabla w_1 + \nabla w_2$$

where w_1 satisfies

$$\begin{cases} w_{1,t} - \Delta w_1 = 0 & \text{in } \Omega\times(0,T) \\ w_1 = 0 & \text{on } \Gamma\times(0,T) \\ w_1(0) = -\beta(-\Delta)^{-1}(\mathrm{div}(u^1)) - \theta^0 & \text{in } \Omega \end{cases} \qquad (A.2)$$

and w_2 verifies

$$\begin{cases} w_{2,t} - \Delta w_2 = -\beta(-\Delta)^{-1}(\mathrm{div}(\tilde{u}_{tt})) & \text{in } \Omega\times(0,T) \\ w_2 = 0 & \text{on } \Gamma\times(0,T) \\ w_2(0) = 0 & \text{in } \Omega. \end{cases} \qquad (A.3)$$

Since $\beta(-\Delta)^{-1}(\mathrm{div} u^1) + \theta^0$ is bounded in $L^2(\Omega)$, because of the regularizing effect of the heat equation (A.2), we deduce that w_1 is bounded in $L^1(0,T;H^{1+s}(\Omega))$ for any $0 < s < 1$.

On the other hand, $(-\Delta)^{-1}(\mathrm{div}(\tilde{u}_{tt}))$ is bounded in $L^2(\Omega\times(0,T))$. Indeed,

$$\mathrm{div}(\tilde{u}_{tt}) = (\lambda+2\mu)\Delta\mathrm{div}(\tilde{u}) - \alpha\beta\mathrm{div}(\tilde{u}_t).$$

Since $\operatorname{div}(\tilde{u}_t)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ it is sufficient to check that $(-\Delta)^{-1}\Delta\operatorname{div}(\tilde{u})$ is bounded in $L^2(\Omega\times(0,T))$. This is easy to check since $\operatorname{div}(\tilde{u})$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and the trace of $\operatorname{div}(\tilde{u})$ on $\partial\Omega\times(0,T)$ is bounded in $L^2(\partial\Omega\times(0,T))$ (see [Li1]). Therefore, w_2 is bounded in $L^2(0,T;H^1(\Omega))$ and in $L^1(0,T;H^{1+\delta}(\Omega))$ for any $0<\delta<1$. This concludes the proof of Theorem A.

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