# NUMERICAL DISPERSIVE SCHEMES FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider semidiscrete approximation schemes for the linear Schrödinger equation and analyze whether the classical dispersive properties of the continuous model hold for these approximations. For the conservative finite difference semi-discretization scheme we show that, as the mesh-size tends to zero, the semidiscrete approximate solutions loose the dispersion property. This fact is proved by constructing solutions concentrated at the points of the spectrum where the second order derivatives of the symbol of the discrete Laplacian vanish. Therefore this phenomenon is due to the presence of numerical spurious high-frequencies.

To recover the dispersive properties of the solutions at the discrete level, we introduce two numerical remedies: Fourier filtering and a two-grid preconditioner. For each of them we prove Strichartz-like estimates and a local space smoothing effect, uniform in the mesh size. The methods we employ are based on classical estimates for oscillatory integrals. These estimates allow us to treat nonlinear problems with $L^2$-initial data, without additional regularity hypotheses. We prove the convergence of the two-grid method for nonlinearities that cannot be handled by energy arguments and which, even in the continuous case, require Strichartz estimates.

## 1. Introduction

Let us consider the linear (LSE) and the nonlinear (NSE) Schrödinger equations:

$$\begin{cases} iu_t + \Delta u = 0,\ x \in \mathbb{R}^d,\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

and

$$\begin{cases} iu_t + \Delta u = F(u),\ x \in \mathbb{R}^d,\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

respectively.

The linear equation (1.1) is solved by $u(t,x) = S(t)\varphi(x)$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator. The linear semigroup has two important properties. First, the conservation of the $L^2$-norm

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.3}$$

and a dispersive estimate of the form:

$$|u(t,x)| = |S(t)\varphi(x)| \leq \frac{1}{(4\pi|t|)^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)},\ x \in \mathbb{R}^d,\ t \neq 0. \tag{1.4}$$

The space-time estimate

$$\|S(\cdot)\varphi\|_{L^{2+4/d}(\mathbb{R},\, L^{2+4/d}(\mathbb{R}^d))} \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{1.5}$$

due to Strichartz [27], is deeper. It guarantees that the solutions decay as $t$ becomes large and that they gain some spatial integrability.

Inequality (1.5) was generalized by Ginibre and Velo [8]. They proved the mixed space-time estimate, well known as Strichartz's estimate:

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(q,r)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.6}$$

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for the so-called $d/2$-admissible pairs $(q,r)$. We recall that the exponent pair $(q,r)$ is $\alpha$-admissible (cf. [14]) if : $2 \le q, r \le \infty$, $(q,r,\alpha) \neq (2,\infty,1)$, and

$$\frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.7}$$

The Strichartz estimates play an important role in the proof of the well-posedness of the nonlinear Schrödinger equation. Typically they are used when the energy methods fail to provide well-posedness results.

The nonlinear problem (1.2) with nonlinearity $F(u) = |u|^p u$, $p < 4/d$ and initial data in $L^2(\mathbb{R}^d)$ was first analyzed by Tsutsumi [30]. The author proved that, in this case, NSE is globally well posed in $L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}^d))$, where $(q,r)$ is a $d/2$-admissible pair depending on the nonlinearity $F$.

The Schrödinger equation has another remarkable property guaranteeing the gain of one half space derivative in $L^2_{x,t}$ (cf. [5] and [15]):

$$\sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-\infty}^{\infty} |(-\Delta)^{1/4} e^{it\Delta}\varphi|^2 dt dx \le C\|\varphi\|^2_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

It has played a crucial role in the study of the nonlinear Schrödinger equation with nonlinearities involving derivatives (see [16]). In particular, it is extremely useful when deriving compactness properties.

For other properties on the Schrödinger equation we refer to [3] and [28].

In this paper we analyze whether semidiscrete schemes for LSE have dispersive properties similar to (1.4), (1.6) and (1.8), uniform with respect to the mesh sizes. The study of these dispersion properties for these approximation schemes is relevant for introducing convergent schemes in the nonlinear context. Indeed, as mentioned above, the proof of the well-posedness of the nonlinear Schrödinger equation requires a fine use of the dispersion properties, and consequently, it seems unlikely that the convergence of the numerical schemes could be proved if these dispersion properties are not verified at the numerical level.

Estimates similar to (1.6) for numerical solutions will allow proving uniform (on the mesh-size parameter) bounds on discrete versions of the space $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d))$. On the other hand, estimates similar to (1.8) on discrete solutions will give sufficient conditions to guarantee their compactness and thus the convergence towards the solution of the nonlinear Schrödinger equation (1.2).

However, as we shall see, standard numerical approximation schemes often fail to satisfy these dispersive estimates, uniformly in the mesh-size parameter, and important work needs to be done to develop numerical schemes that do fulfill these estimates uniformly.

To better illustrate the problems we shall address, let us first consider the conservative semidiscrete numerical scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, & t > 0, \\ u^h(0) = \varphi^h. \end{cases} \tag{1.9}$$

Here $u^h$ stands for the infinite unknown vector $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $u_{\mathbf{j}}(t)$ being the approximation of the solution at the node $x_{\mathbf{j}} = \mathbf{j}h$, and $\Delta_h$ the classical second order finite difference approximation of $\Delta$:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2} \sum_{k=1}^{d} (u^h_{\mathbf{j}+e_k} + u^h_{\mathbf{j}-e_k} - 2u^h_{\mathbf{j}}). \tag{1.10}$$

In the one-dimensional case, the lack of uniform dispersive estimates for the solutions of (1.9) has been observed by the authors in [12, 13]. The symbol of the Laplacian, $\xi^2$, in the numerical scheme (1.9) is replaced by $4/h^2 \sin^2(\xi h/2)$ for the discrete Laplacian (1.10). The first and second derivative of the latter vanish at the points $\pm\pi/h$ and $\pm\pi/2h$ of the spectrum. By building wave packets concentrated at the pathological spectral points $\pm\pi/2h$, it is possible to prove the lack of any uniform estimate of the type (1.4) or (1.6). Similar negative results can be shown to hold concerning (1.8) by building wave packets concentrated at $\pm\pi/h$.

The paper is organized as follows. In Section 2 we analyze the conservative approximation scheme (1.9). We extend the 1-d results mentioned above and prove that this scheme does not ensure the gain of any uniform integrability or local smoothing property of the solutions with respect to the initial data. The behavior of the Fourier symbol of the numerical scheme provides a good insight to this pathological behavior. We then propose a Fourier filtering method allowing recovery of both the integrability and the local smoothing properties of the continuous model. The lack of dispersion properties for the linear scheme makes it of little use to approximate non-linear problems. In fact, in Subsection 2.5, by an explicit construction we see that the solutions of a cubic semi-discrete Schrödinger equation do not satisfy the dispersion property of the continuous one, uniformly in the mesh-size parameter.

We then introduce a numerical scheme for which the dispersion estimates are uniform. The proposed scheme involves a two-grid algorithm to precondition the initial data. Based on this numerical scheme for the LSE we build a convergent numerical scheme for the NSE in the class of $L^2(\mathbb{R}^d)$ initial data.

Section 3 is dedicated to the analysis of the method based on the two-grid preconditioning of the initial data. We analyze the action of the linear semigroup $\exp(it\Delta_h)$ on the subspace of $l^2(h\mathbb{Z}^d)$ consisting of the slowly oscillating sequences generated by the two-grid method. Once we obtain Strichartz-like estimates in this subspace we apply them to approximate the NSE. The nonlinear term is approximated in a such way that it belongs to the class of slowly oscillating data which permits the use of the uniform Strichartz estimates.

The results in this paper should be compared to those in [25]. In that paper the authors analyze the Schrödinger equation on the lattice $\mathbb{Z}^d$ without analyzing the dependence on the mesh-size parameter $h$. They obtain Strichartz-like estimates in a class of exponents $q$ and $r$ larger than in the continuous one. But none of these results are uniform when working on the scaled lattice $h\mathbb{Z}^d$ and letting $h\to 0$ as our results in Section 2 show.

In the context of equations on lattices we also mention [6], [19]. In these papers the authors analyze the dynamics of infinite harmonic lattices in the limit of the lattice distance $\epsilon$ tending to zero.

The analysis in this paper can be adapted to address fully discrete schemes. In [10] necessary and sufficient conditions are given guaranteeing uniform dispersion estimates for fully discrete schemes. The work of Nixon [20] is also worth mentioning. There the 1-d KdV equation is considered and space-time estimates are proved for the implicit Euler scheme.

## 2. A conservative scheme

In this section we analyze the conservative scheme (1.9). This scheme satisfies the classical properties of consistency and stability which imply $L^2$-convergence. We construct pathological explicit solutions for (1.9) for which neither (1.6) nor (1.8) hold uniformly with respect to the mesh-size parameter $h$.

In our analysis we make use of the semidiscrete Fourier transform (SDFT) (we refer to [29] for the main properties of the SDFT). For any $v^h\in l^2(h\mathbb{Z}^d)$ we define its SDFT at the scale $h$ by:

$$\widehat{v}^h(\xi)=(\mathcal{F}_h v^h)(\xi)=h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}e^{-i\xi\cdot\mathbf{j}h}v^h_{\mathbf{j}},\quad \xi\in[-\pi/h,\pi/h]^d. \tag{2.11}$$

We will use the notation $A\lesssim B$ to report the inequality $A\leq \textit{constant}\times B$, where the multiplicative constant is independent of $h$. The statement $A\simeq B$ is equivalent to $A\lesssim B$ and $B\lesssim A$.

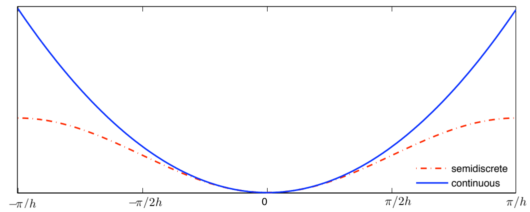


FIGURE 1. The two symbols in dimension one

Taking the SDFT in (1.9) we obtain that $u^h(t) = S^h(t)\varphi^h$ the solution of (1.9) satisfies

$$i\widehat{u}_t^h(t,\xi) + p_h(\xi)\widehat{u}^h(t,\xi) = 0, \quad t \in \mathbb{R}, \quad \xi \in [-\pi/h, \pi/h]^d, \tag{2.12}$$

where the function $p_h : [-\pi/h, \pi/h]^d \to \mathbb{R}$ is defined by

$$p_h(\xi) = \frac{4}{h^2}\sum_{k=1}^{d} \sin^2\big(\frac{\xi_k h}{2}\big). \tag{2.13}$$

Solving the ODE (2.12) we obtain that the Fourier transform of $u^h$ is given by

$$\widehat{u}^h(t,\xi) = e^{-itp_h(\xi)}\widehat{\varphi}^h(\xi), \quad \xi \in [-\pi/h, \pi/h]^d. \tag{2.14}$$

Observe that the new symbol $p_h(\xi)$ is different from the continuous one: $|\xi|^2$. In the one-dimensional case, the symbol $p_h(\xi)$ changes convexity at the points $\xi = \pm\pi/2h$ and has critical points also at $\xi = \pm\pi/h$, two properties that the continuous symbol does not have. Using that

$$\inf_{\xi\in[-\pi/h,\pi/h]} |p_h''(\xi)| + |p_h'''(\xi)| > 0,$$

in [13] (see also [25] for $h = 1$) it has been proved that

$$\|u^h(t)\|_{l^\infty(h\mathbb{Z})} \lesssim \|\varphi^h\|_{l^1(h\mathbb{Z})}\big(|t|^{-1/2} + (|t|h)^{-1/3}\big), \quad t \neq 0. \tag{2.15}$$

Note that estimate (2.15) blows-up as $h \to 0$. Therefore it does not yield uniform Strichartz estimates.

Figure 2 shows that (2.15) could not be improved for large time $t$. In fact when $h = 1$ and $\varphi^1 = \delta_0$ ($\delta_0$ is the discrete Dirac function, $(\delta_0)_0 = 1$ and zero otherwise) the solution $u^1(t)$ behaves as $t^{-1/3}$ for large time $t$ instead of $t^{-1/2}$ in the case of LSE.

In dimension $d$, similar results can be obtained in terms of the number of nonvanishing principal curvatures of the symbol and its gradient. Observe that, at the points $\xi = (\pm\pi/2h, \dots, \pm\pi/2h)$ all the eigenvalues of the hessian matrix $H_{p_h} = (\partial_{ij}p_h)_{ij}$ vanish. Moreover if $k$-components of the vector $\xi$ coincide with $\pm\pi/2h$ the rank of $H_{p_h}$ at this point is $d-k$ instead of $d$, as in the continuous case. This will imply that the solutions of equation (1.9), concentrated at these points of the spectrum, will behave as $t^{-(d-k)/2}(th)^{-k/3}$ instead of $t^{-d/2}$ as $t \to \infty$. This shows that there are no uniform estimates similar to (1.4) or (1.6) at the discrete level. But these inequalities are necessary to prove the uniform boundedness of the semidiscrete solutions in the nonlinear setting.

On the other hand, at the points $\xi = (\pm\pi/h, \dots, \pm\pi/h)$, the gradient of the symbol $p_h(\xi)$ vanishes. As we will see, these pathologies affect the dispersive properties of the semidiscrete scheme (1.9) and its solutions do not fulfill the regularizing property (1.8), uniformly in $h > 0$, which is needed to guarantee the compactness of the semidiscrete solutions. This constitutes an obstacle when passing to the limit as $h \to 0$ in the nonlinear semidiscrete models.

This section is organized as follows. Section 2.1 deals with the analysis of properties (1.4) and (1.6) for the solutions of (1.9). The local smoothing property is analyzed in Section 2.2. In Section 2.3 we prove uniform estimates similar to (1.4) and (1.8), uniformly with respect to the parameter

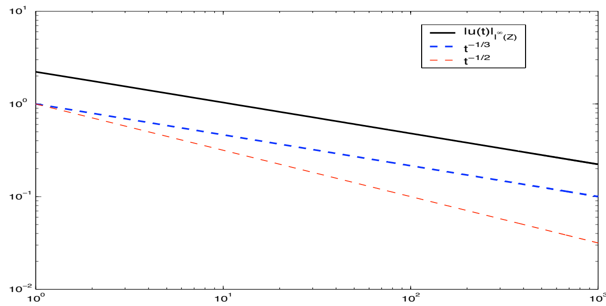


FIGURE 2. Log-log plot of the time evolution of the $l^\infty(\mathbb{Z})$-norm of the fundamental solution $u^1$ for (1.9)

$h$, in the class of initial data whose Fourier spectrum has been filtered conveniently. Strichartz like estimates for filtered solutions are given in Section 2.4.

In Section 2.5 we analyze a numerical scheme for the one-dimensional cubic NSE based on the conservative approximation of the linear Schrödinger semigroup. We prove that its solutions do not remain uniformly bounded in any auxiliary space $L^q_{loc}(\mathbb{R}, L^r(h\mathbb{Z}))$.

2.1. **Lack of uniform dispersive estimates.** First we construct explicit examples of solutions of equation (1.9) for which all the classical estimates of the continuous case (1.6) blow-up.

**Theorem 2.1.** *Let $T > 0$, $r_0 \geq 1$ and $r > r_0$. Then*

$$\sup_{h>0,\, \varphi^h \in l^{r_0}(h\mathbb{Z}^d)} \frac{\|S^h(T)\varphi^h\|_{l^r(h\mathbb{Z}^d)}}{\|\varphi^h\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty \tag{2.16}$$

*and*

$$\sup_{h>0,\, \varphi^h \in l^{r_0}(h\mathbb{Z}^d)} \frac{\|S^h(\cdot)\varphi^h\|_{L^1((0,T),\, l^r(h\mathbb{Z}^d))}}{\|\varphi^h\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty. \tag{2.17}$$

**Remark 2.1.** *A finer analysis can be done. The same result holds if we take the supremum in* (2.16) *and* (2.17) *over the set of functions $\varphi^h \in l^{r_0}(h\mathbb{Z}^d)$ such that the support of their Fourier transform* (2.11) *contains at least one of the points of the set*

$$\mathcal{M}_1^h = \Big\{\xi = (\xi_1, \ldots, \xi_d) \in \Big[-\frac{\pi}{h}, \frac{\pi}{h}\Big]^d : \exists i \in \{1, \ldots, d\} \text{ such that } \xi_i = \frac{\pi}{2h}\Big\}. \tag{2.18}$$

*Observe that at the above points the rank of the hessian matrix $H_{p_h}$ is at most $d-1$.*

**Remark 2.2.** *Let $\mathbf{P}^h$ be an interpolator, piecewise constant or linear. In view of Theorem 2.1, for any fixed $T > 0$, the uniform boundedness principle guarantees the existence of a function $\varphi \in L^2(\mathbb{R}^d)$ and a sequence $\varphi^h$ such that $\mathbf{P}^h\varphi^h \to \varphi$ in $L^2(\mathbb{R}^d)$ and the corresponding solutions $u^h$ of (1.9) satisfy $\|\mathbf{P}^h u^h\|_{L^1((0,T),\, L^r(\mathbb{R}^d))} \to \infty$.*

*Proof of Theorem 2.1.* First, observe that it is sufficient to deal with the one-dimensional case. Indeed, for any sequence $\{\psi_j^h\}_{j\in\mathbb{Z}}$ set $\varphi^h_{\mathbf{j}} = \psi^h_{j_1} \ldots \psi^h_{j_d}$, $\mathbf{j} = (j_1, j_2, \ldots, j_d)$. We are thus considering discrete functions in separated variables. Then, for any $t$ the following holds:

$$(S^h(t)\varphi^h)_{\mathbf{j}} = (S^{1,h}(t)\psi^h)_{j_1}(S^{1,h}(t)\psi^h)_{j_2} \ldots (S^{1,h}(t)\psi^h)_{j_d},$$

where $S^{1,h}(t)$ is the linear semigroup generated by the equation (1.9) in the one-dimensional case. Thus it is obvious that (2.16) and (2.17) hold in dimension $d \geq 2$, once we prove them in the one-dimensional case $d = 1$.

In the following we will consider the one-dimensional case $d = 1$ and prove (2.16) the other being similar. Using the properties of the SDFT it is easy to see that $(S^h(t)\varphi^h)_j = (S^1(t/h^2)\varphi^1)_j$, where $\varphi_j^1 = \varphi_j^h$, $j \in \mathbb{Z}$. A scaling argument in (2.16) shows that

$$\frac{\|S^h(T)\varphi^h\|_{l^q(h\mathbb{Z})}}{\|\varphi^h\|_{l^{q_0}(h\mathbb{Z})}} = h^{\frac{1}{q}-\frac{1}{q_0}} \frac{\|S^1(T/h^2)\varphi^1\|_{l^q(\mathbb{Z})}}{\|\varphi^1\|_{l^{q_0}(\mathbb{Z})}}. \tag{2.19}$$

Let us introduce the operator $S_1(t)$ defined by

$$(S_1(t)\varphi)(x) = \int_{-\pi}^{\pi} e^{-itp_1(\xi)} e^{ix\xi} \widehat{\varphi}(\xi) d\xi, \tag{2.20}$$

which is the extension of the semigroup generated by (1.9) for $h = 1$ to all $x \in \mathbb{R}$.

We point out that for any sequence $\{\varphi_j^1\}_{j\in\mathbb{Z}}$, $S_1(t)\varphi^1$ as in (2.20), which is defined for all $x \in \mathbb{R}$, is in fact the band-limited interpolator of the semi-discrete function $S^1(t)\varphi^1$. The results of Magyar et al. [18] (see also Plancherel and Polya [21]) on band-limited functions show that the following inequalities hold for any $q \geq 1$ and for all continuous functions $\widehat{\varphi}$ supported in $[-\pi, \pi]$:

$$c(q)\|\varphi\|_{l^q(\mathbb{Z})} \leq \|\varphi\|_{L^q(\mathbb{R})} \leq C(q)\|\varphi\|_{l^q(\mathbb{Z})}.$$

Thus for any $q > q_0 \geq 1$ the following holds for all functions $\varphi^1$ whose Fourier transform is supported in $[-\pi, \pi]$:

$$\frac{\|S^1(t)\varphi^1\|_{l^q(\mathbb{Z})}}{\|\varphi^1\|_{l^{q_0}(\mathbb{Z})}} \geq c(q, q_0) \frac{\|S_1(t)\varphi^1\|_{L^q(\mathbb{R})}}{\|\varphi^1\|_{L^{q_0}(\mathbb{R})}}. \tag{2.21}$$

In view of this property it is sufficient to deal with the operator $S_1(t)$.

Denoting $\tau = T/h^2$, by (2.19) the proof of (2.16) is reduced to the proof of the following fact about the new operator $S_1(t)$:

$$\lim_{\tau\to\infty} \tau^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \sup_{\operatorname{supp}(\widehat{\varphi})\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} = \infty. \tag{2.22}$$

The following lemma is the key point in the proof of the last estimate.

**Lemma 2.1.** *There exists a positive constant $c$ such that for all $\tau$ sufficiently large, there exists a function $\varphi_\tau$ such that $\|\varphi_\tau\|_{L^p(\mathbb{R})} \simeq \tau^{1/3p}$ for all $p \geq 1$ and*

$$|(S_1(t)\varphi_\tau)(x)| \geq \frac{1}{2} \tag{2.23}$$

*for all $|t| \leq c\tau$ and $|x - tp_1'(\pi/2)| \leq c\tau^{1/3}$.*

**Remark 2.3.** *Lemma 2.1 shows a lack of dispersion in the semidiscrete setting when compared with the continuous one. In the latter, for any initial data $\varphi_\tau$ such that $\|\varphi_\tau\|_{L^1(\mathbb{R})} \simeq \tau^{1/3}$, the solution $S(t)\varphi_\tau$ of LSE satisfies*

$$\|S(t)\varphi_\tau\|_{L^\infty(\mathbb{R})} \lesssim \frac{\tau^{1/3}}{|t|^{1/2}} \lesssim \frac{1}{\tau^{1/6}}$$

*for all $t \simeq \tau$, which is incompatible with* (2.23).

The proof of Lemma 2.1 will be given later.

Assuming for the moment that Lemma 2.1 holds, we now prove (2.22). In view of Lemma 2.1, given $q > q_0 \geq 1$, for sufficiently large $\tau$ the following holds:

$$\sup_{\operatorname{supp}(\widehat{\varphi})\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim \tau^{\frac{1}{3q}-\frac{1}{3q_0}}.$$

Thus (2.22) holds and the proof is done. □

*Proof of Lemma 2.1.* The techniques used below are similar to those used in [7] to get lower bounds on oscillatory integrals.

We define the relevant initial data through its Fourier transform. Let us first fix a positive function $\widehat{\varphi}$ supported on $(-1,1)$ such that $\int_{-\pi}^{\pi}\widehat{\varphi}=1$. For all positive $\tau$, we set:

$$\widehat{\varphi}_\tau(\xi)=\tau^{1/3}\widehat{\varphi}(\tau^{1/3}(\xi-\pi/2)).$$

We define $\varphi_\tau$ as the inverse Fourier transform of $\widehat{\varphi}_\tau$. Observe that $\widehat{\varphi}_\tau$ is supported in the interval $(\pi/2-\tau^{-1/3},\pi/2+\tau^{-1/3})$ and $\int_{-\pi}^{\pi}\widehat{\varphi}_\tau=1$. Also using that $\varphi_\tau(x)=\varphi_1(\tau^{-1/3}x)$ we get $\|\varphi_\tau\|_{L^p(\mathbb{R})}\simeq\tau^{1/3p}$ for any $p\geq 1$.

The mean value theorem applied to the integral occurring in the right hand side of (2.20) shows that

$$|S_1(t)\varphi_\tau(x)|\geq\Big(1-2\tau^{-1/3}\sup_{\xi\in\,\mathrm{supp}(\widehat{\varphi}_\tau)}|x-tp_1'(\xi)|\Big)\int_{-\pi}^{\pi}\widehat{\varphi}_\tau(\xi)d\xi. \tag{2.24}$$

Using that the second derivative of $p_1$ vanishes at $\xi=\pi/2$ we obtain the existence of a positive constant $c_1$ such that

$$|x-tp_1'(\xi)|\leq|\,x-tp_1'(\pi/2)|+tc_1|\xi-\pi/2|^2,\quad \xi\simeq\pi/2.$$

In particular for all $\xi\in[\pi/2-\tau^{-1/3},\pi/2+\tau^{-1/3}]$ the following holds

$$|x-tp_1'(\xi)|\leq|x-tp_1'(\pi/2)|+tc_1\tau^{-2/3}.$$

Thus there exists a (small enough) positive constant $c$ such that for all $x$ and $t$ satisfying $|x-tp_1'(\pi/2)|\leq c\tau^{1/3}$ and $t\leq c\tau$:

$$2\tau^{-1/3}\sup_{\xi\in\,\mathrm{supp}(\widehat{\varphi}_\tau)}|x-tp_1'(\xi)|\leq\frac{1}{2}.$$

In view of (2.24) this yields (2.23) and finishes the proof. □

2.2. **Lack of uniform local smoothing effect.** In order to analyze the local smoothing effect at the discrete level we introduce the discrete fractional derivatives on the lattice $h\mathbb{Z}^d$. We define for any $s\geq 0$, the fractional derivative $(-\Delta_h)^{s/2}u^h$ at the scale $h$ as:

$$((-\Delta_h)^{s/2}u^h)_{\mathbf{j}}=\int_{[-\pi/h,\pi/h]^d}p_h^{s/2}(\xi)e^{i\mathbf{j}\cdot\xi h}\mathcal{F}_h(u^h)(\xi)d\xi,\ \mathbf{j}\in\mathbb{Z}^d, \tag{2.25}$$

where $p_h(\cdot)$ is as in (2.13) and $\mathcal{F}_h(u^h)$ is the SDFT of the sequence $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ at the scale $h$ .

Concerning the local smoothing effect we have the following result:

**Theorem 2.2.** *Let be $T>0$ and $s>0$. Then*

$$\sup_{h>0,\varphi^h\in l^2(h\mathbb{Z}^d)}\frac{h^d\sum\limits_{|\mathbf{j}|h\leq 1}|((-\Delta_h)^{s/2}S^h(T)\varphi^h)_{\mathbf{j}}|^2}{\|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}}=\infty \tag{2.26}$$

*and*

$$\sup_{h>0,\varphi^h\in l^2(h\mathbb{Z}^d)}\frac{h^d\sum\limits_{|\mathbf{j}|h\leq 1}\int_0^T|((-\Delta_h)^{s/2}S^h(t)\varphi^h)_{\mathbf{j}}|^2dt}{\|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}}=\infty. \tag{2.27}$$

**Remark 2.4.** *The same result holds if we take the supremum in* (2.26) *and* (2.27) *over the set of functions $\varphi^h\in l^2(h\mathbb{Z}^d)$ such that the support of $\varphi^h$ contains at least one of the points of the set*

$$\mathcal{M}_2^h=\Big\{\xi=(\xi_1,\dots,\xi_d)\in\Big[-\frac{\pi}{h},\frac{\pi}{h}\Big]^d:\ \xi_i=\pm\frac{\pi}{h},\ i=1\dots d\Big\}. \tag{2.28}$$

*Observe that at the above points the gradient of $p_h$ vanishes.*

In contrast with the proof of Theorem 2.1 we cannot reduce it to the one-dimensional case. This is due to the extra factor $p_h^{s/2}(\xi)$ which does not allow us to use separation of variables. The proof consists in reducing (2.26) and (2.27) to the case $h=1$ and then using the following lemma.

**Lemma 2.2.** *Let be $s>0$. There is a positive constant $c$ such that for all $\tau$ sufficiently large there exists a function $\varphi_\tau^1$ with $\|\varphi_\tau^1\|_{l^2(\mathbb{Z}^d)}=\tau^{d/2}$ and*

$$|((-\Delta_1)^{s/2}S^1(t)\varphi_\tau^1)_{\mathbf{j}}|\geq 1/2 \tag{2.29}$$

*for all $|t|\leq c\tau^2$, $|\mathbf{j}|\leq c\tau$.*

We postpone the proof of Lemma 2.2 and proceed with the proof of Theorem 2.2.

*Proof of Theorem 2.2.* We prove (2.26), the other estimate (2.27) being similar. As in the previous section we reduce the proof to the case $h=1$. By the definition of $(-\Delta_h)^{s/2}$ for any $\mathbf{j}\in\mathbb{Z}^d$ we have that

$$((-\Delta_h)^{s/2}S^h(t)\varphi^h)_{\mathbf{j}}=h^{-s}((-\Delta_1)^{s/2}S^1(t/h^2)\varphi^1)_{\mathbf{j}},\quad \mathbf{j}\in\mathbb{Z}^d,$$

where $\varphi^h_{\mathbf{j}}=\varphi^1_{\mathbf{j}}$, $\mathbf{j}\in\mathbb{Z}^d$. Thus

$$\frac{h^d\sum\limits_{|\mathbf{j}|h\leq 1}|((-\Delta_h)^{s/2}S^h(T)\varphi^h)_{\mathbf{j}}|^2}{\|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}}=\frac{h^{-2s}\sum\limits_{|\mathbf{j}|\leq 1/h}|((-\Delta_1)^{s/2}S^1(T/h^2)\varphi^1)_{\mathbf{j}}|^2}{\|\varphi^1\|^2_{l^2(\mathbb{Z}^d)}}.$$

With $c$ and $\varphi_\tau$ given by Lemma 2.2 and $\tau$ such that $c\tau^2=T/h^2$, i.e. $\tau=(T/c)^{1/2}h^{-1}$, we have $\|\varphi_\tau^1\|^2_{l^2(\mathbb{Z})}=\tau^d$ and

$$\lim_{\tau\to\infty}\frac{h^{-2s}\sum\limits_{|\mathbf{j}|\leq 1/h}|((-\Delta_1)^{s/2}S^1(T/h^2)\varphi^1_\tau)_{\mathbf{j}}|^2}{\|\varphi^1_\tau\|^2_{l^2(\mathbb{Z}^d)}}\gtrsim\lim_{\tau\to\infty}\frac{\tau^{2s}\tau^d}{\tau^d}=\infty.$$

This finishes the proof. □

*Proof of Lemma 2.2.* We choose a positive function $\widehat{\varphi}$ supported in the unit ball with $\int_{\mathbb{R}^d}\widehat{\varphi}=1$. Set for all $\tau\geq 1$ $\widehat{\varphi}^1_\tau(\xi)=\tau^d\widehat{\varphi}\left(\tau(\xi-\pi_d)\right)$, where $\pi_d=(\pi,\dots,\pi)$. We define $\varphi^1_\tau$ as the inverse Fourier transform at scale $h=1$ of $\widehat{\varphi}^1_\tau$. Thus $\widehat{\varphi}^1_\tau$ is supported in $\{\xi:|\xi-\pi_d|\leq\tau^{-1}\}$, it has mass one and $\|\varphi^1_\tau\|_{l^2(\mathbb{Z}^d)}\simeq\tau^{d/2}$. Applying the mean value theorem to the oscillatory integral occurring in the definition of $(-\Delta_1)^{s/2}S^1(t)\varphi^1_\tau$ and using that $p_1(\xi)$ behaves as a positive constant in the support of $\widehat{\varphi}^1_\tau$ we obtain that for some positive constant $c_0$:

$$\begin{aligned}|((-\Delta_1)^{s/2}S^1(t)\varphi^1_\tau)_{\mathbf{j}}|&\geq\big(1-2\tau^{-1}\sup_{\xi\in\,\mathrm{supp}(\widehat{\varphi}^1_\tau)}|\mathbf{j}-t\nabla p_1(\xi)|\big)\int_{[-\pi,\pi]^d}p_1^{s/2}(\xi)\widehat{\varphi}^1_\tau(\xi)d\xi\\&\geq c_0\big(1-2\tau^{-1}\sup_{\xi\in\,\mathrm{supp}(\widehat{\varphi}^1_\tau)}|\mathbf{j}-t\nabla p_1(\xi)|\big)\int_{[-\pi,\pi]^d}\widehat{\varphi}^1_\tau(\xi)d\xi.\end{aligned}$$

Using that $\nabla p_1$ vanishes at $\xi=\pi_d$ we obtain the existence of a positive constant $c_1$ such that

$$|\mathbf{j}-t\nabla p_1(\xi)|\leq|\mathbf{j}|+tc_1|\xi-\pi_d|,\quad \xi\sim\pi_d.$$

Then there exists a positive constant $c$ such that for all $\mathbf{j}$ and $t$ satisfying $|\mathbf{j}|\leq c\tau$ and $t\leq c\tau^2$ the following holds:

$$2\tau^{-1}\sup_{\xi\in\,\mathrm{supp}(\widehat{\varphi}_\tau)}|\mathbf{j}-t\nabla p_1(\xi)|\leq\frac{1}{2}.$$

Thus for all $t$ and $\mathbf{j}$ as above (2.29) holds. This finishes the proof. □

2.3. **Filtering of the initial data.** As we have seen in the previous section the conservative scheme (1.9) does not reproduce the dispersive properties of the continuous LSE. In this section we prove that a suitable filtering of the initial data in the Fourier space provides uniform dispersive properties and a local smoothing effect. The key point to recover the decay rates (1.4) at the discrete level is to choose initial data with their SDFT supported away from the pathological points $\mathcal{M}_1^h$ in (2.18). Similarly, the local smoothing property holds uniformly on $h$ if the SDFT of the initial data is supported away from the points $\mathcal{M}_2^h$ in (2.28).

For any positive $\epsilon < \pi/2$ we define $\Omega_\epsilon^h$, the set of all the points in the cube $[-\pi/h, \pi/h]^d$ whose distance is at least $\epsilon/h$ from the set in which some of the second order derivatives of $p_h(\xi)$ vanish:

$$\Omega_{\epsilon,d}^h = \Big\{ \xi = (\xi_1, \dots, \xi_d) \in \Big[ -\frac{\pi}{h}, \frac{\pi}{h} \Big]^d : \Big| \xi_i \mp \frac{\pi}{2h} \Big| \geq \frac{\epsilon}{h},\ i = 1, \dots, d \Big\}.$$

Let us define the class of functions $\mathcal{I}_{\epsilon,d}^h \subset l^2(h\mathbb{Z}^d)$, whose SDFT is supported on $\Omega_{\epsilon,d}^h$:

$$\mathcal{I}_{\epsilon,d}^h = \{ \varphi^h \in l^2(h\mathbb{Z}^d) :\ \mathrm{supp}(\widehat{\varphi}^h) \subset \Omega_{\epsilon,d}^h \}. \tag{2.30}$$

We can view this subspace of initial data as a subclass of filtered data in the sense that the Fourier components corresponding to $\xi$ such that $|\xi_i \pm \pi/2h| \leq \epsilon/h$ have been cut-off or filtered out.

The following theorem shows that for initial data in this class the semigroup $S^h(t)$ has the same long time behavior as the continuous one, independently of $h$ in what concerns the $l^{p'}(h\mathbb{Z}^d) - l^p(h\mathbb{Z}^d)$ decay property.

**Theorem 2.3.** *Let be $0 < \epsilon < \pi/2$ and $p \geq 2$. There exists a positive constant $C(\epsilon, p, d)$ such that*

$$\| S^h(t) \varphi^h \|_{l^p(h\mathbb{Z}^d)} \leq C(\epsilon, p, d) |t|^{-\frac{d}{2}\left(1 - \frac{2}{p}\right)} \| \varphi^h \|_{l^{p'}(h\mathbb{Z}^d)},\ t \neq 0 \tag{2.31}$$

*holds for all $\varphi^h \in l^{p'}(h\mathbb{Z}^d) \cap \mathcal{I}_{\epsilon,d}^h$, uniformly on $h > 0$.*

*Proof.* A scaling argument reduces the proof to the case $h = 1$. For any $\varphi^1 \in \mathcal{I}_{\epsilon,d}^1$ the solution of (1.9) is given by $S^1(t)\varphi^1 = K_{\epsilon,d}^1 * \varphi^1$ where

$$K_{\epsilon,d}^1(t, \mathbf{j}) = \int_{\Omega_{\epsilon,d}^1} e^{itp_1(\xi)} e^{i\mathbf{j}\cdot\xi} d\xi, \quad \mathbf{j} \in \mathbb{Z}^d. \tag{2.32}$$

As a consequence of Young's inequality it remains to prove that

$$\| K_{\epsilon,d}^1(t) \|_{l^p(\mathbb{Z}^d)} \leq C(\epsilon, p, d) |t|^{-d/2(1-1/p)} \tag{2.33}$$

for any $p \geq 2$ and for all $t \neq 0$. Observe that it is then sufficient to prove (2.33) in the one-dimensional case. Using that the second derivative of the function $\sin^2(\xi/2)$ is positive on $\Omega_{\epsilon,1}^1$ we obtain by the Van der Corput Lemma (Prop. 2, Ch. 8, p. 332, [26]) that $\| K_{\epsilon,1}^1(t) \|_{l^\infty(\mathbb{Z})} \leq c(\epsilon) |t|^{-1/2}$ which finishes the proof. □

A similar result can be stated for the local smoothing effect. For a positive $\epsilon$, let us define the set $\widetilde{\Omega}_{\epsilon,d}^h$ of all points located at a distance of at least $\epsilon/h$ from the points $(\pm\pi/h)^d$:

$$\widetilde{\Omega}_{\epsilon,d}^h = \Big\{ \xi \in \Big[ -\frac{\pi}{h}, \frac{\pi}{h} \Big]^d : \Big| \xi_i \mp \frac{\pi}{h} \Big| \geq \frac{\epsilon}{h},\ i = 1, \dots, d \Big\}.$$

Observe that on $\widetilde{\Omega}_{\epsilon,d}^h$ the symbol $p_h(\xi)$ has no critical points other than $\xi = 0$. A similar argument as in [15] shows that the linear semigroup $S^h(t)$ gains 1/2-space derivative in $L_{t,x}^2$ with respect to the initial datum filtered as above. More precisely, if $\mathbf{P}_*^h$ denotes the band-limited interpolator (cf. [31], Ch. II):

$$(\mathbf{P}_*^h u^h)(x) = \int_{[-\pi/h,\, \pi/h]^d} \widehat{u}^h(\xi) e^{ix\cdot\xi} d\xi, \quad x \in \mathbb{R}^d, \tag{2.34}$$

the following holds.

**Theorem 2.4.** *Let be $\epsilon > 0$. There exists a positive constant $C(\epsilon, d)$ such that for any $R > 0$*

$$\int_{|x|>R}\int_{-\infty}^{\infty}|(-\Delta)^{1/4}\mathbf{P}_*^h e^{it\Delta_h}\varphi^h)|^2dtdx \le C(\epsilon,d)R\|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}$$

*holds for all $\varphi^h \in l^2(h\mathbb{Z}^d)$ with $\operatorname{supp}(\widehat{\varphi}^h) \subset \widetilde{\Omega}^h_{\epsilon,d}$, uniformly on $h > 0$.*

To prove this result we make use of the following Theorem.

**Theorem 2.5.** *(Theorem 4.1, [15]) Let $\mathcal{O}$ be an open set in $\mathbb{R}^d$, and $\psi$ be a $C^1(\mathcal{O})$ function such that $\nabla\psi(\xi) \neq 0$ for any $\xi \in \mathcal{O}$. Assume that there is $N \in \mathbb{N}$ such that for any $(\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}$ and $r \in \mathbb{R}$ the equations*

$$\psi(\xi_1, \dots, \xi_k, \underline{\xi}, \xi_{k+1}, \dots, \xi_{d-1}) = r,\ \ k = 0, \dots, d-1,$$

*have at most $N$ solutions $\underline{\xi} \in \mathbb{R}$. For $a \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^d)$ define*

$$W(t)f(x) = \int_{\mathcal{O}} e^{i(t\psi(\xi)+x\cdot\xi)}a(x,\psi(\xi))\widehat{f}(\xi)d\xi.$$

*Then for any $R > 0$*

$$\int_{|x|\le R}\int_{-\infty}^{\infty}|W(t)f(x)|^2dtdx \le cRN\int_{\mathcal{O}}\frac{|\widehat{f}(\xi)|^2}{|\nabla\psi(\xi)|}d\xi \tag{2.35}$$

*where $c$ is independent of $R$ and $N$ and $f$.*

**Remark 2.5.** *The result remains true for domains $\mathcal{O}$ where $|\nabla\psi|$ has zeros provided that the right hand side of (2.35) is finite.*

*Proof of Theorem 2.4.* Observe that for any $\varphi^h \in l^2(h\mathbb{Z}^d)$ with $\operatorname{supp}(\widehat{\varphi}^h) \subset \widetilde{\Omega}^h_{\epsilon,d}$ we have:

$$(\mathbf{P}_*^h e^{it\Delta_h}\varphi^h)(x) = \int_{\widetilde{\Omega}^h_{\epsilon,d}} e^{itp_h(\xi)}e^{ix\cdot\xi}\widehat{\varphi}^h(\xi)d\xi, \quad x \in \mathbb{R}^d.$$

Applying Theorem 2.5 with $\mathcal{O} = \widetilde{\Omega}^h_{\epsilon,d}$, $\psi = p_h(\xi)$, $a \equiv 1$ and using that $|\nabla p_h(\xi)| \ge c(\epsilon,d)|\xi|$ for all $\xi \in \widetilde{\Omega}^h_{\epsilon,d}$ we obtain that

$$\int_{|x|<R}\int_{-\infty}^{\infty}|(-\Delta)^{1/4}\mathbf{P}_*^h e^{it\Delta_h}\varphi^h|^2dtdx \lesssim \int_{\widetilde{\Omega}^h_{\epsilon,d}}\frac{|\widehat{\varphi}^h(\xi)|^2|\xi|}{|\nabla p_h(\xi)|}d\xi \lesssim \|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}.$$

This finishes the proof. □

2.4. **Strichartz estimates for filtered data.** In this section we derive Strichartz-like estimates for the operator $S^h(t)$ when it acts on functions belonging to $\mathcal{I}^h_{\epsilon,d}$, the class of functions defined in (2.30).

The main ingredient in obtaining Strichartz estimates is the following result due to Keel and Tao, [14].

**Theorem 2.6.** *([14], Theorem 1.2) Let $H$ be a Hilbert space, $(X, dx)$ be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings, which obey the energy estimate*

$$\|U(t)f\|_{L^2(X)} \le C\|f\|_H \tag{2.36}$$

*and the decay estimate*

$$\|U(t)U(s)^*g\|_{L^\infty(X)} \le C|t-s|^{-\sigma}\|g\|_{L^1(X)} \tag{2.37}$$

*for some $\sigma > 0$. Then*

$$\|U(t)f\|_{L^q(\mathbb{R},\,L^r(X))} \le C\|f\|_H \quad \text{for all} \quad f \in H, \tag{2.38}$$

$$\Big\| \int_{\mathbb{R}} U(s)^* F(s,\cdot)ds \Big\|_H \le C\|F\|_{L^{q'}(\mathbb{R},\,L^{r'}(X))} \quad \text{for all} \quad F \in L^{q'}(\mathbb{R},\,L^{r'}(X)), \tag{2.39}$$

$$\Big\| \int_0^t U(t)U(s)^* F(s,\cdot)ds \Big\|_{L^q(\mathbb{R},\,L^r(X))} \le C\|F\|_{L^{\tilde q'}(\mathbb{R},\,L^{\tilde r'}(X))} \quad \text{for all} \quad F \in L^{\tilde q'}(\mathbb{R},\,L^{\tilde r'}(X)), \tag{2.40}$$

*for any $\sigma$-admissible pairs $(q,r)$ and $(\tilde q, \tilde r)$.*

**Remark 2.6.** *With the same arguments as in* [14]*, the following also holds for all $(q,r)$ and $(\tilde q,\tilde r)$, $\sigma$-admissible pairs:*

$$\Big\| \int_0^t U(t-s)F(s,\cdot)ds \Big\|_{L^q(\mathbb{R},\,L^r(X))} \le C\|F\|_{L^{\tilde q'}(\mathbb{R},\,L^{\tilde r'}(X))}. \tag{2.41}$$

*In the case of the Schrödinger semigroup, $S(t-s) = S(t)S(s)^*$, so* (2.41) *and* (2.40) *coincide. However, in our applications we will often deal with operators that do not satisfy $S(t-s) = S(t)S(s)^*$.*

Let us choose $0 < \epsilon < \pi/2$, $K_d^{1,\epsilon}$ as in (2.32) and $U(t)\varphi^1 = K_d^{1,\epsilon} * \varphi^1$. We apply the above theorem to $U(t)$, with $X = \mathbb{Z}^d$, $dx$ being the counting measure and $H = l^2(\mathbb{Z}^d)$. In this way we obtain Strichartz estimates for the semigroup $S^1(t)$ when acting on $\mathcal{I}^1_{\epsilon,d}$, i.e. when $h = 1$. Then, by scaling, we obtain the following result in the class of filtered initial data.

**Theorem 2.7.** *Let $0 < \epsilon < \pi/2$ and $(q,r)$, $(\tilde q,\tilde r)$ be two $d/2$-admissible pairs.*
*i) There exists a positive constant $C(d,r,\epsilon)$ such that*

$$\|S^h(\cdot)\varphi^h\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))} \le C(d,r,\epsilon)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{2.42}$$

*holds for all functions $\varphi^h \in \mathcal{I}^h_{\epsilon,d}$ and for all $h > 0$.*
*ii) There exists a positive constant $C(d,r,\tilde r,\epsilon)$ such that*

$$\Big\| \int_0^t S^h(t-s)f^h(s)ds \Big\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))} \le C(d,r,\tilde r,\epsilon)\|f^h\|_{L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(h\mathbb{Z}^d))} \tag{2.43}$$

*holds for all functions $f^h \in L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(h\mathbb{Z}^d))$ with $f(t) \in \mathcal{I}^h_{\epsilon,d}$ for a.e. $t \in \mathbb{R}$, and for all $h > 0$.*

2.5. **On the cubic NSE.** In the previous sections we have seen that the linear semidiscrete scheme (1.9) does not satisfy uniform (with respect to $h$) dispersive estimates. Accordingly we can not use it to get numerical approximations for the NSE with uniform bounds on spaces of the form $L^q((0,T), l^r(h\mathbb{Z}^d))$. However, one could agree that, even if a perturbation argument based on the variation of constants formula and the dispersive properties of the linear scheme does not provide uniform bounds for the nonlinear problem, these estimates could still be true.

In this section we give an explicit example showing that a numerical scheme for the cubic NSE based on the conservative scheme (1.9) does not satisfy uniform bounds in $L^q((0,T), l^r(h\mathbb{Z}^d))$. This shows that the conservative scheme (1.9) can not be used neither for the LSE nor for the NSE within the $L^q((0,T), l^r(h\mathbb{Z}^d))$ setting.

We consider an approximation scheme to the one-dimensional NSE with nonlinearity $2|u|^2u$:

$$i\partial_t u^h_n + (\Delta_h u^h)_n = |u^h_n|^2(u^h_{n+1} + u^h_{n-1}). \tag{2.44}$$

In the sequel we shall refer to it as the Ablowitz-Ladik [1] approximation for NSE.

As we shell see, this scheme possesses explicit solutions which blow up in any $L^q_{loc}(\mathbb{R},\,l^r(h\mathbb{Z}))$-norm with $r > 2$ and $q \ge 1$. We point out that this is compatible with the $L^2$-convergence of the numerical scheme (2.44) for smooth initial data [1, 2].

Let us consider $\varphi \in L^2(\mathbb{R})$ as initial data for (1.2) with $F(u) = 2u|u|^2$. As initial condition for (2.44) we take $u^h(0) = \varphi^h$, $\varphi^h$ being an approximation of $\varphi$. Let us assume the existence of a positive $T$ such that for any $h > 0$, there exists $u^h \in L^\infty([0,T],\,l^2(h\mathbb{Z}))$ solution of (2.44). The uniform boundedness of $\{u^h\}_{h>0}$ in $L^\infty([0,T],\, l^2(h\mathbb{Z}))$ does not suffice to prove its convergence to the solution of (1.2). One needs to analyze whether the solutions of (2.44) are uniformly bounded, with respect to $h$, in one of the auxiliary spaces $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$, a property that will guarantee that any possible limit point of $\{u^h\}_{h>0}$ belongs to $L^q((0,T),\, L^r(\mathbb{R}))$. We are going to show that these uniform estimates do no hold in general.

To do that we look for explicit travelling wave solutions of (2.44). By scaling, the problem can be reduced to the case $h=1$. Indeed, $u^h$ is a solution of (2.44) if the scaled function:

$$u^1_n(t) = h u^h_n(th^2),\ n \in \mathbb{Z},\ t \geq 0,$$

solves (2.44) for $h=1$. In this case, $h=1$, there are explicit solutions of (2.44) of the form:

$$u^1_n(t) = A \exp(i(an - bt)) \operatorname{sech}(cn - dt) \tag{2.45}$$

for suitable constants $A, a, b, c, d$ (for the explicit values we refer to [2], p. 84).

In view of the structure of $u^1$ it is easy to see that the solutions of (2.44), obtained from $u^1$ by scaling, are not uniformly bounded as $h \to 0$ in any auxiliary space $L^q((0,T),\, l^r(h\mathbb{Z}))$ with $r > 2$. Indeed, a scaling argument shows that

$$\frac{\|u^h\|_{L^q((0,T),\, l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} = h^{\frac{1}{r}+\frac{2}{q}-\frac{1}{2}} \frac{\|u^1\|_{L^q((0,T/h^2),\, l^r(\mathbb{Z}))}}{\|u^1(0)\|_{l^2(\mathbb{Z})}}.$$

Observe that, for any $t>0$, the $l^r(\mathbb{Z})$-norm behaves as a constant:

$$\|u^1(t)\|_{l^r(\mathbb{Z})} \simeq \Big(\int_{\mathbb{R}} \operatorname{sech}^r(cx - dt)dx\Big)^{1/r} = \Big(\int_{\mathbb{R}} \operatorname{sech}^r(cx)dx\Big)^{1/r}.$$

Thus, for all $T>0$ and $h>0$ the solution $u^1$ satisfies

$$\|u^1\|_{L^q((0,T/h^2),\, l^r(\mathbb{Z}))} \simeq (Th^{-2})^{1/q}.$$

Consequently for any $r>2$ the solution $u^h$ on the lattice $h\mathbb{Z}$ satisfies:

$$\frac{\|u^h\|_{L^q((0,T),\, l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} \simeq h^{\frac{1}{r}-\frac{1}{2}} \to \infty,\ h \to 0.$$

This example shows that, in order to deal with the nonlinear problem, the linear approximation scheme needs to be modified. In the following section we present a method that preserves the dispersion properties and that can be used successfully at the nonlinear level.

## 3. A two-grid algorithm

In this section we present a conservative scheme that preserves the dispersive properties we discuss in the previous sections. In fact, the scheme we shall consider is the standard one (1.9). But, this time, in order to avoid the lack of dispersive properties associated with the high frequency components, the scheme (1.9) will be restricted to the class of filtered data obtained by a two-grid algorithm. The advantage of this filtering method with respect to the Fourier one is that the filtering can be realized in the physical space.

The method, inspired by [9], that extends to several space variables the one introduced in [11], is roughly as follows. We consider two meshes: the coarse one of size $4h$, $4h\mathbb{Z}^d$, and the finer one, the computational one $h\mathbb{Z}^d$, of size $h>0$. The method relies basically on solving the finite-difference semi-discretization (1.9) on the fine mesh $h\mathbb{Z}^d$, but only for slowly oscillating data, interpolated from the coarse grid $4h\mathbb{Z}^d$. As we shall see, the $1/4$ ratio between the two meshes is important to guarantee the convergence of the method. This particular structure of the data cancels the two pathologies of the discrete symbol mentioned in Section 2. Indeed, a careful Fourier analysis of those initial data shows that their discrete Fourier transform vanishes quadratically in each variable at the points $\xi = (\pm\pi/2h)^d$ and $\xi = (\pm\pi/h)^d$. As we shall see, this suffices to recover at the discrete level the dispersive properties of the continuous model.

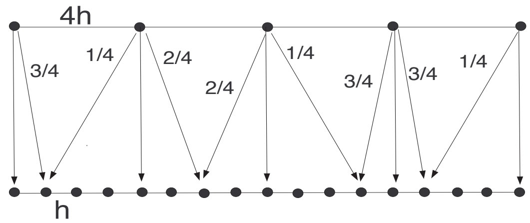


FIGURE 3. The action of the operator $\widetilde{\Pi}$ between the grids $4h\mathbb{Z}$ and $h\mathbb{Z}$

Once the discrete version of the dispersive properties has been proved, we explain how this method can be applied to a semi-discretization of the NSE with nonlinearity $f(u) = |u|^p u$. To do this, the nonlinearity has to be approximated in such a way that the approximate discrete nonlinearities belong to the subspace of filtered data as well.

3.1. **The two-grid algorithm in the linear framework.** To be more precise we introduce the following space of the slowly oscillating sequences. These sequences on the fine one $h\mathbb{Z}^d$ are those which are obtained from the coarse grid $4h\mathbb{Z}^d$ by an interpolation process. Note that, by scaling, any function defined on the lattice $h\mathbb{Z}^d$ can be viewed as a function on the lattice $\mathbb{Z}^d$. Thus it suffices to define this space for $h = 1$.

Let us consider the piecewise and continuous interpolator $\mathbf{P}_1^1$ acting on the coarse grid $4\mathbb{Z}^d$. We define the extension operator $\widetilde{\Pi} : l^2(4\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ by

$$(\widetilde{\Pi} f)_{\mathbf{j}} = (\mathbf{P}_1^1 f)_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^d, \quad f : 4\mathbb{Z}^d \to \mathbb{C}. \tag{3.46}$$

We then define the space of the slowly oscillating sequences, $\widetilde{\Pi}(4h\mathbb{Z}^d)$, as the image of the operator $\widetilde{\Pi}$ acting on functions defined on $4h\mathbb{Z}^d$. We will also make use of $\widetilde{\Pi}^* : l^2(h\mathbb{Z}^d) \to l^2(4h\mathbb{Z}^d)$, the adjoint of $\widetilde{\Pi}$, defined by:

$$(\widetilde{\Pi} g_1^{4h}, g_2^h)_{l^2(h\mathbb{Z}^d)} = (g_1^{4h}, \widetilde{\Pi}^* g_2^h)_{l^2(4h\mathbb{Z}^d)}, \forall\, g_1^{4h} \in l^2(4h\mathbb{Z}^d),\ g_2^h \in l^2(h\mathbb{Z}^d), \tag{3.47}$$

where $(\cdot,\cdot)_{l^2(h\mathbb{Z}^d)}$ and $(\cdot,\cdot)_{l^2(4h\mathbb{Z}^d)}$ are the inner products on $l^2(h\mathbb{Z}^d)$ and $l^2(4h\mathbb{Z}^d)$ respectively.

In the one dimensional case, the explicit expressions of $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ are given by

$$(\widetilde{\Pi} g^{4h})_{4j+r} = \frac{4-r}{4} g_{4j}^{4h} + \frac{r}{4} g_{4j+4}^{4h}, \quad j \in \mathbb{Z},\ r \in \{0,1,2,3\}.$$

and

$$(\widetilde{\Pi}^* g^h)_{4j} = \sum_{r=0}^{3} \frac{4-r}{4} g_{4j+r}^h + \frac{r}{4} g_{4j-4+r}^h, \quad j \in \mathbb{Z}.$$

As we will see, $S^h(t)$ has appropriate decay properties when acts on the subspace $\widetilde{\Pi}(4h\mathbb{Z}^d)$, uniformly on $h > 0$. The main results concerning the gain of integrability are given in the following Theorem.

**Theorem 3.1.** *Let $p \geq 2$ and $(q,r)$, $(\tilde{q},\tilde{r})$ be two $d/2$-admissible pairs. The following hold:*

*i) There exists a positive constant $C(d,p)$ such that*

$$\|S^h(t)\widetilde{\Pi}\varphi^{4h}\|_{l^p(h\mathbb{Z}^d)} \leq C(d,p)|t|^{-d(\frac{1}{2}-\frac{1}{p})}\|\widetilde{\Pi}\varphi^{4h}\|_{l^{p'}(h\mathbb{Z}^d)} \tag{3.48}$$

*for all $\varphi^{4h} \in l^{p'}(4h\mathbb{Z}^d)$, $h > 0$ and $t \neq 0$.*

*ii) There exists a positive constant $C(d,r)$ such that*

$$\|S^h(t)\widetilde{\Pi}\varphi^{4h}\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r)\|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)} \tag{3.49}$$

*for all $\varphi^{4h} \in l^2(4h\mathbb{Z}^d)$ and $h > 0$.*

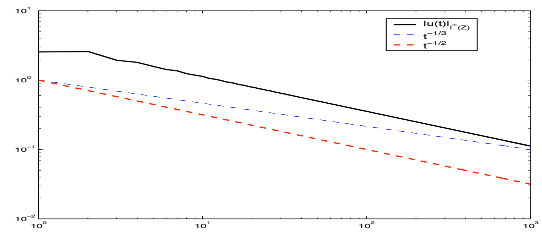

FIGURE 4. Log-log plot of the time evolution of the $l^\infty(\mathbb{Z})$-norm of $S^1(t)\widetilde{\Pi}\delta_0$ where $\delta_0$ is one in zero and vanishes otherwise.

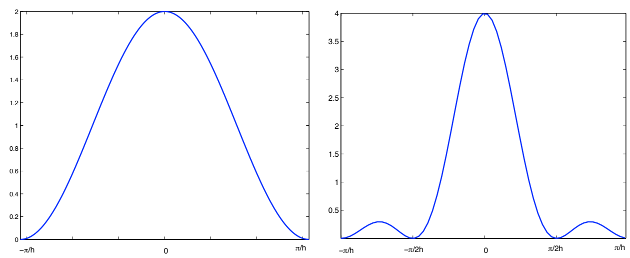

FIGURE 5. Multiplicative factors introduced by the two-grid algorithm in dimension one in the case of mesh ratio 1/2 and 1/4.

*iii) There exists a positive constant $C(d,r)$ such that*

$$\Big\| \int_{-\infty}^{\infty} S^h(t)^* \widetilde{\Pi} f^{4h}(s)ds \Big\|_{l^2(h\mathbb{Z}^d)} \leq C(d,r) \|\widetilde{\Pi} f^{4h}\|_{L^{q'}(\mathbb{R}, l^{r'}(h\mathbb{Z}^d))} \tag{3.50}$$

*for all $f^{4h} \in L^{q'}(\mathbb{R}, l^{\tilde{r}'}(4h\mathbb{Z}^d))$ and $h>0$.*

*iv) There exists a positive constant $C(d,r,\tilde{r})$ such that*

$$\Big\| \int_0^t S^h(t-s)\widetilde{\Pi} f^{4h}(s)ds \Big\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))} \leq C(d,r,\tilde{r}) \|\widetilde{\Pi} f^{4h}\|_{L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))} \tag{3.51}$$

*for all $f^{4h} \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(4h\mathbb{Z}^d))$ and $h>0$.*

**Remark 3.1.** *In the particular case $p=\infty$, estimate* (3.48) *shows that the solution of equation* (1.9) *with initial data in $\widetilde{\Pi}(4h\mathbb{Z}^d)$ decays as $t^{-d/2}$ when $t$ becomes large which agrees with the LSE. This can be seen in Figure 4 when as initial data has been chosen $\widetilde{\Pi}\delta_0$ ($\delta_0$ being the discrete Dirac function defined on the coarse grid $4h\mathbb{Z}$). The solution behaves as $t^{-1/2}$ in contrast with the case presented in Section 2, Figure 2, where the initial data was $\delta_0$ (the discrete Dirac function defined on the fine grid $h\mathbb{Z}$) and the decay was as $t^{-1/3}$.*

The following lemma gives a Fourier characterization of the data that are obtained by this two-grid algorithm involving the meshes $4h\mathbb{Z}^d$ and $h\mathbb{Z}^d$. Its proof uses only the definition of the discrete Fourier transform and we omit it.

**Lemma 3.1.** *Let $\psi^{4h} \in l^2(4h\mathbb{Z}^d)$. Then for all $\xi \in [-\pi/h, \pi/h]^d$*

$$\widehat{\widetilde{\Pi}\psi^{4h}}(\xi) = 4^d \widehat{\Pi\psi^{4h}}(\xi) \prod_{k=1}^{d} \cos^2(\xi_k h)\cos^2\big(\frac{\xi_k h}{2}\big), \tag{3.52}$$

*where $(\Pi\psi^{4h})_{\mathbf{j}} = \psi^{4h}_{\mathbf{j}}$ if $\mathbf{j} \in 4\mathbb{Z}^d$ and vanishes elsewhere.*

**Remark 3.2.** *Observe that the right hand side product in* (3.52) *vanishes on the sets* $\mathcal{M}_1^h$ *and* $\mathcal{M}_2^h$ *defined in Section 2.1 and Section 2.2 respectively. This will allow us to recover the dispersive properties of the numerical scheme introduced in this section.*

**Remark 3.3.** *A simpler two-grid construction could be done by interpolating* $2h\mathbb{Z}^d$ *sequences. We would get for all* $\psi^{2h} \in l^2(2h\mathbb{Z}^d)$ *and* $\xi \in [-\pi/h, \pi/h]^d$

$$\widehat{\widetilde{\Pi}\psi^{2h}}(\xi) = 2^d \widehat{\Pi\psi^{2h}}(\xi) \prod_{k=1}^{d} \cos^2\big(\frac{\xi_k h}{2}\big),$$

*where* $(\Pi\psi^{2h})_{\mathbf{j}} = \psi^{2h}_{\mathbf{j}}$ *if* $\mathbf{j} \in 2\mathbb{Z}^d$ *and vanishes elsewhere. This would cancel the spurious numerical solutions at the frequencies* $\mathcal{M}_2^h$*, but not at* $\mathcal{M}_1^h$*. In this case, as we proved in Section 2, the Strichartz estimates would fail to be uniform on* $h$*. Thus we rather choose* $1/4$ *as the ratio between the grids for the two-grid algorithm. We also point out that* $4$ *is the smallest quotient of the grids for which the decay* $l^1(h\mathbb{Z}^d) - l^\infty(h\mathbb{Z}^d)$ *holds uniformly in the mesh-parameter.*

*Proof of Theorem 3.1.* Let us define the weighted operators $A^h_\beta(t) : l^2(h\mathbb{Z}^d) \to l^2(h\mathbb{Z}^d)$ by

$$(\widehat{A^h_\beta(t)\psi^h})(\xi) = e^{-itp_h(\xi)} |g(\xi h)|^\beta \widehat{\psi^h}(\xi), \quad \xi \in [-\pi/h, \pi/h], \tag{3.53}$$

where

$$g(\xi) = \prod_{k=1}^{d} \cos(\xi_k) \cos\big(\frac{\xi_k}{2}\big).$$

We will prove that for any $\beta \geq 1/4$, $A^h_\beta(t)$ satisfies the hypotheses of Theorem 2.6. Then, according to Lemma 3.1, observing that $S^h(t)\widetilde{\Pi}\varphi^{4h} = 4^d A^h_2(t)\Pi\varphi^{4h}$, we obtain (3.49), (3.50) and (3.51).

It is easy to see that $\|A^h_\beta(t)\psi^h\|_{l^2(h\mathbb{Z}^d)} \leq \|\psi^h\|_{l^2(h\mathbb{Z}^d)}$. According to this, it remains to prove that for any $\beta \geq 1/4$ and $t \neq s$ the following holds:

$$\|A^h_\beta(t) A^h_\beta(s)^* \psi^h\|_{l^\infty(h\mathbb{Z}^d)} \leq c(\beta, d)|t-s|^{-d/2} \|\psi^h\|_{l^1(h\mathbb{Z}^d)}. \tag{3.54}$$

A scaling argument reduces the proof to the case $h = 1$. We claim that (3.54) holds once

$$\|A^1_\gamma(t)\psi^1\|_{l^\infty(\mathbb{Z}^d)} \leq c(\gamma, d)|t|^{-d/2} \|\psi^1\|_{l^1(\mathbb{Z}^d)} \tag{3.55}$$

is satisfied for all $\gamma \geq 1/2$. Indeed, using that the operator $A^1_\alpha(t)$ satisfies $A^1_\alpha(t)^* = A^1_\alpha(-t)$ we obtain

$$\|A^1_\beta(t)A^1_\beta(s)^*\psi^1\|_{l^\infty(\mathbb{Z}^d)} = \|A^1_\beta(t)A^1_\beta(-s)\psi^1\|_{l^\infty(\mathbb{Z}^d)} = \|A^1_{2\beta}(t-s)\psi^1\|_{l^\infty(\mathbb{Z}^d)} \lesssim |t-s|^{-d/2}\|\psi\|_{l^1(\mathbb{Z}^d)},$$

for all $t \neq s$ and $\psi^1 \in l^1(\mathbb{Z}^d)$.

In the following we prove (3.55). We write $A^1_\gamma(t)$ as a convolution $A^1_\gamma(t)\psi^1 = K^t_{d,\gamma} * \psi^1$ where $\widehat{K^t_{d,\gamma}}(\xi) = e^{-itp_1(\xi)}|g(\xi)|^\gamma$. By Young's inequality it is sufficient to prove that for any $\gamma \geq 1/2$ and $t \neq 0$ the following holds:

$$\|K^t_{d,\gamma}\|_{l^\infty(\mathbb{Z}^d)} \leq c(\gamma, d)|t|^{-d/2}. \tag{3.56}$$

We observe that $K^t_{d,\gamma}$ can be written by separation of variables as

$$\widehat{K^t_{d,\gamma}}(\xi) = \prod_{k=1}^{d} e^{-4it\sin^2(\frac{\xi_k}{2})} \big|\cos(\xi_k)\cos\big(\frac{\xi_k}{2}\big)\big|^\gamma \prod_{j=1}^{d} \widehat{K^t_{1,\gamma}}(\xi_j).$$

It remains to prove that (3.56) holds in one space dimension. We make use of the following Lemma:

**Lemma 3.2.** *(Corollary 2.9,* [15]*) Let* $(a,b) \subset \mathbb{R}$ *and* $\psi \in C^3(a,b)$ *be such that* $\psi''$ *changes of monotonicity at finitely many points in the interval* $(a,b)$*. Then*

$$\Big|\int_a^b e^{i(t\psi(\xi) - x\xi)} |\psi''(\xi)|^{1/2} \phi(\xi) d\xi\Big| \leq c_\psi |t|^{-1/2} \Big\{ \|\phi\|_{L^\infty(a,b)} + \int_a^b |\phi'(\xi)| d\xi \Big\}.$$

*holds for all real numbers $x$ and $t$.*

Applying the above Lemma with $\phi(\xi) = |\cos\xi|^{\gamma-1/2}|\cos(\xi/2)|^{\gamma}$, $\gamma \geq 1/2$, and $\psi(\xi) = -4\sin^2(\xi/2)$, we obtain (3.56) for $d = 1$, which finishes the proof. □

3.2. **A conservative approximation of the NSE.** We now build a convergent numerical scheme for the semilinear NSE equation in $\mathbb{R}^d$ :

$$\begin{cases} iu_t + \Delta u = |u|^p u, \ t \neq 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d. \end{cases} \tag{3.57}$$

Our analysis applies for the nonlinearity $f(u) = -|u|^p u$ as well. In fact, the key point for the proof of the global existence of the solutions is that the $L^2$-scalar product $(f(u), u)$ is a real number. All the results extend to more general nonlinearities $f(u)$ satisfying this condition under natural growth assumptions for $L^2$-solutions (see [3], Ch. 4.6, p. 109).

The first existence and uniqueness result for (3.57) with $L^2(\mathbb{R}^d)$ initial data is as follows.

**Theorem 3.2.** *(Global existence in $L^2(\mathbb{R}^d)$, Tsutsumi, [30]). For $0 \leq p < 4/d$ and $\varphi \in L^2(\mathbb{R}^d)$, there exists a unique solution $u$ in $C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$ with $q = 4(p+1)/pd$ that satisfies the $L^2$-norm conservation property and depends continuously on the initial condition in $L^2(\mathbb{R}^d)$.*

The proof uses standard arguments, the key ingredient being to work in the space $C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$. This can only be done using Strichartz estimates. Local existence is proved by applying a fixed point argument to the integral formulation of (3.57) in that space. Global existence holds because of the $L^2(\mathbb{R}^d)$-conservation property which excludes finite-time blow-up.

In order to introduce a numerical approximation of equation (3.57) it is convenient to give the definition of the weak solution of equation (3.57).

**Definition 3.1.** *We say that $u$ is a weak solution of (3.57) if*
*i) $u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$*
*ii) $u(0) = \varphi$ a.e. and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} u(-i\psi_t + \Delta\psi)dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^d} |u|^p u\psi dxdt \tag{3.58}$$

*for all $\psi \in \mathcal{D}(\mathbb{R}, H^2(\mathbb{R}^d))$, where $p$ and $q$ are as in the statement of Theorem 3.2.*

In this section we consider the following numerical approximation scheme for (3.57):

$$i\frac{du^h}{dt} + \Delta_h u^h = \widetilde{\Pi} f(\widetilde{\Pi}^* u^h), \ t \in \mathbb{R}; \qquad u^h(0) = \widetilde{\Pi}\varphi^{4h}, \tag{3.59}$$

with $f(u) = |u|^p u$.

In order to prove the global existence of solutions of (3.59), we will need to guarantee the conservation of the $l^2(h\mathbb{Z}^d)$-norm of solutions, a property that the solutions of NSE satisfy. The choice $\widetilde{\Pi} f(\widetilde{\Pi}^* u^h)$ as an approximation of the nonlinear term $f(u)$ is motivated by the fact that:

$$(\widetilde{\Pi} f(\widetilde{\Pi}^* u^h), u^h)_{l^2(h\mathbb{Z}^d)} = (f(\widetilde{\Pi}^* u^h), \widetilde{\Pi}^* u^h)_{l^2(4h\mathbb{Z}^d)} \in \mathbb{R}, \tag{3.60}$$

that, as mentioned above, guarantees the conservation of the $l^2(h\mathbb{Z}^d)$-norm.

The following holds:

**Theorem 3.3.** *Let $p \in (0, 4/d)$ and $q = 4(p+2)/dp$. Then for all $h > 0$ and for every $\varphi^{4h} \in l^2(4h\mathbb{Z}^d)$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap L^q_{loc}(\mathbb{R}, l^{p+2}(h\mathbb{Z}^d))$ of (3.59). Moreover, $u^h$ satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R}, l^2(h\mathbb{Z}^d))} \leq \|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)} \tag{3.61}$$

*and for all finite interval $I$*

$$\|u^h\|_{L^q(I, l^{p+2}(h\mathbb{Z}^d))} \leq c(I)\|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)}, \tag{3.62}$$

*where the above constants are independent of h.*

*Proof of Theorem 3.3.* The local existence and uniqueness can be proved, as in the continuous case, by a combination of the Strichartz-like estimates in Theorem 3.1 and of a fixed point argument in the space $L^\infty((-T,T), l^2(h\mathbb{Z}^d)) \cap L^q((-T,T), l^{p+2}(h\mathbb{Z}^d))$, $T$ being chosen small enough, depending on the initial data, but independent of $h$. Identity (3.60) guarantees the conservation of the $l^2$-norm of the solutions, and, consequently, the lack of blow-up and the global existence of the solutions. $\square$

3.3. **Convergence of the method.** In the sequel we use the piecewise constant interpolator $\mathbf{P}_0^h$. Given the initial datum $\varphi \in L^2(\mathbb{R}^d)$ for the PDE, we choose the approximating discrete data $(\varphi^{4h}_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$ such that $\mathbf{P}_0^h\widetilde{\Pi}\varphi^{4h}$ converges strongly to $\varphi$ in $L^2(\mathbb{R}^d)$. Thus, in particular, $\|\mathbf{P}_0^h\widetilde{\Pi}\varphi^{4h}\|_{L^2(\mathbb{R}^d)} \leq C(\|\varphi\|_{L^2(\mathbb{R}^d)})$.

The main convergence result is the following:

**Theorem 3.4.** *Let $p$ and $q$ be as in Theorem 3.3 and $u^h$ be the unique solution of (3.59) for the approximate initial data $\widetilde{\Pi}\varphi^{4h}$ as above. Then the sequence $\mathbf{P}_0^h u^h$ satisfies*

$$\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)),\ \mathbf{P}_0^h u^h \rightharpoonup u \text{ in } L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)), \tag{3.63}$$

$$\mathbf{P}_0^h u^h \to u \text{ in } L^2_{loc}(\mathbb{R}^{d+1}),\ \mathbf{P}_0^h\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u \text{ in } L^{q'}_{loc}(\mathbb{R}, L^{(p+2)'}(\mathbb{R}^d)) \tag{3.64}$$

*where $u$ is the unique solution of NSE.*

First we sketch the main ideas of the proof. The main difficulty in the proof of Theorem 3.4 is the strong convergence $\mathbf{P}_0^h u^h \to u$ in $L^2_{loc}(\mathbb{R}^{d+1})$ which is needed to pass to the limit in the nonlinear term. Once it is obtained, the second convergence in (3.64) easily follows. Another technical difficulty comes from the fact that the interpolator $\mathbf{P}_0^h$ is not compactly supported in the Fourier space. Thus we instead consider the band-limited interpolator $\mathbf{P}_*^h$ introduced in (2.34) and prove the compactness for $\mathbf{P}_*^h u^h$. Once this is obtained, the $L^2$-strong convergence of $\mathbf{P}_*^h u^h$ is transferred to $\mathbf{P}_0^h u^h$. This is a consequence of the following property of both interpolators (cf. [22], Th. 3.4.2, p. 90):

$$\|\mathbf{P}_0^h u^h(t) - \mathbf{P}_*^h u^h(t)\|_{L^2(\Omega)} \leq h\|\mathbf{P}_*^h u^h(t)\|_{H^1(\Omega)}, \tag{3.65}$$

which holds for all real $t$ and $\Omega \subset \mathbb{R}^d$.

To prove the $L^2$-strong convergence of $\mathbf{P}_*^h u^h$ we will show that it is uniformly bounded in $L^2_{loc}(\mathbb{R}, H^{1/2}_{loc}(\mathbb{R}^d))$. We shall also obtain estimates on the $L^2_{loc}(\mathbb{R}, H^1_{loc}(\mathbb{R}^d))$-norm which are not uniform on $h$ but, according to (3.65), suffice to ensure that $\mathbf{P}_0^h u^h - \mathbf{P}_*^h u^h$ strongly converges to zero in $L^2_{loc}(\mathbb{R}^{d+1})$. The following lemma provides local estimates for $\mathbf{P}_*^h u^h$ in $H^s$-norm.

**Lemma 3.3.** *Let be $s \geq 1/2$, $I \subset \mathbb{R}$ a bounded interval and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then there is a constant $C(I,\chi)$, independent of $h$, such that*

$$\|\chi \mathbf{P}_*^h(S^h(t)\widetilde{\Pi}\varphi^{4h})\|_{L^2(I,\, H^s(\mathbb{R}^d))} \leq \frac{C(I,\chi)}{h^{s-1/2}}\|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)} \tag{3.66}$$

*holds for all functions $\varphi^{4h} \in l^2(4h\mathbb{Z}^d)$ and $h > 0$. Moreover for any $d/2$-admissible pair $(q,r)$*

$$\Big\|\chi \mathbf{P}_*^h\Big(\int_0^t S^h(t-\tau)\widetilde{\Pi} f^{4h}(\tau)d\tau\Big)\Big\|_{L^2(I,\, H^s(\mathbb{R}^d))} \leq \frac{C(I,\chi)}{h^{s-1/2}}\|\widetilde{\Pi} f^{4h}\|_{L^{q'}(I, l^{r'}(h\mathbb{Z}^d))} \tag{3.67}$$

*for all $f^{4h} \in L^{q'}(I, l^{r'}(4h\mathbb{Z}^d))$ and $h > 0$.*

*Proof of Lemma 3.3.* We divide the proof in two steps. The first one concerns the homogeneous estimate (3.66) and the second one (3.67).

**Step I. Regularity of the homogeneous term.** To prove (3.66) it is sufficient to prove, for any $R>0$, the existence of a positive constant $C(I,R)$ such that

$$\int_I\int_{|x|<R}|(-\Delta)^{s/2}\mathbf{P}^h_*(S^h(t)\widetilde{\Pi}\varphi^{4h})|^2dxdt\leq \frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^{4h}(\xi)|^2d\xi.$$

Let us consider $\psi^h\in l^2(h\mathbb{Z}^d)$. Applying Theorem 2.5 to the function $\mathbf{P}^h_*(S^h(t)\psi^h)$ we obtain

$$\int_I\int_{|x|<R}|(-\Delta)^{s/2}\mathbf{P}^h_*(S^h(t)\psi^h)|^2dxdt\leq C(I,R)\int_{[-\pi/h,\pi/h]^d}\frac{|\xi|^{2s}|\widehat{\mathbf{P}^h_*\psi^h}(\xi)|^2d\xi}{|\nabla p_h(\xi)|}\tag{3.68}$$
$$\leq \frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}\frac{(\sum_{j=1}^d\xi_j^2)^{1/2}|\widehat{\psi^h}(\xi)|^2d\xi}{(\sum_{j=1}^d\sin^2(\xi_jh)/h^2)^{1/2}}\lesssim\frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\psi}^h(\xi)|^2d\xi}{\prod_{j=1}^d|\cos(\xi_jh/2)|},$$

provided that all terms make sense. Note that this estimate holds for all $\psi\in l^2(h\mathbb{Z}^d)$. Observe however that the term in the denominator in the right hand side integral may vanish for the high frequencies $\xi=(\pm\pi/h)^d$. In order to compensate this fact we consider initial data in the class of slowly oscillating sequences $\widetilde{\Pi}(4h\mathbb{Z}^d)$. Now, we apply the last estimates to $\psi^h=\widetilde{\Pi}\varphi^{4h}$. Thus

$$\int_I\int_{|x|<R}|(-\Delta)^{s/2}\mathbf{P}^h_*(S^h(t)\widetilde{\Pi}\varphi^{4h})|^2dxdt\leq\frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\widetilde{\Pi}\varphi^{4h}}(\xi)|^2d\xi}{\prod_{j=1}^d|\cos(\xi_jh/2)|}$$
$$\leq\frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^{4h}(\xi)|^2\prod_{j=1}^d|\cos(\xi_jh/2)|^3d\xi\leq\frac{C(I,R)}{h^{2s-1}}\|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)}.$$

**Step II. Regularity of the inhomogeneous term.** In the following we prove (3.67). This estimate will be reduced to the homogenous one (3.66) by using the argument of Christ and Kiselev [4] (see also [24] in the context of PDE). A simplified version, useful in PDE applications, is given in [24]:

**Lemma 3.4.** *Let $X$ and $Y$ be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(X,Y)$, the space of bounded linear mappings from $X$ to $Y$. Suppose that $-\infty\leq a<b\leq\infty$ and set*

$$Tf(t)=\int_a^bK(t,s)f(s)ds,\quad Wf(t)=\int_a^tK(t,s)f(s)ds.$$

*Assume that $1\leq p<q\leq\infty$ and $\|Tf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}$. Then $\|Wf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}$.*

Without loss of generality we can consider $I=[0,T]$. In view of the above Lemma it is sufficient to prove that the operator

$$Tf^{4h}(t)=\chi\mathbf{P}^h_*\Big(\int_0^TS^h(t-\tau)\widetilde{\Pi}f^{4h}(\tau)d\tau\Big)$$

satisfies

$$\|Tf^{4h}\|_{L^2([0,T],\,H^s(\mathbb{R}^d))}\leq\frac{C(T,\chi)}{h^{s-1/2}}\|\widetilde{\Pi}f^{4h}\|_{L^{q'}([0,T],\,l^{r'}(h\mathbb{Z}^d))}.$$

We write $Tf^{4h}$ as $Tf^{4h}(t)=\chi\mathbf{P}^h_*S^h(t)T_1f^{4h}(t)$ where

$$T_1f^{4h}(t)=\int_0^TS^h(s)^*\widetilde{\Pi}f^{4h}(s)ds.$$

Estimate (3.68) yields

$$\|Tf^{4h}\|_{L^2([0,T],\,H^s(\mathbb{R}^d))} \leq \frac{C(I,\chi)}{h^{s-1/2}}\Big\|\frac{\widehat{T_1 f^{4h}}(\xi)}{\prod_{j=1}^d|\cos(\xi_j h/2)|^{1/2}}\Big\|_{L^2([-\pi/h,\pi/h]^d)}$$
$$\lesssim \frac{C(I,\chi)}{h^{s-1/2}}\Big\|\frac{\widehat{T_1 f^{4h}}(\xi)}{\prod_{j=1}^d|\cos(\xi_j h/2)|^{1/2}|\cos(\xi_j h)|^{1/2}}\Big\|_{L^2([-\pi/h,\pi/h]^d)}$$

provided that all the above integrals are finite.

Explicit computations on $T_1 f^{4h}$ show that

$$\frac{\widehat{T_1 f^{4h}}(\xi)}{\prod_{j=1}^d|\cos(\xi_j h/2)|^{1/2}|\cos(\xi_j h)|^{1/2}} = 4^d\int_0^T e^{isp_h(\xi)}\prod_{j=1}^d|\cos(\frac{\xi_j h}{2})|^{3/2}|\cos(\xi_j h)|^{3/2}\widehat{\Pi f^{4h}}(\xi,s)ds$$
$$= 4^d\Big(\int_0^T (A^h_{3/2}(s))^*\Pi f^{4h}(s)ds\Big)\widehat{\ }(\xi),$$

where the operator $A^h_{3/2}$ is defined in (3.53).

Applying Theorem 2.6 to the operator $A^h_{3/2}$ we obtain, by estimate (2.39), that

$$\Big\|\int_0^T (A^h_{3/2}(s))^*\Pi f^{4h}(s)\Big\|_{l^2(h\mathbb{Z}^d)} \lesssim \|\Pi f^{4h}\|_{L^{q'}([0,T],\,l^{r'}(h\mathbb{Z}^d))} \lesssim \|\widetilde{\Pi} f^{4h}\|_{L^{q'}([0,T],\,l^{r'}(h\mathbb{Z}^d))}.$$

The proof is now complete. □

*Proof of Theorem 3.4.* Using (3.61) we obtain that $\mathbf{P}^h_0 u^h$ is uniformly bounded in $L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))$. This guarantees the existence of a function $u\in L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))$ such that, up to a subsequence, $\mathbf{P}^h_0 u^h \overset{\star}{\rightharpoonup} u$ in $L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))$. By (3.62) we obtain that $u\in L^q(I,\,L^{p+2}(\mathbb{R}^d))$ and, up to a subsequence, $\mathbf{P}^h_0 u^h\rightharpoonup u$ in $L^q(I,\,L^{p+2}(\mathbb{R}^d))$.

In the following we prove the strong convergence of $\mathbf{P}^h_0 u^h$. First we prove that $\mathbf{P}^h_0 u^h - \mathbf{P}^h_* u^h \to 0$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. Second we prove the compactness of $\mathbf{P}^h_* u^h$. Finally we obtain that $\mathbf{P}^h_0 u^h \to u$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$.

For any $\Omega\subset\mathbb{R}^d$, classical properties of the interpolator $\mathbf{P}^h_0 u^h$ (see [22], Th. 3.4.2, p. 90) give us

$$\int_\Omega |\mathbf{P}^h_0 u^h - \mathbf{P}^h_* u^h|^2 dx \leq h^2\|\mathbf{P}^h_* u^h\|^2_{H^1(\Omega)}.$$

Applying Lemma 3.3 with $s=1$ we obtain, for any $\chi\in C^\infty_c(\mathbb{R}^d)$,

$$\begin{aligned}\int_I\int_{\mathbb{R}^d}\chi^2|\mathbf{P}^h_0 u^h - \mathbf{P}^h_* u^h|^2dxdt &\leq h^2\int_I\int_{\mathbb{R}^d}\chi^2|(I-\Delta)^{1/2}\mathbf{P}^h_* u^h|^2dxdt\\ &\leq hC(I,\|\widetilde{\Pi}\varphi^{4h}\|^2_{l^2(h\mathbb{Z}^d)})\to 0,\ h\to 0.\end{aligned}$$

This shows that $\mathbf{P}^h_0 u^h - \mathbf{P}^h_* u^h\to 0$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$.

Using Lemma 3.3 with $s=1/2$ we obtain that for any smooth function $\chi$, $\mathbf{P}^h_* u^h$ satisfies

$$\|\chi\mathbf{P}^h_* u^h\|_{L^2(I,\,H^{1/2}(\mathbb{R}^d))}\leq C(I,\chi,\|\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)}).$$

We can also prove the following uniform boundedness property of its time derivative:

$$\Big\|\frac{d\mathbf{P}^h_* u^h}{dt}\Big\|_{L^1(I,\,H^{-2}(\mathbb{R}^d))}\leq \|\Delta_h\mathbf{P}^h_* u^h\|_{L^1(I,\,H^{-2}(\mathbb{R}^d))} + \|\mathbf{P}^h_*(|u^h|^p u^h)\|_{L^1(I,\,H^{-2}(\mathbb{R}^d))}$$
$$\leq \|\mathbf{P}^h_* u^h\|_{L^1(I,\,L^2(\mathbb{R}^d))} + \|\mathbf{P}^h_*(|u^h|^p u^h)\|_{L^1(I,\,L^{(p+2)'}(\mathbb{R}^d))}\leq C(I,\|\varphi\|_{L^2(\mathbb{R}^d)}).$$

Using the embeddings $H^s(\Omega)\underset{comp}{\hookrightarrow} L^2(\Omega)\hookrightarrow H^{-2}(\Omega)$, $\Omega\subset\mathbb{R}^d$ bounded domain, and the compactness results of [23] we obtain the existence of a function $v$ such that, up to subsequences, $\mathbf{P}^h_* u^h\to v$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. Using the strong convergence of $\mathbf{P}^h_* u^h$ towards $v$ we obtain that $v=u$ and $\mathbf{P}^h_0 u^h\to u$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$.

Let $\Gamma \subset \mathbb{Z}^d$ be a finite set. Thus for any $s \in \Gamma$ we have $\mathbf{P}_0^h u^h(\cdot + sh) \to u$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^d)$ and $\mathbf{P}_0^h u^h(\cdot + sh) \to u$ a.e. in $\mathbb{R} \times \mathbb{R}^d$. The operators $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ involve only a finite number of translations. Then $\mathbf{P}_0^h \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \to |u|^p u$ a.e. in $\mathbb{R}\times\mathbb{R}^d$. and $\mathbf{P}_0^h \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u$ in $L^{q'}(I,\, L^{(p+2)'}(\mathbb{R}^d))$.

Multiplying (3.59) by a function $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, $\mathbf{P}_0^h u^h$ satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{P}_0^h u^h (-i\psi_t + \Delta^h \psi) dx dt = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{P}_0^h \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \psi dx dt. \tag{3.69}$$

All the above weak convergences of $\mathbf{P}_0^h u^h$ and (3.69) show that $u$ satisfies (3.58).

It remains to prove that $u \in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ and $u(0) = \varphi$. To prove that $u \in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ we show its continuity at $t = 0$, the same argument works at any time $t$.

For any positive $0 \leq t \leq T < 1$, the Strichartz estimates in Theorem 3.1 and the Hölder inequality in time variable applied to the variation of constants formula give us:

$$\begin{aligned}
\|u^h(t) - S^h(t)\widetilde{\Pi}\varphi^{4h}\|_{l^2(h\mathbb{Z}^d)} &\leq \Big\| \int_0^t S^h(t-s)\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) ds \Big\|_{L^\infty([0,T],\, l^2(\mathbb{Z}^d))} \\
&\lesssim \| |u^h|^p u^h \|_{L^{q(h)'}([0,T],\, l^{(p+2)'}(h\mathbb{Z}^d))} \leq T^{(q-(p+2))/q} \|u^h\|^{p+1}_{L^q([0,T],\, l^{p+2}(h\mathbb{Z}^d))} \\
&\lesssim T^{1-pd/4} C(\|\varphi\|_{L^2(\mathbb{R}^d)}).
\end{aligned}$$

Using that $\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u$ and $\mathbf{P}_0^h S^h(\cdot)\varphi^h \overset{*}{\rightharpoonup} S(\cdot)\varphi$ in $L^\infty([0,T], L^2(\mathbb{R}^d))$ we get

$$\|u(t) - S(t)\varphi\|_{L^2(\mathbb{R}^d)} \leq \liminf_{h\to 0} \|\mathbf{P}_0^h u^h(\cdot) - \mathbf{P}_0^h S^h(\cdot)\widetilde{\Pi}\varphi^{4h}\|_{L^\infty([0,T],\, L^2(\mathbb{R}^d))} \lesssim T^{1-pd/4} C(\|\varphi\|_{L^2(\mathbb{R}^d)}).$$

This proves that the solution $u$ obtained as limit of $\mathbf{P}_0^h u^h$, satisfies $u(t) \to \varphi$ in $L^2(\mathbb{R}^d)$ as $t \to 0$.

The uniqueness of the limit, a solution of the NSE (3.57), allows us to deduce that the whole sequence $\mathbf{P}_0^h u^h$ converges without extracting subsequences.

The proof of Theorem 3.4 is now complete. ☐

3.4. **The critical case** $p = 4/d$**.** Our method works similarly in the critical case $p = 4/d$ for small initial data. More precisely, the following holds.

**Theorem 3.5.** *There exists a constant $\epsilon$, independent of $h$, such that for all initial data $\varphi^h \in \widetilde{\Pi}(4h\mathbb{Z}^d)$ with $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} < \epsilon$, the semidiscrete critical equation (3.59) with $p = 4/d$ has a unique global solution $u^h \in C(\mathbb{R},\, l^2(h\mathbb{Z}^d)) \cap L^{2+4/d}_{loc}(\mathbb{R},\, l^{2+4/d}(h\mathbb{Z}^d))$. Moreover for any $d/2$-admissible pair $(q,r)$, $u^h \in L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}^d))$ and $\|u^h\|_{L^q(I,\, l^r(h\mathbb{Z}^d)} \leq C(q,I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$ for all finite intervals $I$, uniformly on $h$.*

With the same notations, as in the subcritical case, the following convergence result holds.

**Theorem 3.6.** *Let $p = 4/d$. Under the smallness assumption of Theorem 3.5, the sequence $\mathbf{P}_0^h u^h$ satisfies*

$$\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d)),\ \mathbf{P}_0^h u^h \rightharpoonup u \text{ in } L^{4/d+2}_{loc}(\mathbb{R},\, L^{4/d+2}(\mathbb{R}^d)),$$
$$\mathbf{P}_0^h u^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d),\ \mathbf{P}_0^h \widetilde{\Pi}(f(\widetilde{\Pi}^* u^h)) \rightharpoonup |u|^{4/d} u \text{ in } L^{(4/d+2)'}_{loc}(\mathbb{R},\, L^{(4/d+2)'}(\mathbb{R}^d))$$

*where $u$ is the unique weak solution of the critical NSE with $p = 4/d$.*

In contrast with the viscous numerical scheme introduced in [12] this time we do not need to modify the exponent $4/d$ of the nonlinearity in the numerical scheme. In the present case, the class of Strichartz estimates for the linear semidiscrete semigroup hold for $d/2$-admissible pairs and not for the some $\alpha$-admissible pairs, $\alpha > d/2$. This allows us to use, for the numerical scheme based on the two-grid method, exactly the same nonlinearity as that given by the nonlinear problem after adapting it by means of extension and restriction operators $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ as in (3.59).

We have analyzed here the case of small $L^2$-initial data. In the continuous case, the global wellposedness can be proved under a more general assumption:

$$\|e^{it\Delta}\varphi\|_{L^{2+4/d}(\mathbb{R}, L^{2+4/d}(\mathbb{R}^d))} \leq c_0, \tag{3.70}$$

for some sufficiently small constant $c_0$. Examples of $\varphi$ satisfying (3.70) with large $L^2(\mathbb{R}^d)$-norm are given in [17] (Ch. 5, Section 5.4, p. 108-109).

At the numerical level, condition (3.70) can be replaced by

$$\|S^h(t)\varphi^h\|_{L^{2+4/d}(\mathbb{R},\, l^{2+4/d}(h\mathbb{Z}^d)} \leq c_1, \tag{3.71}$$

where $c_1$ is a positive, small enough constant and $\varphi^h \in \widetilde{\Pi}(4h\mathbb{Z}^d)$. Clearly, for $\varphi^h \in \widetilde{\Pi}(4h\mathbb{Z}^d)$ with small $l^2(h\mathbb{Z}^d)$-norm, estimate (3.49) shows (3.71). The construction of $\varphi^h \in \widetilde{\Pi}(4h\mathbb{Z}^d)$ with large $l^2(h\mathbb{Z}^d)$-norm satisfying (3.71) is an open problem.

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## References

1. M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations*, J. Mathematical Phys. **16** (1975), 598–603.
2. M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Note Series, vol. 302, Cambridge University Press, Cambridge, 2004.
3. T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics 10, Providence, RI: American Mathematical Society, New York, Courant Institute of Mathematical Sciences. xiii, 2003.
4. M. Christ and A. Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), no. 2, 409–425.
5. P. Constantin and J.C. Saut, *Local smoothing properties of dispersive equations*, J. Am. Math. Soc. **1** (1988), no. 2, 413–439.
6. J. Giannoulis, M. Herrmann, and A. Mielke, *Continuum descriptions for the dynamics in discrete lattices: derivation and justification*, Analysis, Modeling and Simulation of Multiscale Problems, A. Mielke editor, vol. 18, Springer, 2006, pp. 435–466.
7. G. Gigante and F. Soria, *On a sharp estimate for oscillatory integrals associated with the Schrödinger equation*, Int. Math. Res. Not. (2002), no. 24, 1275–1293.
8. J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 4, 309–327.
9. R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation*, J. Comput. Phys. **103** (1992), no. 2, 189–221.
10. L.I. Ignat, *Fully discrete schemes for the Schrödinger equation. Dispersive properties*, Mathematical Models and Methods in Applied Sciences **17** (2007), no. 4, 567–591.
11. L.I. Ignat and E. Zuazua, *A two-grid approximation scheme for nonlinear Schrödinger equations: dispersive properties and convergence*, C. R. Acad. Sci. Paris, Ser. I **341** (2005), no. 6, 381–386.
12. ———, *Dispersive properties of a viscous numerical scheme for the Schrödinger equation*, C. R. Acad. Sci. Paris, Ser. I **340** (2005), no. 7, 529–534.
13. ———, *Dispersive properties of numerical schemes for nonlinear Schrödinger equations*, Foundations of Computational Mathematics, Santander 2005, L. M. Pardo et al. eds, vol. 331, London Mathematical Society Lecture Notes, 2006, pp. 181–207.
14. M. Keel and T. Tao, *Endpoint Strichartz estimates*, Am. J. Math. **120** (1998), no. 5, 955–980.
15. C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), no. 1, 33–69.
16. ———, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), no. 3, 255–288.
17. F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, Publicações Matemáticas, IMPA, Rio de Janeiro, 2004.
18. A. Magyar, E. M. Stein, and S. Wainger, *Discrete analogues in harmonic analysis: spherical averages*, Ann. of Math. (2) **155** (2002), no. 1, 189–208.
19. A. Mielke, *Macroscopic behavior of microscopic oscillations in harmonic lattices via Wigner-Husimi transforms*, Arch. Ration. Mech. Anal. **181** (2006), no. 3, 401–448.
20. M. Nixon, *The discretized generalized Korteweg-de Vries equation with fourth order nonlinearity*, J. Comput. Anal. Appl. **5** (2003), no. 4, 369–397.
21. M. Plancherel and G. Pólya, *Fonctions entières et intégrales de Fourier multiples. II*, Comment. Math. Helv. **10** (1937), 110–163.
22. A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer Series in Computational Mathematics. 23, 1994.
23. J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl., IV. Ser. **146** (1987), 65–96.

24. G. Staffilani and D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Commun. Partial Differ. Equations **27** (2002), no. 7-8, 1337–1372.
25. A. Stefanov and P.G. Kevrekidis, *Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations*, Nonlinearity **18** (2005), no. 4, 1841–1857.
26. E.M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series. 43. Princeton, NJ: Princeton University Press , 1993.
27. R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–714.
28. T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006, Local and global analysis.
29. L.N. Trefethen, *Spectral methods in MATLAB*, Software, Environments, and Tools, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
30. Y. Tsutsumi, $L^2$*-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkc. Ekvacioj, Ser. Int. **30** (1987), 115–125.
31. R. Vichnevetsky and J.B. Bowles, *Fourier analysis of numerical approximations of hyperbolic equations*, SIAM Studies in Applied Mathematics, 5. Philadelphia: SIAM - Society for Industrial and Applied Mathematics, 1982.

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