A remark on the observability of conservative linear systems

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Abstract. We consider abstract conservative evolution equations of the form $\dot{z} = Az$, where A is a skew-adjoint operator. We analyze the problem of observability through an operator B, under the assumption that B is bounded in the appropriate functional setting associated to the free dynamics generated by A. Assuming that the pair (A, B) is exactly observable in $L^2(0,T)$ we prove an enhanced observability estimate showing that the weaker $L^1(0,T)$ observations yield the same observed quantities. We use a duality argument and the regularity properties of the free dynamics.

The same result applies for time-discrete models.

The main example of application is the wave equation with internal control. Surprisingly enough, the result fails to be true in the context of boundary control.

We also discuss some connections and consequences in the context of Ingham inequalities. In this frame our results can be interpreted as an improvement of the standard $L^1(0,T)$ version of Ingham's inequality under some averaging of the coefficients entering in the non-harmonic Fourier series.

1. Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator with compact resolvent. Let us consider the following abstract system:

$$\dot{z}(t) = Az(t), \quad z(0) = z_0. \tag{1.1}$$

The dot ($\dot{}$) denotes differentiation with respect to time t. The element $z_0 \in X$ is the *initial state*, and $z = z(t)$ is the *state* of the system.

System is well-known to posses a unique solution $z = z(t) \in C([0,\infty), X)$.

These systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation).

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed

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with the classical operator norm. Let $B \in \mathfrak{L}(X, Y)$ be an observation operator and define the output function

$$\epsilon(t) = Bz(t). \tag{1.2}$$

Note that (1.2) makes sense because of the assumption that B is a bounded operator from X into Y. In fact B trivially fulfills the *admissibility* condition that is required for observation problems (see [**22**]):

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2, \quad \forall\, z_0 \in X. \tag{1.3}$$

The *exact observability* property of system (1.1)-(1.2) can be formulated as follows: System (1.1)-(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|Bz(t)\|_Y^2 \, dt, \quad \forall\, z_0 \in X. \tag{1.4}$$

Moreover, (1.1)-(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that observability issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [**16**]). Indeed, controllability and observability are dual notions, and therefore each statement concerning observability has its counterpart in controllability. In the sequel, we mainly focus on the observability properties of (1.1)-(1.2).

The main result of this paper is as follows:

THEOREM 1.1. *Under the assumption above that $B : X \to Y$ is a bounded operator, the $L^2(0,T;Y)$-observability inequality above (1.4) holds for some constant $k_T > 0$ if and only if the following $L^1(0,T;Y)$ version is satisfied:*

$$k'_T \|z_0\|_X^2 \leq \left[\int_0^T \|Bz(t)\|_Y \, dt \right]^2, \quad \forall\, z_0 \in X, \tag{1.5}$$

for some other constant $k'_T > 0$. Furthermore, k'_T can be taken to be

$$k'_T = \frac{k_T^2}{4||B||^2}. \tag{1.6}$$

In particular, when the observability inequality (1.4) is fulfilled, there exists a constant $C_T > 0$ such that

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq C_T \left[\int_0^T \|Bz(t)\|_Y \, dt \right]^2. \tag{1.7}$$

REMARK 1.2.

- The assumption that A is skew-adjoint is in fact non necessary. The same result holds when A is the generator of a continuous semigroup in X provided B is bounded and the $L^2(0,T;Y)$ observability holds.

  Also the same result holds in the non-autonomous case. In this way, the result can be applied to, for instance, wave equations with added potentials depending in space-time (see for instance [**5**] for an analysis of this problem in the L^2-setting with sharp estimates on the dependence of the observability constants on the potentials).

Note however that the assumption of $L^2(0,T;Y)$-exact observability normally holds only for semigroups with very mild regularizing properties. For instance, in the context of the heat equation one often recovers the observability of the solution z at the final time $t=T$ rather than that of the initial datum. We refer to [**8**] for recent sharp results in this context. However, for heat like equations, the fact that L^2-observations can be replaced by L^1 ones is normally an easy consequence of the regularizing effect of solutions (see [**11**]).

- Obviously, (1.5) implies (1.4). But, of course, the reverse is not true in general. We prove it to be the case as a consequence of the observability and the boundedness of the operator B.
- In view of the main result above the L^p-version of inequality (1.5) holds for all $1 \le p \le \infty$:

$$k_T(p)\,\|z_0\|_X^2 \le \|Bz(t)\|_{L^p(0,T;Y)}^2, \quad \forall\, z_0 \in X. \tag{1.8}$$

- It was proved in [**3, 18**] that system (1.1)-(1.2) is exactly observable for some $T > 0$ in the sense that (1.4) holds if and only if the following assertion holds: *There exist constants* $M, m > 0$ *such that*

$$M^2\,\|(i\omega I - A)z\|^2 + m^2\,\|Bz\|_Y^2 \ge \|z\|^2\,, \forall\ \omega \in \mathbb{R},\ \ z \in \mathcal{D}(A). \tag{1.9}$$

  In view of our main result above, (1.9) is also equivalent to the property (1.5) of $L^1(0,T;Y)$-observation as well. However, the proof in [**3, 18**] using Fourier transform in time does not seem to apply in this case thus making our approach necessary.
- There is an extensive literature providing observability results for wave, plate, Schrödinger and elasticity equations, among other models and by various methods including microlocal analysis, multipliers and Fourier series, etc. All these results yield $L^2(0,T;Y)$ observability inequalities of the form (1.4). As a consequence of the main result above, whenever B is bounded one can readily deduce the $L^1(0,T;Y)$ analog of these inequalities. Note however that, most often, the techniques of proof developed so far do not yield directly the $L^1(0,T)$-observability result above but only its $L^2(0,T;Y)$ counterpart. The $L^1(0,T;Y)$ version is a consequence of our main result.
- Our technique of proof applies as well for conservative time-discrete systems that have been recently analyzed in [**6**] from the numerical analysis viewpoint. This issue will be addressed in section 4 below.

In Section 2 we prove Theorem 1.1. In Section 3 we present two applications to the wave and the Schrödinger equation. In Section 4 we present an application to time-discrete systems. In Section 5 we discuss some consequences in connection with the classical Ingham type inequalities for non-harmonic Fourier series. Finally in Section 6 we discuss some other closely related issues and present some open problems.

2. Proof of the main result

Our proof uses a duality argument and exploits the added regularity properties that the controls of the dual exact controllability problem have when the control operator B is bounded.

Consider the control problem

$$\dot{y}(t) = Ay(t) + B^*u, \quad y(0) = y_0. \tag{2.1}$$

More precisely we consider the exact controllability problem. Given $y_0 \in X'$ (the dual of X) we look for a control function $u \in L^2(0,T;Y')$ such that the solution of (2.4) satisfies:

$$y(T) = 0. \tag{2.2}$$

Note that under the assumption that $B : X \to Y$ is a bounded operator, B^* is a linear bounded operator from Y' to X' so that B^*u belongs to $L^2(0,T;X')$. Then (2.4) possesses a unique solution $y \in C([0,T];X')$.

We also consider the adjoint system:

$$-\dot{\varphi}(t) = A^*\varphi(t), \quad \varphi(T) = \varphi_T. \tag{2.3}$$

Taking into account that A is skew-adjoint, this system coincides with

$$\dot{\varphi}(t) = A\varphi(t), \quad \varphi(T) = \varphi_T. \tag{2.4}$$

In view of the fact that A generates a group of isometries in X, reversible in time, the initial datum for this adjoint system can be taken at $t = 0$, and this leads to the original system (1.1) for the z variable.

It is well known that the exact observability inequality (1.2) implies and it is actually equivalent to the exact controllability property above. More precisely, the exists a positive constant C_T such that the control function u of minimal $L^2(0,T;Y')$-norm fulfills

$$||u||_{L^2(0,T;Y')} \leq C_T ||y_0||_{Y'}, \quad \forall y_0 \in Y'. \tag{2.5}$$

Moreover, following J. L. Lions' HUM method [**16**] it is well-known that this control is of the form

$$u = B\hat{z} \tag{2.6}$$

where $\hat{z}$ is the solution of (1.1) associated to the initial datum $z_0 \in X$ that minimizes the following functional $J : X \to \mathbb{R}$:

$$J(z_0) = \frac{1}{2}\int_0^T ||Bz||_Y^2 dt + < z_0, y_0 >, \tag{2.7}$$

where $< \cdot, \cdot >$ stands for the duality pairing between X and X'. In (2.7) Y has been identified with its dual Y' so that $B\hat{z}$ is an element of $L^2(0,T;Y')$.

The fact that J achieves its minimum on H is a consequence of the observability inequality (1.3). Actually

$$||z_0||_X \leq \frac{2}{k_T}||y_0||_{X'}. \tag{2.8}$$

But the well-posedness of the system (1.1) also ensures that

$$||\hat{z}||_{L^\infty(0,T;X)} = ||\hat{z}_0||_X \leq \frac{2}{k_T}||y_0||_{X'}, \tag{2.9}$$

and, as a consequence of the fact that the operator $B : X \to Y$ is bounded, this implies that

$$||u||_{L^\infty(0,T;Y')} = ||B^*\hat{z}||_{L^\infty(0,T;Y')} \leq \frac{2||B||}{k_T}||y_0||_{X'}. \tag{2.10}$$

This shows that, because of the boundedness of the observation operator B, the control has the added integrability property of being in $L^\infty(0,T;Y')$. This fact can now be used to improve the observability inequality and prove its $L^1(0,T;Y)$-version.

Let now z be any solution of (1.1) with initial datum z_0 in X. The duality between the controlled state y and z leads to the identity

$$-< z_0, y_0 >= \int_0^T < u, Bz > dt, \tag{2.11}$$

where in the left hand side term $< \cdot, \cdot >$ stands for the duality between X and X' and on the right hand side between $L^2(0,T;Y')$ and $L^2(0,T;Y)$. Then

$$||z_0||_X = \sup_{y_0 \in X'} \frac{< z_0, y_0 >}{||z_0||_X} \leq \sup_{y_0 \in X'} \frac{\int_0^T < u, Bz > dt}{||z_0||_X} \tag{2.12}$$

$$\leq \sup_{y_0 \in X'} \frac{||u||_{L^\infty(0,T;Y')}||Bz||_{L^1(0,T;Y)}}{||z_0||_X} \leq \frac{2||B||}{k_T}||Bz||_{L^1(0,T;Y)}.$$

This concludes the proof of the $L^1(0,T;Y)$-version of the observability inequality.

3. Applications

3.1. The wave equation with distributed control. Let Ω be a bounded domain of $\mathbb{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the homogeneous wave equation

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in} \quad Q \\ z = 0 & \text{on} \quad \Sigma \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

The exact observability inequality for (3.1) reads as follows:

$$||(\varphi(0), \varphi_t(0))||^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{3.2}$$

for all solutions of (3.1).

C. Bardos, G. Lebeau and J. Rauch [**2**] proved that, in the class of C^∞ domains, the observability inequality (2.5) holds if and only if (ω, T) satisfy the following *geometric control condition (GCC)* in Ω*: Every ray of geometric optics that propagates in* Ω *and is reflected on its boundary* Γ *enters* ω *in time less than* T. A simpler proof of this result using multiplier techniques was given in ([**16**]) in the particular case where ω is a neighborhood of a suitable subset of the boundary.

The main result of this paper shows that, whenever (3.2) holds, the following stronger version holds as well:

$$||(\varphi(0), \varphi_t(0))||_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \left[\int_\omega \varphi^2 dx\right]^{1/2} dt. \tag{3.3}$$

As far as we know this result is new.

Note that, the same does not hold in the case of boundary control. Indeed when Γ_0 is a subset of the boundary $\partial\Omega$ fulfilling the GCC, the following observability inequality holds in the energy space:

$$||(\varphi(0), \varphi_t(0))||^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_{\partial\Gamma_0} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt. \tag{3.4}$$

Note however that the normal derivative operator is not bounded within the energy space. Actually, the fact that $\partial\varphi/\partial\nu$ belongs to $L^2(\Gamma_0 \times (0, T))$ for finite energy solutions of the wave equation is a consequence of a hidden regularity result. Thus, the main result of this paper can not be applied in this case and the $L^1(0, T; L^2(\Gamma_0))$ version of the observability inequality above can not been guaranteed to hold. In fact in $1 - d$ the explicit formula of solutions obtained by the d'Alembert formula shows that this $L^1(0, T; L^2(\Gamma_0))$ fails to hold. Indeed, in $1 - d$, $L^1(0, T)$ measurements of the normal derivative on one point (or on both of them) of the boundary, only yields the observation of the $W^{1,1}(\Omega) \times L^1(\Omega)$ norm of the initial data.

This issue is also closely related to the L^1-versions of Ingham's inequality we shall discuss below. A numerical study of these issues can be found in [**1**].

3.2. The Schrödinger equation with distributed control. It is by now well-known that, when the GCC holds, the Schrödinger equation is exactly observable in any time $T > 0$ (see [**16**], [**15**]). The same can be said for fourth order plate and beam equations. The main result of this paper can be applied in this context too.

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Consider the equation

$$\begin{cases} iz_t = \Delta z, \quad (t, x) \in (0, T) \times \Omega, \\ z(0) = z_0, \ x \in \Omega, \quad z(t, x) = 0, \ (t, x) \in (0, T) \times \partial\Omega. \end{cases} \tag{3.5}$$

where $z_0 \in L^2(\Omega)$ is the initial data.

When ω an open subset of Ω satisfies the GCC for any $T > 0$, there exist positive constants $k_T > 0$ and $K_T > 0$ such that for any $z_0 \in L^2(\Omega)$, the solution of (3.5) satisfies

$$k_T \|z_0\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega |z|^2 \, dxdt \leq K_T \|z_0\|^2_{L^2(\Omega)} . \tag{3.6}$$

As a consequence of the main result of this paper, in these circumstances, the following $L^1(0, T; L^2(\omega))$-version holds as well:

$$k'_T \|z_0\|^2_{L^2(\Omega)} \leq \left[\int_0^T \left[\int_\omega |z|^2 \, dx\right]^{1/2} dt\right]^2 \leq K'_T \|z_0\|^2_{L^2(\Omega)} . \tag{3.7}$$

3.3. Other systems. Similar results can be established for other relevant systems of conservative PDEs. In particular for the linearized KdV equation in a bounded domain for which we know that exact observability holds for all $T > 0$ and from any open non-empty subset of the interval where the equation holds (see [**21**], [**19**]).

4. Time-discrete conservative systems

In this section, we adopt the framework of [**6**] to present our main result for time discrete systems.

Consider a time-discretization of the continuous system under study. For any $\Delta t > 0$, we denote by z^k and ϵ^k respectively the approximations of the solution z and the output function y of system (1.1)–(1.2) at time $t_k = k\Delta t$ for $k \in \mathbb{Z}$. Consider the following *implicit midpoint* time discretization of system (1.1):

$$\frac{z^{k+1}-z^k}{\Delta t} = A\left(\frac{z^{k+1}+z^k}{2}\right), \quad \text{in } X, \quad k \in \mathbb{Z}, \quad z^0 \text{ given.} \tag{4.1}$$

The output function of (4.1) is given by

$$\epsilon^k = Bz^k, \quad k \in \mathbb{Z}. \tag{4.2}$$

Note that (4.1)–(4.2) is a discrete version of (1.1)–(1.2).

Taking into account that A is skew-adjoint, it is easy to show that $\left\|z^k\right\|_X$ is conserved in the discrete time variable $k \in \mathbb{Z}$, i.e. $\left\|z^k\right\|_X = \left\|z^0\right\|_X$. Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting towards the solution of (1.1) as $\Delta t \to 0$.

The uniform exact observability problem for system (4.1) is formulated as follows: *To find a positive constant k_T, independent of Δt, such that the solutions z^k of system* (4.1) *satisfy:*

$$k_T \left\|z^0\right\|_X^2 \leq \Delta t \sum_{k\in(0,T/\Delta t)} \left\|y^k\right\|_Y^2, \tag{4.3}$$

for all initial data z^0 in an appropriate class.

Clearly, (4.3) is a discrete version of (1.4).

Of course, this problem makes sense for a fixed mesh size Δt. But here, as in [**6**], we analyze whether observability inequalities hold uniformly on the mesh size parameter as $\Delta t \to 0$, but this time for observations done in $L^1(0,T;Y)$.

According to the results in [**6**] it is well-known that, under the assumption that exact observability (1.2) holds for the time continuous system (1.1) then it holds uniformly for the time-discrete systems (4.1) within a subspace of filtered solutions in which high frequency components have been eliminated. This result is sharp since high frequency wave packets and eigenfunction clusters may fail to be observable because of numerical dispersion.

To be more precise, we assume that A is skew-adjoint with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : j \in \mathbb{N}\}$, where $(\mu_j)_{j\in\mathbb{N}}$ is a sequence of real numbers. Set $(\Phi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{N}}$, that is:

$$A\Phi_j = i\mu_j\Phi_j. \tag{4.4}$$

Define

$$\mathcal{C}_s = \text{span}\{\Phi_j : |\mu_j| \leq s\}. \tag{4.5}$$

Then inequality (4.3) holds uniformly (with respect to $\Delta t > 0$) in the class $\mathcal{C}_{\delta/\Delta t}$ for any $\delta > 0$ and for T_δ large enough, depending on the filtering parameter δ. We refer to ([**6**] for further details).

As a consequence of the method of proof developed in Section 2 we can readily deduce that the same occurs at the level of $\ell^1(Y)$ observations. Here by $\ell^1(Y)$ we refer to the time-discrete version of $L^1(0,T;Y)$.

The following holds:

THEOREM 4.1. *Let A be a skew-adjoint operator with compact resolvent in the Hilbert space X. Let $B : X \to Y$ be a bounded linear operator. Assume that the exact observability inequality (1.2) holds for the time-continuous system (4.1).*

Then, for any $\delta > 0$ there exists T_δ large enough, depending on the filtering parameter δ, and a constant $k'_T(\delta)$ such that the $\ell^1(Y)$-observation inequality below holds

$$k'_T(\delta)\left\|z^0\right\|_X^2 \le \left[\Delta t \sum_{k\in(0,T/\Delta t)} \left\|y^k\right\|_Y\right]^2, \quad \forall z^0 \in \mathcal{C}_s, \tag{4.6}$$

uniformly with respect to Δt.

REMARK 4.1. The proof of the $\ell^2(Y)$ observability inequality for the time-discrete system in [**6**] relies on the use of the resolvent estimate and of the Fourier transform to reduce the time observation problem to the frequency domain. Therefore this proof naturally works in the $L^2(0,T;Y)$ (resp. $\ell^2(Y)$) setting for the time continuous (resp. time-discrete) models. The proof of our main result, ensuring observability in $L^1(0,T;Y)$ (resp. $\ell^1(Y)$) both in the continuous and discrete setting does not follow this path. It is rather a consequence of the gain of integrability of the control that turns out to be in $L^\infty(0,T;Y)$ (resp. $\ell^\infty(Y)$) and a duality argument as in the proof of Theorem 1.1. This proof at the time-discrete level yields the same observation time T_δ as in [**6**] and uniform results with respect to the time-discrete parameter Δt.

PROOF. The proof is similar to the one of Theorem 1.1.

Indeed, in view of the uniform (with respect to Δt) observability $\ell^2(Y)$ inequality (4.3) the uniform controllability of the following system holds as well:

$$\frac{y^{k+1}-y^k}{\Delta t} = A\left(\frac{y^{k+1}+y^k}{2}\right) + B^*u^k, \quad \text{in } X, \quad k \in \mathbb{Z}, \quad y^0 \text{ given.} \tag{4.7}$$

But, this time, the fact that the observability inequality is satisfied in the class of filtered solutions $\mathcal{C}_{\delta/\Delta t}$ ensures only a partial controllability result of the form

$$\pi_\delta y^K = 0, \tag{4.8}$$

where π_δ stands for the projection onto the subspace $\mathcal{C}_{\delta/\Delta t}$ and K is the final discrete time-step such that $K\Delta t = T_\delta$.

Moreover, the control $\{u^k\}$ is of the form $u^k = B\hat{z}^k$, where $\{\hat{z}^k\}$ stands for the minimizer of the time-discrete version of the functional J in (2.7). Using the arguments of the continuous case it is easy to see that, actually, because B is a bounded linear operator and the system generated by A is conservative $\{u^k\}$ is bounded above in $\ell^\infty(Y')$ by the norm of the initial datum y^0 in X'. By duality this implies the $\ell^1(Y)$ version of the observability inequality. □

5. Ingham type inequalities

In this section we show how the gain, from $L^2(0,T;Y)$ into $L^1(0,T;Y)$, on the observability inequalities of conservative systems can be interpreted in the context of the classical Ingham inequalities for non-harmonic Fourier series. Ingham inequalities play a key role in the obtention of observability inequalities for conservative systems and in particular for $1-d$ vibrating structures such as strings and beams. We refer to the original paper by Ingham [**13**] and also to [**17**] where a beginners introduction about how to use Ingham type inequalities for the control of conservative systems is given.

5.1. Classical Ingham inequalities. Ingham's inequality generalizes the classical Parseval's equality for orthogonal trigonometric polynomials. It reads as follows:

THEOREM 5.1. *(Ingham [**13**]) Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{5.1}$$

For any real T with

$$T > 2\pi/\gamma \tag{5.2}$$

there exist positive constants $c_1 = c_1(T,\gamma), C_1 = C_1(T,\gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$c_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \leq C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 . \tag{5.3}$$

Furthermore, the following sharp $L^1(0,T)$ version holds as well:

$$\frac{\gamma}{\pi} \sup_{n\in\mathbb{Z}} \mid a_n \mid \leq \int_0^{2\pi/\gamma} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right| dt. \tag{5.4}$$

REMARK 5.1. Some comments are in order:

- Both results are sharp.
- It is important to observe that the constants in the inequalities only depend on γ and T. Thus, they are uniform within the classes of sequences $\{\lambda_n\}$ with gap γ.
- The sharp multiplicative constant γ/π of the inequality (5.4) was obtained in Ingham's second paper [**14**].
- Inequality (5.4) is stronger than (5.3) in the sense that the non-harmonic Fourier series is estimated in $L^1(0,T)$, which is a weaker norm than $L^2(0,T)$. On the other hand, the observed discrete norm in the coefficients $\{a_n\}$ is weaker in (5.3) as one could expect since the ℓ^∞-norm is weaker than the ℓ^2 one.
- There are many extensions of this inequality, for instance, to classes of $\{\lambda_n\}$ fulfilling weaker gap conditions. There are also many applications to observation and control problems, mainly to $1-d$ systems of vibrations such as strings and beams, but we will not discuss this here. We merely refer to the book [**4**] where these inequalities are applied to the control of networks of flexible strings.

5.2. An improved averaged Ingham inequality. In the previous subsection we have presented the two main versions of the Ingham inequality: The $L^2(0,T)$ and the $L^1(0,T)$ one. The fact that the $L^2(0,T)$ inequality can be improved into the $L^1(0,T)$ one is somehow related to the improvement of the observability properties of conservative systems from $L^2(0,T;Y)$ into $L^1(0,T,Y)$ that we proved in previous sections. The goal of this section is to establish such a connection.

As a consequence of our main result in Theorem 1.1 the following holds.

THEOREM 5.2. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma>0$ be such that the gap condition (5.1) holds.*

Consider the space-time dependent function:

$$f(x,t)=\sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}\sin(n\pi x),\ t\in\mathbb{R},\ x\in(0,1). \tag{5.5}$$

Let (α,β) be any non-empty subinterval of $(0,1)$.

Then, for all T as in (5.2) there exists a positive constants $k=k(T,\gamma,\alpha,\beta)>0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$k\sum_{n\in\mathbb{Z}} \mid a_n\mid^2\leq \|f\|^2_{L^1(0,T;L^2(\alpha,\beta))}. \tag{5.6}$$

Some remarks are in order:

REMARK 5.2.

- Inequality (5.6) can be viewed as an intermediate version of (5.3) and (5.4). It is stronger than both results in the sense that we can get estimates on the ℓ^2 norm of the coefficients by using $L^1(0,T)$-norms. But this occurs at the price of taking norms in $L^1(0,T;L^2(\alpha,\beta))$, which involves some averaging in space.
- In view of this result one could try to make the interval (α,β) to collapse into a point x_0. But, of course, inequality (5.6) would not pass to the limit. Otherwise, one would obtain an inequality combining the best aspects of both (5.3) and (5.4) in the sense that the ℓ^2-norm of the coefficients would be estimated in terms of the $L^1(0,T)$-norm of the non-harmonic Fourier series. But this is of course impossible. The proof of the result we present now clearly indicates how the inequality (5.6) blows up as α and β collapse into a single point x_0. This is simply related to the fact that the lower bound $\int_\alpha^\beta \sin^2(n\pi x)dx\geq c>0$ fails to hold uniformly with respect to n as α and β collapse on a point. In fact, the behavior of $\sin^2(n\pi x_0)$ as $n\to\infty$ is a wellknown issue in the context of pointwise control and related to the classical theory of Diophantine approximation. But, whatever x_0 is, the infimum of $\sin^2(n\pi x_0)$ vanishes as n tends to infinity (see [**4**]).
- Inequality (5.6) can be rewritten in the following manner

$$\sqrt{k}\left[\sum_{n\in\mathbb{Z}}\mid a_n\mid^2\right]^{1/2}\leq\int_0^T\left|\sum_{n,m}\gamma_{n,m}a_n\bar{a}_m e^{i(\lambda_n-\lambda_m)t}\right|^{1/2}dt \tag{5.7}$$

where

$$\gamma_{n,m}=\int_\alpha^\beta \sin(n\pi x)\sin(m\pi x)dx. \tag{5.8}$$

When $(\alpha, \beta) = (0, 1)$ the matrix $\gamma_{n,m}$ is diagonal and the inequality (5.7) is obvious.

PROOF. The proof is a direct consequence of our main Theorem.

Indeed, we view the function $f(x,t)$ in (5.5) as a solution of the abstract equation associated to the operator A defined as follows:

$$D(A) = \{g(x) = \sum_{n\in\mathbb{Z}} g_n \sin(n\pi x) : \sum_{n\in\mathbb{Z}} \lambda_n^2 |g_n|^2 < \infty\}, \tag{5.9}$$

$$Ag = \sum_{n\in\mathbb{Z}} \lambda_n g_n \sin(n\pi x), \forall g = \sum_{n\in\mathbb{Z}} g_n \sin(n\pi x) \in D(A). \tag{5.10}$$

In view of the ℓ^2 version of Ingham's inequality, integrating in (α, β) with respect to x and using the fact that $\int_\alpha^\beta \sin^2(n\pi x)dx \geq c > 0$ uniformly with respect to $n \in \mathbb{Z}$, we deduce that

$$\sum_{n\in\mathbb{Z}} | a_n |^2 \leq C\|f\|^2_{L^2((0,T)\times(\alpha,\beta))} \tag{5.11}$$

for some positive constant $C > 0$.

Inequality (5.6) is then a consequence of (5.11) and the main result of Theorem 1.1. □

6. Further comments an open problems

- In [**12**] it was shown, in the context of bounded observation operators B, that classical $L^2(0,T;Y)$-observability estimates leads to an exponential stabilization result of the dissipative semigroup:

$$\dot{w}(t) = Aw(t) - B^*Bw, \quad w(0) = w_0. \tag{6.1}$$

In other words, for some $C, \gamma > 0$ it follows that

$$\|w(t)\|_X \leq Ce^{-\gamma t}\|w_0\|_X, \quad \forall t > 0, \forall w_0 \in X. \tag{6.2}$$

Indeed, for this system the energy dissipation law reads:

$$\frac{d}{dt}||w(t)||^2_X = -||Bw(t)||^2_Y. \tag{6.3}$$

In view of this, to prove the exponential decay of the semigroup generated by (6.4) it suffices to prove that the $L^2(0,T)$-observability estimate (1.3) also holds for the dissipative dynamics associated with (6.4). This is done by a perturbation argument, as a consequence of the observability of the conservative one, thanks to the fact that B is bounded.

In view of the $L^1(0,T)$-observability estimates we have derived in this paper it would be natural to consider the dissipated dynamics:

$$\dot{v}(t) = Av(t) - \frac{B^*Bv}{||v||_Y}, \quad v(0) = v_0. \tag{6.4}$$

In this case the energy dissipation law is as follows:

$$\frac{d}{dt}||v(t)||^2_X = -||Bv(t)||_Y. \tag{6.5}$$

Note however that whether some explicit decay estimate can be obtained for this system in view of the $L^1(0,T;Y)$-observability results of this paper is an open problem.

- In [**20**] the problem of the optimal placement of $L^2(\omega \times (0,T))$-observers has been considered for the $1-d$ wave equation with observations on non-trivial measurable sets ω of the domain where the equation evolves. In view of the results of this paper it would make sense to consider the same problem in the context of the $L^1(0,T;L^2(\omega))$-observation problem.
- In view of the $L^1(0,T;Y)$-observability results of this paper it is natural to look for the controls obtained by minimizing the functional

$$J(z_0) = \frac{1}{2}\left[\int_0^T ||Bz||_Y dt\right]^2 + < z_0, y_0 >, \tag{6.6}$$

rather than the one in (2.7). The minimizer $\hat{z}_0 \in X$ for this functional exists and is unique. The corresponding control is of the form

$$u = \int_0^T ||B\hat{z}||_Y dt \frac{B\hat{z}}{||B\hat{z}||_Y}, \tag{6.7}$$

provided $||B\hat{z}||_Y \neq 0$ *a.e.* $t \in (0,T)$. This is often the case for parabolic equations due to the time-analyticity of solutions. But this condition can not be guaranteed to hold for conservative semigroups. In fact, for the wave equation, due to the finite velocity of propagation it is easy to show that there are solutions for which the set of time instances in which $||B\hat{z}||_Y$ vanishes is of positive measure.

However, regardless of whether the identity (6.7) holds or not, the controls obtained in this way are of minimal $L^\infty(0,T;Y)$-norm (see [**9**], [**24**]).

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