TIMOSHENKO'S BEAM EQUATION AS A LIMIT OF A NONLINEAR ONE-DIMENSIONAL VON KÁRMÁN SYSTEM

G. PERLA MENZALA

National Laboratory of Scientific Computation, LNCC/CNPq,

Rua Getulio Vargas 333, Quitandinha,

Petrópolis, RJ, CEP 25651-070, RJ, Brasil

and

Institute of Mathematics,

UFRJ, P.O.Box 68530,

Rio de Janeiro , RJ, Brasil.

perla@lncc.br

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and

Enrique ZUAZUA

Universidad Complutense de Madrid

Dpto. de Matemática Aplicada,

28040 Madrid, España.

zuazua@eucmax.sim.ucm.es

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Abstract

We consider a dynamical $1-D$ nonlinear von Kármán model depending on one parameter $\varepsilon > 0$ and study its weak limit as $\varepsilon \to 0$. We analyse various boundary conditions and prove that the nature of the limit system is very sensitive to them. The models we obtain do not coincide with the classical Timoshenko equation we obtained as limit in the case of Dirichlet boundary conditions. We prove that, depending on the type of boundary condition we consider, the nonlinearity of Timoshenko's model may v anish or, by the contrary, may become a nonlinearity concentrated on the extremes of the beam.

§1) Introduction

A wide accepted dynamical model describing large deflections of thin plates is given by the so-called von Kármán system of equations. There is a large literature on this model specially in the last ten years or so, when several authors considered prob lems of existence, uniqueness, asymptotic behavior in time (when some damping effect is considered) as well as some other important properties (see [2], [3], [6] and the references therein).

In a recent work J. Lagnese and G. Leugering [4] considered a one-dimensional version of the von Kármán system describing approximately the planar motion of a uniform prismatic beam of lenght L. More precisely, in [4] the following system was considered

$$(1.1)\quad \begin{gathered} v_{tt} - \left[v_x + \frac{1}{2} w_x^2 \right]_x = 0 \\ w_{tt} + w_{xxxx} - h\, w_{xxtt} - \left[w_x \left(v_x + \frac{1}{2} w_x^2 \right) \right]_x = 0 \end{gathered}$$

where $0 < x < L$ and $t > 0$. In (1.1) subcripts mean partial derivatives and $h > 0$ is a parameter related to the rotational inertia of the beam. The quantities $v = v(x,t)$ and $w = w(x,t)$ represent, respectively, the longitudinal and transversal dis placement of the point x at time t. Suitable boundary conditions at $x = 0$, $x = L$ and initial conditions at $t = 0$ were given.

On the other side, when we consider a uniform beam of lenght L and study the transverse deflections (represented by $u = u(x,t)$) of its centerline at the point x at time t, then, the following model can be deduced (named Timoshenko's equation):

$$u_{tt} + u_{xxxx} - h\,u_{xxtt} - \frac{1}{2L}\left(\int_0^L u_x^2\,dx\right)u_{xx} = 0 \tag{1.2}$$

(see [1], [8] and the references therein).
Evidently, since both models ((1.1) and (1.2)) try to describe approximately the same phenomenon intuition tells us, that there should be certain "proximity" between (1.1) and the solution $u = u(x,t)$ of (1.2). This paper is devoted to the anal ysis of the convergence of (1.1) towards (1.2) or its variations when an appropriate parameter tends to zero. To make the problem more precise, we need to recall briefly our previous work [7].
In [7] we considered the following problem: Let $\varepsilon > 0$ and $v = v^\varepsilon$, $w = w^\varepsilon$ solving the problem

$$\varepsilon\,v_{tt} - \left[v_x + \frac{1}{2}\,w_x^2\right]_x = 0 \tag{1.3}$$

$$w_{tt} + w_{xxxx} - h\,w_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}\,w_x^2\right)\right]_x = 0 \tag{1.4}$$

in the interval $\Omega = (0,L)$ and $t > 0$ with boundary conditions

$$\begin{aligned} &v(0,t) = v(L,t) = 0 \qquad \forall\, t > 0 \\ &w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \forall\, t > 0 \end{aligned} \tag{1.5}$$

and initial conditions

$$\begin{aligned} &v(x,0) = v_0(x) \qquad w(x,0) = w_0(x) \\ &v_t(x,0) = v_1(x) \qquad w_t(x,0) = w_1(x) \end{aligned} \tag{1.6}$$

The following result was proved in [7]: *Assume that* $(v_0, v_1, w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\Omega)$, *then*

$$(w^\varepsilon, w_t^\varepsilon) \quad \overset{\varepsilon \to 0}{\rightharpoonup} \quad (u, u_t)$$

weakly in $L^2(0,T; H_0^2(\Omega)) \times L^2(\Omega \times (0,T))$ *where* u *solves (1.2) with* $u(x,0) = w_0(x)$, $u_t(x,0) = w_1(x)$ *and*

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0 \qquad \forall\, t > 0.$$

Here $H^m(\Omega)$ and $H_0^m(\Omega)$ denote the usual Sobolev spaces. This result guarantees that, as the velocity of propagation of longitudinal vibrations tends to infinity, solutions of (1.3)-(1.4) converge weakly to the solutions of Timoshenko's beam mode l under Dirichlet boundary conditions (1.5).

In simple situations, the above result could be expected: For example, suppose that v in (1.3) only depends on x, i.e. $v = v(x)$ then, (1.3) implies that $v_x + \dfrac{1}{2}\, w_x^2 = \eta(t)$ for some function $\eta = \eta(t)$. Integration (in x) from zero to L and using (1.5) give us that

$$\frac{1}{2}\int_0^L w_x^2\, dx = L\,\eta(t).$$

Substitution in (1.4) give us that

$$w_{tt} + w_{xxxx} - h\, w_{xxtt} - \big[w_x\, \eta(t)\big]_x = 0$$

that is

$$w_{tt} + w_{xxxx} - h\, w_{xxtt} - \frac{1}{2L}\Big(\int_0^L w_x^2\, dx\Big) w_{xx} = 0.$$

In the above motivation we can see how crucial were the boundary conditions (1.5) on v. The boundary conditions play also a key role in the rigorous proof of [7].

The main purpose of this paper is to study the general case, in which, of course $v = v(x,t)$ and see how sensitive to the boundary conditions is this limit process as $\varepsilon \to 0$.

The main results could be summarized as follows:

Consider problem (1.3)-(1.4) with initial conditions (1.6) and Neumann boundary conditions on v and clamped ends for w, then, as $\varepsilon \to 0$, the (weak) limit model turns out to be a linear equation. More precisely the non-linearity of the limit vanis hes when passing to the limit. If we consider hinged boundary conditions for w, the rest being unchanged, at the limit a non-linear time-dependent potential arises and depends only on the values of the limit solution at the extremes of the interval. The se results are in contrast with previously known results in the framework of Dirichlet boundary conditions in which the limit was shown to be the Timoshenko nonlinear beam equation in which the nonlinearity is equidistributed all along the beam.

Classical energy estimates provide easily uniform bounds on the solutions. The main difficulty when passing to the limit is the identification of the limit of the nonlinear term. This is done by using ad-hoc test functions which depend on the boundary con ditions on a sensitive way and that, as indicated above, may led to rather drastic changes on the nature of the limit system.

We point out that many other important situations could be treated using the main ideas of this paper. For example, instead of the boundary conditions (1.5) or the ones worked out in this paper we could also consider the following ones

$$v(0,t) = v(L,t) = 0 \qquad \forall\, t > 0$$
$$w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 \quad \forall\, t > 0$$

or

$$v(0,t) = v_x(L,t) = 0 \qquad \forall\, t > 0$$
$$w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \forall\, t > 0$$

Our notations in this paper are standard and can be found in the book of J.L. Lions [5].

Let us briefly describe all sections in this paper: In all Sections we will consider the coupled system (1.3)-(1.4) with initial conditions (1.6). In Section 2 we study the well-posedness of system (1.3)-(1.4) in one of the above situations (Neumann condi tions on v and clamped ends for w). In Section 3, we briefly recall from [7] the main steps to prove the (weak) convergence as $\varepsilon \to 0$ in the case of Dirichlet conditions on v and clamped ends for w. In Section 4 we pass to the limit in the ca se of Neumann conditions on v and clamped ends for w. Finally, in Section 5 we describe the remaining case which in our opinion is the most surprising one.

§2) Global well-posedness

As we mentioned in the introduction, we want to consider system (1.3)-(1.4) with initial conditions (1.6) subject to boundary conditions other than (1.5). We will consider two cases:

(I) (Neumann conditions on v and clamped ends for w)

$$\begin{aligned} &v_x(0,t) = v_x(L,t) = 0 \qquad \forall\, t > 0 \\ &w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \forall\, t > 0; \end{aligned} \tag{2.1}$$

(II) (Neumann conditions on v and hinged ends for w)

$$\begin{aligned} &v_x(0,t) = v_x(L,t) = 0 \qquad \forall\, t > 0 \\ &w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 \quad \forall\, t > 0. \end{aligned} \tag{2.2}$$

In order to study the well-posedness of system (1.3)-(1.4)-(1.6) with either one of the above boundary conditions we formulate the system as an abstract evolution equation in a suitable Hilbert space. Since all cases are quite similar we just give the det ails in case (I) i.e. we consider problem (1.3)-(1.4)-(1.6) with $\varepsilon > 0$ $(h > 0)$ and boundary conditions (2.1). We introduce the Hilbert space

$$X = V \times L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\Omega)$$

where V is the Sobolev space $H^1(\Omega)$ with null average

$$V = \left\{ v \in H^1(\Omega), \int_0^L v\, dx = 0 \right\}.$$

The norm in X is given by

$$||(v,y,w,z)||_X^2 = ||v_x||^2 + ||y||^2 + ||w_{xx}||^2 + ||z||^2 + ||z_x||^2$$

for any $(v, y, w, z) \in X$. Here $|| \cdot ||$ denotes the norm in $L^2(\Omega)$. We write our problem in the form

$$(2.3) \qquad \begin{cases} DU_t = AU + N(U) \\ U(0) = U_0 \in X \end{cases}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \left(1 - h\,\frac{\partial^2}{\partial x^2}\right) \end{bmatrix}, \qquad U = \begin{bmatrix} v \\ y \\ w \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix}, \qquad U_0 = \begin{bmatrix} v_0 \\ v_1 \\ w_0 \\ w_1 \end{bmatrix}$$

and

$$N(U) = \begin{bmatrix} 0 \\ \frac{1}{2}(w_x^2)_x \\ 0 \\ \left[w_x\left(v_x + \frac{1}{2}\,w_x^2\right)\right]_x \end{bmatrix}$$

It is easy to see that $D^{-1} A$ with domain

$$[H_0^2(\Omega) \cap V] \times V \times [H^4(\Omega) \cap H_0^2(\Omega)] \times [H^3(\Omega) \cap H_0^2(\Omega)]$$

is the infinitesimal generator of a semigroup of contractions in X. This implies that in order to show local existence of problem (2.3) it is enough to show that $D^{-1} N(U)$ is locally Lipschitz continuous in X. Clearly, if $U = \begin{pmatrix} v \\ y \\ w \quad z \end{pmatrix}$ and $\widetilde{U} = \begin{pmatrix} \tilde{v} \\ \tilde{y} \\ \tilde{w} \\ \tilde{z} \end{pmatrix}$ belong to X then

$$D^{-1}[N(U) - N(\widetilde{U})] = \begin{pmatrix} 0 \\ f \\ 0 \\ g \end{pmatrix}$$

where

$$f = \frac{1}{2\varepsilon}\,[w_x^2 - \tilde{w}_x^2]_x \quad \text{and} \quad g = \left(1 - h\,\frac{\partial^2}{\partial x^2}\right)^{-1}\left[w_x\left(v_x + \frac{1}{2}\,w_x^2\right) - \tilde{w}_x\left(\tilde{v}_x + \frac{1}{2}\,\tilde{w}_x^2\right)\right]_x .$$

Consequently

$$||D^{-1}[N(U) - N(\widetilde{U})]||_X^2 = ||f||_{L^2(\Omega)}^2 + ||g||_{H^1(\Omega)}^2 .$$

Using the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ we can easily show that

$$||f||_{L^2(\Omega)} \leq c(\varepsilon)[1 + ||U||_X + ||\widetilde{U}||_X]\,||U - \widetilde{U}||_X \tag{2.4}$$

for some constant $c(\varepsilon) > 0$. Since the operator $\left(1 - h\,\frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}$ is bounded from $L^2(\Omega) \to H_0^1(\Omega)$, then

$$||g||_{H^1(\Omega)} \leq c\left\|w_x\left(v_x + \frac{1}{2}\,w_x^2\right) - \tilde{w}_x\left(\tilde{v}_x + \frac{1}{2}\,\tilde{w}_x^2\right)\right\|_{L^2(\Omega)} . \tag{2.5}$$

Adding and substracting the term $\left(v_x + \frac{1}{2}\,w_x^2\right)\tilde{w}_x$ (inside the norm in (2.5)) and using the triangle inequality we obtain that

$$||g||_{H^1(\Omega)} \leq c||w_x - \tilde{w}_x||_{L^\infty}\,||v_x + \frac{1}{2}\,w_x^2||_{L^2} +$$

$$+ c||\tilde{w}_x||_{L^\infty}\left\{||v_x - \tilde{v}_x||_{L^2}\,||v_x + \tilde{v}_x||_{L^\infty} + \frac{1}{2}\,||w_x - \tilde{w}_x||_{L^\infty}\,||w_x + \tilde{w}_x||_{L^2}\right\} \tag{2.6}$$

Again, we use the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and deduce from (2.6) that

$$||g||_{H^1(\Omega)} \leq c\big(||U||_X, ||\widetilde{U}||_X\big)||U - \widetilde{U}||_X$$

which together with (2.4) shows that $D^{-1}\,N(U)$ is locally Lipschitz continuous in X. In order to obtain global existence we need an a priori estimate. In our case this is not difficult because the total energy associated with problem (1.3)-(1.4) is given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L \left\{\varepsilon v_t^2 + \left[v_x + \frac{1}{2}\,w_x^2\right]^2 + w_t^2 + w_{xx}^2 + h\,w_{xt}^2\right\}dx \tag{2.7}$$

and we can easily verify that the derivative of $E_\varepsilon(t)$ is given by

$$\frac{d}{dt}\,E_\varepsilon(t) = \left[w_t w_{xtt} + w_{xt} w_{xx} - w_t w_{xxx} - \left(v_x + \frac{1}{2}\,w_x^2\right) v_t + w_t w_x \left(v_x + \frac{1}{2}\,w_x^2\right)\right]_0^L$$

which is identically equal to zero due to the boundary conditions (2.1). Consequently, the energy is conserved, i.e., we have that $E_\varepsilon(t) = E_\varepsilon(0)$ for all $t \geq 0$. This implies that $||U(t)||_X$ is bounded in each interval where the solution exist s. Therefore, the solution exists globally in time. Uniqueness is proved in the usual way using Gronwall's inequality. Therefore, the following result holds:

Theorem 1. *Let* $\varepsilon > 0$, $(h > 0)$ *and* $(v_0, v_1, w_0, w_1) \in X$. *Then, problem* (1.3), (1.4), (1.6) *with boundary conditions* (2.1) *has a (unique) global weak solutions such that*

$$(v^\varepsilon, v_t^\varepsilon, w^\varepsilon, w_t^\varepsilon) \in C([0,\infty); X)$$

and the total energy $E_\varepsilon(t)$ *given by* (2.7) *satisfies*

$$E_\varepsilon(t) = E_\varepsilon(0) \quad \text{for all} \quad t \geq 0.$$

Remark. We refer to the article of J.E. Lagnese and G. Leugering [4] for a proof of Theorem 1 in a more complicated case where the boundary conditions are nonlinear and dissipative.

§3) The asymptotic limit with Dirichlet boundary conditions

For the sake of completeness we briefly recall the main steps of the convergence result in [7].

Let $\varepsilon > 0$, $h > 0$ and consider the initial boundary value problem (1.3), (1.4), (1.5), (1.6) with initial conditions (v_0, v_1, w_0, w_1) belonging to the Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\mathcal{O}m)$. Similar discussion as the one given in the previous section shows that problem (1.3), (1.4), (1.5), (1.6) is globally well posed in the space X and the total energy $E_\varepsilon(t)$ given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L \left[\varepsilon(v_t^\varepsilon)^2 + \left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right]^2 + (w_t^\varepsilon)^2 + (w_{xx}^\varepsilon)^2 + h(w_{xt}^\varepsilon)^2\right] dx$$

is constant, i.e. $E_\varepsilon(t) = E_\varepsilon(0)$ for all $t \geq 0$. Here $\{v^\varepsilon, w^\varepsilon\}$ denote the solution-pair of system (1.3), (1.4), (1.5), (1.6). Thus the following sequences (in ε):

$$\{\sqrt{\varepsilon}\, v_t^\varepsilon\}, \qquad \left\{v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right\}, \qquad \{w_t^\varepsilon\}, \qquad \{w_{xt}^\varepsilon\}, \qquad \{w_{xx}^\varepsilon\}$$

and $\{w_{xt}^\varepsilon\}$ remain bounded in $L^2(\Omega\times(0,T))$. Extracting subsequences (that we still denote by the index ε in order to simplify notations) we deduce that there exist $\xi(x,t)$, $\eta(x,t)$ and $u(x,t)$ such that

$$\sqrt{\varepsilon}\, v_t^\varepsilon \rightharpoonup \xi \qquad \text{weakly} -^* \text{ in} \quad L^\infty(0,T;L^2(\Omega)) \tag{3.1}$$

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2 \rightharpoonup \eta \qquad \text{weakly} -^* \text{ in} \quad L^\infty(0,T;L^2(\Omega)) \tag{3.2}$$

and

$$w^\varepsilon \rightharpoonup u \quad \text{weakly} -^* \text{ in} \quad L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \tag{3.3}$$

as $\varepsilon \to 0$.

Clearly, the weak convergence in (3.3) suffices to pass to the limit in the linear terms of (1.4). It remains to identify the weak limit of the nonlinear term

$$\left[w_x^\varepsilon\left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right)\right]_x$$

as $\varepsilon \to 0$.

Since $E_\varepsilon(t)$ is bounded then $\{w^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega))$ then, we can use Aubin-Lions' compactness criteria to deduce that

$$w^\varepsilon \to u \quad \text{strongly in} \quad L^\infty(0,T;H_0^{2-\delta}(\Omega)) \tag{3.4}$$

as $\varepsilon \to 0$, for any $\delta > 0$. Combining (3.2) with (3.4) it follows that

$$w_x^\varepsilon\left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right] \rightharpoonup u_x\,\eta \quad \text{weakly in} \quad L^2(\Omega\times(0,T)) \tag{3.5}$$

as $\varepsilon \to 0$.

In order to conclude our result it suffices to identify the weak limit η in (3.2). Again, we use the boundedness of $E_\varepsilon(t)$ to observe that $\{v_x^\varepsilon\}$ is bounded in $L^2(\Omega \times (0,T))$. Consequently, we can extract a subsequence such that

$$v_x^\varepsilon \rightharpoonup \rho \quad \text{weakly in} \quad L^2(\Omega \times (0,T)) \tag{3.6}$$

as $\varepsilon \to 0$, for some $\rho = \rho(x,t)$. From (3.4) and (3.6) we deduce that

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2 \rightharpoonup \rho + \frac{1}{2}\,u_x^2 \quad \text{weakly in} \quad L^2(\Omega \times (0,T)). \tag{3.7}$$

Together with (3.2) this implies that

$$\eta = \rho + \frac{1}{2}\,u_x^2. \tag{3.8}$$

We claim that η is independent of x. In fact, due to (3.1) we have that

$$\varepsilon\, v_{tt}^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad H^{-1}(0,T;L^2(\Omega)) \tag{3.9}$$

as $\varepsilon \to 0$. From (1.3), (3,9) and (3.7) it follows that

$$\eta_x = \left[\rho + \frac{1}{2}\,u_x^2\right]_x = 0$$

which proves our claim. Thus, $\eta = \eta(t)$. Integrating the identity (3.8) from $x = 0$ to $x = L$ give us that

$$L\,\eta(t) = \int_0^L \rho\,dx + \frac{1}{2}\int_0^L u_x^2\,dx = \frac{1}{2}\int_0^L u_x^2\,dx$$

because $\int_0^L \rho\,dx = 0$. Indeed, $\int_0^L \rho\,dx = \operatorname*{Lim}_{\varepsilon\to 0} \int_0^L v_x^\varepsilon\,dx = 0$ since $v^\varepsilon(0,t) = v^\varepsilon(L,t) = 0$ and (3.6) holds. Hence,

$$\eta\,u_x = \left(\frac{1}{2L}\int_0^L u_x^2\,dx\right)u_x\,.$$

Consequently

$$\left[w_x^\varepsilon\left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right)\right]_x \rightharpoonup \left(\frac{1}{2L}\int_0^L u_x^2\,dx\right)u_{xx}$$

as $\varepsilon \to 0$. The convergences above hold along suitable subsequences. However, taking into account that the limit u has been identified as the unique solution of

$$u_{tt} + u_{xxxx} - h\, u_{xxtt} - \left(\frac{1}{2L}\int_0^L u_x^2\, dx\right) u_{xx} = 0 \text{ in } \Omega \times (0,T) \tag{3.10}$$

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0, \quad t > 0 \tag{3.11}$$

$$u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \quad x \in (0,L) \tag{3.12}$$

we deduce that the whole family converges as $\varepsilon \to 0$.

We can summarize the above result.

Theorem 2 ([7]). *Let* $(v_0, v_1, w_0, w_1) \in X$ *where* $X = H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\Omega)$, *be fixed. Let* $h > 0$ *and consider the solution* $\{w^\varepsilon, v^\varepsilon\}$ *of system* (1.3), (1.4), (1.5), (1.6). l Then, the following convergences hold as $\varepsilon \to 0$:

$$w^\varepsilon \rightharpoonup u \qquad \text{weakly in} \quad L^2(0,T;H_0^2(\Omega))$$

$$v_x^\varepsilon \rightharpoonup \frac{1}{2L}\int_0^L u_x^2\, dx - \frac{1}{2} u_x^2 \quad \text{weakly in} \quad L^2(\Omega \times (0,T))$$

The function $u = u(x,t)$ satisfies (3.10), (3.11), (3.12).

§4) The asymptotic limit with Neumann boundary conditions on v^ε and clamped end condition on w^ε

In this section we consider $\varepsilon > 0$ $(h > 0)$ and analyse the asymptotic behavior as $\varepsilon \to 0$ of the global solution of problem (1.3), (1.4) with boundary conditions (2.1) and initial conditions (1.6). The existence of solution is guaranteed by Theorem 1.

Our main purpose now is to study the asymptotic limit of $\{w^\varepsilon, v^\varepsilon\}$ as $\varepsilon \to 0$. We shall use the method of [7] described in Section 3 just pointing out the extra steps needed in this case due to the new boundary conditions.

The total energy $E_\varepsilon(t)$ given by (2.7) is constant for all $t \geq 0$. Therefore the following sequences

$$\{\sqrt{\varepsilon}\, v_t^\varepsilon\}, \quad \left\{v_x^\varepsilon + \frac{1}{2}(w_x^\varepsilon)^2\right\}, \quad \{w_x^\varepsilon\}, \quad \{w_{xx}^\varepsilon\} \quad \text{and} \quad \{w_{xt}^\varepsilon\}$$

are bounded in $L^2(\Omega \times (0,T))$. Extracting subsequences we deduce the existence of functions $\xi(x,t),\ \eta(x,t)$ and $u(x,t)$ such that

$$\sqrt{\varepsilon}\, v_t^\varepsilon \rightharpoonup \xi \ \text{ weakly} -^* \text{ in } \quad L^\infty(0,T;L^2(\Omega)) \tag{4.1}$$

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon) \rightharpoonup \eta \quad \text{weakly} -^* \text{ in } \quad L^\infty(0,T;L^2(\Omega)) \tag{4.2}$$

$$w^\varepsilon \rightharpoonup u \quad \text{weakly} -^* \text{ in } \quad L^\infty(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \tag{4.3}$$

as $\varepsilon \to 0$.

Using (4.3) we can pass to the limit in the linear terms of (1.4). It remains to identify the limit of the nonlinear term

$$\left[w_x^\varepsilon\left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right)\right]_x$$

as $\varepsilon \to 0$. Since $E_\varepsilon(t)$ is bounded then $\{w^\varepsilon\}$ is uniformly bounded in

$$L^\infty(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)).$$

Using the classical Aubin-Lions compactness lemma we deduce that

$$w^\varepsilon \to u \quad \text{strongly in } \quad L^\infty(0,T;H_0^{2-\delta}(\Omega)) \tag{4.4}$$

as $\varepsilon \to 0$, for any $\delta > 0$. Combining (4.2) with (4.4) we obtain that

$$w_x^\varepsilon\left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right] \rightharpoonup u_x\eta \quad \text{weakly in } \quad L^2(\Omega \times (0,T)). \tag{4.5}$$

We want to identify η in (4.5). First, we claim that $\{v^\varepsilon\}$ is bounded in $L^\infty(0,T;H^1(\Omega))$. In fact, due to the boundedness of the energy and Sobolev embedding theorem we know that $v_x^\varepsilon \in L^\infty(0,T;L^2(\Omega))$. Therefore, we ca n extract a subsequence such that

$$v_x^\varepsilon \rightharpoonup \rho \quad \text{weakly in } \quad L^2(\Omega \times (0,T)) \tag{4.6}$$

as $\varepsilon \to 0$, for some $\rho = \rho(x,t)$. From (4.4) and (4.6) it follows that

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon) \rightharpoonup \rho + \frac{1}{2}\,u_x^2 \quad \text{weakly in } \quad L^2(\Omega \times (0,T)) \tag{4.7}$$

as $\varepsilon \to 0$. Together with (4.2) this says that

$$\eta = \rho + \frac{1}{2}\, u_x^2 \, .$$

In view of (4.1) we also know that

$$\varepsilon\, v_{tt}^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad H^{-1}(0,T;L^2(\Omega)) \tag{4.8}$$

as $\varepsilon \to 0$. From (4.8), (1.3) and (4.7) it follows that

$$\eta_x = \left[\rho + \frac{1}{2}\, u_x^2\right]_x = 0$$

i.e. $\eta = \eta(t)$. Let us identify $\eta(t)$. We take the derivative in x of (1.3), multiply the result by $a(x) = \dfrac{L^2}{4} - \left(x - \dfrac{L}{2}\right)^2$. Integration (in space) from zero to L followed by integration by parts give us

$$\begin{aligned} \varepsilon\, \frac{d^2}{dt^2} \int_0^L v_x^\varepsilon a(x)\, dx &= \int_0^L \left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right]_{xx} a(x)\, dx = \\ &= -2 \int_0^L \left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right] dx. \end{aligned} \tag{4.9}$$

Note that when integrating by parts, no boundary terms appear since $a = 0$ at $x = 0, L$ and also because of the boundary conditions that v^ε and w^ε satisfy.
Letting $\varepsilon \to 0$ in (4.9) and using (4.7) we obtain that

$$\varepsilon\, \frac{d^2}{dt^2} \int_0^L v_x^\varepsilon a(x)\, dx \rightharpoonup -2L\, \eta(t). \tag{4.10}$$

On the other side, since $a \in L^2(\Omega)$ then

$$\int_0^L v_x^\varepsilon a(x)\, dx \rightharpoonup \int_0^L \rho(x) a(x)\, dx \quad \text{weakly in} \quad L^2(0,T)$$

therefore

$$\varepsilon\, \frac{d^2}{dt^2} \int_0^L v_x^\varepsilon a(x)\, dx \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(0,T)$$

as $\varepsilon \to 0$. This information together with (4.10) implies that

$$-2L\,\eta(t) = 0$$

that is, $\eta(t) = 0$. Consequently

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2 \rightharpoonup 0 = \eta \quad \text{weakly in} \quad L^2(\Omega \times (0,T))$$

as $\varepsilon \to 0$. Returning to (4.5) we conclude that

$$\left[w_x^\varepsilon \left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right)\right]_x \rightharpoonup 0 \quad \text{as} \quad \varepsilon \to 0.$$

We have identified the weak limit of v_x^ε. However, in order to have a complete description of the limiting behavior of v^ε we have to analyse its average. Integrating equation (1.3) with respect to $x \in [0, L]$ se get

$$\varepsilon\,\frac{d^2}{dt^2} \int_0^L v^\varepsilon\,dx = \int_0^L \left[v_x^\varepsilon + \frac{1}{2}\left(w_x^\varepsilon\right)^2\right]_x dx = 0,$$

Therefore

$$\int_0^L v^\varepsilon\,dx = \int_0^L v_0\,dx + t \int_0^L v_1\,dx.$$

We have proved the following theorem.

Theorem 3. *Let $(v_0, v_1, w_0, w_1) \in X$ where $X = V \times V \cap L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\Omega)$, where*

$$V = \left\{ f \in H^1(\Omega),\, \int_0^L f\,dx = 0 \right\}, \quad \varepsilon > 0, \quad h > 0$$

and $\{w^\varepsilon, v^\varepsilon\}$ be the (unique) global solution of problem (1.3), (1.4), (2.1), (1.6). *Then,*

$$(w^\varepsilon, w_t^\varepsilon) \rightharpoonup (u, u_t) \quad \text{weakly in} \quad L^2(0,T; H_0^2(\Omega)) \times L^2(\Omega \times (0,T))$$

as $\varepsilon \to 0$, where $u = u(x,t)$ is the solution of

$$u_{tt} + u_{xxxx} - h\,u_{xxtt} = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0 \quad \forall\, t > 0$$

$$u(x,0) = w_0(x), \;\; u_t(x,0) = w_1(x), \quad 0 < x < L$$

Moreover

$$v_x^\varepsilon \rightharpoonup -\frac{1}{2}\, u_x^2 \quad \textit{weakly in} \quad L^2(\Omega \times (0,T))$$

as $\varepsilon \to 0$ *and* $\int_0^L v^\varepsilon\, dx = \int_0^L v_0\, dx + t\int_0^L v_1\, dx$, *for all* $\varepsilon > 0$.

Remark The final result of Theorem 3 is kind of unexpected when compared with the result given in Theorem 2 (see also [7]) in the case of Dirichlet boundary conditions for v^ε. In the present case, the limit is completely linear.

§5) The asymptotic limit: Hinged ends for w and Neumann boundary conditions for v.

In this section we will consider a different boundary condition on w^ε than in the previous section. Let $\varepsilon > 0$ (and $h > 0$) and consider problem (1.3), (1.4), (1.6) with boundary condition (2.2). In this case, integration of (1.3) from zero to $x = L$ give us that

$$\varepsilon\,\frac{d^2}{dt^2}\int_0^L v^\varepsilon(x,t)\, dx = \frac{1}{2}\big[\big(w_x^\varepsilon(L,t)\big)^2 - \big(w_x^\varepsilon(0,t)\big)^2\big].$$

Therefore, in general we could have that $\int_0^L v^\varepsilon\, dx \neq 0$. Due to this observation instead of the space X we considered in Section 2 we introduce the Hilbert space Y

$$Y = H^1(\Omega) \times L^2(\Omega) \times W \times H_0^1(\Omega)$$

where

$$V = \left\{ f \in H^1(\Omega),\, \int_0^L f\, dx = 0 \right\}$$

and

$$W = H^2(\Omega) \cap H_0^1(\Omega)$$

WE can rewrite problem (1.3), (1.4), (1.6) as a first order system like we did in (2.3) with $U_0 \in Y$ where D, A, U and $N(U)$ were already given in Section 2. Direct calculation shows that $D^{-1}(A)$ with domain

$$\{f \in H^2(\Omega), f_x = 0, x = 0, x = L\} \times H^1(\Omega) \times H^4(\Omega) \cap H_0^1(\Omega) \times H^3(\Omega) \cap H_0^1$$

is the infinitesimal generator of a semigroup of contractions in Y.

Theorem 4. *Let $\varepsilon > 0$ $(h > 0)$ and $(v_0, v_1, w_0, w_1) \in Y$ consider problem (1.3), (1.4), (1.6) with boundary conditions (2.2). Then, there exists $T_0 > 0$ and a mild solution $\{v^\varepsilon, w^\varepsilon\}$ of the above problem such th at*

$$(v^\varepsilon, v_t^\varepsilon, w^\varepsilon, w_t^\varepsilon) \in C([0, T_0), Y).$$

In this case we do not have a simple a priori estimate. For example, the natural quantity

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L \left\{\varepsilon|v_t^\varepsilon|^2 + \left[v_x^\varepsilon + \frac{1}{2}(w_x^\varepsilon)^2\right]^2 + |w_t^\varepsilon|^2 + |w_{xx}^\varepsilon|^2 + h|x_{xt}^\varepsilon|^2\right\}dx \tag{5.1}$$

is not conserved. In fact, direct calculation, using the boundary conditions (2.2) give us that

$$\frac{dE_\varepsilon(t)}{dt} = \frac{1}{2}|w_x^\varepsilon(L,t)|^2\, v_t^\varepsilon(L,t) - \frac{1}{2}\,|w_x^\varepsilon(0,t)|^2\, v_t^\varepsilon(0,t). \tag{5.2}$$

The right hand side of (5.1) will be (in general) different from zero.

We claim that $\{v^\varepsilon\}$ is bounded in $L^\infty(0, T_0; H^1(\Omega))$. In fact, due to the boundedness of $E_\varepsilon(t)$ we know that $v_x^\varepsilon \in L^\infty(0, T_0; L^2(\Omega))$. Integration from zero to $x = L$ of equation (1.3) and the boundary conditions (2.2) giv e us that

$$\varepsilon\,\frac{d^2}{dt^2}\int_0^L v^\varepsilon(x,t)\,dx = \frac{1}{2}\left[\big(w_x^\varepsilon(L,t)\big)^2 - \big(w_x^\varepsilon(0,t)\big)^2\right]. \tag{5.3}$$

Using the Sobolev embedding theorem, the right hand side of (5.3) can be bounded by a constant C (because of the boundedness of $E_\varepsilon(t)$) which allow us to conclude from (5.3) that $v^\varepsilon \in L^\infty(0, T_0; L^2(\Omega))$. This proves our claim.

Next, since $\{w^\varepsilon\}$ is uniformly bounded in

$$L^\infty(0, T; W) \cap W^{1,\infty}(0, T_0; H_0^1(\Omega))$$

then the classical Aubin-Lions compactness lemma applies and we deduce that

$$w^\varepsilon \to u \quad \text{strongly in} \quad L^\infty(0, T_0; H_0^{2-\delta}(\Omega)) \tag{5.4}$$

as $\varepsilon \to 0$ for any $\delta > 0$. This implies that

$$w_x^\varepsilon \left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2 \right] \rightharpoonup u_x\,\eta \quad \text{weakly in} \quad L^2(\Omega \times (0,T_0)). \tag{5.5}$$

Our objective is to identify η. Proceeding like in Section 4 (see (4.6), (4.7) and (4.8)) we show that

$$v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon) \rightharpoonup \eta(t) = \rho + \frac{1}{2}\,u_x^2 \quad \text{weakly in} \quad L^2(\Omega \times (0,T_0)) \tag{5.6}$$

as $\varepsilon \to 0$. Now we consider $a(x) = \dfrac{L^2}{4} - \left(x - \dfrac{L}{2}\right)^2$. We take the derivative in x of (1.3), multiply the result by $a(x)$ and integrate (in space) from $x = 0$ to $x = L$. Integration by parts of the resulting identity give us

$$\varepsilon\,\frac{d^2}{dt^2}\int_0^L v_x^\varepsilon a(x)\,dx = \int_0^L \left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)\right)_{xx} a(x)\,dx =$$

$$= \frac{1}{2}\left[\left(w_x^\varepsilon(L,t)\right)^2 + \left(w_x^\varepsilon(0,t)\right)^2\right] - 2\int_0^L \left(v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right) dx. \tag{5.7}$$

Using (5.4) and (5.6) we deduce from (5.7) that

$$\begin{aligned} \varepsilon\,\frac{d^2}{dt^2}\int_0^L v_x^\varepsilon\,a(x)dx \rightharpoonup & \frac{L}{2}\left[u_x^2(L,t) + u_x^2(0,t)\right] - 2\int_0^L \left(\rho + \frac{1}{2}\,u_x^2\right) dx \\ = & \frac{L}{2}\left[u_x^2(L,t) + u_x^2(0,t)\right] - 2\,\eta(t)L \end{aligned} \tag{5.8}$$

Reasoning as in Section 4 (see (4.8)) we have that

$$\varepsilon\,\frac{d^2}{dt^2}\int_0^L v_x^\varepsilon\,a(x)dx \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(0,T_0)$$

as $\varepsilon \to 0$. This together with (5.8) implies that

$$\eta(t) = \frac{1}{4}\left[u_x^2(0,t) + u_x^2(L,t)\right].$$

Consequently, from (5.5) we conclude that

$$\left[w_x^\varepsilon\left[v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2\right]\right]_x \rightharpoonup \big(u_x\,\eta(t)\big)_x =$$

$$= \frac{1}{4}\left[u_x^2(0,t) + u_x^2(L,t)\right]u_{xx}$$

weakly in $L^2(\Omega\times(0,T_0))$ as $\varepsilon\to 0$.

We proved the following

Theorem 5. *Let $(v_0, v_1, w_0, w_1)\in Y$ be fixed. Then*

$$(w^\varepsilon, w_t^\varepsilon) \rightharpoonup (u, u_t) \quad \textit{weakly in} \quad L^2(0,T_0; H_0^2(\Omega))\times L^2(\Omega\times(0,T))$$

as $\varepsilon\to 0$, where $u = u(x,t)$ is the solution of

(5.9)
$$u_{tt} + u_{xxxx} - h\,u_{xxtt} - \frac{1}{4}\left[u_x^2(L,t) + u_x^2(0,t)\right]u_{xx} = 0$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0 \quad 0\le t < T_0$$

$$u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \qquad 0 < x < L$$

Moreover,

$$v_x^\varepsilon \rightharpoonup -\frac{1}{2}\,u_x^2 + \frac{1}{4}\left[u_x^2(L,t) + u_x^2(0,t)\right]$$

weakly in $L^2(\Omega\times(0,T_0))$ as $\varepsilon\to 0$.

§6) Further comments and results

In this section we describe some possible extensions of our results and also indicate open problems on the subject.

(I) System (1.3), (1.4) could be considered under the presence of thermal effects. For example, we can consider the model

$$\varepsilon\,v_{tt} - \left[v_x + \frac{1}{2}\,w_x^2\right]_x = 0$$

$$w_{tt} + w_{xxxx} - h\,w_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}\,(w_x)^2\right)\right]_x + \alpha\theta_{xx} = 0$$

$$\theta_t - \theta_{xx} - \alpha w_{xxt} = 0$$

in $\Omega \times (0,T)$, $\Omega = \{0 < x < L\}$, with boundary conditions

$$v_x(0,t) = v_x(L,t) = \theta(0,t) = \theta(L,t) = 0 \quad \forall\, t > 0$$

$$w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \forall\, t > 0$$

and initial conditions

$$v(x,0) = v_0(x) \qquad\qquad w(x,0) = w_0(x)$$

$$v_t(t,x) = v_1(x) \qquad\qquad w_t(x,0) = w_1(x)$$

$$\theta(x,0) = \theta_0(x)$$

The same arguments allow us to describe the limit of $\{w^\varepsilon, v^\varepsilon, \theta^\varepsilon\}$ as $\varepsilon \to 0$. The limit (u,θ) of $(w^\varepsilon, \theta^\varepsilon)$ satisfies a linear system of equations modelling a thermoelastic beam.

(II) One may show that when the initial data satisfy suitable compatibility conditions, the convergences in Theorem 3 hold in the strong topologies. By weak lower semicontinuity of the L^2-norm we have

$$\operatorname*{Lim\,inf}_{\varepsilon \to 0} E_\varepsilon(t) \geq \frac{1}{2}\int_0^L \left\{|\xi|^2 + u_t^2 + u_{xx}^2 + h\,u_{xt}^2\right\} dx \tag{6.1}$$

where $E_\varepsilon(t)$ is given by (2.7) and ξ was found in (4.1). Using the conservation of energy we also know that

$$\begin{aligned} E_\varepsilon(t) = E_\varepsilon(0) \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\int_0^L \Bigg\{ \left[\frac{dv_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]^2 + \\ + w_1^2 + \left(\frac{d^2 w_0}{dx^2}\right)^2 + h\left(\frac{dw_1}{dx}\right)^2 \Bigg\} dx = E(0). \end{aligned} \tag{6.2}$$

The energy for the limit system in Theorem 3 is given by

$$F(t) = \frac{1}{2}\int_0^L [u_t^2 + u_{xx}^2 + h\,u_{xt}^2]\, dx \tag{6.3}$$

and it is conserved along time, i.e., $F(t) = F(0)$ for all $t > 0$. Combining (6.1) with (6.3) we deduce that

$$\operatorname*{Lim\,inf}_{\varepsilon \to 0} E_\varepsilon(t) \geq F(t) \qquad \forall\, 0 \leq t \leq T. \tag{6.4}$$

Since, obviously,

$$\frac{1}{2}\int_0^L \left[|\xi|^2 + |u_t|^2 + u_{xx}^2 + h\, u_{xt}^2\right] dx \geq F(t).$$

Suppose that the initial data $\{w_0, w_1\}$ are such that

$$E(0) = F(0). \tag{6.5}$$

Then, combining (6.4)-(6.5) we would have

$$\operatorname*{Lim\,inf}_{\varepsilon\to 0} E_\varepsilon(t) = F(t) \qquad \forall\, 0 \leq t \leq T$$

which amounts to say that $\sqrt{\varepsilon}\, v_t^\varepsilon$ and $v_x^\varepsilon + \frac{1}{2}\,(w_x^\varepsilon)^2$ converge strongly to zero in L^2 and, in view of the weak convergence $(w^\varepsilon, w_t^\varepsilon) \rightharpoonup (u, u_t)$ in $L^2(0,T; H_0^2(\Omega)) \times L^2(\Omega\times(0,T))$ as $\varepsilon \to 0$, then

$$(w^\varepsilon, w_t^\varepsilon) \to (u, u_t) \quad \text{strongly in} \quad L^2(0,T; H_0^2(\Omega)) \times L^2(\Omega\times(0,T))$$

as $\varepsilon \to 0$. However, (6.5) holds if and only if

$$\frac{dv_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 = 0 \quad \text{almost everywhere in} \quad 0 < x < L.$$

Otherwise $E(0) > F(0)$ and the above arguments do not apply. Similar discussion also works in all other cases we studied in the previous sections.

(III) In the proof of Theorems 2, 3 and 5 we considered the case when the initial data (v_0, v_1, w_0, w_1) is fixed. The same result holds if we consider the case when they do depend on ε provided we assume that $(v_0^\varepsilon, v_1^\varepsilon, w_0^\varepsilon, w_1^\varepsilon)$ are such that

$$\{\sqrt{\varepsilon}\, v_1^\varepsilon\}_{\varepsilon>0} \quad \text{and} \quad \left\{\frac{dv_0^\varepsilon}{dx}\right\}_{\varepsilon>0} \quad \text{are bounded in } L^2(\Omega)$$

and

$$(w_0^\varepsilon, w_1^\varepsilon) \rightharpoonup (w_0, w_1) \text{ weakly in } W \times H_0^1(\Omega)$$

as $\varepsilon \to 0$.

(IV) Our approach does not seem to work in the two-dimensional case when we consider the full nonlinear von Kármán equations or related models (see [2], [3] and the references therein) and try to show that it remains "close" to the $2-D$ Timoshenko's model:

$$u_{tt} + \Delta^2 u - h\,\Delta u_{tt} - \left(\iint_\Omega |\nabla u|^2\,dxdy\right)\Delta u = 0.$$

In the engineering literature there is a formal procedure named Berger's approximation where such proximity is claimed.

(V) The case when $h = 0$. The limiting process we describe in all previous sections seem to work well with minor modifications, however, existence (and uniqueness) of global weak solutions of such models is not quite clear (to us).

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