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Stabilization and Periodic Solutions of a Hybrid System Arising in the Control of Noise

SORIN MICU Facultatea de Matematica-Informatica, Universitatea din Craiova, 1100, Romania and Universidad Carlos III de Madrid, 28913 Leganés, Spain.

ENRIQUE ZUAZUA Departamento de Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain.

1 INTRODUCTION

In the last years, the construction of new fuel efficient turboprop engines has motivated the development of an active control methodology for interior pressure field chambers. Generally, the active control of noise consists in the generation of an appropriate secondary pressure wave which optimally interferes with the primary one.

In this paper we study a simplified model for this problem, introduced in Banks et al. (1990), consisting of a two-dimensional interior cavity with a flexible boundary. This can be considered as a hybrid system in which the acoustic vibrations of an interior fluid are coupled with the mechanical vibrations of a string. For other examples of hybrid systems, such as those coupling strings or beams with rigid bodies, see Littman and Markus (1988 b) or Hansen and Zuazua (1992). For simplicity we modify slightly the model introduced in Banks et al. (1990) considering on the boundary an active string instead of an Euler-Bernoulli beam.

In Sections 2 and 3 we present the mathematical formulation of the problem and we give a result of existence, uniqueness and stability of solutions.

The aymptotic properties are studied in Sections 4 and 5. We prove the convergence to zero of the energy of the system for every initial data and we note one of the most important mathematical features of the model: the very weak dissipation leads to a non uniform decay rate of the energy. This is a relevant mathematical property of a hybrid system.

Finally, in Section 6, we study the non-homogeneous problem with a second term acting on the flexible part of the boundary. This part transmits noise or vibrations from the exterior field to the interior cavity via fluid-structure interaction. It is natural to consider that the non-homogeneous term is periodic in time. Under this assumption, we study the existence of time-periodic solutions which is equivalent to the fact that all trajectories are bounded. The non uniform decay rate does not allow to apply a standard fix point argument. To overcome this difficulty we use a Fourier decomposition argument.

2 THE MATHEMATICAL MODEL

We consider the two-dimensional cavity $\Omega = (0,1) \times (0,1)$ in which the interior acoustic vibrations are coupled with the mechanical vibration of a string located in the subset $\Gamma_0 = \{(x,0) : x \in (0,1)\}$ of the boundary of Ω.

To describe the acoustic wave motion let $\vec{v}$ be the velocity, p the pressure and ρ the density of the fluid in our domain. Also, we consider that, at rest, the pressure p_0 and the density ρ_0 are constant. The linearized equations for the propagation of sound in an inviscid, elastic and compressible fluid, describing small disturbances, are:

$$\begin{cases} \rho' + \rho_0 \, div \, \vec{v} = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \rho_0 \vec{v}\,' + \nabla p = 0 & \text{in} \quad \Omega \times (0,\infty). \end{cases} \tag{1}$$

We denote by $'$ the time derivative.

Let W be the transversal displacement (in the plane of Ω) of the string which is assumed to be dissipative and with Neumann boundary conditions. On it is acting the interior pressure p of the fluid:

$$\begin{cases} W'' - W_{xx} - W'_{xx} = p - p_0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0) = W_x(1) = 0 & \text{for} \quad t \in (0,\infty). \end{cases} \tag{2}$$

On Γ_0 we impose the condition of continuity of velocity which results from the assumption that the string is impenetrable to the fluid. The part $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ of the boundary of Ω is rigid and impenetrable, thus leading to zero normal velocity. We obtain the following boundary conditions:

$$\begin{cases} \vec{v} \cdot \nu = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \vec{v} \cdot \nu = W' & \text{on} \quad \Gamma_0 \times (0,\infty). \end{cases} \tag{3}$$

In studying sound waves, it is usual to assume that $p = f(\rho)$. In the case of small perturbations, we can consider that the relation between p and ρ is linear:

$$p - p_0 = c_0^2(\rho - \rho_0) \tag{4}$$

where c_0 is the speed of sound in our fluid.

We introduce now a velocity potential Φ, such that $\nabla\Phi = \vec{v}$. The second equation of (1) implies that $p - p_0 = -\rho_0\Phi'$ and consequently, from (4) we get that $\rho' = -\frac{1}{c_0^2}\rho_0\Phi''$.

Considering that the speed of sound, c_0, and the density at rest, ρ_0, are 1, we obtain the following system in Φ and W:

$$\begin{cases} \Phi'' - \triangle\Phi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \dfrac{\partial\Phi}{\partial y} = -W' & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W'' - W_{xx} - W'_{xx} + \Phi' = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & \text{for} \quad t \in (0,\infty) \\ \Phi(0) = \Phi^0, \quad \Phi'(0) = \Phi^1 & \text{in} \quad \Omega \\ W(0) = W^0, \quad W'(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases} \tag{5}$$

In Micu and Zuazua (1994) we have studied a similar system in which the dissipative term was W' instead of $-W'_{xx}$.

We define the energy associated with this system by:

$$E(t) = \frac{1}{2}\int_\Omega (\mid \nabla\Phi \mid^2 + (\Phi')^2) + \frac{1}{2}\int_{\Gamma_0}((W_x)^2 + (W')^2). \tag{6}$$

The system has a dissipative nature. Indeed, multiplying in (5) the first equation by Φ', the fourth equation by W' and integrating by parts, we get, formally, that:

$$dE(t)/dt = -\int_{\Gamma_0}(W'_x)^2 \leq 0.$$

The initial data $(\Phi^0, \Phi^1, W^0, W^1)$ is considered in the space of finite energy:

$$X = H^1(\Omega) \times L^2(\Omega) \times H^1(\Gamma_0) \times L^2(\Gamma_0). \tag{7}$$

For $f^i = (f_1^i, f_2^i, f_3^i, f_4^i) \in X$ with $i \in \{1,2\}$, we define the inner product in X by:

$$\langle f^1, f^2\rangle = \int_\Omega (\nabla f_1^1 \nabla f_1^2 + f_1^1 f_1^2) + \int_\Omega f_2^1 f_2^2 + \int_{\Gamma_0}((f_3^1)_x(f_3^2)_x + f_3^1 f_3^2) + \int_{\Gamma_0} f_4^1 f_4^2.$$

Define now the following space:

$$\mathcal{D}(\mathcal{A}) = \{(\Phi^0, \Phi^1, W^0, W^1) \in H^2(\Omega)\times H^1(\Omega)\times H^1(\Gamma_0)\times H^1(\Gamma_0) : \ (W^0 + W^1) \in H^2(\Gamma_0),$$

$$\frac{\partial\Phi^0}{\partial\nu} = 0 \text{ on } \Gamma_1, \ \frac{\partial\Phi^0}{\partial y} = -W^1 \text{ on } \Gamma_0, \ (W^0 + W^1)_x(0) = (W^0 + W^1)_x(1) = 0\}$$

and the operator $\mathcal{A}$ defined in $\mathcal{D}(\mathcal{A})$ as follows:

$$\mathcal{A}(\Phi, \Psi, W, V) = (-\Psi \ , -\Delta\Phi \ , -V \ , -(W + V)_{xx} + \Psi).$$

We can consider now the following abstract Cauchy problem:

$$\begin{cases} U' + \mathcal{A}U = 0 \\ U(0) = U_0 \\ U(t) = (\Phi, \Phi', W, W')(t) \in \mathcal{D}(\mathcal{A}). \end{cases} \tag{8}$$

At the end of the next section we will discuss the relation between the original system (5) and (8).

3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

First, we have a classical result of existence, uniqueness and stability for the system (8):

THEOREM 1 *i) Strong solutions: If* $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{D}(\mathcal{A})$ *then there exists a unique strong solution of (8)* $(\Phi, \Phi', W, W') \in \mathcal{C}([0,\infty), \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0,\infty), X)$.

Moreover, for any strong solution, the associated energy (6) satisfies:

$$\frac{dE}{dt}(t) = -\int_{\Gamma_0} (W_x')^2. \tag{9}$$

ii) Weak solutions: If $(\Phi^0, \Phi^1, W^0, W^1) \in X$ *then there exists a unique solution of (8) with the property:* $(\Phi, \Phi', W, W') \in \mathcal{C}([0,\infty), X)$.

For two weak solutions (Φ, W) *and* $(\bar{\Phi}, \bar{W})$ *we have the following stability property:*

$$\begin{aligned} &\int_\Omega (\mid \nabla(\Phi - \bar{\Phi}) \mid^2 + \mid (\Phi - \bar{\Phi})' \mid^2) + \int_{\Gamma_0} (\mid (W - \bar{W})_x \mid^2 + \mid (W - \bar{W})' \mid^2) \leq \\ &\leq \int_\Omega (\mid \nabla(\Phi^0 - \bar{\Phi}^0) \mid^2 + \mid \Phi^1 - \bar{\Phi}^1 \mid^2) + \int_{\Gamma_0} (\mid (W^0 - \bar{W}^0)_x \mid^2 + \mid W^1 - \bar{W}^1 \mid^2). \end{aligned} \tag{10}$$

Proof: We prove that the operator $\mathcal{A} + \mathcal{I}$ is maximal monotone in X and we apply the Hille-Yosida theory (see Cazenave and Haraux (1990), Theorem 3.1.1, p.37).

Indeed, if $U = (\Phi, \Psi, W, V) \in \mathcal{D}(\mathcal{A})$ then $\langle (\mathcal{A} + \mathcal{I})U, U \rangle \geq \int_{\Gamma_0} (V_x)^2 \geq 0$, which means that $\mathcal{A} + \mathcal{I}$ is monotone.

On the other hand, for all $F = (f_1, f_2, f_3, f_4) \in X$ we can find a unique solution $U = (\Phi, \Psi, W, V) \in \mathcal{D}(\mathcal{A})$ for the equation $(\mathcal{A} + \mathcal{I})U = F$. This is equivalent to solve the following system:

$$\begin{cases} -\Psi + \Phi = f_1 \\ -\Delta\Phi + \Psi = f_2, \ \dfrac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma_1 \text{ and } \dfrac{\partial \Phi}{\partial \nu} = V \text{ on } \Gamma_0 \\ -V + W = f_3 \\ -(W + V)_{xx} + \Psi + V = f_4 \text{ and } (W + V)_x(0) = (W + V)_x(1) = 0 \end{cases} \tag{11}$$

First, we consider the variational equation corresponding to (11), which consists in finding (Φ, W) in $H^1(\Omega) \times H^1(\Gamma_0)$ such that, for all $(\varphi, u) \in H^1(\Omega) \times H^1(\Gamma_0)$:

$$\begin{aligned} &\int_\Omega \nabla\Phi\nabla\varphi + \int_\Omega \Phi\varphi - \int_{\Gamma_0} W\varphi + 2\int_{\Gamma_0} W_x u_x + \int_{\Gamma_0} \Phi u + \int_{\Gamma_0} W u = \\ &\int_\Omega (f_1 + f_2)\varphi - \int_{\Gamma_0} f_3\varphi + \int_{\Gamma_0} (f_3)_x u_x + \int_{\Gamma_0} (f_1 + f_3 + f_4)u. \end{aligned} \tag{12}$$

The left side of the equation (12) defines a continuous and coercive bilinear form in $(H^1(\Omega)\times H^1(\Gamma_0))^2$ while the right side defines a continuous linear form in $H^1(\Omega)\times H^1(\Gamma_0)$.

Applying Lax-Milgram's Lemma it results that (12) has a unique solution (Φ, W) in $H^1(\Omega)\times H^1(\Gamma_0)$. Finally, in view of the classical regularity results for the Laplace's operator, this implies that $\mathcal{A}+\mathcal{I}$ is maximal.

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Define now the space $\tilde{X}=\tilde{H}^1(\Omega)\times L^2(\Omega)\times\tilde{H}^1(\Gamma_0)\times L^2(\Gamma_0)$ where $\tilde{H}^1$ is the quotient space $\tilde{H}^1=H^1/P_0$ and P_0 is the set of all constant functions and $\tilde{\mathcal{D}}(\mathcal{A})$ the subspace of $\tilde{X}$ defined in the same way as $\mathcal{D}(\mathcal{A})$:

$$\tilde{\mathcal{D}}(\mathcal{A})=\{(\Phi^0,\Phi^1,W^0,W^1)\in\tilde{H}^2(\Omega)\times H^1(\Omega)\times\tilde{H}^1(\Gamma_0)\times H^1(\Gamma_0):(W^0+W^1)\in H^2(\Gamma_0),$$
$$\frac{\partial\Phi^0}{\partial\nu}=0 \text{ on } \Gamma_1,\ \frac{\partial\Phi^0}{\partial y}=-W^1 \text{ on } \Gamma_0,\ (W^0+W^1)_x(0)=(W^0+W^1)_x(1)=0\}.$$

Observing that if (Φ,Φ',W,W') is the solution of (8) with initial data (Φ^0,Φ^1,W^0,W^1) then $(\Phi+c_1,\Phi_t,W+c_2,W_t)$ is the solution with initial data $(\Phi^0+c_1,\Phi^1,W^0+c_2,W^1)$ we can obtain a similar theorem of existence uniqueness and stability in $\tilde{X}$ but in addition, in view of the coercivity of the energy in this space, we have that for all strong solutions:

$$\begin{aligned}&\|(\Phi,\Phi',W,W')(t)\|_{\tilde{X}}\le\|(\Phi^0,\Phi^1,W^0,W^1)\|_{\tilde{X}},\quad\forall t\ge0,\\&\|\mathcal{A}(\Phi,\Phi',W,W')(t)\|_{\tilde{X}}\le\|\mathcal{A}(\Phi^0,\Phi^1,W^0,W^1)\|_{\tilde{X}},\quad\forall t\ge0\end{aligned}\tag{13}$$

which means that the trajectories are bounded in $\tilde{\mathcal{D}}(\mathcal{A})$.

We want to see now when the boundary conditions $W_x(0)=W_x(1)=0$ make sense and for which initial data the solution is relatively compact in $\tilde{X}$. To do this we have to ask for more regularity of W^0.

Property 1 *If the initial data* (Φ^0,Φ^1,W^0,W^1) *belongs to the space:*

$$\tilde{\mathcal{D}}=\{(\Phi^0,\Phi^1,W^0,W^1)\in\tilde{\mathcal{D}}(\mathcal{A}):W^0\in\tilde{H}^2(\Gamma_0)\ \textit{and}\ W^0_x(0)=W^0_x(1)=0\}$$

the corresponding solution of (8), (Φ,Φ',W,W')*, is in* $\mathcal{C}_b([0,\infty),\tilde{\mathcal{D}})$*, where* $\mathcal{C}_b([0,\infty),\tilde{\mathcal{D}})=\mathcal{C}([0,\infty),\tilde{\mathcal{D}})\cap L^\infty(0,\infty;\tilde{\mathcal{D}})$*. Moreover,* $\tilde{\mathcal{D}}$ *is dense and compact in* $\tilde{X}$*.*

Proof: We have from (13) that $W'(t)+W(t)=V(t)\in\mathcal{C}_b([0,\infty),H^2(\Gamma_0))$. Thus $W(t)=e^{-t}W_0+\int_0^t e^{s-t}V(s)ds$ and, as soon as W^0 is considered in $H^2(\Gamma_0)$, we deduce that $W\in\mathcal{C}_b([0,\infty),H^2(\Gamma_0))$. The same argument shows that the boundary conditions $W_x(0)=W_x(1)=0$ are fulfilled for all $t\ge0$ if $W^0_x(0)=W^0_x(1)=0$.

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Remark: Under the hypothesis that the initial data of the problem (8) belongs to $\tilde{\mathcal{D}}$, this problem is equivalent to the original system (5).

4 STRONG STABILIZATION

Concerning the asymptotic behavior of solutions we prove first the strong stabilization in the energy space.

THEOREM 2 *For each initial data in X the corresponding weak solution of (8) tends asymptotically towards zero in $\tilde{X}$, that is,* $\lim_{t\to\infty} E(t) = 0$.

Proof: The main tools of our analysis is an extension of the well known Invariance Principle of La Salle and Holmgren's Uniqueness Theorem.

Observe first that it is sufficient to consider the case of initial data $(\Phi^0, \Phi^1, W^0, W^1)$ in $\tilde{\mathcal{D}}$. A standard density argument and the property of stability (10) enable us to complete the proof.

In this case Theorem 1 and Property 1 give an unique strong solution $U = (\Phi, \Phi', W, W')$ for the equation (8), with $\{U(t)\}_{t\geq 0}$ bounded in $\tilde{\mathcal{D}}$. Since $\tilde{\mathcal{D}} \subseteq \tilde{X}$ with compact inclusion, we have that $\{U(t)\}_{t\geq 0}$ is relatively compact in $\tilde{X}$. So, all we have to prove is that the unique limit point in $\tilde{X}$ of the trajectory $\{U(t)\}_{t\geq 0}$ as $t \to \infty$ is $(0, 0, 0, 0)$.

For this take $(t_n)_{n\geq 0}$ with $U(t_n) \to (z^0, z^1, v^0, v^1)$ in $\tilde{X}$ when $n \to \infty$ and show that

$$z^0 = z^1 = 0, v^0 = v^1 = 0 \text{ in } \tilde{X} \tag{14}$$

i.e. there exist two constants c_1 and c_2 such that $z^0 = c_1$, $z^1 = 0$, $v^0 = c_2$ and $v^1 = 0$.

For some $T > 0$, that will be fixed later on, define, for all $n \geq 0$, the family of translated solutions,

$$\Phi_n(t) = \Phi(t + t_n), \quad W_n(t) = W(t + t_n), \quad t \in [0, T].$$

By the Theorem of Ascoli Arzela the sequence $(\Phi_n, \Phi_n', W_n, W_n')_{n\geq 0}$ is relatively compact in $\mathcal{C}([0, T]; \tilde{X})$ and hence there is a subsequence, also denoted by $(\Phi_n, \Phi_n', W_n, W_n')_{n\geq 0}$ and $(z, z', v, v') \in \mathcal{C}([0, T]; \tilde{X})$, such that $(\Phi_n, \Phi_n', W_n, W_n') \to (z, z', v, v')$ in $\mathcal{C}([0, T]; \tilde{X})$.

Observe that $z(0) = z^0, v(0) = v^0, z'(0) = z^1, v'(0) = v^1$ and so if we prove that z and v are constant in space and time we obtain the stated result.

Since E is a non increasing and non negative function it has a finite limit when t tends to infinite. On the other hand since

$$E(t_n) - E(t_n + T) = \int_{t_n}^{t_n+T} \int_{\Gamma_0} (W_x')^2 = \int_0^T \int_{\Gamma_0} ((W_n)_x')^2$$

we can pass to limit and obtain that

$$\int_0^T \int_{\Gamma_0} (v_x')^2 = 0 \tag{15}$$

It results that (z, z', v, v') satisfies:

$$\begin{cases} z'' - \triangle z = 0 & \text{in} \quad \Omega \times (0, T) \\ \frac{\partial z}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0, T) \\ \frac{\partial z}{\partial y} = -v' & \text{on} \quad \Gamma_0 \times (0, T) \\ v'' - v_{xx} + z' = 0 & \text{on} \quad \Gamma_0 \times (0, T) \\ v_x(0, t) = v_x(1, t) = 0 & \text{for} \quad t \in (0, T). \end{cases} \tag{16}$$

Consider now the functions $Y = z_x$ and $\varphi = v_x$ which satisfy:

$$\begin{cases} Y'' - \triangle Y = 0 & \text{in} \quad \Omega \times (0,T) \\ Y = 0 & \text{for} \quad x \in \{0,1\}, (y,t) \in (0,1) \times (0,T) \\ Y_y = 0 & \text{for} \quad y \in \{0,1\}, (x,t) \in (0,1) \times (0,T) \\ \varphi_{xx} = Y' & \text{on} \quad \Gamma_0 \times (0,T) \\ \varphi(0,t) = \varphi(1,t) = 0 & \text{for} \quad t \in (0,T). \end{cases} \tag{17}$$

Finally define the function $\eta = Y''$ which satisfies:

$$\begin{cases} \eta'' - \triangle \eta = 0 & \text{in} \quad \Omega \times (0,T) \\ \eta_y = 0 & \text{on} \quad \Gamma_0 \times (0,T) \\ \eta = 0 & \text{on} \quad \Gamma_0 \times (0,T). \end{cases} \tag{18}$$

We can apply now Holmgren's Uniqueness Theorem (see Hörmander (1990), Theorem 8.6.5, p.309 and Lions (1988), Theorem 8.1, p.88) which implies that $\eta = 0$ in $\Omega \times (1, T-1)$ if $T > 2$ and so

$$Y(t,x,y) = Y_1(x,y)t + Y_2(x,y) \text{ in } \Omega \times (1, T-1).$$

We deduce that, for $i \in \{1,2\}$,

$$\begin{cases} \Delta Y_i = 0 & \text{in } \Omega \\ \dfrac{\partial Y_i}{\partial n} = 0 & \text{for } y \in \{0,1\}, x \in (0,1) \\ Y_i = 0 & \text{for } x \in \{0,1\}, y \in (0,1) \end{cases} \tag{19}$$

which implies that $Y = 0$ in $\Omega \times (1, T-1)$ and $\varphi_{xx} = 0$ on $\Gamma_0 \times (1, T-1)$.

It follows that $z_x = 0$ and $v_{xxx} = 0$ implying that z and v are constant in time and space. Hence $z = 0$ and $v = 0$ in $\tilde{X}$ which concludes the proof.

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Remark: We have proved the strong stabilization of the system in $\tilde{X}$ but not in X because the energy is not coercive in X. We have that $\nabla\Phi, \Phi_t, W_x$ and W_t tend to zero in the L^2 norm when t goes to infinity but not necessarily Φ and W. Observe that $\Phi = 1$ and $W = 1$ is a constant solution in time of our system. However, from a physical point of view, only the convergence in $\tilde{X}$ is relevant, since it represents the fact that the pressure and velocity of the interior fluid tend to its rest values.

5 THE LACK OF UNIFORM DECAY

In this paragraph we observe that the rate of decay is not uniform. Results like this are typical for linear hybrid systems in which the dissipation is very weak: it can force the strong stabilization but it cannot assure the uniform decay.

THEOREM 3 *The rate of decay of the energy is not exponential in $\tilde{X}$, i.e. there is no $C > 0$ and $\omega > 0$ such that $E(t) \leq CE(0)e^{-\omega t}$ for all weak solution.*

Proof: It consists in constructing solutions with decay rates slower than any preassigned exponential function. To do this we look for a sequence of solutions $\{(\Phi_n, W_n)\}_{n\geq 1}$ for the equation (5) of the type $(\Phi_n, W_n) = e^{\lambda_n t}(\varphi_n, v_n)$ where $\varphi_n = \varphi_n(y)$ and $v_n \in I\!R$ and $\mathcal{R}e\, \lambda_n \to 0$ when $n \to \infty$.

We can see that the equation (5) has solutions of this form if (φ_n, v_n) satisfies

$$\begin{cases} \lambda^2\varphi - \varphi_{yy} = 0 \text{ for } y \in (0,1) \\ \varphi_y(1) = 0,\ \varphi_y(0) = -\lambda v \\ \lambda^2 v + \lambda v + \lambda\varphi(0) = 0. \end{cases} \tag{20}$$

It results that $\varphi(y) = cosh\,(\lambda(y-1))$ and $v = sinh\,\lambda$ is solution of (20) if λ is solution of the algebric equation: $e^{2\lambda} - 1 = -\dfrac{2}{\lambda+2}$.

To study this equation we consider, for $n \in I\!N$, the squares γ_n with centers $\alpha_n = n\pi\, i$ and sides $\frac{3}{n\pi} < \varepsilon_n < \frac{4}{n\pi}$.

We can apply now Rouché' s Theorem and we find that, for n large enough, there is a solution of this equation in each square γ_n. This concludes the proof.

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Remark: Taking $(\Phi_{k,n}, W_{k,n}) = e^{\lambda_{k,n}t}(\varphi_{k,n}, v_{k,n})$ where $\varphi_{k,n}(x,y) = \cos(k\pi x)\xi_{k,n}(y)$ and $v_{k,n}(x) = \cos(k\pi x)u_{k,n}$ with $u_{k,n} \in I\!R$, we can prove that, for each $k \in I\!N$, there is a sequence of eigenvalues $\lambda_{n,k}$ with $\mathcal{R}e\,\lambda_{k,n} \to 0$ as n tends to infinity.

Remark: It is well known that the lack of exponential decay implies the lack of uniform decay. In this case the norm of the semigroup $S(t)$ in $\mathcal{L}(\tilde{X}, \tilde{X})$ is one for all $t > 0$. Littman and Markus (1988 a) showed that, in this situation, the rate of decay of a particular trajectory may be slower than any preassigned function that decreases to zero as $t \to \infty$.

6 EXISTENCE OF PERIODIC SOLUTIONS

In this section we consider a non-homogeneous time-periodic term and we study the existence of periodic solutions for the system:

$$\begin{cases} \Phi'' - \triangle\Phi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \frac{\partial\Phi}{\partial\nu} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \frac{\partial\Phi}{\partial y} = -W' & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W'' - W_{xx} - W'_{xx} + \Phi' = f & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & \text{for} \quad t \in (0,\infty). \end{cases} \tag{21}$$

The non-homogeneous periodic term f represents a perturbing force on the active part of the boundary due to an exterior noise source. We give a positive answer to this problem in the case of a smooth function f.

THEOREM 4 *Assume that $f \in W^{1,2}(0,T; L^2(0,1))$ is periodic with respect to t, of period T. If $f(t) = \sum_{n=0}^{\infty} f_n(t) cos(n\pi x)$ and:*

$$\begin{cases} \sum_{n=0}^{\infty} \| (f_n)' \|_{L^2(0,T)} \, exp(\frac{3\pi}{4}n) < \infty, \\ \sum_{n=0}^{\infty} (n+1) \| f_n \|_{L^2(0,T)} \, exp(\frac{3\pi}{4}n) < \infty \end{cases} \quad (22)$$

then there exists a unique weak solution of (21) periodic in $\tilde{X}$ with period T.

Proof: First we observe that since the decay rate of the energy is not uniform it is not possible to use directly a fix point method for the Poincaré's map. To avoid this difficulty we reduce the system to an infinite sequence of one-dimensional problems. To treat the one-dimensional problem we first introduce an artificial dissipation which produces the uniform decay of the energy. Then by classical arguments we obtain a periodic solution. Then we let the perturbative term go to zero. In this way it is proved that each one-dimensional problem has a unique time-periodic solution.

To acomplish this we write $f(t,x) = \sum_{n=0}^{\infty} f_n(t) cos(n\pi x)$ and study, for each n, the problem of the existence of time-periodic solutions for the one-dimensional system:

$$\begin{cases} \psi'' - \psi_{yy} + n^2\pi^2\psi = 0 & \text{for } y \in (0,1), t \in (0,\infty) \\ \frac{\partial\psi}{\partial y}(1) = 0 & \text{for } t \in (0,\infty) \\ \frac{\partial\psi}{\partial y}(0) = -v' & \text{for } t \in (0,\infty) \\ v'' + n^2\pi^2 v' + n^2\pi^2 v + \psi'(0) = f_n & \text{for } t \in (0,\infty). \end{cases} \quad (23)$$

We prove the existence of a unique time-periodic solution of (23) in three steps.

Step 1. For ε sufficiently small we consider the problem

$$\begin{cases} \psi'' - \psi_{yy} + \varepsilon\psi' + n^2\pi^2\psi = 0 & \text{for } y \in (0,1), t \in (0,\infty) \\ \frac{\partial\psi}{\partial y}(1) = 0 & \text{for } t \in (0,\infty) \\ \frac{\partial\psi}{\partial y}(0) = -v' & \text{for } t \in (0,\infty) \\ v'' + n^2\pi^2 v' + n^2\pi^2 v + \psi'(0) = f_n & \text{for } t \in (0,\infty). \end{cases} \quad (24)$$

It is easy to see that the term $\varepsilon\psi'$ produces an uniform decay rate for the energy of the homogeneous system:

$$F(t) = \frac{1}{2}\int_0^1 (| \psi_y |^2 + | \psi' |^2 + n^2\pi^2 | \psi |^2) \; + \frac{1}{2}((v')^2 + n^2\pi^2 v^2).$$

Indeed, if we consider the following perturbation of F:

$$G(t) = F(t) + \delta(\int_0^1 (\psi_t\psi) \; + w_t w + \psi(0)w), \text{ for all } t \geq 0$$

we deduce that, for δ small enough,

$$\frac{1}{2}F(t) < G(t) < \frac{3}{2}F(t) \text{ for all } t \geq 0 \text{ and}$$

$$G'(t) \leq -\beta F(t) \text{ for all } t \geq 0 \text{ and for } \beta < \varepsilon.$$

The last two inequalities imply the exponential decay of the energy $F(t)$.

In this case for $k = k(\varepsilon) \in I\!\!N$ large enough, we can apply the Banach's Fix Point Theorem for the contractive operator:

$$\mathcal{J}^k_{\varepsilon,n} : X \to X,\ \mathcal{J}^k_{\varepsilon,n}(\psi^0, \psi^1, v^0, v^1) = (\psi(kT), \psi'(kT), v(kT), v'(kT)).$$

Since $\mathcal{J}^k_{\varepsilon,n}$ is contractive it has a unique fix point u_0. It results that

$$\mathcal{J}^1_{\varepsilon,n} u_0 = \mathcal{J}^1_{\varepsilon,n}(\mathcal{J}^k_{\varepsilon,n} u_0) = \mathcal{J}^k_{\varepsilon,n}(\mathcal{J}^1_{\varepsilon,n} u_o).$$

Since $\mathcal{J}^k_{\varepsilon,n}$ has a unique fix point we get that u_0 is the unique fix point for $\mathcal{J}^1_{\varepsilon,n}$. This fix point gives us a unique periodic solution $(\psi_{\varepsilon,n}, v_{\varepsilon,n})$.

Step 2. We obtain some estimates for the periodic solutions found on the first step which are independent of ε .

Integrating in time the derivative of the energy of the system and the derivative of the energy of the system obtained by taking a time-derivative in (24), we get that:

$$\begin{aligned} n^2\pi^2 \int_0^T (v'_{\varepsilon,n})^2\, dt\ &\le \int_0^T f_n^2\, dt, \\ n^2\pi^2 \int_0^T (v''_{\varepsilon,n})^2\, dt\ &\le \int_0^T (f'_n)^2\, dt \end{aligned} \tag{25}$$

The equation on Γ_0 of (24) implies that $n^2\pi^2 \int_0^T v_{\varepsilon,n} = \int_0^T f$. Applying now Poincaré's inequality we deduce that:

$$\int_0^T v^2_{\varepsilon,n} \le \frac{C}{n^2\pi^2} \int_0^T f^2. \tag{26}$$

We go back to the equation for v in (24) and we obtain:

$$\int_0^T (\psi'_{\varepsilon,n})^2(t,0)\, dt\ \le C((n^2\pi^2+1)\int_0^T f_n^2\, dt\ + \frac{1}{n^2\pi^2}\int_0^T (f'_n)^2\, dt\ ). \tag{27}$$

Since $\int_0^T \psi_{\varepsilon,n}(t,y) dt\ = 0$, for all $y \in (0,1)$ we can apply Poincaré's inequality

$$\int_0^T (\psi_{\varepsilon,n})^2(t,0)\, dt\ \le C\int_0^T (\psi'_{\varepsilon,n})^2(t,0)\, dt. \tag{28}$$

We can apply now multiplier techniques (see Lions (1988), Lemma 1.3, p.139). Multiplying the first equation of (24) by $(1-y)\Psi_y$, integrating by parts and applying Gronwall's Lemma we obtain that

$$\int_0^T (\psi_t^2 + \psi_y^2 + n^2\pi^2\psi^2)\, dt\ (y) \le \int_0^T (\psi_t^2 + \psi_y^2 + \alpha\psi^2)\, dt\ (0)\ \ exp((\frac{3\pi n}{2} + \frac{\varepsilon}{2})y). \tag{29}$$

With the estimates (25), (27) and (28) for $\psi_{\varepsilon,n}(t,0)$, $(\psi_{\varepsilon,n})_y(t,0)$ and $\psi'_{\varepsilon,n}(t,0)$ in $L^2(0,T)$, we deduce that

$$\begin{aligned} &\int_0^T ((\psi'_{\varepsilon,n})^2 + ((\psi_{\varepsilon,n})_y)^2 + n^2\pi^2\psi^2_{\varepsilon,n})(t,y)\, dt\ \le \\ &\le c'\int_0^T [(f'_n)^2 + (1+n^2)(f_n)^2]\, dt\ exp(\frac{3\pi n}{2}y),\ \ \text{for all } y \in (0,1). \end{aligned} \tag{30}$$

Step 3. The last estimates show that the set of periodic solutions corresponding to each ε, $\{(\psi_{\varepsilon,n}, \psi'_{\varepsilon,n}, v_{\varepsilon,n}, v'_{\varepsilon,n})\}_{\varepsilon>0}$, is bounded in $L^2(0,T;Y)$ with $Y = H^1(0,1)\times L^2(0,1)\times I\!R \times I\!R$. Since our problem is linear we can pass to the limit when $\varepsilon \to 0$ and we obtain for each $n \in I\!N$ a weak periodic solution (ψ_n, v_n) of equation (23).

We turn now to our two dimensional problem and consider the series:

$$\Phi = \sum_{n=0}^{\infty} \psi_n(t,y)cos(n\pi x) \text{ and } W = \sum_{n=0}^{\infty} v_n(t)cos(n\pi x)$$

where, for each $n \in I\!N, (\psi_n, v_n)$ is the periodic solution found on step 3. The two series converge in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)$ and in $H^1(0,T;H^1(\Gamma_0))$ respectively, if f verifies (22). On the other hand (Φ, W) is the periodic solution we were looking for. The uniqueness follows from Theorem 2 since the difference of two solutions of the non-homogeneous problem is a solution of the homogeneous problem.

■

Remark: Condition (22) holds for functions f with the property that $f(t)$ belongs to the space of Gevrey's functions of exponent one defined by: $\partial_x^j f(t,x)$ are continuous on $0 \le t < \infty$, $x \in [0,1]$ and, in addition, for each compact set $K \subset [0,\infty)\times[0,1]$ and each constant $\theta > 0$ there is a constant C_θ such that $\mid \partial_x^j f(t,x) \mid \le C_\theta\, \theta^j\, j^j$ for all $j = 1, 2, ..$ when $(t,x) \in K$. This is a class of analytic functions (see Hörmander (1990), p. 281).

7 COMMENTS

In Banks et al. (1990) a two-dimensional model is presented in which, on the subset Γ_0 of the boundary, a fixed end Euler-Bernoulli beam is considered. The methods developed in this paper can be adapted to this type of problems.

It is possible to give results like Theorem 1 and Theorem 2 (with very similar proofs) for the case of a string located in Γ_0 but with Dirichlet boundary conditions. Nevertheless, the methods used for proving Theorem 3 and 4, based on the separation of variables, cannot be applied in this case.

The results of Sections 2 and 3 can be generalized for any domain. If Ω is a bounded open set in $I\!R^2$ with smooth boundary and Γ_0 is a part of the boundary of the domain it is sufficient to change in (1) the equation on Γ_0 by

$$W'' - \frac{d^2W}{d\tau^2} + \frac{d^2W'}{d\tau^2} + \Phi' = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty)$$

where $\frac{d}{d\tau}$ is the derivative in the tangential direction. The rest of the results of this paper may be extended to some particular geometries, for instance when Ω is a ball of $I\!R^2$.

The last result given in Theorem 4 requests a lot of regularity for the non-homogeneous term f. This phenomena may be expected since the dissipation is very weak and is located in a relatively small part of the boundary. The assumptions of Theorem 4 are very probably sharp. Indeed, we have assumed that f and f' belong to $L^2(0,T;L^2(\Gamma_0))$. However, when dealing with classical damped wave equations like, for instance,

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} + u' = f & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{31}$$

the fact that $f \in L^2(0,T;L^2(\partial\Omega))$ suffices to guarantee that the energy of the trajectories remains bounded. In Theorem 4 we have imposed an extra assumption on f'. This is due to the hybrid boundary conditions we are dealing with, which are roughly of second order. We also assumed that $f(t)$ belongs to some Gevrey class. This is reasonable in view of the structure of the dissipation region which implies a very weak decay rate of high frecuencies. Indeed, as Ralston (1969) observed, (and later on Bardos, Lebeau and Rauch (1988) applied in the context of the control and stabilization of the wave equation in bounded domains) if one characteristic ray escapes to the dissipative region we can not expect an uniform decay. In this case every segment $\{(x,y_0) : x \in (0,1)\}$, for any $y_0 \in (0,1)$, constitutes a ray with such a property.

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