A multi-body system coupling wave equations in n and $n-1$ space dimensions

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Abstract

We analyze the fine properties of a model arising in multi-body or built-up structures constituted by two n-dimensional wave equations coupled with a $(n-1)$-dimensional one along a flexible interface. When $n=2$ this system models the coupling of two $2-d$ wave equations through a $1-d$ elastic string. The material properties (such as rigidity, density and velocity of propagation of waves) in each of the three media (the two n-dimensional domains and the interface) do not need to be the same.

This system is well-posed in the classical energy space. We investigate the possibility that this system to be well-posed in asymmetric spaces in which the solutions have a different regularity (in terms of the number of derivatives which are in L^2) on the two sides of the interface, a property that is well-known to hold for the $1-d$ analogue of this model.

We first perform a plane wave analysis which allows predicting the ranges of coefficients in which well-posedness in asymmetric spaces may hold. We then give a rigorous proof of this property, based on decomposition and microlocal arguments, in the case in which the velocity of propagation on the interface is smaller than that on the medium where the initial data are less regular. The optimal gap on the regularity of solutions to both sides of the interface is of one L^2 derivative in space-time.

Contents

1 Introduction

This work is devoted to analyze fine properties of some simple but relevant models arising in multi-body or built-up structures. We consider a model describing the vibrations of such a structure in $n-d$ constituted by two n-dimensional domains connected through a flexible $(n-1)$-dimensional interface. The model is a system of two $n-d$ wave equations coupled through a $(n-1)$-dimensional one on the interface. The material properties (such as rigidity, density and velocity of propagation of waves) in each of the three media (the two n-dimensional domains and the $(n-1)$-dimensional interface) do not need to be the same. Thus these coefficients may vary from one medium to another.

The problem will be addressed in any space dimension since the mathematical tools we shall develop apply in a similar way. In $2-d$ the model is relevant for vibrations of networks of membranes and strings, and in $3-d$ for the propagation of acoustic waves.

The models we shall analyze fit perfectly in the class of second order (in time) systems with a good variational characterization in which classical methods like energy ones, semigroups, etc. apply. Consequently, existence and uniqueness of finite-energy solutions holds automatically. However, as it has been observed in a number of closely related systems, the complexity of the interaction of the various components of the system may lead to some unexpected results and, in particular, to the well-posedness of the system in asymmetric spaces which cannot be seen by classical energy estimates. More precisely we consider spaces of solutions which are H^1 on one side of the interface and H^2 in the other one.

This property of well-posedness in asymmetric spaces has been observed in a number of $1-d$ models involving the coupling of elastic strings and beams with point masses (see [2], [3], [4], [5] and [6]), using non-harmonic Fourier series and D'Alembert's formula. As far as we know, this paper is the first one dealing with this topic in several space dimensions.

The main result of this paper says that, under suitable assumptions on the velocities of propagation in the three media, the system is indeed well-posed in an asymmetric space which is characterized by the fact that solutions are more regular (by one derivative in L^2) to one side of the interface. The main assumption is, roughly, that the velocity of propagation on the interface is smaller than that on the medium in which the initial are less regular. This result extends the previous ones mentioned above on the $1-d$ analogue.

This surprising property may be predicted by a plane wave analysis to which we devote the first part of this article. This analysis suggests that, under suitable conditions on the ratios of the velocities of propagation in the three media and, in particular, in the case above, there are directions of propagation of incident waves for which part of its energy penetrates into the second medium, without any regularization. This is incompatible with the property of well-posedness in asymmetric spaces. However, the same plane wave analysis suggests that, for some values of the velocities of propagation, total reflection of waves on the interface occurs to first order, with a gain of one derivative on the wave transfered to the second medium. This is evidence for the possibility of well-posedness of the system in asymmetric spaces, with a gap of one derivative in L^2 from one side of the interface to the other. The second part of the article is devoted to prove this result, using microlocal techniques.

To perform the plane wave analysis we consider the system in a very particular geometric configuration: the case where the whole space $\mathbb{R}^n$ is split in two subdomains by a hyperplane (the interface) that, without loss of generality, we assume to be $x_1 = 0$. In this way the space $\mathbb{R}^n$ is split in Ω^- and Ω^+, which are respectively the half spaces $x_1 < 0$ and $x_1 > 0$. The deformation on $x_1 < 0$ is denoted by u^-, u^+ in $x_1 > 0$ and w on the interface. See Figure 1.

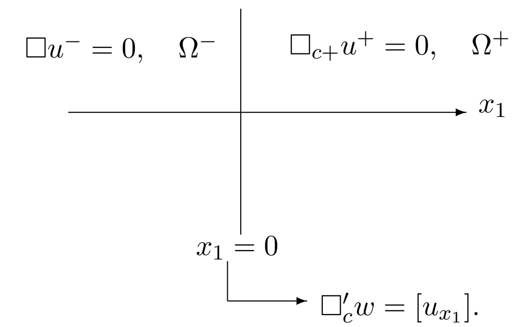

Figure 1: The whole space $\mathbb{R}^n$ split in two subdomains $\Omega^\pm$ by the interface $x_1 = 0$.

This analysis is useful to identify the ranges of coefficients for which well-posedness in asymmetric spaces may hold. But the well-posedness result in asymmetric spaces we prove holds in a much more general setting in which a smooth domain of $\mathbb{R}^n$ is split in two subdomains by means of a smooth compact manifold without boundary. See Figure 2.

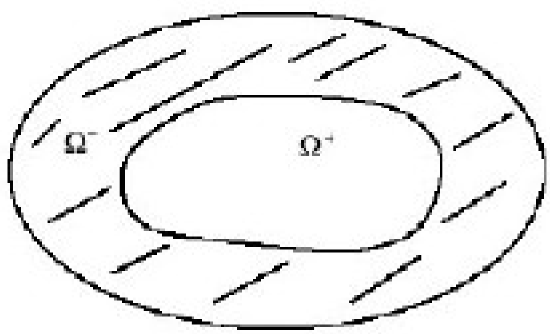

Figure 2: Smooth bounded domain split into two subdomains Ω^+ and Ω^- by a smooth compact hypersurface.

The methods we employ are quite general. For instance, the plane wave analysis applies to all constant coefficient models. We refer for instance to [17] where the interaction of a wave-heat model is analyzed. It is important to note that, as developed by J. Ralston [15, 16] (see also the more recent work [17]), the case of curved boundaries and interfaces needs to be addressed by means of Gaussian beams. On the other hand, the microlocal tools we apply to prove the well-posedness in asymmetric spaces can also be applied in much more general situations. But a systematic analysis of these problems for more sophisticated models remains to be done. In particular, it would be also desirable to address the system of elasticity in $3-d$ with a flexible $2-d$ interface. The problem in that case is more complex because of the additional interaction between transversal and longitudinal waves (see [1] and [11]).

The model addressed in this article is a particular one in a large class of interaction systems coupling PDEs of different nature. They arise not only in the context of the vibrations of multi-structures (to which the model addressed in this paper belongs), but also in the frame of thermoelasticity ([8], [11]) in which the relevant systems are constituted by the coupling between heat and wave-like models that are fulfilled in the same domain, or also in the context of fluid-structure interaction in which parabolic and hyperbolic-like equations are coupled through a possibly moving or free interface ([17]). All these models together constitute an interesting and widely open subject for future research.

The paper is organized as follows. In section 2 we describe the model and its main properties. In sections 3 and 4 we perform the plane wave analysis in the cases where the velocities of propagation in both n-dimensional media coincide and are different, respectively. In section 5 we prove some fine microlocal estimates. Section 6 is devoted to the statement and proof of the main result of well-posedness in asymmetric spaces. We end with some further comments and open problems in section 7.

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2 The model and its basic properties

2.1 Description of the model

Let Ω be a domain of $\mathbb{R}^n$ (non-necessarily bounded) with smooth boundary $\partial\Omega$. Let $\Gamma \subset \Omega$ be a smooth hyper-surface which divides Ω into Ω^+ and Ω^-. We call $\partial^+\Omega = \partial\Omega \cap \partial\Omega^+$ and $\partial^-\Omega = \partial\Omega \cap \partial\Omega^-$.

To simplify the discussion we assume that Γ does not intersect the exterior boundary of Ω. This avoids the generation of possible singularities on the boundary (see Figure 2).

We study the following problem

$$\begin{cases} u^-_{tt} - \Delta u^- = 0 & \text{in } \mathbb{R} \times \Omega^- \\ u^+_{tt} - c_+^2 \Delta u = 0 & \text{in } \mathbb{R} \times \Omega^+ \\ w_{tt} - c^2 \Delta_\Gamma w = [\partial_\nu u]_c & \text{on } \mathbb{R} \times \Gamma \\ u = w & \text{on } \mathbb{R} \times \Gamma \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega \\ u(0) = u_0,\, u_t(0) = u_1 & \text{in } \Omega^- \cup \Omega^+ \\ w(0) = w_0,\, w_t(0) = w_1 & \text{on } \Gamma. \end{cases} \tag{1}$$

In this model the velocity of propagation is 1 in Ω^-, c on the interface and c_+ on the second medium Ω^+. By Δ_Γ we denote the Laplace-Beltrami operator on the interface Γ. The jump on the interface $[\partial_\nu u]_c$ is that of the conormal derivatives so that $[\partial_\nu u]_c = c_+ \partial u^+/\partial\nu - \partial u^-/\partial\nu$. We will sometimes denote the jump simply by $[\cdot]$ when $c_+ = 1$, i. e. when the velocities of propagation are the same to both sides of the interface Γ. By $\partial_\nu u = \partial u/\partial\nu$ we denote the normal derivative on the interface. The normal vector ν is taken to be of unit norm and exterior to Ω^+, thus pointing to the interior of Ω^-. To shorten the notation, we shall often write u for the pair (u^-, u^+). Accordingly, we shall sometimes write the d'Alembert equation for u in the domain $\Omega^- \cup \Omega^+$. This means that each of the components of u, u^- and u^+, satisfies the d'Alembert equation in each subdomain Ω^- and Ω^+. But this will only be done when the velocity of propagation in both media $\Omega^\pm$ is the same (i. e. when $c_+ = 1$). Also when we write that u coincides with w on the interface, this means that each component of u, u^- and u^+, coincides with w over Γ. A similar notation is employed for the initial data.

We also consider the inhomogeneous problem

$$\begin{cases} u^-_{tt} - \Delta u^- = h^- & \text{in } \mathbb{R}\times\Omega^- \\ u^+_{tt} - c_+^2\Delta u = h^+ & \text{in } \mathbb{R}\times\Omega^+ \\ w_{tt} - c^2\Delta_\Gamma w = [\partial_\nu u]_c + h_\Gamma & \text{on } \mathbb{R}\times\Gamma \\ u = w & \text{on } \mathbb{R}\times\Gamma \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega \\ u(0)=u_0,\ u_t(0)=u_1 & \text{in } \Omega^-\cup\Omega^+ \\ w(0)=w_0,\ w_t(0)=w_1 & \text{on } \Gamma. \end{cases} \tag{2}$$

Obviously, in this system, there is no a priori restriction between the applied force $h = (h^-, h^+)$ in the interior $\Omega^-\cup\Omega^+$ and on the interface Γ. Therefore h_Γ is not necessarily restricted to be the trace of h over Γ which will be denoted by $h|_\Gamma$.

In the rest of this section we state and prove some of the basic properties of these models.

2.2 Finite energy solutions

Consider system (2). The formal energy estimate

$$\frac{d}{dt}\left(\int_\Omega u_t^2dx + \int_{\Omega^-}|\nabla u^-|^2dx + c_+^2\int_{\Omega^+}|\nabla u^+|^2dx + \int_\Gamma(w_t^2 + c^2|\nabla_\Gamma w|^2)d\sigma\right) \\ = 2\int_\Omega hu_tdx + 2\int_\Gamma h_\Gamma w_td\sigma \tag{3}$$

holds.

In other words, denoting by $E(t)$ the total energy of the system,

$$E(t) = \frac{1}{2}\left(\int_\Omega u_t^2dx + \int_{\Omega^-}|\nabla u^-|^2dx + c_+^2\int_{\Omega^+}|\nabla u^+|^2dx + \int_\Gamma(w_t^2 + c^2|\nabla_\Gamma w|^2)d\sigma\right), \tag{4}$$

which is the addition of the energy of the vibrations $u^\pm$ in $\Omega^\pm$ and of w over the interface Γ, we observe that this energy satisfies the law

$$\frac{dE(t)}{dt} = \int_\Omega hu_tdx + \int_\Gamma h_\Gamma w_td\sigma. \tag{5}$$

On the other hand, the energy is equivalent to the square of the canonical norm in the energy space:

$$H = \{(u_0,u_1,w_0,w_1)\in[H_0^1(\Omega)\times L^2(\Omega)]\times[H_0^1(\Gamma)\times L^2(\Gamma)] : u_0|_\Gamma = w_0\}.$$

In this notation we understand, once more, that u stands for the pair (u^-,u^+) so that $u\in H_0^1(\Omega)$ means that both $u^\pm$ belong to H^1 in their corresponding subdomains and that the homogeneous Dirichlet boundary condition is satisfied on the exterior boundary and that the continuity of traces holds on the interface.

Using classical semigroup or energy methods it is easy to see that the homogeneous system generates a group of isometries in H. More precisely, the following holds:

Proposition 1. *Suppose that*

$$(u_0, u_1, w_0, w_1) \in H \tag{6}$$

and

$$h \in L^1(0, T; L^2(\Omega)), \quad h_\Gamma \in L^1(0, T; L^2(\Gamma)). \tag{7}$$

Then there exists a unique weak finite energy solution

$$u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

$$w \in C([0, T]; H^1(\Gamma)) \cap C^1([0, T]; L^2(\Gamma)).$$

In other words,

$$(u, u_t, w, w_t) \in C([0, T]; H). \tag{8}$$

2.3 Smooth solutions

The following variant also holds:

Proposition 2. *Suppose that, as above, (6)-(7) hold.*

Suppose further that

$$u_0|_{\Omega^+} \in H^2(\Omega^+), \quad u_0|_{\Omega^-} \in H^2(\Omega^-), \tag{9}$$

$$w_0 = u_0|_\Gamma \in H^2(\Gamma), \tag{10}$$

$$u_1 \in H_0^1(\Omega), \quad w_1 = u_1|_\Gamma \in H^1(\Gamma), \tag{11}$$

$$h \in W^{1,1}(0; T; L^2(\Omega)),\ h_\Gamma \in W^{1,1}(0; T; L^2(\Gamma)). \tag{12}$$

Then the weak solution satisfies the further conditions

$$u \in C([0, T]; H_0^1(\Omega) \cap H^2(\Omega^\pm)) \cap C^1([0, T]; H^1(\Omega)) \tag{13}$$

$$w \in C([0, T]; H^2(\Gamma)) \cap C^1([0, T]; H^1(\Gamma)) \cap H^2((0, T) \times \Gamma). \tag{14}$$

Remark 3. *One of the key points of this result, that will be used in an essential way in the proof of well-posedness in asymmetric spaces, is that the right hand side terms* h *and* h_Γ*, are not required to satisfy any compatibility condition. In other words, we do not impose the trace of* h *over* Γ *to coincide with* h_Γ*. But, to compensate that lack of continuity of traces we need information of their time derivatives as in (12).*

It is also interesting to observe that we only get $w \in H^2((0,T) \times \Gamma)$ *and not* $w \in C([0,T]; H^2(\Gamma))$ *as we do for the components* $u^\pm$*. This is so, as we shall see in the proof of the Proposition, because of the presence of the term* $[\partial_\nu u]_c$ *in the equation satisfied by* w*.*

Proof. We first apply the previous Proposition and deduce that there exists an unique finite energy solution satisfying (8).

We then apply the same Proposition but for the system under time derivation. One then sees that $(\tilde{u}, \tilde{w}) = (u_t, w_t)$ are also finite energy solutions. To check this we observe that $(\tilde{h}, \tilde{h}_\Gamma) = (h_t, h_{\Gamma,t})$ fulfill the requirements in Proposition 1. Furthermore, the initial data for $(\tilde{u}, \tilde{w})$ are also of finite energy. Indeed, $\tilde{u}(0) = u_t(0) = u_1 \in H_0^1(\Omega)$, $\tilde{u}_t^-(0) = u_{tt}^-(0) = \Delta u_0^- + h^-(0) \in L^2(\Omega^-)$, $\tilde{u}_t^+(0) = u_{tt}^+(0) = c_+^2 \Delta u_0^- + h^+(0) \in L^2(\Omega^+)$, $\tilde{w}(0) = w_t(0) = w_1 \in H^1(\Gamma)$, $\tilde{w}_t(0) = w_{tt}(0) = \Delta_\Gamma w_0 + [\partial u_0 / \partial \nu]_c + h_\Gamma(0) \in L^2(\Gamma)$. Finally, $\tilde{u}(0)|_\Gamma = u_1|_\Gamma = w_1 = \tilde{w}(0)$.

In view of this, we deduce that, in addition to

$$u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$

$$w \in C([0,T]; H^1(\Gamma)) \cap C^1([0,T]; L^2(\Gamma)),$$

we also have

$$u_t \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$

$$w_t \in C([0,T]; H^1(\Gamma)) \cap C^1([0,T]; L^2(\Gamma)).$$

In order to conclude the proof we have to show that

$$u^\pm \in C([0,T]; H^2(\Omega^\pm)) \tag{15}$$

and

$$w \in H^2((0,T) \times \Gamma)). \tag{16}$$

To show (16) we write $\Delta_\Gamma w = w_{tt} + [\partial u / \partial \nu]_c - h_\Gamma$. Then, (16) holds by elliptic regularity, as a consequence of this elliptic equation, the assumptions on h_Γ, the fact that $w_{tt} \in C([0,T]; L^2(\Gamma))$ and that $[\partial u / \partial \nu]_c \in L^2((0,T) \times \Gamma)$. The latter is true because of the first statement of Lemma 4 and the regularity results we got for w and u.

The regularity (15) of $u^{\pm}$ is a consequence of (16) and the second statement of Lemma 4. Rigorously speaking, this Lemma applies in the case $h \equiv 0$. The general one can be handeled similarly since, when the boundary data vanish and $h \in W^{1,1}(0,T; L^2(\Omega^{\pm}))$, standard semigroup theory guarantees that (15) is true. $\square$

2.4 The non-homogeneous Dirichlet problem for the wave equation

In this subsection we recall a well-known result on the existence, uniqueness and regularity of solutions of the Dirichlet problem for the wave equation in a bounded smooth domain Ω with non-homogeneous boundary data on $\partial\Omega$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = g & \text{on } [0,T] \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega. \end{cases} \tag{17}$$

Note that, here, the domain Ω is not split by an intermediate interface.

The following holds:

Lemma 4. *Suppose that $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ and $g \in H^1((0,T) \times \partial\Omega)$ with the compatibility condition that $u_0|_{\partial\Omega} = g|_{t=0}$. Then the solution u is of finite energy and the following estimate is satisfied:*

$$\begin{aligned} \|(u(t), u_t(t))\|_{L^\infty(0,T;H^1(\Omega)\times L^2(\Omega))} + \|\partial_\nu u\|_{L^2((0,T)\times\partial\Omega)} \\ \leq C[\|(u_0, u_1)\|_{H^1(\Omega)\times L^2(\Omega)} + \|g\|_{H^1((0,T)\times\partial\Omega)}]. \end{aligned} \tag{18}$$

Now, suppose that $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ and $g \in H^2((0,T) \times \partial\Omega)$ with the compatibility conditions that $u_0|_{\partial\Omega} = g|_{t=0}$ and $u_1|_{\partial\Omega} = g_t|_{t=0}$. Then the solution u has one more derivative in L^2 and the following estimate holds:

$$\begin{aligned} \|(u(t), u_t(t))\|_{L^\infty(0,T;H^2(\Omega)\times H^1(\Omega))} + \|\partial_\nu u\|_{H^1((0,T)\times\partial\Omega)} \\ \leq C[\|(u_0, u_1)\|_{H^2(\Omega)\times H^1(\Omega)} + \|g\|_{H^2((0,T)\times\partial\Omega)}]. \end{aligned} \tag{19}$$

Remark 5. *Obviously, the same result holds if the velocity of propagation in the wave equation under consideration is c_+ instead of 1 as in (17).*

Proof. This type of result is by now well-known (see [10], Theorems 2.1 and 2.2). Let us recall the main steps of the proof.

We first consider (18). It suffices to prove an apriori estimate for smooth data. First we establish the following energy identity

$$\frac{1}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} = \int_0^t \int_{\partial\Omega} g_t \partial_\nu u \, d\sigma dt + \frac{1}{2}\|u_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)}.$$

The classical proof of the fact that $\partial_\nu u$ belongs to $L^2((0,T)\times\Gamma)$, relies on the use of the Rellich multiplier $q(x)\cdot\nabla u$, where $q(x)$ is a smooth extension of the normal vector field to the domain Ω (see [12]). When applying this multiplier and integrating by parts one gets a second identity relating the energy in the domain Ω with the L^2-norm on the boundary of the normal derivative. Combining these two identities one shows simultaneously both that u is of finite energy and that $\partial_\nu u$ belongs to $L^2((0,T)\times\Gamma)$. This completes the proof of (18).

For the proof of (19) we apply the previous argument to u_t and combine it with elliptic estimates for fixed t. □

3 Plane waves: Same velocity in both n-dimensional media

This section is devoted to develop the plane wave analysis for the model above, in which both velocities of propagation are the same to the left and right of the interface ($c_+ = 1$). We consider the whole space $\mathbb{R}^n$ with the interface being $x_1 = 0$.

3.1 Case 1: Same velocity of propagation in the three media

To begin with we consider the case where the velocity of propagation on the interface and to both sides of it are the same, i. e. $c = 1$.

Take a characteristic direction (ξ,τ) with

$$\tau^2 = \mid \xi \mid^2 = 1;\ \xi = \left(\xi_1, \xi'\right);\ \xi_1 < 0. \tag{20}$$

We also set

$$\widetilde{\xi} = (-\xi_1, \xi)\,, \tag{21}$$

which is the direction of reflection of a planar wave moving in the direction ξ, when reaching the interface.

Consider an incoming high-frequency wave of the form $e^{i(x\cdot\xi+\tau t)/\varepsilon}$ in the medium Ω^- that travels towards the interface. This produces generically a reflected wave, an interface wave and, by transmission, a wave that enters the second medium Ω^+ (see Figure 3 below). The parameter $\varepsilon > 0$ is devoted to tend to zero. But in the computations we develop it is kept fixed.

We make the following ansatz. In the domains Ω^- and Ω^+ the corresponding solutions take the form:

$$u^- = e^{i(x\cdot\xi+\tau t)/\varepsilon} + a e^{i(x\cdot\widetilde{\xi}+\tau t)/\varepsilon};\quad u^+ = b e^{i(x\cdot\xi+\tau t)/\varepsilon}. \tag{22}$$

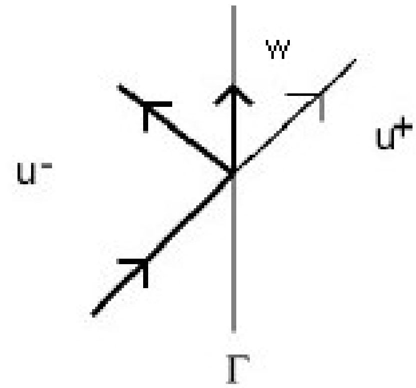

Figure 3: Incoming wave u^- that propagates in Ω^-, bounces back on the interface, generates an interface wave w, and a wave u^+ that penetrates in the second medium Ω^+.

This means that the reflection coefficient is a and b is the transmission coefficient, the amplitude of the wave transfered to the medium Ω^+.

Due to the continuity of motions at the interface, necessarily, the interface wave is

$$w = be^{i(x'\cdot\xi'+\tau t)/\varepsilon}.$$

The continuity on deformations imposes also that the amplitudes of u^+ and u^- coincide. Therefore,

$$b = 1 + a \tag{23}$$

We now perform the analysis on the interface. We have

$$\Box' w = -\frac{b(\tau^2 - \mid \xi' \mid^2)}{\varepsilon^2} e^{i(\tau t+\xi'\cdot x')/\varepsilon} = -\frac{\xi_1^2 b}{\varepsilon^2} e^{i(\tau t+\xi'\cdot x')/\varepsilon}$$

and

$$[u_{x_1}] = \frac{i\xi_1}{\varepsilon}(b + a - 1)e^{i(\tau t+\xi'\cdot x')/\varepsilon} = \frac{i\xi_1 2a}{\varepsilon} e^{i(\tau t+\xi'\cdot x')/\varepsilon}. \tag{24}$$

The interface equation then reads

$$-\frac{\xi_1 b}{\varepsilon} = 2ai$$

or, equivalently,

$$-\frac{\xi_1}{\varepsilon}(1+a) = 2ai \Rightarrow a\left(2i + \frac{\xi_1}{\varepsilon}\right) = -\xi_1/\varepsilon.$$

This gives the following explicit formula for the reflection coefficient a:

$$a = \frac{-\xi_1}{\xi_1 + 2i\varepsilon}.$$

According to it, for ξ_1 fixed, as $\varepsilon \to 0$, we see that $a \to -1$ and therefore pure Dirichlet reflection occurs. More precisely, for all directions of incidence, as the frequency of the oscillations increases, the incoming wave is more and more reflected on the interface.

A finer look provides a more precise description when $\xi_1 \sim \varepsilon$. Set $\xi_1 = \omega_1 \varepsilon$. Then

$$a = -\frac{\omega_1}{\omega_1 + 2i} = -1 + \frac{2i}{\omega_1 + 2i}.$$

Similarly, the transmission coefficient takes the form

$$b = 1 + a = \frac{2i\varepsilon}{\xi_1 + 2i\varepsilon} = \frac{2i}{\omega_1 + 2i}.$$

Observe that b is of order one in this case.

This analysis indicates that no regularization of waves should occur when crossing the interface if the waves enter the interface on directions of propagation that are nearly tangent. In other words, in view of the previous analysis, one may conjecture that finite energy solutions may not remain in the class $(u^+(t), u_t^+(t)) \in H^{1+s}(\Omega^+) \times H^s(\Omega^+)$, with $s > 0$, when the initial data are of finite energy and have that additional regularity of s derivatives in the energy space on Ω^+ and on the interface. On the basis of the plane wave analysis above one can build an heurestic argument to support that conjecture along the following lines. Consider the particular case of initial data vanishing on the interface and in the domain Ω^+. One could then take an initial datum in Ω^- of the form above, a plane wave, where the direction of propagation ξ has been chosen in the region where $\xi_1 \sim \varepsilon$, suitably localized in space away from the interface. According to the previous analysis, when reaching the interface this incoming wave should produce another one in the domain Ω^+ with amplitude $b \sim 2i/(2i+\omega_1)$, in addition to the reflected wave. This wave should penetrate the domain Ω^+ and have a H^1-norm of the same order as that of the incoming wave and, this should contradict uniform estimates of the $H^{1+s}(\Omega^+)$-norm.

Note however that this argument is formal and that, in order to rigorously conclude that well-posedness in asymmetric spaces fails, it has to be made rigorous. For that one has to localize the construction developed above on the basis of plane waves. We refer to [17] for full developments of this kind in the context of a model of interaction of heat and wave equations. The fact that the direction of propagation of waves penetrating the second medium depends on ε, probably, adds extra technical difficulties with respect to the case where the direction of propagation is independent of ε. Below, in the following subsection, we shall obtain some cases in which this occurs in directions which are independent of ε.

Transmission of waves has been conjectured for waves travelling parallel to the interface. This could be related to the existence of an infinite sequence of modes of vibration, exponentially concentrated along the interface, the so called interface . The existence of an infinite number of interface modes would very likely be also a way of proving lack of well-posedness in asymmetric spaces. Those interface modes were found in [13] for a closely related model of fluid-structure interaction in which the $2-d$ wave equation on a square cavity is coupled with a $1-d$ wave equation on its boundary. However, the coupling conditions in [13] are different from those we consider here. Consequently, the existence of those interface modes in the present model remains to be investigated.

3.2 Case 2: Different speed of propagation on the interface

We now set

$$\Box'_c w = w_{tt} - c^2 \Delta' w = [u_{x_1}],$$

so that the velocity of propagation in the interface is $c > 0$.

We pursue the arguments above with the same ansatz. In this case, the quantities entering in the interface equation read

$$\begin{aligned} \Box'_c w &= -\frac{b}{\varepsilon^2}\left(\tau^2 - c^2 \mid \xi' \mid^2\right) e^{i(\tau t + \xi' \cdot x')/\varepsilon} \\ &= -\frac{b}{\varepsilon^2}\left(\mid \xi_1 \mid^2 + (1-c^2) \mid \xi' \mid^2\right) e^{i(\tau t + \xi' \cdot x')/\varepsilon}, \end{aligned} \tag{25}$$

while the jump of the normal derivatives at the interface does not change (see (24)).

Thus, the interface equation yields:

$$\frac{2i\xi_1 a}{\varepsilon} = -\frac{b}{\varepsilon^2}\left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right),$$

or, in other words,

$$2i\xi_1 a = -\frac{(1+a)}{\varepsilon}\left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right),$$

or

$$\left[2i\xi_1 + \frac{1}{\varepsilon}\left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right)\right] a = -\frac{\xi_1^2 + (1-c^2) \mid \xi' \mid^2}{\varepsilon}.$$

Thus,

$$a = -\frac{\xi_1^2 + (1-c^2) \mid \xi' \mid^2}{2i\varepsilon\xi_1 + \left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right)}.$$

Once more, as $\varepsilon \to 0$, for ξ_1 fixed, $a \to -1$ and, consequently, pure reflection occurs.

The transmission wave has amplitude

$$b = \frac{2i\varepsilon\xi_1}{2i\varepsilon\xi_1 + \left[\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right]}.$$

Obviously, for ξ fixed, as $\varepsilon \to 0$, $b \to 0$, and the amplitude of the transmission wave vanishes.

Let us now analyze whether for suitable choices of directions ξ and frequencies $1/\varepsilon$, waves may penetrate the medium Ω^+. We consider the regime

$$2i\varepsilon\xi_1 \sim \xi_1^2 + (1-c^2) \mid \xi' \mid^2, \tag{26}$$

for which b of the order 1 and, consequently, transmission of waves should occur.

Obviously, the left hand side term vanishes as $\varepsilon \to 0$ and, therefore, for (26) to hold, $\xi_1^2 + (1-c^2) \mid \xi' \mid^2$ has to tend to zero too. In order to analyze whether this is possible we have to distinguish two cases:

Case 1: $c < 1$.

In this case, if $\xi_1^2 + (1-c^2) \mid \xi' \mid^2 \to 0$, then $\xi_1 \to 0$ and $\mid \xi' \mid \to 0$ (which would be a contradiction to the restriction $\tau^2 = \mid \xi_1 \mid^2 + \mid \xi' \mid^2$, τ being fixed). Consequently, for $c < 1$ there are no waves in this regime. Thus, we conjecture pure reflection of all waves, which is compatible with well-posedness in asymmetric spaces.

Case 2: $c > 1$.

If $c > 1$, then there are non trivial directions ξ in which the symbol

$$\xi_1^2 + (1-c^2) \mid \xi' \mid^2$$

vanishes. Indeed, for $|\tau| = 1$ we have

$$\begin{aligned} \xi_1^2 + (1-c^2) \mid \xi' \mid^2 &= \xi_1^2 + (c^2-1)\left(\xi_1^2 - \tau^2\right) = c^2\xi_1^2 - (c^2-1)\tau^2 \\ &= c^2\xi_1^2 - (c^2-1), \end{aligned}$$

and it vanishes in the directions

$$\mid \xi_1 \mid = \sqrt{\frac{c^2-1}{c^2}}. \tag{27}$$

Note that, when this happens, $a = 0$ and $b = 1$ and the wave is completely absorbed by the second medium and reflection does not occur. This is incompatible with well-posedness in asymmetric spaces.

In order to make this argument rigorous one has to build solutions which are concentrated along a ray that, when hiting the interface along the critical incidence direction (27), is purely transmitted to the second medium and the interface. Taking into account that the direction of incidence of the ray is now independent of the frequency $1/\varepsilon$, a classical Geometric Optics construction as in [17] will probably work. But the details need to be developed.

3.3 Conclusion

We conclude that,

- Both when $c = 1$ and $c > 1$ we are in the same situation in which waves penetrate from one medium to the other one and, accordingly, one conjectures that *there is no asymmetric space in which the coupled system is well-posed.*

  Note however that, as mentioned above, one expects two different phenomena to happen. When $c = 1$ one expects the existence of modes of vibration which are exponentially concentrated along the interface, while when $c > 1$ one rather expects that a Geometric Optics construction will lead to solutions that are concentrated along the critical rays satisfying (27) and for which the total energy will be transfered to the second medium.

- When $c < 1$ all waves are purely reflected and one can conjecture some decoupling at the level of regularity of the two media to both sides of the interface, and well-posedness in some asymmetric spaces.

  A more careful look to the reflection coefficient b leads to the conjecture that the maximal gap on the regularity of solutions to both sides of the interface is 1. Indeed, b has the form:

$$b = \frac{2i\varepsilon\xi_1}{2i\varepsilon\xi_1 + \left[\xi_1^2 + (1 - c^2) \mid \xi' \mid^2\right]}.$$

  Then, clearly, taking into account that $c < 1$ and the ellipticity of the symbol $\left[\xi_1^2 + (1 - c^2) \mid \xi' \mid^2\right]$ we immediately deduce that b vanishes at order $O(\varepsilon)$. This clearly shows that the waves which cross the interface should gain one derivative in L^2, but no more than that. This is the analogue of the $1-d$ result proved in [6]. One of the main results of this paper is the rigorous formulation and proof of this result in section 6.

4 Plane wave analysis: Different velocities of propagation in the n-dimensional media

4.1 Plane waves

We now assume that the velocity of propagation in the medium Ω^- is 1, it is c on the interface and c_+ on the medium Ω^+.

We perform the same analysis as above.

The system under consideration now reads

$$\begin{aligned} u^-_{tt} - \Delta u^- &= 0 && \text{for } x_1 < 0 \\ w_{tt} - c^2 \Delta_\Gamma u &= [\partial_\nu u]_c && \text{for } x_1 = 0 \\ u^+_{tt} - c_+^2 \Delta u^+ &= 0 && \text{for } x_1 > 0 \\ u &= w && \text{for } x_1 = 0. \end{aligned}$$

This time the jump $[\cdot]_c$ stands for that of the conormal derivatives so that: $[\partial_\nu u]_c = c_+ u^+_{x_1} - u^-_{x_1}$.

We have:

$$u^- = e^{i(x\cdot\xi+\tau t)/\varepsilon} + a e^{i(x\cdot\widetilde{\xi}+\tau t)/\varepsilon} \tag{28}$$

$$u^+ = b e^{i(x'\cdot\xi'+\tau t+\eta x_1)/\varepsilon} \tag{29}$$

$$w = b e^{i(x'\cdot\xi'+\tau t)/\varepsilon}. \tag{30}$$

The main difference with respect to the previous case is that the direction (ξ', η) of propagation of the wave that is transferred to the medium Ω^+ has to be computed carefully. Indeed, in order for u^+ as above to be a solution of the corresponding wave equation ($u^+_{tt} - c_+^2 \Delta u^+ = 0$) one needs

$$\tau^2 - c_+^2[|\xi'|^2 + \eta^2] = 0$$

and therefore

$$\eta = -\sqrt{\tau^2/c_+^2 - |\xi'|^2}.$$

This corresponds to the classical Snell's law. This formula indicates, in particular, that when $c_+ > 1$ there are situations in which there is no compatible direction (η, ξ') for the ray to be transmitted to the second medium. This happens when

$$\tau^2/c_+^2 - |\xi'|^2 < 0$$

or

$$\tau^2/|\xi'|^2 < c_+^2,$$

i.e. when the angle of incidence is close to tangent. In that case the whole wave necessarily bounces back and, in the second medium Ω^+, we only recover a wave that decays exponentially away form the interface. But we need to analyze all directions of incidence.

As in the previous section, by the continuity of traces we have

$$b = 1 + a. \tag{31}$$

The value of $\Box_c' w$ on the interface is the same as in (25). But the jump of the derivative in x_1 changes:

$$[u_{x_1}]_c = \frac{i\xi_1}{\varepsilon}(a - 1 + (c_+\eta b)/\xi_1)e^{i(\tau t + \xi' \cdot x')/\varepsilon}.$$

In this case the interface equation reads:

$$i\xi_1(a - 1 + (c_+\eta(1+a))/\xi_1) = -\frac{(1+a)}{\varepsilon}\left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right).$$

Thus, the reflection coefficient is

$$a = \frac{i\varepsilon(\xi_1 - c_+\eta) - \xi_1^2 - (1-c^2) \mid \xi' \mid^2}{i\varepsilon(\xi_1 + c_+\eta) + \left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right)},$$

while the transmission one is

$$b = \frac{2i\xi_1\varepsilon}{i\varepsilon(\xi_1 + c_+\eta) + \left(\xi_1^2 + (1-c^2) \mid \xi' \mid^2\right)}.$$

It is easy to note that this formula generalizes those of the previous section that correspond to the case $c_+ = 1$, $\eta = \xi_1$.

We now distinguish the following cases:

Case 1: $c < 1$**.** In this case all waves are purely reflected, regardless of the value of c_+, b being of order $O(\varepsilon)$. This is compatible with well-posedness in an asymmetric space, the regularity being larger in the right domain, with a gap of one derivative in L^2 from left to right.

Case 2: $c = 1$**.** In this case, by taking $\xi_1 \sim \omega_1\varepsilon$ we deduce that

$$b = \frac{2i}{i(1 + c_+\eta/\xi_1) + \omega_1},$$

and

$$a = \frac{i(1 - c_+\eta/\xi_1) - \omega_1}{i(1 + c_+\eta/\xi_1) + \omega_1} = -1 + \frac{2i}{i(1 + c_+\eta/\xi_1) + \omega_1}.$$

On the other hand,

$$c_+\eta/\xi_1 = -\frac{\sqrt{(1-c_+^2)|\xi'|^2 + \varepsilon^2\omega_1^2}}{\varepsilon\omega_1}.$$

When $c_+ \neq 1$, $|c_+\eta/\xi_1| \to \infty$ as $\varepsilon \to 0$. Accordingly, $b \to 0$ and therefore pure reflection is expected.

This analysis is meaningful provided the direction of propagation in the second medium, η, is real. This is obviously true when $c_+ < 1$. This is compatible with the well-posedness in an asymmetric space. When $c_+ > 1$ and $\xi' \sim 1$, the value of η is purely imaginary and this suggests that, even crossing the interface the solution takes rather the form of a decaying (away from the interface) exponential wave. This predicts also smoothing when crossing the interface.

The case $c_+ = 1$ has been treated in subsection 3.1 and we concluded that we may not expect well-posedness in asymmetric spaces.

Therefore, when $c = 1$ one may not exclude well-posedness in asymmetric spaces when $c_+ \neq 1$.

Case 3: $c > 1$. In this case there are critical directions of incidence such that

$$\xi_1^2 + (1 - c^2) \mid \xi' \mid^2 = 0. \tag{32}$$

In this direction

$$b = \frac{2i\xi_1}{i(\xi_1 + c_+\eta)} \neq 0.$$

This is incompatible with well-posedness in asymmetric spaces whenever the direction of propagation η in the second medium is real. That happens when, fixing $\tau = 1$, we have

$$1/c_+^2 - |\xi'|^2 > 0,$$

together with (32) and

$$\xi_1^2 + |\xi'|^2 = 1.$$

That is possible whenever $c_+ < c$. In this case well-posedness in asymmetric spaces should be excluded. When $c_+ > c$, all η-s are imaginary and this predicts smoothing when crossing the boundary.

The case $c_+ = c$ seems to be critical an a closer look seems to be needed. Our analysis yileds $\eta = 0$ which indicates that the transfered waves should travel parallel to the interface. This could be compatible with well-posedness in asymmetric spaces as well.

4.2 Conclusion

According to the previous analysis we may conclude that:

- Well-posedness in asymmetric may not be excluded in the following cases.

  a) $c < 1$ and any c_+;

  b) $c = 1$ and any $c_+ \neq 1$;

  c) $c > 1$ and $c_+ \geq c$.

  Recall that we assume the velocity of propagation in the medium Ω^- ($x_1 < 0$) to be 1, that of the interface to be c and on the second medium Ω^+ ($x_1 > 0$) to be c_+.

- We do not expect well-posedness in asymmetric spaces to hold for:

  a) $c = c_+ = 1$;

  b) $c > 1$ and $c_+ < c$.

The rest of this paper is devoted to rigorously prove well-posedness in asymmetric spaces. This will be achieved when $c < 1$.

Whether well-posedness occurs or not in the other cases remains to be proved rigorously.

5 Microlocal estimates

In this section we suppose that $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary $\partial\Omega$. We analyze some fine properties of solutions of the non-homogeneous Dirichlet problem (17) for the wave equation in the domain Ω. We also consider the wave equation over the boundary of Ω.

5.1 The non-homogenous Dirichlet problem

We first state estimates in the microlocally elliptic regime. Recall that a point (t, x, τ, ξ) is in the characteristic set of the wave operator if $|\tau| = |\xi|$. If $x \in \partial\Omega$ the restriction to the tangent space at $\partial\Omega$ defines the point $(t, x, \tau, \xi - (\xi, \nu)\nu)$ in the cotangent space of $\mathbb{R} \times \partial\Omega$ and the characteristic set at (t, x) is mapped to the full cone $(\tau, \xi) \in T_{t,x}\mathbb{R} \times \partial\Omega$ with $|\xi| \le |\tau|$, in which we distinguish the hyperbolic regime $|\xi| < |\tau|$ and the glancing one $|\xi| = |\tau|$.

Lemma 6. *Let P be a zero order pseudodifferential operator on $\mathbb{R} \times \partial\Omega$, invariant with respect to translations in t, vanishing in a conical neighborhood of all vectors $\{(\tau, \xi) : |\tau| \le c|\xi|\}$, with $0 < c < 1$, identically 1 in a conical neighborhood of $\{(\tau, \xi) : |\tau| \ge |\xi|\}$. Then*

$$\|(1 - P)\partial_\nu u\|_{H^1((0,T)\times\partial\Omega)} \le C[\|g\|_{H^2((0,T)\times\partial\Omega)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}]$$

provided u is a solution to the problem (17), with initial data $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, and right hand side $g \in H^2((0, T) \times \partial\Omega)$ such that

$$(g(0), g_t(0)) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma); \quad g|_{t=0} = u_0|_\Gamma. \tag{33}$$

Proof. Taking into account that the system under consideration is linear, we can proceed in two steps considering separately the two cases:

Case 1: Reduction to the case $g \equiv 0$. We can determine $(\hat{u}_0, \hat{u}_1)$ so that the compatibility conditions in Lemma 4 are satisfied. Indeed, it suffices to choose

$(\hat{u}_0, \hat{u}_1)$ in $H^2(\Omega) \times H^1(\Omega)$ such that $g(0) = \hat{u}_0|_\Gamma$ and $g_t(0) = \hat{u}_1|_\Gamma$. This is possible since $g(0) \in H^{3/2}(\Gamma)$ and $g_t(0) \in H^{1/2}(\Gamma)$, by assumption. When replacing the initial data (u_0, u_1) by $(\hat{u}_0, \hat{u}_1)$, the result is a consequence of the previous Lemma 4.

Case 2: According to the analysis in the previous case, it is sufficient to consider the case where $g = 0$ with initial data $(u_0 - \hat{u}_0, u_1 - \hat{u}_1)$ which belong to $(H_0^1(\Omega), L^2(\Omega))$. In this case the estimate follows from the following one. □

Lemma 7. *Suppose $g = 0$. Then the map*

$$(u_0, u_1) \to (1 - P)\partial_\nu u|_{\partial\Omega}$$

is bounded from $H_0^1(\Omega) \times L^2(\Omega)$ to $C^N([-2T, 2T] \times \partial\Omega)$ for all $N > 0$.

Proof. This follows from Theorem 20.1.14 of L. Hörmander [7]. □

5.2 The wave equation on the boundary

We consider the wave equation over the manifold $\partial\Omega$. This will be applied later in the proof of well-posedness in asymmetric spaces to both sides of the interface.

The same analysis applies in the case where $\partial\Omega$ is replaced by one of the connected components of the boundary of Ω.

Lemma 8. *Let $0 < c < 1$ and P as above. Suppose that*

$$\begin{cases} w_{tt} - c^2 \Delta_{\partial\Omega} w = h_{\partial\Omega} & \text{in } \mathbb{R} \times \partial\Omega, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } \partial\Omega. \end{cases} \tag{34}$$

Then

$$\begin{aligned} \|u\|_{H^2((0,T)\times\partial\Omega)} \leq C \quad & [\|w_0\|_{H^2(\partial\Omega)} + \|w_1\|_{H^1(\partial\Omega)} \\ & + \|(1-P)h_{\partial\Omega}\|_{L^1(0,T;H^1(\partial\Omega))} + \|Ph_{\partial\Omega}\|_{L^2((0,T)\times\partial\Omega)}] \end{aligned}$$

Proof. When $h_{\partial\Omega} \equiv 0$ the bound is consequence of the energy estimate for the wave equation over the smooth closed hypersurface $\partial\Omega$ which is well-posed in $H^2(\partial\Omega) \times H^1(\partial\Omega)$.

In the case where $w_0 \equiv w_1 \equiv 0$, the bound in terms of $(1 - P)h_{\partial\Omega}$ follows by an energy estimate, and the one in terms of $Ph_{\partial\Omega}$ since the characteristic set of the wave operator under consideration is disjoint from the support of the symbol of P. □

6 Well-posedness in asymmetric spaces

In this section we address the problem of the well-posedness in asymmetric spaces. Our main result asserts that well-posedness in asymmetric spaces holds provided the velocity of propagation on the interface $c < 1$, whatever the velocity of propagation in Ω^+, c_+, is.

The main idea of the proof is that the possible H^1-singularities of solutions in the medium Ω^- fall in the elliptic regime of the interface wave operator because of the fact that $c < 1$. Thus, these singularities may not penetrate the interface neither the second medium Ω^+. Accordingly, the solution keeps one extra derivative in L^2 both in the interface and in Ω^+.

Therefore, we study the problem

$$\begin{cases} u^-_{tt} - \Delta u^- = 0 & \text{in } \mathbb{R} \times \Omega^- \\ u^+_{tt} - c_+^2 \Delta u^+ = 0 & \text{in } \mathbb{R} \times \Omega^+ \\ w_{tt} - c^2 \Delta_\Gamma w = [\partial_\nu u]_c & \text{in } \mathbb{R} \times \Gamma \\ u = w & \text{on } \mathbb{R} \times \Gamma \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega \\ u(0) = u_0,\, u_t(0) = u_1 & \text{in } \Omega^- \cup \Omega^+ \\ w(0) = w_0,\, u_t(0) = w_1 & \text{in } \Gamma, \end{cases} \tag{35}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$ and $\Gamma \subset \Omega$ a smooth hyper-surface without boundary that splits Ω into two subdomains $\Omega^\pm$.

Theorem 9. *Assume that $c < 1$. Suppose that*

$$u_0 \in H_0^1(\Omega), \qquad u_0|_{\Omega^+} \in H^2(\Omega^+), \tag{36}$$

$$u_1 \in L^2(\Omega), \qquad u_1|_{\Omega^+} \in H^1(\Omega^+), \qquad u_1|_{\partial^+\Omega} = 0, \tag{37}$$

$$w_0 = u_0|_\Gamma \in H^2(\Gamma), \qquad w_1 = u_1|_\Gamma \in H^1(\Gamma). \tag{38}$$

Then, for all values of $c_+ > 0$, the unique finite energy solution of (35) satisfies the further regularity properties

$$u^+ \in C([0,T]; H^2(\Omega^+)), \qquad \partial_t u^+ \in C([0,T]; H^1(\Omega^+)) \tag{39}$$

$$w \in H^2((0,T) \times \Gamma). \tag{40}$$

Remark 10. *According to this result the system is well-posed in an asymmetric space in which there is a gap of one L^2-derivative on the regularity of solutions on the two subdomains $\Omega^\pm$. Obviously, for that being true, one needs to assume one more L^2-derivative on the initial data in Ω^+ and on the interface.*

By symmetry, a similar result holds when this further regularity is imposed on the left domain Ω^-, provided $c < c_+$.

The same result is true if we replace $H^2(\Omega^+)$ and $H^2(\Gamma)$ in the Theorem by $H^{1+s}(\Omega^+)$ and $H^{1+s}(\Gamma)$ with $0 \le s \le 1$. In other words, the flow is invariant in a space with s more derivatives in L^2 in the domain Ω^+ provided $0 \le s \le 1$. But the gap of one derivative stated in the Theorem is optimal, and corresponds to the limit case $s = 1$. This fact was conjectured by means of the plane wave analysis of the previous sections, and is clarified and confirmed in the proof below in which we clearly see the need of having $0 \le s \le 1$.

Proof. We proceed in several steps. The problems on $\Omega_\pm$ and Γ are solved iteratively. In each step we gain some regularity until after two steps we conclude by applying Proposition 2.

1. Let $w^1 : \mathbb{R}\times\Gamma \to \mathbb{R}$ be defined as solution to the wave equation on the interface

$$w^1_{tt} - c^2\Delta_\Gamma w^1 = 0$$

$$w^1(0) = w_0, \qquad \partial_t w^1(0) = w_1.$$

   Clearly by energy conservation

$$\|w^1(t)\|^2_{L^2(\Gamma)} + c^2\|\nabla_\Gamma w(t)\|^2_{L^2(\Gamma)} = \|w_1\|^2_{L^2(\Gamma)} + c^2\|\nabla_\Gamma w_0\|^2_{L^2(\Gamma)}$$

   and the system is well-posed in $H^1(\Gamma) \times L^2(\Gamma)$.

   A similar estimate holds for second order derivatives and well-posedness also holds in $H^2(\Gamma) \times H^1(\Gamma)$. Therefore $w \in C([0,T]; H^2(\Gamma)) \cap C^1([0,T]; H^1(\Gamma)) \cap C^2([0,T]; L^2(\Gamma))$.

2. We define now $u^1 : \mathbb{R} \times \Omega \to \mathbb{R}$ as solution to

$$u^{1,-}_{tt} - \Delta u^{1,-} = 0 \qquad \text{in } \mathbb{R} \times \Omega^-$$

$$u^{1,+}_{tt} - c_+^2\Delta u^{1,+} = 0 \qquad \text{in } \mathbb{R} \times \Omega^+$$

$$u^1 = 0 \qquad \text{on } \mathbb{R} \times \partial\Omega$$

$$u^1 = w^1 \qquad \text{on } \mathbb{R} \times \Gamma$$

   with initial data

$$u^1(0) = u_0, \quad u^1_t(0) = u_1 \qquad \text{in } \Omega^+ \cup \Omega^-.$$

We now apply Lemma 4 in each domain $\Omega^\pm$ and see that the solution satisfies the estimate

$$\sup_{|t|\le T}\big[\|u^1(t)\|_{H^1(\Omega)}+\|u^1_t(t)\|_{L^2(\Omega)}+\|u^{1,+}(t)\|_{H^2(\Omega^+)}+\|\partial_t u^{1,+}(t)\|_{H^1(\Omega^+)}\big]$$
$$\le C[\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|u_0^+\|_{H^2(\Omega^+)}+\|u_1^+\|_{H^1(\Omega^+)}$$
$$+\|w^1\|_{H^2((-T,T)\times\Gamma)}].$$

Note in particular that the compatibility conditions on the data needed to apply Lemma 4 are fulfilled because of the assumptions of the Theorem and due to the fact that w^1 is a solution of the wave equation on the interface with initial data w_0 and w_1, which are the traces over Γ of the initial data u_0 and u_1 in the domains $\Omega^\pm$. Here and in the sequel we denote by $u^\pm$ the restriction of u to $\Omega^\pm$.

Let P be as in Lemma 6. Applying Lemma 6 and Lemma 4 in $\Omega^\pm$ we have

$$\begin{aligned}\|(1-P)[\partial_\nu u^1]_c\|_{H^1((-T,T)\times\Gamma)}&+\|P[\partial_\nu u^1]_c\|_{L^2((-T,T)\times\Gamma)}\\ &\le C(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|u_0^+\|_{H^2(\Omega^+)}+\|u_1^+\|_{H^1(\Omega^+)}\\ &+\|w^1\|_{H^2((-T,T)\times\Gamma)})\\ &\le C(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|w_0\|_{H^2(\Gamma)}+\|w_1\|_{H^1(\Gamma)}\\ &\quad+\|u_0^+\|_{H^2(\Omega^+)}+\|u_1^+\|_{H^1(\Omega^+)}).\end{aligned} \tag{41}$$

To be more precise, because of the second statement in Lemma 3, $\partial u^{1,+}/\partial\nu\in H^1((0,T)\times\Gamma)$. Therefore, in order to show (41) it is sufficient to prove that $(1-P)\partial_\nu u^{1,-}\in H^1((-T,T)\times\Gamma)$ and $P\partial_\nu u^{1,-}\in L^2((-T,T)\times\Gamma)$. This is so because of Lemma 3 and Lemma 4. Note that, in particular, the compatibility conditions on the data needed to apply Lemma 4 are guaranteed since w^1 is a solution of the interface wave equation.

3. We define w^2 by

$$w^2_{tt}-c^2\Delta_\Gamma w^2=[\partial_\nu u^1]_c=P[\partial_\nu u^1]_c+(1-P)[\partial_\nu u^1]_c$$

with initial data

$$w^2(0)=w_0,\qquad w^2_t(0)=w_1\qquad \text{on }\Gamma.$$

By Lemma 8,

$$\begin{aligned}\|w^2\|_{H^2([-T,T]\times\Gamma)}\le& C(\|w_0\|_{H^2(\Gamma)}+\|w_1\|_{H^1(\Gamma)}\\ &+\|(1-P)[\partial_\nu u^1]_c\|_{L^1(-T,T;H^1(\Gamma))}+\|P[\partial_\nu u^1]_c\|_{L^2((-T,T)\times\Gamma)}).\end{aligned} \tag{42}$$

We observe that the terms on the right hand side are controlled by the data, in view of (41).

4. Next we define u^2 as solution to

$$
\begin{aligned}
u^{2,-}_{tt} - \Delta u^{2,-} = 0 &\qquad \text{in } \mathbb{R}\times\Omega^- \\
u^{2,+}_{tt} - c_+^2\Delta u^{2,+} = 0 &\qquad \text{in } \mathbb{R}\times\Omega^+ \\
u^2 = 0 &\qquad \text{on } \mathbb{R}\times\partial\Omega \\
u^2 = w^2 &\qquad \text{on } \mathbb{R}\times\Gamma
\end{aligned}
$$

with initial data

$$
u^2(0) = u_0, \quad u^2_t(0) = u_1 \qquad \text{in } \Omega^+\cup\Omega^-.
$$

Then, by Lemma 4 and Lemma 6,

$$
\begin{aligned}
&\sup_{|t|\le T}\Big[\|(u^2(t), u^2_t(t))\|_{H^1(\Omega)\times L^2(\Omega)} + \|(u^{2,+}(t), u^{2,+}_t(t))\|_{H^2(\Omega^+)\times H^1(\Omega^+)}\Big] \\
&\quad + \|(1-P)[\partial_\nu u^2]_c\|_{H^1((-T,T)\times\Gamma)} + \|P[\partial_\nu u^2]_c\|_{L^2((-T,T)\times\Gamma)} \\
\le\, & C(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0^+\|_{H^2(\Omega^+)} + \|u_1^+\|_{H^1(\Omega^+)} \\
&\quad + \|w^2\|_{H^2((-T,T)\times\Gamma)}).
\end{aligned} \tag{43}
$$

From the previous step we know that the right hand side of (43) is bounded above by the norms of the data. More precisely, arguing as in Step 2 above, when checking that

$$
(1-P)[\partial_\nu u^2]_c \in H^1((-T,T)\times\Gamma); \quad P[\partial_\nu u^2]_c \in L^2((-T,T)\times\Gamma)
$$

we distinguish between Ω^- and Ω^+. In Ω^+, we have $\partial_\nu u^{2,+} \in H^1((0,T)\times\Gamma)$ because of the regularity of the data and Lemma 3. However, in Ω^- we can only gurantee that $\partial_\nu u^{2,-} \in L^2((0,T)\times\Gamma)$ and $(1-P)\partial_\nu u^{2,-} \in H^1((-T,T)\times\Gamma)$ in view of the structure of P and that $c < 1$.

5. We search the desired solution (u, w) as $u = u^2 + y$ and $w = w^2 + \eta$. In this way we obtain the equivalent system for (y, η):

$$
\begin{aligned}
y^-_{tt} - \Delta y^- = 0 &\qquad \text{in } \mathbb{R}\times\Omega^- \\
y^+_{tt} - c_+^2\Delta y^+ = 0 &\qquad \text{in } \mathbb{R}\times\Omega^+ \\
\eta = y|_\Gamma &\qquad \text{on } \Gamma \\
\eta_{tt} - c^2\Delta_\Gamma\eta = [\partial_\nu y]_c + [\partial_\nu(u^2 - u^1)]_c &\qquad \text{on } \mathbb{R}\times\Gamma \\
y = 0 &\qquad \text{on } \mathbb{R}\times\partial\Omega \\
y(0) = \partial_t y(0) = 0 &\qquad \text{in } \Omega \\
\eta(0) = \partial_t\eta(0) = 0 &\qquad \text{on } \Gamma.
\end{aligned}
$$

By Proposition 2 this system has a solution (y,η) and it satisfies

$$\begin{aligned}&\|y^+\|_{C([-T,T];H^2(\Omega^+))}+\|y^-\|_{C([-T,T];H^2(\Omega^-))}+\|\partial_t y\|_{C([-T,T];H^1(\Omega))}\\&\quad+\|\eta\|_{H^2((-T,T)\times\Gamma)}\leq C\|[\partial_\nu(u^2-u^1)]_c\|_{W^{1,1}(0,T;L^2(\Gamma))}.\end{aligned}\tag{44}$$

Let $z=u^2-u^1$. In view of (44) and estimate (43) on u^2, Theorem 9 will follow from the bound

$$\begin{aligned}\|[\partial_\nu(u^2-u^1)]_c\|_{W^{1,1}(0,T;L^2(\Gamma))}&\leq C\|[\partial_\nu z]_c\|_{H^1((-T,T)\times\Gamma)}\\&\leq C\|w^2-w^1\|_{H^2((-T,T)\times\Gamma)},\end{aligned}\tag{45}$$

and the fact that, as shown above, both w^1 and w^2 are bounded above in $H^2((-T,T)\times\Gamma)$.

The function z is the solution of

$$\begin{array}{rcll}z^-_{tt}-\Delta z^-&=&0&\text{in }\mathbb{R}\times\Omega^-\\z^+_{tt}-c_+^2\Delta z^+&=&0&\text{in }\mathbb{R}\times\Omega^+\\z(0)=z_t(0)&=&0&\text{in }\Omega^+\cup\Omega^-\\z&=&0&\text{on }\mathbb{R}\times\partial\Omega\\z&=&w^2-w^1&\text{on }\mathbb{R}\times\Gamma.\end{array}$$

Hence, (45) follows from Lemma 4. This concludes the proof.

□

7 Further comments and open problems

1. The plane wave analysis developed in sections 3 and 4 needs to be further completed in order to show that well-posedness in asymmetric spaces fails in the cases in which this was conjectured. Note that our arguments based on plane waves are formal and that, to make them rigorous, one needs to use cut-off functions to localize the support of solutions on bounded domains to fulfill the Dirichlet boundary conditions. Similar arguments have been fully developed in [17] in the context of a problem of wave-heat interaction.

   A similar construction should be done for curved interfaces. But this would be tecnically more involved since it would require to use Gaussian beams constructions (see [15, 16] and [17]).

   As we mentioned above, very likely, the case where the pathological directions of propagation are tangent to the interface will be me more difficult from a technical point of view. That case could also correspond to the existence of an infinite sequence of eigenfunctions concentrated along the interface. But this issue, which may also be related to the geometry of the interface, needs to be further investigated.

2. The well-posedness in asymmetric spaces has only been proved in the previous section in the case where $c < 1$, for all c_+. However, in section 4.2 we predicted that a similar well-posednees property could hold for other values of the parameters c and c_+ as well, and, more precisely, when $c = 1$ and $c_+ \neq 1$, and $c > 1$ and $c_+ \geq c$. These cases need to be further investigated.

3. Even in the cases where we have predicted the lack of well-posedness in asymmetric spaces, we have observed that the waves that approach the interface in some directions are purely reflected. This suggests the possibility of having well-posedness of the system on some asymmetric spaces defined in microlocal terms. But, in this case the space would not be as simple as prescribing a different degree of regularity to each side of the interface. This issue needs also to be further analysed.

4. When the system is well-posed in asymmetric spaces, it would be interesting to investigate the possibility of using domain decomposition techniques to approximate those solutions. Our analysis leads to the asymptotic form of the reflection and transmission coefficient, and this could be used to derive interface conditions for the convergence of the domain decomposition method to hold in asymmetric spaces. We refer to the book by [9] for the analysis of domain decomposition methods for wave like equations.

5. In those situations in which the system is well-posed in asymmetric spaces, it would be interesting to design numerical schemes guaranteeing convergence in those spaces. The most natural problem to be addressed in this context would be the Cauchy problem in the whole space with a flat interface, by means of the classical conservative semi-discrete finite-difference approximation scheme.

6. It would be also interesting to investigate all these problems by means of semiclassical and defect measures. We refer to [14] for a study of the classical problem of refraction by means of those tools.

7. The models considered in this article are linear. The extension of the results on well-posedness in asymmetric spaces for nonlinear problems is widely open. Obviously, when considering semilinear perturbations, one can prove local (in time) well-posedness results. But the lack of energy estimates in asymmetric

spaces makes global well-posedness uncertain. As we mentioned in the introduction, however, more realistic models should be free-boundary ones, in the sense that the deformation of the elastic interface should produce also the deformation of both domains $\Omega^{\pm}$. Very little is known about the existence and uniqueness of solutions for the free-boundary analogues of the models discussed in this article. Even less is known about the possible well-posedness in asymmetric spaces.

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