Polynomial decay and control of a $1-d$ hyperbolic-parabolic coupled system

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Dedicated to the memory of Xunjing Li

Abstract

In this paper we consider a linearized model for fluid-structure interaction in one space dimension. The domain where the system evolves consists in two parts in which the wave and heat equations evolve respectively, with transmission conditions at the interface. First of all we develop a careful spectral asymptotic analysis on high frequencies for the underlying semigroup. It is shown that the semigroup governed by the system can be split into a parabolic and a hyperbolic projection. The dissipative mechanism of the system in the domain where the heat equation holds produces a slow decay of the hyperbolic component of solutions. According to this analysis we obtain sharp polynomial decay rates for the whole energy of smooth solutions. Next, we discuss the problem of null-controllability of the system when the control acts on the boundary of the domain where the heat equation holds. The key observability inequality of the dual system with observation on the heat component is derived though a new Ingham-type inequality, which in turn, thanks to our spectral analysis, is a consequence of a known observability inequality of the same system but with observation on the wave component.

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Contents

1 Introduction

This paper is devoted to analyze a linearized model for fluid-structure interaction in one space dimension. More precisely, we consider the following hyperbolic-parabolic coupled system:

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\infty)\times(0,1), \\ z_{tt} - z_{xx} = 0 & \text{in } (0,\infty)\times(-1,0), \\ y(t,1) = 0 & t\in(0,\infty), \\ z(t,-1) = 0 & t\in(0,\infty), \\ y(t,0) = z(t,0), \quad y_x(t,0) = z_x(t,0) & t\in(0,\infty), \\ y(0) = y_0 & \text{in } (0,1), \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } (-1,0). \end{cases} \tag{1.1}$$

This system consists of a wave equation, arising on the interval $(-1,0)$ with state (z,z_t), and a heat equation, that holds on the interval $(0,1)$ with state y. The wave and heat components are coupled through an interface, the point $x=0$, with transmission conditions imposing the continuity of (y,z) and (y_x,z_x). System (1.1) is a linearized $1-d$ version of a system for fluid-structure interaction. More realistic models should consist, for instance, in the coupling of the Navier-Stokes equations with the equation of elasticity along a free interface of contact. But for these more sophisticated situations, basic questions concerning existence and uniqueness of solutions are still open (We refer to [3] for a result on existence of solutions for a variant of this system).

System (1.1) is a simpler one to be analyzed. Nevertheless, the understanding of its long time behavior and its control properties is far from trivial, as we will see.

We introduce the following two Hilbert spaces

$$W_1 \stackrel{\triangle}{=} \Big\{ f \in H^1(0,1) \ \Big| \ f(1) = 0 \Big\}, \qquad W_2 \stackrel{\triangle}{=} \Big\{ f \in H^1(-1,0) \ \Big| \ f(-1) = 0 \Big\}. \tag{1.2}$$

Throughout this paper, the norms in W_1 and W_2 are given respectively by

$$|f|_{W_1} = \sqrt{\int_0^1 |f_x(x)|^2 dx}, \quad \forall\, f \in W_1; \qquad |f|_{W_2} = \sqrt{\int_{-1}^0 |f_x(x)|^2 dx}, \quad \forall\, f \in W_2.$$

Clearly,

$$H \stackrel{\triangle}{=} W_1 \times W_2 \times L^2(-1,0) \tag{1.3}$$

is a Hilbert space with the norm

$$|(f,g,h)|_H = \sqrt{|f|_{W_1}^2 + |g|_{W_2}^2 + |h|_{L^2(-1,0)}^2}, \qquad \forall\, (f,g,h) \in H. \tag{1.4}$$

It is easy to show that system (1.1) is well-posed in the Hilbert space H.

The energy of system (1.1) is defined by

$$E(t) \stackrel{\triangle}{=} E(y,z,z_t)(t) = \frac{1}{2} |(y(t), z(t), z_t(t))|_H^2. \tag{1.5}$$

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt} E(t) = -\frac{1}{2} \int_0^1 |y_t|^2 dx = -\frac{1}{2} \int_0^1 |y_{xx}|^2 dx. \tag{1.6}$$

This formula indicates clearly that the only dissipation acting on the system is through the heat equation in $(0,1)$. Naturally, one hopes to know whether the dissipation is strong enough to produce the exponential decay of the energy of solutions of system (1.1). This is the first topic we shall address in this paper.

At the first glance, it seems reasonable to expect the answer to the above problem to be positive since the dissipative mechanism looks very efficient. Indeed, the energy of system (1.1) is dissipated at a rate which is proportional to the H^2-norm of the parabolic component of the solution. Recall that for the pure heat equation, the energy decays exponentially as $t \to \infty$. On the other hand, it is also well-known that the energy of solutions of the wave equation with dissipation localized in a subinterval decays exponentially as $t \to \infty$, too.

However, as we shall show later, the answer is negative. That is, the energy of system (1.1) does not decay uniformly as $t \to \infty$. In fact, the semigroup governed by the system can be split into a parabolic and a hyperbolic projection. The dissipative mechanism of the system in the domain where the heat equation holds produces a slow decay of the hyperbolic component of solutions. To show this, we will develop a careful spectral asymptotic analysis on high frequencies for the generator of the underlying semigroup.

When analyzing the spectral asymptotic behavior, several difficulties arise, most of which are due to the different behaviors of the parabolic and hyperbolic eigenvalues and/or eigenvectors, and the transmission conditions on the interface. Let us explain this a little more:

(a) The characteristic equation of the system is easy to get, as well as its leading term $\varrho(\lambda)$, whose roots provide an approximation of the high frequency eigenvalues. The roots of $\rho(\lambda)$ are split into two families, $\{\lambda_\ell^0\}_{\ell=1}^\infty$ and $\{\lambda_k^1\}_{|k|=0}^\infty$, with different asymptotic behaviors. To locate the position of the eigenvalues, we develop a fixed point argument, which differs from the most standard one based on use of Rouchés theorem.

(b) This analysis yields two branches of eigenvalues, λ_ℓ^p and λ_k^h (*see* Lemmas 2.2 and 2.3), respectively with quite different qualitative properties. We refer to them as parabolic and hyperbolic eigenvalues respectively. We also obtain the asymptotic form of the corresponding eigenvectors. We then need to show that each branch of eigenvectors constitutes a Riesz basis of the corresponding space.

(c) We need of course to combine carefully those two bases in the final analysis to show that the whole family of eigenvectors constitutes a Riesz basis in the energy space H. For this, the transmission conditions on the interface must be considered and play a crucial role.

This analysis allows obtaining sharp polynomial decay rates for the whole energy of smooth solutions.

Our spectral analysis results are also crucial for the second topic addressed in this paper, the control problem that we describe below.

Fix a $T > 0$, and consider the null controllability problem of the following system

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,T) \times (0,1), \\ v_{tt} - v_{xx} = 0 & \text{in } (0,T) \times (-1,0), \\ u(t,1) = g_1(t) & t \in (0,T), \\ v(t,-1) = 0 & t \in (0,T), \\ u(t,0) = v(t,0), \quad u_x(t,0) = v_x(t,0) & t \in (0,T), \\ u(0) = u_0 & \text{in } (0,1), \\ v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0) \end{cases} \tag{1.7}$$

by means of the boundary control $g_1(\cdot) \in L^2(0,T)$. The state space of system (1.7) is the Hilbert space

$$\mathcal{H} \triangleq \Big\{ (f,g,h) \;\Big|\; (h,f) \in H^{-1}(-1,1), \quad g \in L^2(-1,0) \Big\} \tag{1.8}$$

with the norm:

$$|(f,g,h)|_{\mathcal{H}} = \sqrt{|(h,f)|^2_{H^{-1}(-1,1)} + |g|^2_{L^2(-1,0)}}. \tag{1.9}$$

Here and henceforth, by writing $(w_1,w_2) \in H^s(-1,1)$ (*resp.* $H_0^s(-1,1)$) for $s \in \mathbb{R}$, we mean that the function $w \triangleq w_1\chi_{(-1,0)} + w_2\chi_{(0,1)}$ belongs to $H^s(-1,1)$ (*resp.* $H_0^s(-1,1)$).

System (1.7) is said to be null controllable in $\mathcal{H}$ by means of controls in $L^2(0,T)$ if for any $(u_0, v_0, v_1) \in \mathcal{H}$, we may find a control $g_1 \in L^2(0,T)$ such that the solution $(u, v, v_t) \in C([0,T]; \mathcal{H})$ of (1.7), (whose well-posedness is a consequence of the estimate (4.1) in Theorem 4.1 via the transposition method), satisfies $u(T) = 0$ in $(0,1)$ and $v(T) = v_t(T) = 0$ in $(-1,0)$.

We recall that the same problem but with boundary control acting through the wave component at $x = -1$ was solved in [19]. More precisely, consider the controlled system

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,T) \times (0,1), \\ v_{tt} - v_{xx} = 0 & \text{in } (0,T) \times (-1,0), \\ u(t,1) = 0 & t \in (0,T), \\ v(t,-1) = g_2(t) & t \in (0,T), \\ u(t,0) = v(t,0), \quad u_x(t,0) = v_x(t,0) & t \in (0,T), \\ u(0) = u_0 & \text{in } (0,1), \\ v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0). \end{cases} \tag{1.10}$$

The following result was proved in [19]:

Theorem 1.1 *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in \mathcal{H}$, there exists a control $g_2 \in L^2(0,T)$ such that the solution (u, v, v_t) of system (1.10) satisfies $u(T) = 0$ in $(0,1)$ and $v(T) = v_t(T) = 0$ in $(-1,0)$.*

The proof of Theorem 1.1 is based on the following observability estimate on equation (1.1), which will also play a key role in this paper.

Lemma 1.1 ([19]) *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of equation (1.1) satisfies*

$$|(y(T), z(T), z_t(T))|_H^2 \le C |z_x(\cdot, -1)|_{L^2(0,T)}^2, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{1.11}$$

Now, it is natural to expect that the same null controllability result still holds for system (1.7) in which the control acts on the heat component instead of the wave one. However, this is not the case.

Indeed, the classical duality argument shows that the null controllability property of system (1.7) is equivalent to the following similar observability inequality

$$|(y(T), z(T), z_t(T))|_H^2 \le C |y_x(\cdot, 1)|_{L^2(0,T)}^2, \qquad \forall\, (y_0, z_0, z_1) \in H \tag{1.12}$$

for every solution of equation (1.1). Note that in (1.12) the boundary measurement is done through the parabolic component, while in (1.11) it is done through the hyperbolic one.

However, as we shall see in Subsection 4.2, inequality (1.12) fails. Indeed, the norm of the left hand side of inequality (1.12) is too strong even if we replace the norm of its right hand side by

$$|y_x(\cdot, 1)|_{H^s(0,T)}^2$$

for any given (large) $s > 0$. This fact has the following two consequences:

(1) System (1.7) is not null controllable in the same Hilbert space as for system (1.10) but rather in a much smaller Hilbert space $\mathcal{S}^{-1}V(\subset \mathcal{H})$, which will be defined in (4.18) and (4.23);

(2) The only possibility to derive an observability estimate for (1.1) by boundary observation at $x = 1$, the external endpoint of the heat domain, is to use a much weaker norm than $|\cdot|_H$ in the left hand side of (1.12).

Consequently, we need to construct a Hilbert space $V' \supset H$, such that the following weak observability inequality

$$|(y(T), z(T), z_t(T))|_{V'}^2 \leq C|y_x(\cdot, 1)|_{L^2(0,T)}^2, \qquad \forall\ (y_0, z_0, z_1) \in H \tag{1.13}$$

holds for every solution of equation (1.1).

For this, we note that, according to our spectral analysis, the desired inequality can be interpreted as an Ingham-type inequality (*see* [8] and [15]). However, due to the mixed character of the spectrum of the system, the Ingham-type inequality we need, involving both real and complex exponentials, is not available in the literature. We will give a proof of this inequality by means of a nonstandard method. Indeed, this inequality holds as a consequence of our spectral analysis and the known observability inequality in Lemma 1.1, which is about the boundary observation of the system at the endpoint of the wave domain. The same method yields the structure of the Hilbert space V' in the observability inequality (1.13).

So far we have only considered the unit intervals $(-1, 0)$ and $(0, 1)$ for the wave and heat domains, respectively. The general case, in which these intervals have different lengths, can be treated in the same way. Note that the observability result of Lemma 1.1 in [19] holds in this more general case, too. Thus, roughly speaking, in order to develop the programme of this paper in the general case, it is sufficient to give the asymptotic spectral results. This is done in Appendix C at the end of this paper.

Note also that, from the point of view of fluid-structure interaction, it would be more natural to replace the first transmission conditions $y(t, 0) = z(t, 0)$ at the interface $x = 0$ by

$$y(t, 0) = z_t(t, 0), \qquad t \in (0, \infty), \tag{1.14}$$

since y may be viewed as the velocity of the fluid; while z_t represents the velocity of the deformation of the structure. This problem can be handled using similar techniques. We refer to [17] for a brief description of the main results about this case and to [18] for the details of the proofs.

Of course, these problems make sense for similar models in several space dimensions. The results in this paper and those in [17] show the complexity of the decay and control problems of fluid-structure interaction even in one space dimension. In [13] we analyze the problem of decay of the energy in several space dimensions. Energy and trace estimates, and the existing results on the observability of the wave equation ([2] and [11]), yield the polynomial decay of the smooth solutions of the system under suitable geometric conditions on the subset where the heat equation evolves. Roughly speaking, the heat domain needs to satisfy the Geometric Control Condition in [2] with respect to the wave domain. The decay rate we obtain is polynomial but slower than that of the $1-d$ case. We also show that this geometric assumption and decay rate we obtain are sharp by means of Geometric Optics constructions of approximate solutions of system (1.1).

The rest of this paper is organized as follows. Section 2 is devoted to the spectral analysis of the generator of the underlying semigroup of system (1.1). In Section 3, we show the polynomial decay result. Section 4 is devoted to the analysis of the negative and the positive boundary controllability and observability through the heat component. In Appendices A and B, we give the proofs of some technical results that will be used along the paper. In Appendix C, as we mentioned before, we will list the main spectral analysis result for the case that the heat and wave intervals have different lengths.

The main results in this paper have been announced in [16] without proofs.

2 Spectral analysis

This section is devoted to the spectral analysis of the generator of the underlying semigroup of system (1.1).

The generator $\mathcal{A}: \ D(\mathcal{A}) \subset H \to H$ is defined as the unbounded operator

$$\mathcal{A}Y = (f_{xx}, h, g_{xx}), \tag{2.1}$$

for $Y = (f, g, h) \in D(\mathcal{A})$, with domain

$$\begin{aligned} D(\mathcal{A}) \triangleq \Big\{(f,g,h) \in H \ \Big| \ \ (g,f) \in H^2(-1,1), \quad h \in H^1(-1,0), \quad f \in H^3(0,1), \\ f_{xx}(1) = h(-1) = 0 \ \text{ and } \ f_{xx}(0) = h(0)\Big\}. \end{aligned} \tag{2.2}$$

It is easy to show that the operator $\mathcal{A}$ defined above does generate a contractive C_0-semigroup in H with compact resolvent. We refer to [13] for the proof of this result in the more general multi-dimensional case.

Let G be a linear operator in a Hilbert space V. We denote by $\sigma_p(G)$ the point spectrum of G (i.e. the set of eigenvalues of G). We recall that a non-zero vector $\eta \in V$ is called a generalized eigenvector of G, corresponding to some $\lambda \in \sigma_p(G)$, if $(\lambda I - G)^m \eta = 0$ for some positive integer m. Further, we recall that a Riesz basis of V is obtained from an orthonormal basis by means of a bounded invertible operator transformation in V. Also, we recall that a sequence of vectors $\{v_j\}_{j=1}^{\infty}$ in V is said to be $\omega-$linearly independent if

$$\sum_{j=1}^{\infty} c_j v_j = 0, \qquad c_j \in \mathbb{C} \text{ for } j \in \mathbb{N}$$

is not possible with

$$0 < \sum_{j=1}^{\infty} |c_j v_j|^2 < \infty.$$

Throughout this paper, for any $\lambda = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, 2\pi)$, its square root is defined by $\sqrt{\lambda} = \sqrt{r}e^{i\theta/2}$.

The rest of this section is divided into 3 subsections. The first subsection is devoted to the asymptotic behavior of the eigenvalues of $\mathcal{A}$. The second subsection is devoted to the asymptotic behavior of the corresponding eigenvectors and the related Riesz basis property. The Riesz basis property of the generalized eigenvectors of $\mathcal{A}$ will be displayed in the third subsection.

2.1 Asymptotic behavior of eigenvalues

First of all, we have the following simple but crucial result.

Lemma 2.1 *The point spectrum of $\mathcal{A}$ is characterized by*

$$\sigma_p(\mathcal{A}) = \{\lambda \neq 0 \mid \kappa(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}, \tag{2.3}$$

where

$$\kappa(\lambda) \triangleq \sqrt{\lambda}(e^{-2\sqrt{\lambda}} - 1)(e^{2\lambda} + 1) - (e^{-2\sqrt{\lambda}} + 1)(e^{2\lambda} - 1). \tag{2.4}$$

Furthermore, for any $\lambda \in \sigma_p(\mathcal{A})$, it holds

$$1 + e^{2\sqrt{\lambda}} \neq 0, \tag{2.5}$$

and the corresponding eigenvector is of the form

$$\mu(p, q, r), \qquad \mu \in \mathbb{C} \setminus \{0\}, \tag{2.6}$$

where

$$\begin{aligned}
p = p(x, \lambda) &\triangleq \frac{\sqrt{\lambda}(1 + e^{-2\lambda})}{1 + e^{2\sqrt{\lambda}}}\Big(e^{\sqrt{\lambda}x} - e^{\sqrt{\lambda}(2-x)}\Big), \qquad && x \in (0, 1), \\
q = q(x, \lambda) &\triangleq e^{\lambda x} - e^{-\lambda(x+2)}, && x \in (-1, 0), \\
r = r(x, \lambda) &\triangleq \lambda\Big[e^{\lambda x} - e^{-\lambda(x+2)}\Big], && x \in (-1, 0).
\end{aligned} \tag{2.7}$$

The proof of Lemma 2.1 is given in Appendix A. This lemma reduces the eigenvalue problem of A to the problem of finding nonzero roots of $\kappa(\lambda)$. Thus it is important to analyze the asymptotic behavior of "large" roots λ of $\kappa(\lambda)$.

When λ is large, the leading term of $\kappa(\lambda)$ is

$$\varrho(\lambda) \triangleq \sqrt{\lambda}(e^{-2\sqrt{\lambda}} - 1)(e^{2\lambda} + 1). \tag{2.8}$$

Obviously, $\varrho(\lambda)$ has two classes of non-zero roots, i.e., the roots of "$e^{-2\sqrt{\lambda}} - 1$" and those of "$e^{2\lambda} + 1$", respectively.

It is easy to check that "$e^{-2\sqrt{\lambda}} - 1$" and "$e^{2\lambda} + 1$" have the following (non-zero) roots

$$\lambda_\ell^0 = -\ell^2\pi^2, \qquad \ell = 1, 2, \cdots \tag{2.9}$$

and

$$\lambda_k^1 = (1/2 + k)\pi i, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{2.10}$$

respectively.

Remark 2.1 *Note that* $\{\lambda_\ell^0\}_{\ell=1}^\infty$ *are typically the eigenvalues of the classical heat equation with Dirichlet boundary conditions in an interval of unit length; while* $\{\lambda_k^1\}_{|k|=0}^\infty$ *are those of the classical wave equation with mixed Neumann-Dirichlet type boundary conditions.*

We denote by $B_r(z)$ the (closed) disk in $\mathbb{C}$ centered at z and with radius r. The following lemma shows that every "large" root λ of $\varrho(\lambda)$ is very close to some "large" root of $\kappa(\lambda)$ and vice-versa.

Lemma 2.2 *There exist* $\ell_1 \in \mathbb{N}$ *and* $k_1 \in \mathbb{N}$ *such that* $\kappa(\lambda)$ *has two sequences of roots,* $\{\lambda_\ell^p\}_{\ell=\ell_1}^\infty$ *and* $\{\lambda_k^h\}_{|k|=k_1}^\infty$*, which satisfy respectively*

$$\sqrt{\lambda_\ell^p} \in B_{\ell^{-1}}\Big(\sqrt{\lambda_\ell^0}\Big), \qquad \ell \geq \ell_1 \tag{2.11}$$

and

$$\lambda_k^h \in B_{|k|^{-1/2}}(\lambda_k^1), \qquad |k| \geq k_1. \tag{2.12}$$

In the sequel, $\{\lambda_\ell^p\}$ will be referred to as the *parabolic eigenvalues* and $\{\lambda_k^h\}$ as the *hyperbolic eigenvalues* since they are close to the sequences $\{\lambda_\ell^0\}$ and $\{\lambda_k^1\}$ respectively whose nature was discussed in Remark 2.1. In order to prove Lemma 2.2, we need the following simple result.

Proposition 2.1 *For any* $\mu \in B_{|k|^{-1/2}}(0)$*, we have*

$$\sqrt{\mu + \lambda_k^1} = \sqrt{\lambda_k^1} + O(k^{-1}) \tag{2.13}$$

and

$$\frac{e^{-2\sqrt{\mu+\lambda_k^1}} + 1}{e^{-2\sqrt{\mu+\lambda_k^1}} - 1} = -\mathrm{sgn}\,(k) + O(|k|^{-2}). \tag{2.14}$$

The proof of Proposition 2.1 will be given in Appendix B.

Proof of Lemma 2.2. We distinguish two cases.

The first case is when "$\mathrm{Re}\,\lambda \to -\infty$". In this case, let us show that $\kappa(\lambda)$ has a sequence of roots $\{\lambda_\ell^p\}$ satisfying (2.11). For this purpose, we put

$$K(\lambda) \triangleq (e^{-2\sqrt{\lambda}} - 1) - \frac{(e^{-2\sqrt{\lambda}} + 1)(e^{2\lambda} - 1)}{\sqrt{\lambda}(e^{2\lambda} + 1)}. \tag{2.15}$$

Obviously, $\lambda \neq 0$ is a root of $\kappa(\lambda)$ if and only if it is a root of $K(\lambda)$. Also, for any fixed $\ell \in \mathbb{N}$, we denote

$$\mu = \sqrt{\lambda} - \ell\pi i. \tag{2.16}$$

Then,

$$K(\lambda)=K(\mu^2+2\mu\ell\pi i-\ell^2\pi^2)=(e^{-2\mu}-1)-\frac{(e^{-2\mu}+1)(e^{2(\mu^2+2\mu\ell\pi i-\ell^2\pi^2)}-1)}{(\mu+\ell\pi i)(e^{2(\mu^2+2\mu\ell\pi i-\ell^2\pi^2)}+1)}. \tag{2.17}$$

Now, let us define a function

$$G(\mu)=\mu+\frac{1}{2}K(\mu^2+2\mu\ell\pi i-\ell^2\pi^2). \tag{2.18}$$

Then for any $\mu\in B_{\ell^{-1}}(0)$, we have

$$\begin{aligned}|G(\mu)|&\le\Big|\mu+\frac{1}{2}(e^{-2\mu}-1)\Big|+\frac{1}{2}\Big|\frac{(e^{-2\mu}+1)(e^{2(\mu^2+2\mu\ell\pi i-\ell^2\pi^2)}-1)}{(\mu+\ell\pi i)(e^{2(\mu^2+2\mu\ell\pi i-\ell^2\pi^2)}+1)}\Big|\\&\le\Big|\mu\int_0^1\Big(1-e^{-2s\mu}\Big)ds\Big|+\frac{1}{2}\Big|\frac{[2+O(\ell^{-1})][-1+O(\ell^{-2})]}{[\ell\pi i+O(\ell^{-1})][1+O(\ell^{-2})]}\Big|.\end{aligned}$$

It is easy to see that

$$\mu\int_0^1\Big(1-e^{-2s\mu}\Big)ds=\mu^2\sum_{j=1}^{\infty}\frac{(-2)^j\mu^{j-1}}{(j+1)!}=O(\ell^{-2}),\qquad\forall\ \mu\in B_{\ell^{-1}}(0)$$

and

$$\frac{[2+O(\ell^{-1})][-1+O(\ell^{-2})]}{[\ell\pi i+O(\ell^{-1})][1+O(\ell^{-2})]}=\frac{-2+O(\ell^{-1})}{\ell\pi i+O(\ell^{-1})}=\frac{-2}{\ell\pi i}+O(\ell^{-2}).$$

Thus,

$$|G(\mu)|\le\frac{1}{\ell\pi}+O(\ell^{-2}). \tag{2.19}$$

Now, by (2.19), we see that that there is a sufficiently large $\ell_1\in\mathbb{N}$ such that

$$G(\mu)\in B_{\ell^{-1}}(0),\qquad\forall\ \mu\in B_{\ell^{-1}}(0),\ \ \ell\ge\ell_1. \tag{2.20}$$

Thus, by means of Browder fixed point theorem, we conclude that there exists a $\mu_\ell^0\in B_{\ell^{-1}}(0)$ such that $G(\mu_\ell^0)=\mu_\ell^0$. It is easy to see that $\lambda_\ell^p=(\mu_\ell^0)^2+2\mu_\ell^0\ell\pi i-\ell^2\pi^2$ is a root of $\kappa(\lambda)$, and (2.11) holds.

The second case is when "$\mathrm{Im}\,\lambda\to\infty$". In this case, we are going to show that $\kappa(\lambda)$ has a sequence of roots $\{\lambda_k^h\}$ satisfying (2.12). For this purpose, we set

$$H(\lambda)\triangleq(e^{2\lambda}+1)-\frac{(e^{-2\sqrt{\lambda}}+1)(e^{2\lambda}-1)}{(e^{-2\sqrt{\lambda}}-1)\sqrt{\lambda}}. \tag{2.21}$$

Obviously, $\lambda\neq0$ is a root of $\kappa(\lambda)$ if and only if it is a root of $H(\lambda)$. Also, for any fixed $k=0,\pm1,\pm2,\cdots$, we denote

$$\mu=\lambda-\lambda_k^1. \tag{2.22}$$

Then,

$$H(\lambda)=H(\mu+\lambda_k^1)=(-e^{2\mu}+1)+\frac{(e^{-2\sqrt{\mu+\lambda_k^1}}+1)(e^{2\mu}+1)}{(e^{-2\sqrt{\mu+\lambda_k^1}}-1)\sqrt{\mu+\lambda_k^1}}. \tag{2.23}$$

We define a function

$$J(\mu) = \mu + \frac{1}{2} H(\mu + \lambda_k^1). \tag{2.24}$$

Now, from (2.23) and (2.24), using (2.13) and (2.14) in Proposition 2.1, we see that for any $\mu \in B_{|k|^{-1/2}}(0)$, it holds

$$\begin{aligned} |J(\mu)| &\le \left|\mu + \frac{1}{2}(1 - e^{2\mu})\right| + \frac{1}{2}\left|\frac{(e^{-2\sqrt{\mu+\lambda_k^1}} + 1)(e^{2\mu} + 1)}{(e^{-2\sqrt{\mu+\lambda_k^1}} - 1)\sqrt{\mu + \lambda_k^1}}\right| \\ &\le \left|\mu \int_0^1 \left(1 - e^{2s\mu}\right) ds\right| + \frac{1}{2}\left|\Big(-\operatorname{sgn}(k) + O(|k|^{-2})\Big)\frac{2 + O(|k|^{-1/2})}{\sqrt{\lambda_k^1} + O(k^{-1})}\right|. \end{aligned}$$

It is easy to see that

$$\mu \int_0^1 \left(1 - e^{2s\mu}\right) ds = -\mu^2 \sum_{j=1}^{\infty} \frac{2^j \mu^{j-1}}{(j+1)!} = O(k^{-1}), \qquad \forall\, \mu \in B_{|k|^{-1/2}}(0)$$

and

$$\begin{aligned} &\Big(-\operatorname{sgn}(k) + O(|k|^{-2})\Big)\frac{2 + O(|k|^{-1/2})}{\sqrt{\lambda_k^1} + O(k^{-1})} = \frac{-2\operatorname{sgn}(k) + O(|k|^{-1/2})}{\sqrt{\lambda_k^1} + O(k^{-1})} \\ &= \frac{-2\operatorname{sgn}(k)}{\sqrt{\lambda_k^1} + O(k^{-1})} + O(k^{-1}) = -\frac{2\operatorname{sgn}(k)}{\sqrt{\lambda_k^1}} + O(k^{-1}). \end{aligned}$$

Thus, we get

$$|J(\mu)| \le \frac{1}{\sqrt{|\lambda_k^1|}} + O(k^{-1}) = \frac{1}{\sqrt{|k|\pi}} + O(k^{-1}), \qquad \forall\, \mu \in B_{|k|^{-1/2}}(0). \tag{2.25}$$

By (2.25), it is obvious that there is a sufficiently large $k_1 \in \mathbb{N}$ such that

$$J(\mu) \in B_{|k|^{-1/2}}(0), \qquad \forall\, \mu \in B_{|k|^{-1/2}}(0),\ |k| \ge k_1. \tag{2.26}$$

Thus, by means of Browder fixed point theorem, we conclude that there exists a $\mu_k^1 \in B_{|k|^{-1/2}}(0)$ such that $J(\mu_k^1) = \mu_k^1$. It is easy to see that $\lambda_k^h = \lambda_k^1 + \mu_k^1$ is a root of $\kappa(\lambda)$, and (2.12) holds. This completes the proof of Lemma 2.2. $\square$

Remark 2.2 *Obviously, (2.20) (resp. (2.26)) shows that G (resp. J) is a map from $B_{\ell^{-1}}(0)$ (resp. $B_{|k|^{-1/2}}(0)$) into itself. It can be further shown that these two maps are actually contractive. Therefore, by means of the contractive mapping principle, it is easy to see that there exists one and only one root λ_ℓ^p (resp. λ_k^h) of $\kappa(\lambda)$ such that $\sqrt{\lambda_\ell^p} \in B_{\ell^{-1}}(\sqrt{\lambda_\ell^0})$ (resp. $\lambda_k^h \in B_{|k|^{-1/2}}(\lambda_k^1)$) whenever ℓ (resp. $|k|$) is large. However, we will not use the uniqueness in the sequel. Therefore, for simplicity, we show only the existence.*

Lemma 2.2 gives also asymptotic estimates on the parabolic eigenvalues $\{\lambda_\ell^p\}_{\ell=\ell_1}^{\infty}$ and the hyperbolic eigenvalues $\{\lambda_k^h\}_{|k|=k_1}^{\infty}$. However, in the sequel, we will need more precise asymptotic estimates on λ_ℓ^p and λ_k^h. The desired estimates are shown in the following lemma.

Lemma 2.3 *The following asymptotic estimates hold:*

$$\sqrt{\lambda_\ell^p} = \sqrt{\lambda_\ell^0} + \frac{1}{\sqrt{\lambda_\ell^0}} + O(\ell^{-2}), \qquad \ell \to \infty, \tag{2.27}$$

$$\lambda_k^h = \lambda_k^1 - \frac{\operatorname{sgn}(k)}{\sqrt{\lambda_k^1}} + O(|k|^{-1}), \qquad |k| \to \infty. \tag{2.28}$$

Remark 2.3 *By (2.27) and (2.9), it is easy to get the following asymptotic estimate on* λ_ℓ^p*:*

$$\lambda_\ell^p = -\ell^2\pi^2 + 2 + O(\ell^{-1}). \tag{2.29}$$

Also, by (2.28) and (2.10), we get the following asymptotic estimate on λ_k^h*:*

$$\lambda_k^h = -\frac{1}{\sqrt{|1+2k|\pi}} + \left(\frac{1}{2} + k\right)\pi i + \frac{\operatorname{sgn}(k)}{\sqrt{|1+2k|\pi}} i + O(|k|^{-1}). \tag{2.30}$$

Note that, by (2.3) in Lemma 2.1, we have $\operatorname{Re}(\lambda_\ell^p) < 0$ *and* $\operatorname{Re}(\lambda_k^h) < 0$ *for all* k *and* ℓ*. Asymptotic estimates (2.29) and (2.30) reproduce this result (at least for large* k *and* ℓ*) although, when* $k \leq 0$*, the first term of (2.30) in its definition goes into its imaginary part while the third one goes into the real one.*

Remark 2.4 *By Remark 2.3, it is easy that the asymptotic behaviors of the parabolic and hyperbolic eigenvalues,* λ_ℓ^p *and* λ_k^h*, are quite different. Obviously,* λ_ℓ^p *is close to* λ_ℓ^0 *in a very simple way; while* λ_k^h *is also close to* λ_k^1 *but in a more complicated way. Consequently the construction of the hyperbolic component of the corresponding approximate (hyperbolic) "explicit" eigenvectors,* (q_k^1, r_k^1) *(see (2.50)), is also more delicate (since we need to show further that the approximate eigenvectors are quadratically close to the original one). On the other hand, we also need to show that* $\{(q_k^1, r_k^1)\}$ *forms a Riesz basis of* $W_2 \times L^2(-1,0)$ *(see Lemma 2.5). The proof of this property is based on a generalized Kadec's* $\frac{1}{4}$*-theorem.*

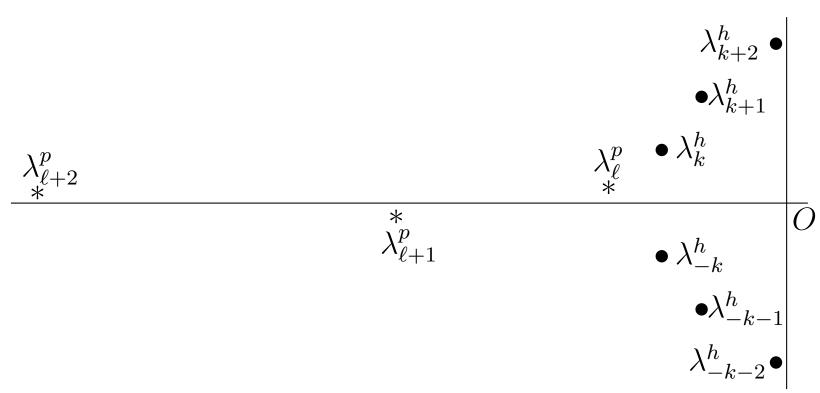

Figure 1: parabolic eigenvalues λ_ℓ^p and hyperbolic eigenvalues λ_k^h in the complex plane $\mathbb{C}$

In order to prove Lemma 2.3, we need the following calculus result.

Proposition 2.2 *(i) For any $j \in \mathbb{N}$, it holds*

$$e^{\lambda_\ell^p x} = O(\ell^{-j}), \qquad \forall\, x \in (0,2]. \tag{2.31}$$

(ii) The following three estimates

$$\frac{e^{x\mathrm{Re}\,\lambda_\ell^p} - e^{x\mathrm{Re}\,\lambda_\ell^0}}{e^{x\mathrm{Re}\,\lambda_\ell^0}} = O(1), \qquad \forall\, x \in [-4,4], \tag{2.32}$$

$$e^{\sqrt{\lambda_\ell^p}x} - e^{\sqrt{\lambda_\ell^0}x} = O(\ell^{-1}), \qquad \forall\, x \in [-2,2] \tag{2.33}$$

and

$$e^{\lambda_k^h x} - e^{\lambda_k^1 x} = O(|k|^{-1/2}), \qquad \forall\, x \in [-2,2] \tag{2.34}$$

hold uniformly with respect to x.

The proof of Proposition 2.2 will be given in Appendix B.

Proof of Lemma 2.3. We proceed as in the proof of Lemma 5.2 in [14]. From (2.4), we see that λ_ℓ^p satisfies

$$e^{-2\sqrt{\lambda_\ell^p}} = 1 + \frac{\left(e^{-2\sqrt{\lambda_\ell^p}}+1\right)(e^{2\lambda_\ell^p}-1)}{\sqrt{\lambda_\ell^p}\left(e^{2\lambda_\ell^p}+1\right)}. \tag{2.35}$$

By (2.31) and (2.33) in Proposition 2.2, we get

$$e^{2\lambda_\ell^p} = O(\ell^{-2}), \qquad e^{-2\sqrt{\lambda_\ell^p}} = 1 + O(\ell^{-1}). \tag{2.36}$$

Thus, by (2.35) and (2.36), we get

$$\begin{aligned} e^{-2(\sqrt{\lambda_\ell^p}-\sqrt{\lambda_\ell^0})} &= 1 + \frac{\left(e^{-2\sqrt{\lambda_\ell^p}}+1\right)(e^{2\lambda_\ell^p}-1)}{\sqrt{\lambda_\ell^p}\left(e^{2\lambda_\ell^p}+1\right)} = 1 + \frac{[2+O(\ell^{-1})][-1+O(\ell^{-2})]}{\left[\sqrt{\lambda_\ell^0}+O(\ell^{-1})\right][1+O(\ell^{-2})]} \\ &= 1 - \left(\frac{2}{\sqrt{\lambda_\ell^0}} + O(\ell^{-2})\right). \end{aligned} \tag{2.37}$$

Taking logarithms in (2.37), and noting that $\ln(1-z) = -z + O(|z|^2)$ when $|z| < 1$, we conclude that (2.27) holds for any ℓ large enough.

Similarly, from (2.4), we see that λ_k^h satisfies

$$e^{2\lambda_k^h} = -1 + \frac{\left(e^{-2\sqrt{\lambda_k^h}}+1\right)(e^{2\lambda_k^h}-1)}{\left(e^{-2\sqrt{\lambda_k^h}}-1\right)\sqrt{\lambda_k^h}}. \tag{2.38}$$

By (2.12) in Lemma 2.2, and using (2.13) and (2.14) in Proposition 2.1, we get

$$\sqrt{\lambda_k^h} = \sqrt{\lambda_k^1} + O(|k|^{-1}) \tag{2.39}$$

and

$$\frac{e^{-2\sqrt{\lambda_k^h}}+1}{e^{-2\sqrt{\lambda_k^h}}-1} = -\mathrm{sgn}\,(k) + O(|k|^{-2}). \tag{2.40}$$

On the other hand, by (2.10) and using (2.34) in Proposition 2.2, it is easy to see that

$$e^{2\lambda_k^h} = -1 + O(|k|^{-1/2}). \tag{2.41}$$

Now, by (2.38)–(2.41), we get

$$\begin{aligned} e^{2(\lambda_k^h-\lambda_k^1)} &= 1 - \Big(- \mathrm{sgn}\,(k) + O(|k|^{-2})\Big)\frac{\Big(-2+O(|k|^{-1/2})\Big)}{\sqrt{\lambda_k^1}+O(|k|^{-1})} \\ &= 1 - \left(\frac{2\mathrm{sgn}\,(k)}{\sqrt{\lambda_k^1}} + O(|k|^{-1})\right). \end{aligned} \tag{2.42}$$

Taking logarithms in (2.42), and noting that $\ln(1-z) = -z + O(|z|^2)$ when $|z| < 1$, we conclude that (2.28) holds for any $|k|$ large enough. $\square$

2.2 Asymptotic behavior of eigenvectors and approximate Riesz bases

First, for the parabolic eigenvalues λ_ℓ^p with $\ell \geq \ell_1$ (recall Lemma 2.2 for ℓ_1), we choose the corresponding eigenvector of $\mathcal{A}$ as follows (recall (2.7) for p, q and r)

$$\begin{aligned} &p_\ell^p(x) \triangleq -\frac{p(x,\lambda_\ell^p)}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}, \qquad x\in(0,1), \\ &q_\ell^p(x) \triangleq -\frac{q(x,\lambda_\ell^p)}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}, \qquad r_\ell^p(x) \triangleq -\frac{r(x,\lambda_\ell^p)}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}, \qquad x\in(-1,0). \end{aligned} \tag{2.43}$$

In order to describe the asymptotic behavior of $(p_\ell^p, q_\ell^p, r_\ell^p)$, we need to introduce the following functions:

$$p_\ell^0(x) \triangleq \begin{cases} \dfrac{\sin \ell\pi x}{\ell\pi}, & \ell = 1, 2\cdots, \ell_1 - 1, \\ \dfrac{1+e^{-2\lambda_\ell^p}}{\ell\pi(1+e^{2\ell^2\pi^2})}\sin\ell\pi x, & \ell \geq \ell_1, \end{cases} \tag{2.44}$$

where $x \in (0,1)$. Obviously, p_ℓ^0 ($\ell \in \mathbb{N}$) are eigenfunctions of the heat equation in the interval $(0,1)$ with Dirichlet boundary conditions.

We have the following key result:

Proposition 2.3 *It holds*

$$\sum_{\ell=\ell_1}^{\infty}\Big[|p_\ell^p - p_\ell^0|_{W_1}^2 + |q_\ell^p|_{W_2}^2 + |r_\ell^p|_{L^2(-1,0)}^2\Big] < \infty. \tag{2.45}$$

The proof of Proposition 2.3 will be given in Appendix B. Proposition 2.3 shows that the energy of the parabolic eigenvector is concentrated in the "heat" component (*see* Figure 2 below).

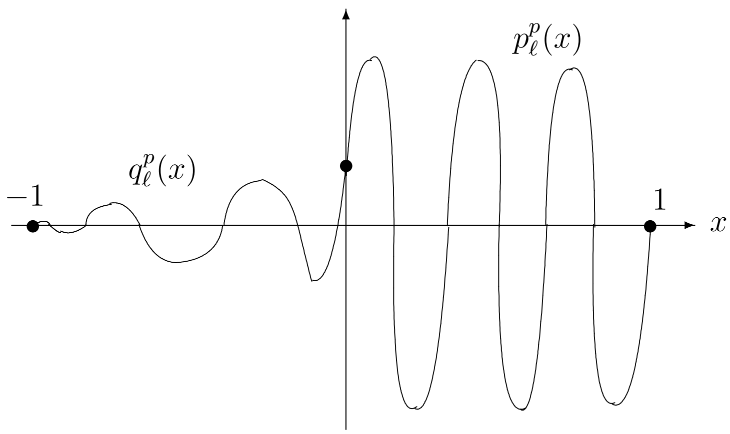

FIGURE 2: ASYMPTOTIC FORM OF PARABOLIC EIGENVECTORS

On the other hand, we have the following result.

Lemma 2.4 *The sequence* $\{p_\ell^0\}_{k=1}^\infty$ *forms a Riesz basis of* $H_0^1(0,1)$.

Proof. Noting that $\{\frac{\sin \ell\pi x}{\ell\pi}\}_{\ell=1}^\infty$ is a orthogonal basis of $H_0^1(0,1)$, it suffices to show that there is a positive constant C such that

$$\frac{1}{C} \leq \left|\frac{1+e^{-2\lambda_\ell^p}}{1+e^{2\ell^2\pi^2}}\right| \leq C, \qquad \forall\, \ell \geq \ell_1. \tag{2.46}$$

By (2.29) in Remark 2.3, the right hand side of (2.46) is obvious.

Using (2.29) again, we have

$$\begin{aligned}&\left|1+e^{-2\lambda_\ell^p}\right| = \left|1+e^{-2(\lambda_\ell^0+2)} - 2(\lambda_\ell^p-\lambda_\ell^0-2)e^{-2(\lambda_\ell^0+2)}\int_0^1 e^{-2s(\lambda_\ell^p-\lambda_\ell^0-2)}ds\right|\\ &= \left|1+[1+O(\ell^{-1})]e^{-2(\lambda_\ell^0+2)}\right| \geq \frac{e^{-4}}{2}e^{2\ell^2\pi^2}-1,\end{aligned}$$

which yields the left hand side of (2.46) immediately. $\square$

Next, for the hyperbolic eigenvalues λ_k^h with $|k| \geq k_1$ (recall Lemma 2.2 for k_1), we choose the corresponding eigenvector of $\mathcal{A}$ as follows (recall (2.7) for p, q and r)

$$\begin{aligned}&p_k^h(x) \triangleq -\frac{p(x,\lambda_k^h)}{2i\lambda_k^h}, \qquad x\in(0,1),\\ &q_k^h(x) \triangleq -\frac{q(x,\lambda_k^h)}{2i\lambda_k^h}, \qquad r_k^h(x) \triangleq -\frac{r(x,\lambda_k^h)}{2i\lambda_k^h}, \qquad x\in(-1,0).\end{aligned} \tag{2.47}$$

For $k = 0, \pm 1, \pm 2, \cdots$, we put

$$\tilde{\lambda}_k^h \triangleq \left[\Big(\frac{1}{2} + k\Big)\pi + \frac{\mathrm{sgn}\,(k)}{\sqrt{(1+2k)\pi}} + \frac{\mathrm{sgn}\,(k)}{\sqrt{(1+2k)\pi}} i \right] i. \tag{2.48}$$

It is easy to check that

$$\tilde{\lambda}_k^h = \lambda_k^1 - \frac{\mathrm{sgn}\,(k)}{\sqrt{\lambda_k^1}}, \qquad \forall\, |k| \geq k_1. \tag{2.49}$$

In order to describe the asymptotic behavior of (p_k^h, q_k^h, r_k^h), we need to introduce the following functions:

$$q_k^1(x) \triangleq -\frac{q(x, \tilde{\lambda}_k^h)}{2i\tilde{\lambda}_k^h}, \qquad r_k^1(x) \triangleq -\frac{r(x, \tilde{\lambda}_k^h)}{2i\tilde{\lambda}_k^h}, \qquad x \in (-1, 0), \tag{2.50}$$

where $q(x, \tilde{\lambda}_k^h)$ and $r(x, \tilde{\lambda}_k^h)$ are defined in (2.7). The first order approximation of these functions are eigenfunctions of the wave equation in the interval $(-1, 0)$ with Dirichlet boundary condition at $x = -1$ and Neumann boundary condition at $x = 0$.

We have the following key result:

Proposition 2.4 *It holds*

$$\sum_{|k|=k_1}^{\infty} [|p_k^h|_{W_1}^2 + |q_k^h - q_k^1|_{W_2}^2 + |r_k^h - r_k^1|_{L^2(-1,0)}^2] < \infty. \tag{2.51}$$

The proof of Proposition 2.4 will be given in Appendix B. Proposition 2.4 shows that the energy of the hyperbolic eigenvector is concentrated in the "wave" component (*see* Figure 3 below).

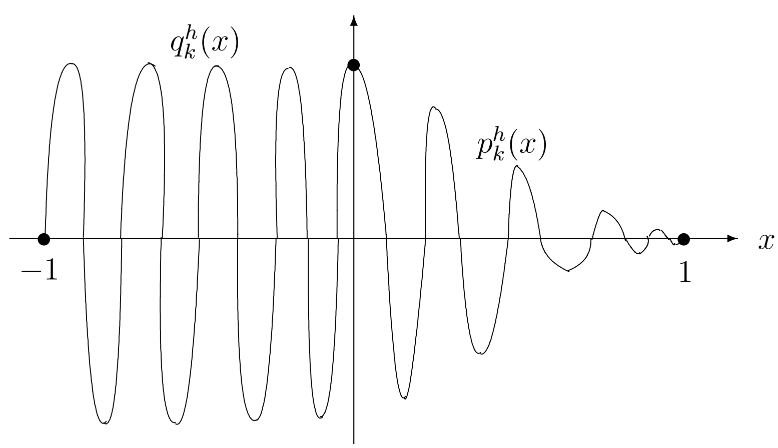

FIGURE 3: ASYMPTOTIC FORM OF HYPERBOLIC EIGENVECTORS

On the other hand, we have the following result (recall (1.2) for W_2).

Lemma 2.5 *The sequence $\{(q_k^1, r_k^1)\}_{k=-\infty}^{\infty}$ constitutes a Riesz basis of $W_2 \times L^2(-1,0)$.*

In order to prove Lemma 2.5, we need the following generalization of Kadec's $\frac{1}{4}$-theorem ([10], pp. 196 in [15]).

Lemma 2.6 *If $\{\sigma_k\}_{k=-\infty}^{\infty}$ is a sequence in $\mathbb{C}$ for which*

$$\sup_k |\mathrm{Re}\,\sigma_k - k| < \frac{1}{4} \quad \text{and} \quad \sup_k |\mathrm{Im}\,\sigma_k| < \infty,$$

then the system $\{e^{i\sigma_k x}\}_{k=-\infty}^{\infty}$ is a Riesz basis for $L^2(-\pi,\pi)$.

Proof of Lemma 2.5. The proof is divided into several steps.

Step 1. Denote

$$\sigma_k = k + \frac{\mathrm{sgn}\,(k)}{\pi\sqrt{(1+2k)\pi}} + \frac{\mathrm{sgn}\,(k)}{\pi\sqrt{(1+2k)\pi}} i, \qquad k = 0, \pm1, \pm2, \cdots.$$

Here and in the sequel we adopt the convention that $\mathrm{sgn}\,(0) = 0$. It is easy to check that $\{\sigma_k\}_{k=-\infty}^{\infty}$ satisfies the condition in Lemma 2.6. Thus the system $\{e^{i\sigma_k x}\}_{k=-\infty}^{\infty}$ is a Riesz basis for $L^2(-\pi,\pi)$. Note that $\tilde{\lambda}_k^h = (1/2 + \sigma_k)\pi i$. Hence, by scaling, it is easy to see that system $\{e^{\tilde{\lambda}_k^h x}\}_{k=-\infty}^{\infty}$ is a Riesz basis for $L^2(-1,1)$. Also, by a simple transformation $x \to -x$, we see that $\{e^{-\tilde{\lambda}_k^h x}\}_{k=-\infty}^{\infty}$ is a Riesz basis for $L^2(-1,1)$.

Step 2. Put

$$e_k(x) \triangleq \frac{e^{\tilde{\lambda}_k^h x} + e^{-\tilde{\lambda}_k^h x}}{2}\chi_{(-1,0)}(x) + \frac{e^{\tilde{\lambda}_k^h x} - e^{-\tilde{\lambda}_k^h x}}{2}\chi_{(0,1)}(x), \qquad x \in (-1,1). \tag{2.52}$$

We claim that $\{e_k(x)\}_{k=-\infty}^{\infty}$ forms a Riesz basis in $L^2(-1,1)$.

In fact, for any $(f,g) \in L^2(-1,1)$, define

$$F(x) = f(x)\chi_{(-1,0)}(x) + f(-x)\chi_{(0,1)}(x), \qquad x \in [-1,1] \tag{2.53}$$

and

$$G(x) = g(x)\chi_{(0,1)}(x) - g(-x)\chi_{(-1,0)}(x), \qquad x \in [-1,1]. \tag{2.54}$$

Obviously, $F \in L^2(-1,1)$ and $G \in L^2(-1,1)$. Therefore, using the Riesz basis property of $\{e^{\tilde{\lambda}_k^h x}\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$, we conclude that there exist two sequences $\{a_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$ and $\{b_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$ such that

$$F(x) = \sum_{k=-\infty}^{\infty} a_k e^{\tilde{\lambda}_k^h x} \qquad \text{in } L^2(-1,1) \tag{2.55}$$

and

$$G(x) = \sum_{k=-\infty}^{\infty} b_k e^{\tilde{\lambda}_k^h x} \qquad \text{in } L^2(-1,1). \tag{2.56}$$

Furthermore,

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}(|a_k+b_k|^2) \le 2\sum_{k=-\infty}^{\infty}(|a_k|^2+|b_k|^2) \\
&\le C\Big(|F|^2_{L^2(-1,1)}+|G|^2_{L^2(-1,1)}\Big) \le C|(f,g)|^2_{L^2(-1,1)}.
\end{aligned} \tag{2.57}
$$

However, by (2.53) and (2.54), we have

$$
F(x)=F(-x), \quad G(x)=-G(-x), \qquad x\in(-1,1). \tag{2.58}
$$

Therefore, it follows from (2.55), (2.56) and (2.58) that

$$
\sum_{k=-\infty}^{\infty} a_k e^{\tilde\lambda_k^h x}=\sum_{k=-\infty}^{\infty} a_k e^{-\tilde\lambda_k^h x}, \quad \sum_{k=-\infty}^{\infty} b_k e^{\tilde\lambda_k^h x}=-\sum_{k=-\infty}^{\infty} b_k e^{-\tilde\lambda_k^h x}, \qquad x\in(-1,1). \tag{2.59}
$$

Now, by (2.59) and (2.55), we see that

$$
\begin{aligned}
\frac12\sum_{k=-\infty}^{\infty}(a_k+b_k)\Big(e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}\Big) &= \frac12\Big[\sum_{k=-\infty}^{\infty}(a_k+b_k)e^{\tilde\lambda_k^h x}+\sum_{k=-\infty}^{\infty}(a_k+b_k)e^{-\tilde\lambda_k^h x}\Big] \\
&= \frac12\Big[\sum_{k=-\infty}^{\infty}(a_k+b_k)e^{\tilde\lambda_k^h x}+\sum_{k=-\infty}^{\infty}(a_k-b_k)e^{\tilde\lambda_k^h x}\Big] \\
&= \sum_{k=-\infty}^{\infty} a_k e^{\tilde\lambda_k^h x}=F(x), \qquad x\in(-1,1).
\end{aligned} \tag{2.60}
$$

Similarly, by (2.59) and (2.56), we have

$$
\frac12\sum_{k=-\infty}^{\infty}(a_k+b_k)\Big(e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h x}\Big)=\sum_{k=-\infty}^{\infty} b_k e^{\tilde\lambda_k^h x}=G(x), \qquad x\in(-1,1). \tag{2.61}
$$

Consequently, combining (2.60) and (2.61), noting the definition of $e_k(x)$ in (2.52), and recalling (2.53) and (2.54), we conclude that

$$
(f,g)=\sum_{k=-\infty}^{\infty}(a_k+b_k)e_k(x), \qquad x\in(-1,1). \tag{2.62}
$$

Now, by (2.62) and (2.52), using the Riesz basis property of $\{e^{\tilde\lambda_k^h x}\}_{k=-\infty}^{\infty}$ and $\{e^{-\tilde\lambda_k^h x}\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$, it is easy to check that

$$
\begin{aligned}
|(f,g)|^2_{L^2(-1,1)} &= \frac14\left(\int_{-1}^{0}\left|\sum_{k=-\infty}^{\infty}(a_k+b_k)\Big(e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}\Big)\right|^2dx\right. \\
&\qquad\left.+\int_0^1\left|\sum_{k=-\infty}^{\infty}(a_k+b_k)\Big(e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h x}\Big)\right|^2dx\right) \\
&\le \int_{-1}^{1}\left|\sum_{k=-\infty}^{\infty}(a_k+b_k)e^{\tilde\lambda_k^h x}\right|^2dx+\int_{-1}^{1}\left|\sum_{k=-\infty}^{\infty}(a_k+b_k)e^{-\tilde\lambda_k^h x}\right|^2dx \\
&\le C\sum_{k=-\infty}^{\infty}|a_k+b_k|^2.
\end{aligned} \tag{2.63}
$$

Further, let us show the $\omega-$linearly independence property of $\{e_k\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$. For this purpose, we assume that there is a sequence $\{c_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$ with $\sum\limits_{k=-\infty}^{\infty} |c_k|^2 < \infty$ such that

$$\sum_{k=-\infty}^{\infty} c_k e_k(x) = 0, \qquad x \in (-1,1). \tag{2.64}$$

From (2.64) and the definition of e_k in (2.52), we conclude that

$$\sum_{k=-\infty}^{\infty} c_k \Big(e^{\tilde{\lambda}_k^h x} + e^{-\tilde{\lambda}_k^h x} \Big) = 0, \qquad x \in (-1,0) \tag{2.65}$$

and

$$\sum_{k=-\infty}^{\infty} c_k \Big(e^{\tilde{\lambda}_k^h x} - e^{-\tilde{\lambda}_k^h x} \Big) = 0, \qquad x \in (0,1). \tag{2.66}$$

Obviously, (2.65) is equivalent to

$$\sum_{k=-\infty}^{\infty} c_k \Big(e^{\tilde{\lambda}_k^h x} + e^{-\tilde{\lambda}_k^h x} \Big) = 0, \qquad x \in (0,1). \tag{2.67}$$

Now, by adding (2.66) to (2.67), we get

$$\sum_{k=-\infty}^{\infty} c_k e^{\tilde{\lambda}_k^h x} = 0, \qquad x \in (0,1). \tag{2.68}$$

Similarly, by subtracting (2.67) from (2.66), we get

$$\sum_{k=-\infty}^{\infty} c_k e^{-\tilde{\lambda}_k^h x} = 0, \qquad x \in (0,1). \tag{2.69}$$

However, (2.69) is equivalent to

$$\sum_{k=-\infty}^{\infty} c_k e^{\tilde{\lambda}_k^h x} = 0, \qquad x \in (-1,0). \tag{2.70}$$

Combining (2.68) and (2.70), we get

$$\sum_{k=-\infty}^{\infty} c_k e^{\tilde{\lambda}_k^h x} = 0, \qquad x \in (-1,1). \tag{2.71}$$

Now, from (2.71) and by the Riesz basis property of system $\{e^{\tilde{\lambda}_k^h x}\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$, we conclude that $c_k = 0$ for all $k = 0, \pm 1, \pm 2, \cdots$. This gives the $\omega-$linear independence of $\{e_k\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$.

Finally, by (2.62), (2.57) and (2.63), and noting the $\omega-$linear independence of $\{e_k\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$, we conclude that $\{e_k(x)\}_{k=-\infty}^{\infty}$ forms a Riesz basis in $L^2(-1,1)$.

Step 3. We claim that

$$\left\{\left(\frac{e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}}{2},\frac{e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h x}}{2}\right)\right\}_{k=-\infty}^{\infty} \tag{2.72}$$

forms a Riesz basis in $L^2(0,1)\times L^2(0,1)$.

In fact, choose any $(f,g)\in L^2(0,1)\times L^2(0,1)$. Define $\hat f(x)=f(-x),\quad x\in(-1,0)$. Then $(\hat f,g)\in L^2(-1,1)$. Thus, by the Riesz basis property of $\{e_k(x)\}_{k=-\infty}^{\infty}$ in $L^2(-1,1)$, we conclude that there is a sequence $\{c_k\}_{k=-\infty}^{\infty}\subset\mathbb{C}$ such that

$$(\hat f,g)=\sum_{k=-\infty}^{\infty}c_ke_k(x)\qquad\text{in } L^2(-1,1). \tag{2.73}$$

By (2.63) and the definition of $e_k(x)$ in (2.52), it is easy to see that

$$f(-x)=\hat f(x)=\frac{1}{2}\sum_{k=-\infty}^{\infty}c_k\Big(e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}\Big),\qquad x\in(-1,0) \tag{2.74}$$

and

$$g(x)=\frac{1}{2}\sum_{k=-\infty}^{\infty}c_k\Big(e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h x}\Big),\qquad x\in(0,1). \tag{2.75}$$

However, (2.74) is equivalent to

$$f(x)=\frac{1}{2}\sum_{k=-\infty}^{\infty}c_k\Big(e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}\Big),\qquad x\in(0,1). \tag{2.76}$$

In view of (2.75) and (2.76), we conclude that

$$(f,g)=\sum_{k=-\infty}^{\infty}c_k\left(\frac{e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h x}}{2},\frac{e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h x}}{2}\right),$$

which yields easily the Riesz basis property of the system (2.72) in $L^2(0,1)\times L^2(0,1)$.

By translation, we see that

$$\left\{\left(\frac{e^{\tilde\lambda_k^h (x+1)}+e^{-\tilde\lambda_k^h (x+1)}}{2},\frac{e^{\tilde\lambda_k^h (x+1)}-e^{-\tilde\lambda_k^h (x+1)}}{2}\right)\right\}_{k=-\infty}^{\infty} \tag{2.77}$$

forms a Riesz basis in $L^2(-1,0)\times L^2(-1,0)$.

On the other hand, by (2.48), it is easy to see that there is a constant $C>0$ such that

$$\frac{1}{C}\le|e^{-\tilde\lambda_k^h}|\le C,\qquad k=0,\pm1,\pm2,\cdots.$$

Thus

$$\left\{\left(\frac{e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h (x+2)}}{2},\frac{e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h (x+2)}}{2}\right)\right\}_{k=-\infty}^{\infty} \tag{2.78}$$

is a Riesz basis for $L^2(-1,0)\times L^2(-1,0)$ since every element in system (2.78) is obtained from that in system (2.77) by simple multiplication by $e^{-\tilde{\lambda}_k^h}$.

Step 4. We claim that

$$\left\{\left(\frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{2\tilde{\lambda}_k^h}, \frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{2}\right)\right\}_{k=-\infty}^{\infty} \tag{2.79}$$

is a Riesz basis for $W_2\times L^2(-1,0)$ (recall (1.2) for W_2). To see this, we take any $(f,g)\in W_2\times L^2(-1,0)$. Then $(f_x,g)\in L^2(-1,0)\times L^2(-1,0)$. Thus, there exists a sequence $\{d_k\}_{k=-\infty}^{\infty}\subset\mathbb{C}$ such that

$$(f_x,g)=\sum_{k=-\infty}^{\infty} d_k\left(\frac{e^{\tilde{\lambda}_k^h x}+e^{-\tilde{\lambda}_k^h(x+2)}}{2}, \frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{2}\right) \tag{2.80}$$

in $L^2(-1,0)\times L^2(-1,0)$. Especially, we have

$$f_x=\frac{1}{2}\sum_{k=-\infty}^{\infty} d_k\left(e^{\tilde{\lambda}_k^h x}+e^{-\tilde{\lambda}_k^h(x+2)}\right) \qquad \text{in } L^2(-1,0).$$

Integrating the above equality from -1 to x, noting that $f(-1)=0$, we get

$$f(x)=\frac{1}{2}\sum_{k=-\infty}^{\infty} d_k\frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{\tilde{\lambda}_k^h} \qquad \text{in } W_2. \tag{2.81}$$

From (2.80) and (2.81), we see that

$$(f,g)=\sum_{k=-\infty}^{\infty} d_k\left(\frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{2\tilde{\lambda}_k^h}, \frac{e^{\tilde{\lambda}_k^h x}-e^{-\tilde{\lambda}_k^h(x+2)}}{2}\right) \qquad \text{in } W_2\times L^2(-1,0).$$

which yields easily the Riesz basis property of system (2.79) in $W_2\times L^2(-1,0)$.

Finally, from the definition of q_k^1 and r_k^1 in (2.50), it is easy to see that (q_k^1,r_k^1) may be obtained from the corresponding element in system (2.79) by simple multiplication by i. Therefore, $\{(q_k^1,r_k^1)\}_{k=-\infty}^{\infty}$ constitutes a Riesz basis of $W_2\times L^2(-1,0)$. This completes the proof of Lemma 2.5. $\square$

We have analyzed the asymptotic behavior of parabolic and hyperbolic eigenvectors and the Riesz basis property of the corresponding approximate eigenvectors. Now, we need to combine these two results together.

For this purpose, we put

$$p_k^1=p_k^1(x)\stackrel{\triangle}{=}q_k^1(0)(1-x), \qquad x\in(0,1),\ k=0,\pm1,\pm2,\cdots. \tag{2.82}$$

By means of a direct computation, it is easy to check the following result (We omit the details):

Proposition 2.5 *It holds*

$$\sum_{k=-\infty}^{\infty}|p_k^1|_{W_1}^2=\sum_{k=-\infty}^{\infty}|q_k^1(0)|^2<\infty. \tag{2.83}$$

Now, combining Proposition 2.3, Proposition 2.4 and Proposition 2.5, and noting (1.4), we conclude immediately that

Lemma 2.7 *The following estimate holds*

$$\sum_{\ell=\ell_1}^{\infty}\left[|(p_\ell^p, q_\ell^p, r_\ell^p) - (p_\ell^0, 0, 0)|_H^2\right] + \sum_{|k|=k_1}^{\infty}\left[|(p_k^h, q_k^h, r_k^h) - (p_k^1, q_k^1, r_k^1)|_H^2\right] < \infty.$$

On the other hand, we have

Lemma 2.8 *System*

$$\{(p_\ell^0, 0, 0)\}_{\ell=1}^{\infty}\bigcup\{(p_k^1, q_k^1, r_k^1)\}_{k=-\infty}^{\infty} \tag{2.84}$$

forms a Riesz basis in H.

Proof. First of all, let us show that system (2.84) is complete in H. For this purpose, we fix any $(f, g, h) \in H$. By Lemma 2.5, $\{(q_k^1, r_k^1)\}_{k=-\infty}^{\infty}$ is a Riesz basis of $W_2 \times L^2(-1, 0)$. Thus, there is a sequence $\{a_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$ with $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$ such that

$$(g, h) = \sum_{k=-\infty}^{\infty} a_k(q_k^1, r_k^1) \qquad \text{in } W_2 \times L^2(-1, 0). \tag{2.85}$$

Especially, we have

$$g = \sum_{k=-\infty}^{\infty} a_k q_k^1 \qquad \text{in } W_2. \tag{2.86}$$

Now, by (2.86) and the definitions of H and p_k^1, we see that

$$f(0) = g(0) = \sum_{k=-\infty}^{\infty} a_k q_k^1(0) = \sum_{k=-\infty}^{\infty} a_k p_k^1(0). \tag{2.87}$$

On the other hand, obviously,

$$f(1) = \sum_{k=-\infty}^{\infty} a_k p_k^1(1) = 0. \tag{2.88}$$

Also, by (2.83), it is easy to see that $\sum_{k=-\infty}^{\infty} a_k p_k^1 \in H^1(0, 1)$. Hence, combining (2.87) and (2.88), we conclude that

$$f - \sum_{k=-\infty}^{\infty} a_k p_k^1 \in H_0^1(0, 1).$$

Therefore, by Lemma 2.4, there exists a sequence $\{b_\ell\}_{\ell=1}^{\infty} \subset \mathbb{C}$ with $\sum_{\ell=1}^{\infty} |b_\ell|^2 < \infty$ such that

$$f - \sum_{k=-\infty}^{\infty} a_k p_k^1 = \sum_{\ell=1}^{\infty} b_\ell p_\ell^0 \qquad \text{in } H_0^1(0, 1). \tag{2.89}$$

Now, combining (2.85) and (2.89), we arrive at

$$(f,g,h)=\sum_{k=-\infty}^{\infty}a_k(p_k^1,q_k^1,r_k^1)+\sum_{\ell=1}^{\infty}b_\ell(p_\ell^0,0,0)\qquad \text{in } H,$$

which yields the completeness of system (2.84) in H.

Next, we claim that there is a constant $C>0$ such that for all sequences $\{\tilde{a}_k\}_{k=-\infty}^{\infty}\subset\mathbb{C}$ and $\{\tilde{b}_\ell\}_{\ell=1}^{\infty}\subset\mathbb{C}$ with $\sum\limits_{k=-\infty}^{\infty}|\tilde{a}_k|^2+\sum\limits_{\ell=1}^{\infty}|\tilde{b}_\ell|^2<\infty$, it holds

$$\begin{aligned}\frac{1}{C}\left(\sum_{k=-\infty}^{\infty}|\tilde{a}_k|^2+\sum_{\ell=1}^{\infty}|\tilde{b}_\ell|^2\right)&\le\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(p_k^1,q_k^1,r_k^1)+\sum_{\ell=1}^{\infty}\tilde{b}_\ell(p_\ell^0,0,0)\right|_H^2\\&\le C\left(\sum_{k=-\infty}^{\infty}|\tilde{a}_k|^2+\sum_{\ell=1}^{\infty}|\tilde{b}_\ell|^2\right).\end{aligned}\tag{2.90}$$

In fact, in view of (1.4) and (2.82), we get

$$\begin{aligned}&\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(p_k^1,q_k^1,r_k^1)+\sum_{\ell=1}^{\infty}\tilde{b}_\ell(p_\ell^0,0,0)\right|_H^2\\&=\left|\sum_{k=-\infty}^{\infty}\tilde{a}_kp_k^1+\sum_{\ell=1}^{\infty}\tilde{b}_\ell p_\ell^0\right|_{W_1}^2+\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(q_k^1,r_k^1)\right|_{W_2\times L^2(-1,0)}^2\\&=\int_0^1\Big|-\sum_{k=-\infty}^{\infty}\tilde{a}_kq_k^1(0)+\sum_{\ell=1}^{\infty}\tilde{b}_\ell\partial_xp_\ell^0(x)\Big|^2dx+\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(q_k^1,r_k^1)\right|_{W_2\times L^2(-1,0)}^2.\end{aligned}\tag{2.91}$$

However, by (2.44), it is easy to see that $\int_0^1\partial_xp_\ell^0(x)dx=0$. Therefore,

$$\begin{aligned}&\int_0^1\Big|-\sum_{k=-\infty}^{\infty}\tilde{a}_kq_k^1(0)+\sum_{\ell=1}^{\infty}\tilde{b}_\ell\partial_xp_\ell^0(x)\Big|^2dx\\&=\Big|\sum_{k=-\infty}^{\infty}\tilde{a}_kq_k^1(0)\Big|^2-2\mathrm{Re}\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=1}^{\infty}\overline{\tilde{a}_kq_k^1(0)}\,\tilde{b}_\ell\int_0^1\partial_xp_\ell^0(x)dx\right)\\&\quad+\int_0^1\Big|\sum_{\ell=1}^{\infty}\tilde{b}_\ell\partial_xp_\ell^0(x)\Big|^2dx\\&=\Big|\sum_{k=-\infty}^{\infty}\tilde{a}_kq_k^1(0)\Big|^2+\Big|\sum_{\ell=1}^{\infty}\tilde{b}_\ell p_\ell^0\Big|_{H_0^1(0,1)}^2.\end{aligned}\tag{2.92}$$

Combining (2.91) and (2.92), we get

$$\begin{aligned}&\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(p_k^1,q_k^1,r_k^1)+\sum_{\ell=1}^{\infty}\tilde{b}_\ell(p_\ell^0,0,0)\right|_H^2\\&=\Big|\sum_{k=-\infty}^{\infty}\tilde{a}_kq_k^1(0)\Big|^2+\Big|\sum_{\ell=1}^{\infty}\tilde{b}_\ell p_\ell^0\Big|_{H_0^1(0,1)}^2+\left|\sum_{k=-\infty}^{\infty}\tilde{a}_k(q_k^1,r_k^1)\right|_{W_2\times L^2(-1,0)}^2.\end{aligned}\tag{2.93}$$

Recall that, by Lemmas 2.4 and 2.5, $\{p_\ell^0\}_{\ell=1}^\infty$ and $\{(q_k^1, r_k^1)\}_{k=-\infty}^\infty$ are Riesz basis in $H_0^1(0,1)$ and $W_2 \times L^2(-1,0)$, respectively. Therefore, there is a constant $C > 0$ such that

$$\frac{1}{C}\sum_{\ell=1}^\infty |\tilde{b}_\ell|^2 \le \Big|\sum_{\ell=1}^\infty \tilde{b}_\ell p_\ell^0\Big|^2_{H_0^1(0,1)} \le C\sum_{\ell=1}^\infty |\tilde{b}_\ell|^2 \tag{2.94}$$

and

$$\frac{1}{C}\sum_{k=-\infty}^\infty |\tilde{a}_k|^2 \le \left|\sum_{k=-\infty}^\infty \tilde{a}_k(q_k^1, r_k^1)\right|^2_{W_2\times L^2(-1,0)} \le C\sum_{k=-\infty}^\infty |\tilde{a}_k|^2. \tag{2.95}$$

Also, by Proposition 2.5, it is easy to see that

$$\Big|\sum_{k=-\infty}^\infty \tilde{a}_k q_k^1(0)\Big|^2 \le C\sum_{k=-\infty}^\infty |\tilde{a}_k|^2. \tag{2.96}$$

Now, combining (2.93), (2.94), (2.95) and (2.96), one gets (2.90) immediately. This completes the proof of Lemma 2.8. □

2.3 Riesz basis property of the generalized eigenvectors

The main result of this section is the following theorem.

Theorem 2.1 *There exist positive integers n_0, $\tilde{\ell}_1 \ge \ell_1$ and $\tilde{k}_1 \ge k_1$ such that*

$$\{u_{j,0},\cdots,u_{j,m_j-1}\}_{j=1}^{n_0}\bigcup\{(p_\ell^p, q_\ell^p, r_\ell^p)\}_{\ell=\tilde{\ell}_1}^\infty\bigcup\{(p_k^h, q_k^h, r_k^h)\}_{|k|=\tilde{k}_1}^\infty \tag{2.97}$$

form a Riesz basis of H, where $u_{j,0}$ is an eigenvector of $\mathcal{A}$ with respect to some eigenvalue μ_j of $\mathcal{A}$ ($j = 1, 2, \cdots, n_0$) with algebraic multiplicity m_j, $\{u_{j,1},\cdots,u_{j,m_j-1}\}$ is the associated Jordan chain of the corresponding generalized eigenvectors of $\mathcal{A}$ with respect to μ_j and $u_{j,0}$, i.e., $\mathcal{A}u_{j,0} = \mu_j u_{j,0}$ and $\mathcal{A}u_{j,k} = \mu_j u_{j,k} + u_{j,k-1}$ ($k = 1,\cdots m_j - 1$).

In order to prove Theorem 2.1, we need the following known result ([6]).

Lemma 2.9 *Let V be a Hilbert space and G be a densely defined linear operator with compact resolvent in V. Let $\{f_n\}_{n=1}^\infty$ be a Riesz basis of V. Suppose a sequence of generalized eigenvectors $\{g_n\}_{n=N+1}^\infty$ of G satisfies*

$$\sum_{n=N+1}^\infty |g_n - f_n|_V^2 < \infty$$

for some $N \in \mathbb{N}$. Then one can find an integer $M \ge N$ and some generalized eigenvectors $\{g_{n0}\}_{n=1}^M$ of G such that

$$\{g_{n0}\}_{n=1}^M\bigcup\{g_n\}_{n=M+1}^\infty$$

forms a Riesz basis of V.

Proof of Theorem 2.1. By Lemmas 2.7, 2.8 and 2.9, we conclude that there exist positive integers m_0, $\tilde{\ell}_1 \geq \ell_1$ and $\tilde{k}_1 \geq k_1$, and generalized eigenvectors (p_m, q_m, r_m) $(m = 1, \cdots, m_0)$ of $\mathcal{A}$ such that

$$\{(p_m, q_m, r_m))\}_{m=1}^{m_0} \bigcup \{(p_\ell^p, q_\ell^p, r_\ell^p)\}_{\ell=\tilde{\ell}_1}^{\infty} \bigcup \{(p_k^h, q_k^h, r_k^h)\}_{|k|=\tilde{k}_1}^{\infty}$$

forms a Riesz basis of H.

We set

$$M = \operatorname{span}\{(p_m, q_m, r_m) \mid m = 1, 2, \cdots, m_0\}.$$

Obviously, $\dim M = m_0 < \infty$ and M is an invariant subspace of $\mathcal{A}$. Thus $\mathcal{A}|_M$ can be represented as a matrix. Hence, $\mathcal{A}|_M$ admits n_0 eigenvalues $\mu_1, \mu_2, \cdots, \mu_{n_0}$, where $n_0 \in \{1, 2 \cdots, m_0\}$. We denote by $m_j (\geq 1)$ the algebraic multiplicity of the eigenvalue μ_j $(j = 1, 2, \cdots, n_0)$. Applying Jordan's decomposition theorem (in matrix theory), one concludes that $m_1+m_2+\cdots+m_{n_0} = m_0$, and there are m_0 generalized eigenvectors $u_{1,0}, \cdots, u_{1,m_1-1}, \cdots$, $u_{n_0,0}, \cdots, u_{n_0,m_{n_0}-1}$ of $\mathcal{A}|_M$, which form a basis of M, where $\{u_{j,0}, u_{j,1}, \cdots, u_{j,m_j-1}\}$ is the associated Jordan chain of the corresponding generalized eigenvectors of $\mathcal{A}_M$ with respect to μ_j, i.e., $\mathcal{A}|_M u_{j,0} = \mu_j u_{j,0}$ and $\mathcal{A}|_M u_{j,k} = \mu_j u_{j,k} + u_{j,k-1}$ $(k = 1, \cdots m_j - 1)$.

It is easy to see that

$$\{u_{j,0}, \cdots, u_{j,m_j-1}\}_{j=1}^{n_0} \bigcup \{(p_\ell^p, q_\ell^p, r_\ell^p)\}_{\ell=\tilde{\ell}_1}^{\infty} \bigcup \{(p_k^h, q_k^h, r_k^h)\}_{|k|=\tilde{k}_1}^{\infty}$$

forms a Riesz basis of H.

Finally, noting that μ_k is also an eigenvalue of $\mathcal{A}$, and $\{u_{j,0}, u_{j,1}, \cdots, u_{j,m_j-1}\}$ is the associated Jordan chain of the corresponding generalized eigenvectors of $\mathcal{A}$ with respect to μ_j, we obtain the desired result immediately. This completes the proof of Theorem 2.1. $\square$

As a simple consequence of Theorem 2.1, we have the following result.

Corollary 2.1 *For any $(y_0, z_0, z_1) \in H$, there exist $\{a_{j,0}, \cdots, a_{j,m_j-1}\}_{j=1}^{n_0} \subset \mathbb{C}$, and two sequences $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty} \subset \mathbb{C}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty} \subset \mathbb{C}$ with*

$$\sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} |b_k|^2 < \infty \tag{2.98}$$

such that

$$(y_0, z_0, z_1) = \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell (p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k (p_k^h, q_k^h, r_k^h). \tag{2.99}$$

Furthermore, the corresponding solution of system (1.1) is given by

$$\begin{aligned}(y(t), z(t), z_t(t)) = & \sum_{j=1}^{n_0} e^{\mu_j t} \sum_{k=0}^{m_j-1} a_{j,k} \sum_{s=0}^{k} \frac{t^{k-s}}{(k-s)!} u_{j,s} \\ & + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell (p_\ell^p, q_\ell^p, r_\ell^p) e^{\lambda_\ell^p t} + \sum_{|k|=\tilde{k}_1}^{\infty} b_k (p_k^h, q_k^h, r_k^h) e^{\lambda_k^h t},\end{aligned} \tag{2.100}$$

where n_0, $\tilde{\ell}_1$, $\tilde{k}_1$, μ_j, m_j, $u_{j,0}, \cdots, u_{j,m_j-1}$ $(j = 1, 2, \cdots, n_0)$ are given in Theorem 2.1.

Remark 2.5 *Since $u_{j,s}$ $(s = 0, 1, \cdots, m_j - 1;\ j = 1, 2, \cdots, n_0)$ are generalized eigenvectors of $\mathcal{A}$, we may denote them as follows*

$$u_{j,s} = (p_{j,s}, q_{j,s}, r_{j,s}). \tag{2.101}$$

Obviously, $(p_{j,0}, q_{j,0}, r_{j,0})$ is an eigenvector of $\mathcal{A}$ corresponding to its eigenvalue μ_j. Therefore, $(p_{j,0}, q_{j,0}, r_{j,0})$ can be expressed as (2.6) with the corresponding eigenvalue λ replaced by μ_j. We will use this fact in the sequel.

In view of (2.100) in Corollary 2.1, one sees that any solution of system (1.1) can be decomposed into the sum of three parts, i.e. a *low-frequency finite-dimensional component*

$$(y^{low}(t), z^{low}(t), z_t^{low}(t)) \triangleq \sum_{j=1}^{n_0} e^{\mu_j t} \sum_{k=0}^{m_j-1} a_{j,k} \sum_{s=0}^{k} \frac{t^{k-s}}{(k-s)!} u_{j,s}, \tag{2.102}$$

a *parabolic component*

$$(y^p(t), z^p(t), z_t^p(t)) \triangleq \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell (p_\ell^p, q_\ell^p, r_\ell^p) e^{\lambda_\ell^p t}, \tag{2.103}$$

and a *hyperbolic component*

$$(y^h(t), z^h(t), z_t^h(t)) \triangleq \sum_{|k|=\tilde{k}_1}^{\infty} b_k (p_k^h, q_k^h, r_k^h) e^{\lambda_k^h t}. \tag{2.104}$$

Note that the parabolic (*resp.* hyperbolic) component is not strictly supported in the interval where the heat (*resp.* wave) equation holds. However, the reminder is smaller and smaller when ℓ (*resp.* $|k|$) becomes large. On the other hand, from (2.30), we see that the distance between the hyperbolic eigenvalue λ_k^h and the imaginary axis tends to 0 as $k \to \infty$. Therefore, by (2.100), it is easy to see that the hyperbolic component of the solution of system (1.1) does not decay exponentially, while the rest does. However, assuming more regularity on the initial data, we can show that the energy of the hyperbolic component decays polynomially. This will be shown in Section 3.

3 Polynomial decay rate

This section is devoted to showing the polynomial decay of the energy of solutions of system (1.1). We have the following result.

Theorem 3.1 *There is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of (1.1) satisfies*

$$|(y(t), z(t), z_t(t))|_H \leq \frac{C}{t^2} |(y_0, z_0, z_1)|_{D(\mathcal{A})}, \qquad \forall\, t > 0. \tag{3.1}$$

Proof. We take any $(y_0, z_0, z_1) \in D(\mathcal{A})$. Since $D(\mathcal{A}) \subset H$, we conclude that there exist complex numbers $\{a_{j,0}, \cdots, a_{j,m_j-1}\}_{j=1}^{n_0}$, and two sequences $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty} \subset \mathbb{C}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty} \subset \mathbb{C}$ satisfying (2.98) such that (2.99) holds. Then, by Corollary 2.1, the corresponding solution of system (1.1) is given by (2.100).

Recall that μ_j $(j = 1, 2, \cdots, n_0)$, λ_ℓ^p $(\ell \geq \tilde{\ell}_1)$ and λ_k^h $(|k| \geq \tilde{k}_1)$ in (2.100) are eigenvalues of $\mathcal{A}$. By (2.3) in Lemma 2.1, we have

$$\max(\operatorname{Re}\mu_1, \operatorname{Re}\mu_2, \cdots, \operatorname{Re}\mu_{n_0}) < 0. \tag{3.2}$$

On the other hand, by $(y_0, z_0, z_1) \in D(\mathcal{A})$ and using the Riesz basis property shown in Theorem 2.1, we have

$$\begin{aligned}
\infty > |(y_0, z_0, z_1)|^2_{D(\mathcal{A})} &= |(y_0, z_0, z_1)|^2_H + |\mathcal{A}(y_0, z_0, z_1)|^2_H \\
&= \Big|\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1} a_{j,k}u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k(p_k^h, q_k^h, r_k^h)\Big|^2_H \\
&\quad + \Big|\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1} a_{j,k}\mathcal{A}u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell\mathcal{A}(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k\mathcal{A}(p_k^h, q_k^h, r_k^h)\Big|^2_H \\
&= \Big|\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1} a_{j,k}u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k(p_k^h, q_k^h, r_k^h)\Big|^2_H \\
&\quad + \Big|\sum_{j=1}^{n_0}\Big[a_{j,m_j-1}\mu_{j,m_j-1}u_{j,m_j-1} + \sum_{k=0}^{m_j-2}(a_{j,k+1} + a_{j,k}\mu_{j,k})u_{j,k}\Big] \\
&\quad + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell\lambda_\ell^p(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k\lambda_k^h(p_k^h, q_k^h, r_k^h)\Big|^2_H \\
&\geq C\sum_{|k|=\tilde{k}_1}^{\infty}(1 + |\lambda_k^h|^2)|b_k|^2.
\end{aligned} \tag{3.3}$$

Using again the Riesz basis property shown in Theorem 2.1, from (2.100), we end up with

$$\begin{aligned}
&|(y(t), z(t), z_t(t))|^2_H \\
&\leq C\left[\sum_{j=1}^{n_0} e^{2t\operatorname{Re}\mu_j}\sum_{k=0}^{m_j-1}|a_{j,k}|^2\sum_{s=0}^{k} t^{2(k-s)} + \sum_{\ell=\tilde{\ell}_1}^{\infty} e^{2t\operatorname{Re}\lambda_\ell^p}|a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} e^{2t\operatorname{Re}\lambda_k^h}|b_k|^2\right].
\end{aligned} \tag{3.4}$$

However, by (2.30), we see that

$$\operatorname{Re}\lambda_k^h = -\frac{1}{\sqrt{|1 + 2k|\pi}} + O(|k|^{-1}) = -\frac{1}{\sqrt{2|k|\pi}} + O(|k|^{-1}).$$

Thus, for $|k|$ large enough, the following estimate holds:

$$-\operatorname{Re}\lambda_k^h > \frac{1}{\sqrt{4|k|\pi}} (> 0).$$

Similarly, for $|k|$ large enough, we get

$$|\lambda_k^h|^2 = \Big(\frac{1}{2} + k\Big)^2 \pi^2 + O(|k|^{1/2}) > k^2.$$

Therefore, for all $t > 0$, it holds

$$\begin{aligned} \frac{e^{2t\mathrm{Re}\,\lambda_k^h}}{|\lambda_k^h|^2} &= \frac{1}{|\lambda_k^h|^2 e^{-2t\mathrm{Re}\,\lambda_k^h}} = \frac{1}{|\lambda_k^h|^2 \sum\limits_{j=0}^{\infty} (j!)^{-1}(2t)^j(-\mathrm{Re}\,\lambda_k^h)^j} \\ &\le \frac{1}{(4!)^{-2}(2t)^4(-\mathrm{Re}\,\lambda_k^h)^4|\lambda_k^h|^2} \le Ct^{-4}. \end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we have

$$\sum_{|k|=\tilde{k}_1}^{\infty} e^{2t\mathrm{Re}\,\lambda_k^h}|b_k|^2 \le Ct^{-4} \sum_{|k|=\tilde{k}_1}^{\infty} |\lambda_k^h|^2|b_k|^2 \le Ct^{-4}|(y_0, z_0, z_1)|^2_{D(\mathcal{A})}. \tag{3.6}$$

On the other hand, by (2.29) and (3.2), it is easy to see that there is a constant $\delta_0 > 0$ such that

$$\sup\Big(\{\mathrm{Re}\,\lambda_\ell^p \mid \ell \ge \tilde{\ell}_1\} \bigcup \{\mathrm{Re}\,\mu_1, \mathrm{Re}\,\mu_2, \cdots, \mathrm{Re}\,\mu_{n_0}\}\Big) < -\delta_0.$$

Therefore, for t large enough, it holds

$$\sum_{j=1}^{n_0} e^{2t\mathrm{Re}\,\mu_j} \sum_{k=0}^{m_j-1} |a_{j,k}|^2 \sum_{s=0}^{k} t^{2(k-s)} + \sum_{\ell=\tilde{\ell}_1}^{\infty} e^{2t\mathrm{Re}\,\lambda_\ell^p}|a_\ell|^2 \le Ce^{-\delta_0 t}|(y_0, z_0, z_1)|^2_H. \tag{3.7}$$

Finally, combining (3.6) and (3.7), one obtains (3.1). This completes the proof of Theorem 3.1. $\square$

Remark 3.1 *From the proof of Theorem 3.1, we see that the solution of system (1.1) can be split into the sum of two parts, i.e., the parabolic component plus the finite-dimensional one which decays exponentially to zero and the hyperbolic component for which we have the polynomial decay of smooth solutions.*

Remark 3.2 *Obviously, for any $j \in \mathbb{N}$, $D(\mathcal{A}^j)$ is a Hilbert space (with graph norm). Similar to the proof of Theorem 3.1, one can show that for any $(y_0, z_0, z_1) \in D(\mathcal{A}^j)$, the solution of (1.1) satisfies $|(y(t), z(t), z_t(t))|_H \le Ct^{-2j}|(y_0, z_0, z_1)|_{D(\mathcal{A}^j)}$ for all $t > 0$. Therefore, solutions of (1.1) decay faster when initial data are smoother.*

4 Boundary control and observation through the heat component

This section is devoted to analyzing the null controllability property of system (1.7) and the boundary observation property of system (1.10) through the heat component.

We will divide this section into 5 subsections. In the first subsection, we will show a regularity result for system (1.1). In the second subsection, we will give a negative observability result for system (1.1) in the natural energy space H, which implies a negative controllability result for system (1.7). In the third subsection, we will give a description on the controllability subspace of system (1.7) and state the corresponding observability estimate in Theorems 4.3 and 4.4, respectively. In order to prove the observability result, we need a key new Ingham-type inequality, which will be shown in the fourth subsection. The fifth subsection is devoted to proving the observability result. In the sixth subsection, we will prove the controllability result.

4.1 A regularity result for the adjoint system

We begin with the following regularity result for system (1.1).

Theorem 4.1 *Let $T > 0$. Then for any $(y_0, z_0, z_1) \in H$, the corresponding solution of system (1.1) satisfies $y_x(\cdot, 1) \in L^2(0, T)$, and there is a constant $C = C(T)$ such that*

$$|y_x(\cdot, 1)|_{L^2(0,T)} \le C|(y_0, z_0, z_1)|_H, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{4.1}$$

Furthermore, for any $s \ge 0$, the hyperbolic component y^h of y, defined in (2.104), satisfies $y^h_x(\cdot, 1) \in H^s(0, T)$, and there is a constant $C = C(T, s)$ such that

$$|y^h_x(\cdot, 1)|_{H^s(0,T)} \le C|(y_0, z_0, z_1)|_H, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{4.2}$$

Similarly, the parabolic component z^p of z, defined in (2.103), satisfies $z^p_x(\cdot, -1) \in H^s(0, T)$, and there is a constant $C = C(T, s)$ such that

$$|z^p_x(\cdot, -1)|_{H^s(0,T)} \le C|(y_0, z_0, z_1)|_H, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{4.3}$$

Remark 4.1 *Theorem 4.1 and, especially, (4.2) (resp. (4.3)) provides a regularity result for solutions of system (1.1). The smoothing effect in (4.2) and (4.3) is quite typical for solutions of heat or more generally, parabolic equations, but not hyperbolic models. Note however that (4.2) (resp. (4.3)) guarantees some smoothing effect on the hyperbolic component of y (resp. the parabolic component of z), too. This is due to the fact that, although the hyperbolic component of the solution is dissipated very weakly, the restriction of the hyperbolic eigenvectors to the parabolic interval $(0, 1)$ (resp. the restriction of the parabolic eigenvectors to the hyperbolic interval $(-1, 0)$) are asymptotically very small. This makes that, despite of the very weak damping, the restriction of the normal derivative of the hyperbolic component (resp. the parabolic component) of finite energy solutions to the parabolic extreme $x = 1$ (resp. the hyperbolic extreme $x = -1$) lies in $H^s(0, T)$ for all $s \ge 0$ and $T \ge 0$.*

In order to prove Theorem 4.1, we need the following technical result.

Proposition 4.1 *There is a constant $C > 0$ such that*

$$\frac{1}{C}|k|e^{\sqrt{2|k|\pi}} \le \frac{\left|1+e^{2\sqrt{\lambda_k^h}}\right|^2}{\left|e^{\sqrt{\lambda_k^h}}\left(1+e^{-2\lambda_k^h}\right)\right|^2} \le C|k|e^{\sqrt{2|k|\pi}} \tag{4.4}$$

holds for all $|k| \ge \tilde{k}_1$.

The proof of Proposition 4.1 is given in Appendix B.

Proof of Theorem 4.1. Since the "low frequency" component of the solution does not affect the argument below, we may simply assume that $(y_0, z_0, z_1) \in H$ is of the form

$$(y_0, z_0, z_1) = \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} b_k(p_k^h, q_k^h, r_k^h),$$

where $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty}$ satisfy (2.98). Then by (2.100), the corresponding solution of system (1.1) is given by

$$(y(t), z(t), z_t(t)) = \sum_{\ell=\tilde{\ell}_1}^{\infty} e^{\lambda_\ell^p t} a_\ell(p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} e^{\lambda_k^h t} b_k(p_k^h, q_k^h, r_k^h). \tag{4.5}$$

By (2.43), (2.47) and (2.7), it is easy to check that

$$\begin{aligned} \partial_x p_\ell^p(x) &= -\frac{\lambda_\ell^p\left(1+e^{-2\lambda_\ell^p}\right)\left(e^{\sqrt{\lambda_\ell^p}x}+e^{\sqrt{\lambda_\ell^p}(2-x)}\right)}{\ell^2\pi^2\left(1+e^{2\ell^2\pi^2}\right)\left(1+e^{2\sqrt{\lambda_\ell^p}}\right)}, \\ \partial_x p_k^h(x) &= -\frac{\left(1+e^{-2\lambda_k^h}\right)\left(e^{\sqrt{\lambda_k^h}x}+e^{\sqrt{\lambda_k^h}(2-x)}\right)}{2i\left(1+e^{2\sqrt{\lambda_k^h}}\right)}. \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} |y_x(\cdot,1)|^2_{L^2(0,T)} = \int_0^T \Bigg| &\sum_{\ell=\tilde{\ell}_1}^{\infty} \frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\left(1+e^{-2\lambda_\ell^p}\right)a_\ell}{\ell^2\pi^2\left(1+e^{2\ell^2\pi^2}\right)\left(1+e^{2\sqrt{\lambda_\ell^p}}\right)} e^{\lambda_\ell^p t} \\ &+ \sum_{|k|=\tilde{k}_1}^{\infty} \frac{e^{\sqrt{\lambda_k^h}}\left(1+e^{-2\lambda_k^h}\right)b_k}{i\left(1+e^{2\sqrt{\lambda_k^h}}\right)} e^{\lambda_k^h t}\Bigg|^2 dt. \end{aligned} \tag{4.7}$$

Similarly, by the definition of y^h in (2.104), for $s = 0, 1, 2\cdots$, we have

$$|y_x^h(\cdot,1)|^2_{H^s(0,T)} = \sum_{j=0}^{s}\int_0^T \left| \sum_{|k|=\tilde{k}_1}^{\infty} \frac{e^{\sqrt{\lambda_k^h}}\left(1+e^{-2\lambda_k^h}\right)b_k(\lambda_k^h)^j}{i\left(1+e^{2\sqrt{\lambda_k^h}}\right)} e^{\lambda_k^h t}\right|^2 dt. \tag{4.8}$$

By (2.29), we see that $\lambda_\ell^p = -\ell^2\pi^2 + O(1)$. Also by (2.33) in Proposition 2.2 and recalling that $\lambda_\ell^0 = -\ell^2\pi^2$, it is easy to see that $e^{\sqrt{\lambda_\ell^p}} = (-1)^\ell + O(\ell^{-1})$ and $e^{2\sqrt{\lambda_\ell^p}} = 1 + O(\ell^{-1})$. Therefore

$$\left| \frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\Big(1+e^{-2\lambda_\ell^p}\Big)}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)\Big(1+e^{2\sqrt{\lambda_\ell^p}}\Big)} \right| \le C, \qquad \forall\, \ell \in \mathbb{N}. \tag{4.9}$$

On the other hand, by Proposition 4.1 and recalling that $\lambda_k^h = (1/2+k)\pi i + O(|k|^{-1/2})$ (*see* (2.12)), we conclude that for any given $j \in \mathbb{N}$, it holds

$$\left| \frac{e^{\sqrt{\lambda_k^h}}\Big(1+e^{-2\lambda_k^h}\Big)(\lambda_k^h)^j}{1+e^{2\sqrt{\lambda_k^h}}} \right| \le C|k|^{-1/2} e^{-\frac{1}{2}\sqrt{2|k|\pi}} |\lambda_k^h|^j \le C e^{-\sqrt{|k|}} \tag{4.10}$$

as $|k| \to +\infty$.

Now, by (4.7), (4.8), (4.9) and (4.10), using (2.29) and (2.30), we get

$$\begin{aligned}
&|y_x(\cdot,1)|^2_{L^2(0,T)} + |y_x^h(\cdot,1)|^2_{H^s(0,T)} \\
&\le C\left[\int_0^T \left| \sum_{\ell=\tilde{\ell}_1}^\infty \frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\Big(1+e^{-2\lambda_\ell^p}\Big)a_\ell}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)\Big(1+e^{2\sqrt{\lambda_\ell^p}}\Big)} e^{\lambda_\ell^p t} \right|^2 dt \right.\\
&\quad \left. + \sum_{j=0}^s \int_0^T \left| \sum_{|k|=\tilde{k}_1}^\infty \frac{e^{\sqrt{\lambda_k^h}}\Big(1+e^{-2\lambda_k^h}\Big) b_k (\lambda_k^h)^j}{i\Big(1+e^{2\sqrt{\lambda_k^h}}\Big)} e^{\lambda_k^h t} \right|^2 dt \right]\\
&\le C\left[\sum_{\ell=\tilde{\ell}_1}^\infty |a_\ell|^2 \sum_{\ell=\tilde{\ell}_1}^\infty \int_0^T e^{2t\mathrm{Re}\,\lambda_\ell^p} dt + \sum_{|k|=\tilde{k}_1}^\infty |b_k|^2 \sum_{|k|=\tilde{k}_1}^\infty \int_0^T e^{-2\sqrt{|k|}} e^{2t\mathrm{Re}\,\lambda_k^h} dt \right]\\
&\le C\left[\sum_{\ell=\tilde{\ell}_1}^\infty |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^\infty |b_k|^2 \right] \le C|(y_0,z_0,z_1)|^2_H,
\end{aligned}$$

which gives (4.1) and (4.2) immediately.

Estimate (4.3) can be proved in a similar way and we omit the details. This completes the proof of Theorem 4.1. $\square$

4.2 Negative observability and controllability results

Now, let us show the following negative result on the observability for system (1.1) in the natural energy space H.

Theorem 4.2 *Let $T > 0$ and $s \ge 0$. Then*

$$\sup_{(y_0,z_0,z_1)\in H\setminus\{0\}} \frac{\Big|(y(T),z(T),z_t(T))\Big|_H}{|y_x(\cdot,1)|_{H^s(0,T)}} = +\infty, \tag{4.11}$$

where (y,z,z_t) is the solution of system (1.1) with initial data (y_0,z_0,z_1), and we agree that $|y_x(\cdot,1)|_{H^s(0,T)} = +\infty$ whenever $y_x(\cdot,1) \notin H^s(0,T)$.

Remark 4.2 *Theorem 4.2 implies the lack of boundary observability from the heat component with a defect of infinite order (see (4.11)). The reason for this is that the hyperbolic eigenvectors of the underlying semigroup of system (1.1) are very weakly dissipated and very much concentrated in the wave interval* $(-1,0)$.

Proof of Theorem 4.2. It suffices to find a sequence of initial data $\{(y_0^k, z_0^k, z_1^k)\}_{k=1}^{+\infty} \subset H\setminus\{0\}$ such that for any fixed $s\in\mathbb{N}$, the corresponding solutions (y^k, z^k, z_t^k) of system (1.1) satisfies $y_x^k(\cdot,1)\in H^s(0,T)$ and

$$\lim_{k\to+\infty}\frac{\Big|(y^k(T), z^k(T), z_t^k(T))\Big|_H}{|y_x^k(\cdot,1)|_{H^s(0,T)}} = +\infty. \tag{4.12}$$

For this purpose, we choose the hyperbolic eigenvectors (for $k>0$ large enough)

$$(y_0^k, z_0^k, z_1^k) = (p_k^h, q_k^h, r_k^h), \tag{4.13}$$

where (p_k^h, q_k^h, r_k^h), associated with λ_k^h, is defined by (2.47). Then, by (4.2) in Theorem 4.1, we see that the corresponding solutions (y^k, z^k, z_t^k) of system (1.1) satisfies $y_x^k(\cdot,1)\in H^s(0,T)$. On the other hand, by (2.100), it is easy to see that

$$(y^k(t), z^k(t), z_t^k(t)) = e^{\lambda_k^h t}(p_k^h, q_k^h, r_k^h), \qquad \forall\, t\geq 0. \tag{4.14}$$

According to Theorem 2.1, $\{(p_k^h, q_k^h, r_k^h)\}_{k=1}^{\infty}$ is a subsequence of a Riesz basis in H. Thus there is a constant $c>0$, independent of k, such that

$$\Big|(y^k(T), z^k(T), z_t^k(T))\Big|_H = \Big|e^{\lambda_k^h T}(p_k^h, q_k^h, r_k^h)\Big|_H \geq c|e^{\lambda_k^h T}| = ce^{T\mathrm{Re}\,\lambda_k^h}. \tag{4.15}$$

On the other hand, by (4.6), we get

$$\begin{aligned} |y_x^k(\cdot,1)|^2_{H^s(0,T)} &= \sum_{j=0}^{s}\left|\frac{e^{\sqrt{\lambda_k^h}}(1+e^{-2\lambda_k^h})(\lambda_k^h)^j}{1+e^{2\sqrt{\lambda_k^h}}}\right|^2\int_0^T e^{2t\mathrm{Re}\,\lambda_k^h}dt \\ &\leq T\sum_{j=0}^{s}\left|\frac{e^{\sqrt{\lambda_k^h}}(1+e^{-2\lambda_k^h})(\lambda_k^h)^j}{1+e^{2\sqrt{\lambda_k^h}}}\right|^2. \end{aligned} \tag{4.16}$$

By (4.16) and (4.10), we conclude that there is a constant $C_s>0$ such that

$$|y_x^k(\cdot,1)|^2_{H^s(0,T)} \leq C_s e^{-2\sqrt{k}} \tag{4.17}$$

when k is sufficiently large.

Finally, by (4.15) and (4.17), and noting that $\{\mathrm{Re}\,\lambda_k^h\}_{|k|=\tilde{k}_1}^{\infty}$ is bounded, we conclude that there exists a constant $c_0>0$, independent of k, such that for k large enough, it holds

$$\frac{|(y^k(T), z^k(T), z_t^k(T))|_H}{|y_x^k(\cdot,1)|_{H^s(0,T)}} \geq c_0 e^{\sqrt{k}},$$

which yields (4.12). This completes the proof of Theorem 4.2. $\square$

According to the negative observability result in Theorem 4.2, one may expect the observability inequality to be true provided we put suitable exponentially large weights on the observed quantities. This is precisely the result we will show in the next subsection.

By means of the well-known duality relationship between controllability and observability, from Theorem 4.2, one concludes that system (1.7) is not null controllable in $\mathcal{H}$ with $L^2(0,T)$-controls at $x=1$ neither, with controls in $H^{-s}(0,T)$ for any given $s>0$.

4.3 Positive controllability and observability results

In this subsection, we will follow [7] to give a description on the controllability subspace of system (1.7) and state the corresponding observability result in terms of nonharmonic Fourier series.

We put (recall Theorem 2.1 for $u_{j,k}$ and so on)

$$V=\left\{\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}a_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell(p_\ell^p,q_\ell^p,r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}b_k(p_k^h,q_k^h,r_k^h)\ \middle|\ a_{j,k},a_\ell,b_k\in\mathbb{C},\quad \sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}|k|e^{\sqrt{2|k|\pi}}|b_k|^2<\infty\right\} \tag{4.18}$$

and

$$V'=\left\{\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}a_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell(p_\ell^p,q_\ell^p,r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}b_k(p_k^h,q_k^h,r_k^h)\ \middle|\ a_{j,k},a_\ell,b_k\in\mathbb{C},\quad \sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}\frac{|b_k|^2}{|k|e^{\sqrt{2|k|\pi}}}<\infty\right\}. \tag{4.19}$$

Then V and V' are Hilbert spaces respectively with the norms

$$\left|\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}a_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell(p_\ell^p,q_\ell^p,r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}b_k(p_k^h,q_k^h,r_k^h)\right|_V=\sqrt{\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}|k|e^{\sqrt{2|k|\pi}}|b_k|^2} \tag{4.20}$$

and

$$\left|\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}a_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell(p_\ell^p,q_\ell^p,r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}b_k(p_k^h,q_k^h,r_k^h)\right|_{V'}=\sqrt{\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}\frac{|b_k|^2}{|k|e^{\sqrt{2|k|\pi}}}}\ . \tag{4.21}$$

It is easy to show that $V \subset H \subset V'$ topologically and algebraically, and V' is the dual space of V with respect to the pivot space H.

Note that there is a certain asymmetry in the definition of V and V'. Indeed, the heat component is weighted as in H while the wave component is weighted exponentially. This is natural in view of the negative result we developed in the proof of Theorem 4.2. These spaces are those in which naturally the control problem (1.7) is well-posed. This is due to the different observability properties of the hyperbolic and parabolic eigenvectors. On the other hand, in the absence of control, system (1.7) is well-posed in those spaces due to the Riesz basis property shown in subsection 2.3.

Next, we put

$$\begin{aligned} W = \operatorname{span} \Big\{ & \Big\{ u_{j,k} \;\Big|\; k = 0, \cdots, m_j - 1; j = 1, \cdots, n_0 \Big\} \\ & \bigcup \Big\{ (p_\ell^p, q_\ell^p, r_\ell^p) \;\Big|\; \ell = \tilde{\ell}_1, \tilde{\ell}_1 + 1, \cdots \Big\} \bigcup \Big\{ (p_k^h, q_k^h, r_k^h) \;\Big|\; |k| = \tilde{k}_1, \tilde{k}_1 + 1, \cdots \Big\} \Big\}, \end{aligned} \tag{4.22}$$

i.e., the (linear)space spanned by the low-frequency eigenvectors (or generalized eigenvectors), the parabolic eigenvectors and the hyperbolic eigenvectors. Obviously, we have $W \subset V$ and W is dense in V, H and V'.

Also, we denote by A the operator in $L^2(-1,1)$:

$$\begin{cases} D(A) = H_0^1(-1,1) \cap H^2(-1,1), \\ Au = -u_{xx}, \quad \forall\, u \in D(A). \end{cases}$$

We introduce the map $\mathcal{S}: \mathcal{H} \to H$ as follows (recall (1.8) and (1.9) for $\mathcal{H}$):

$$\mathcal{S}(f,g,h) = \Big(A^{-1}(h,f), -g \Big), \qquad \forall\, (f,g,h) \in \mathcal{H}. \tag{4.23}$$

It is easy to show that $\mathcal{S}$ is an isometric isomorphism from $\mathcal{H}$ onto H.

We have the following null controllability result on system (1.7):

Theorem 4.3 *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in \mathcal{S}^{-1}V$, there exists a control $g_1 \in L^2(0,T)$ such that the solution (u, v, v_t) of system (1.7) satisfies $u(T) = 0$ in $(0,1)$ and $v(T) = v_t(T) = 0$ in $(-1,0)$.*

Remark 4.3 *By the transposition method, we know that for any $(u_0, v_0, v_1) \in \mathcal{H}$ and $g_1 \in L^2(0,T)$, system (1.7) admits a unique solution $(u, v, v_t) \in C([0,T]; \mathcal{H})$. However, even if $(u_0, v_0, v_1) \in \mathcal{S}^{-1}V$, the solution (u, v, v_t) of system (1.7) does not need to take values continuously in $\mathcal{S}^{-1}V$.*

Remark 4.4 *Theorem 4.3 guarantees the null-controllability of system (1.7) with initial data in space $\mathcal{S}^{-1}V$ in which the Fourier coefficients of the hyperbolic component of the solutions need to be, roughly speaking, exponentially small for high frequencies. This fact is not needed for the wave equation that is exactly controllable in $L^2(-1,0) \times H^{-1}(-1,0)$ with L^2-boundary control. This is neither needed for the heat equation which is null controllable, say, in*

$L^2(0,1)$. The need for taking the initial data to be controlled in $\mathcal{S}^{-1}V$ comes from the behavior of the hyperbolic eigenvalues and eigenvectors. Indeed, the fact that most of the energy of the hyperbolic eigenvectors is mainly concentrated in the subinterval where the wave equation holds produces they to be badly observable from the parabolic extreme $x = 1$. On the other hand, the fact that the real part of the hyperbolic eigenvalue vanishes asymptotically makes the corresponding eigenvectors to be very weakly dissipated. Consequently, the $L^2(0,T)$-control g_1 at $x = 1$ has a very weak effect on the hyperbolic high frequency components. These facts force the controllable data to be in $\mathcal{S}^{-1}V$.

The proof of Theorem 4.3 is given in subsection 4.6. By means of the well-known duality argument ([11]), in order to prove Theorem 4.3, we need to derive the following key observability estimate and regularity result.

Theorem 4.4 *The following two conclusions hold:*

(1) For any $T > 2$, there is a constant $C > 0$ such that every solution of equation (1.1) satisfies

$$|(y(T), z(T), z_t(T))|^2_{V'} \le C|y_x(\cdot, 1)|^2_{L^2(0,T)}, \qquad \forall\, (y_0, z_0, z_1) \in V'. \tag{4.24}$$

(2) For any $T > 0$, there is a constant $C > 0$ such that the solution of equation (1.1) satisfies

$$|y_x(\cdot, 1)|^2_{L^2(0,T)} \le C|(y_0, z_0, z_1)|^2_{V'}, \qquad \forall\, (y_0, z_0, z_1) \in V'. \tag{4.25}$$

The proof of Theorem 4.4 will be given in subsection 4.5. We will see that the observability estimate (4.24) in Theorem 4.4 is a consequence of Lemma 1.1.

Remark 4.5 *We recall that the proof of Lemma 1.1 (in [19]) is based on sidewise energy estimates for the $1 - d$ wave equation and Carleman estimates for the heat equation. As pointed in [19], the same argument does not apply to the proof of Theorem 4.4 because of the lack of sidewise energy estimates for the heat equation. Our proof of Theorem 4.4 is based on Lemma 4.1 in the next subsection, which is based on Lemma 1.1 and the spectral analysis we have developed. Therefore the proof of Theorem 4.4 involves (indirectly) Carleman estimates. It would be interesting to give a direct proof of Theorem 4.4 by means of Carleman estimates. But this is by now an open problem. In this respect, we would like to mention the paper [5], where the null controllability problem for the semilinear heat equation with discontinuous coefficients and interfaces is addressed by Carleman estimates under some monotonicity conditions on the coefficients at the interfaces.*

4.4 A new Ingham-type inequality

We observe that Lemma 1.1 implies the following new Ingham-type inequality, which will play a key role in the proof of Theorem 4.4.

Lemma 4.1 *Let $T > 2$. Then there is a constant $C = C(T) > 0$ such that*

$$
\begin{aligned}
&\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2e^{2(T-1)\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{\infty}|b_k|^2\\
&\le C\int_0^T\left|\sum_{j=1}^{n_0}e^{\mu_j t}\sum_{k=0}^{m_j-1}a_{j,k}t^k+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell e^{\lambda_\ell^p t}+\sum_{|k|=\tilde{k}_1}^{\infty}b_k e^{\lambda_k^h t}\right|^2dt
\end{aligned}
\tag{4.26}
$$

holds for all complex numbers $a_{j,k}$ ($k = 0, 1, \cdots, m_j - 1$; $j = 1, 2, \cdots, n_0$), and all sequences $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty}$ in $\mathbb{C}$ with

$$
\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}|b_k|^2<\infty. \tag{4.27}
$$

Remark 4.6 *The original Ingham inequality was given in [8]. It is concerned with the estimate on the series of exponentials via its coefficients. There exists a large literature on variants and extensions of this type of inequalities (see for example, [15], [1], [4], [9] and the references therein). However, as far as we know, none of the existing results on series of exponentials combining real and imaginary ones seem to be sufficient to yield inequality (4.26). On the other hand, the proof of (4.26) we will develop below is not based on techniques of the theory of series of exponentials but rather on the observability inequality (1.11) on the observation of the system from the wave extreme $x = -1$. Thus, the Ingham-type inequality (4.27) is derived as a consequence of inequalities obtained by PDE techniques and the spectral analysis in Section 2.*

Proof of Lemma 4.1. For any given complex numbers $\tilde{a}_{j,k}$ ($k = 0, 1, \cdots, m_j - 1$; $j = 1, 2, \cdots, n_0$), and any given sequences $\{\tilde{a}_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{\tilde{b}_k\}_{|k|=\tilde{k}_1}^{\infty}$ in $\mathbb{C}$ satisfying

$$
\sum_{\ell=\tilde{\ell}_1}^{\infty}|\tilde{a}_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}|\tilde{b}_k|^2<\infty,
$$

we put

$$
(y_0, z_0, z_1) \triangleq \sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}\tilde{a}_\ell(p_\ell^p, q_\ell^p, r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}\tilde{b}_k(p_k^h, q_k^h, r_k^h)(\in H).
$$

By Corollary 2.1, the corresponding solution of equation (1.1) with this initial datum is given by (2.100).

Now, by (2.100), using the Riesz basis property shown in Theorem 2.1, and noting that $\{\mathrm{Re}\,\mu_j\}_{j=1}^{n_0}$ and $\{\mathrm{Re}\,\lambda_k^h\}_{|k|=\tilde{k}_1}^{\infty}$ are bounded, it is easy to conclude that there is a constant

$C > 0$ such that

$$
\begin{aligned}
&|(y(T), z(T), z_t(T))|_H^2 \\
&= \left| \sum_{j=1}^{n_0} e^{\mu_j T} \sum_{k=0}^{m_j-1} \tilde{a}_{j,k} \sum_{s=0}^{k} \frac{T^{k-s}}{(k-s)!} u_{j,s} + \sum_{\ell=\tilde{\ell}_1}^{\infty} \tilde{a}_\ell e^{\lambda_\ell^p T} (p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} \tilde{b}_k e^{\lambda_k^h T} (p_k^h, q_k^h, r_k^h) \right|_H^2 \\
&= \left| \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} e^{\mu_j T} \sum_{s=k}^{m_j-1} \frac{\tilde{a}_{j,s} T^{s-k}}{(s-k)!} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} \tilde{a}_\ell e^{\lambda_\ell^p T} (p_\ell^p, q_\ell^p, r_\ell^p) + \sum_{|k|=\tilde{k}_1}^{\infty} \tilde{b}_k e^{\lambda_k^h T} (p_k^h, q_k^h, r_k^h) \right|_H^2 \\
&\geq C \left[\sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} e^{2T \mathrm{Re}\, \mu_j} \left| \sum_{s=k}^{m_j-1} \frac{\tilde{a}_{j,s} T^{s-k}}{(s-k)!} \right|^2 + \sum_{\ell=\tilde{\ell}_1}^{\infty} |\tilde{a}_\ell|^2 e^{2T \mathrm{Re}\, \lambda_\ell^p} + \sum_{|k|=\tilde{k}_1}^{\infty} |\tilde{b}_k|^2 e^{2T \mathrm{Re}\, \lambda_k^h} \right] \\
&\geq C \left[\sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} \left| \sum_{s=k}^{m_j-1} \frac{\tilde{a}_{j,s} T^{s-k}}{(s-k)!} \right|^2 + \sum_{\ell=\tilde{\ell}_1}^{\infty} |\tilde{a}_\ell|^2 e^{2T \mathrm{Re}\, \lambda_\ell^p} + \sum_{|k|=\tilde{k}_1}^{\infty} |\tilde{b}_k|^2 \right].
\end{aligned} \tag{4.28}
$$

By the compactness of the unit ball in the finite-dimensional space, we deduce that for any $j = 1, 2, \cdots, n_0$, there is a constant $C = C(m_j, T) > 0$, independent of $\tilde{a}_{j,k}$, such that

$$
\sum_{k=0}^{m_j-1} \left| \sum_{s=k}^{m_j-1} \frac{\tilde{a}_{j,s} T^{s-k}}{(s-k)!} \right|^2 \geq C \sum_{k=0}^{m_j-1} |\tilde{a}_{j,k}|^2. \tag{4.29}
$$

Combining (4.28) and (4.29), we arrive at

$$
|(y(T), z(T), z_t(T))|_H^2 \geq C \left[\sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} |\tilde{a}_{j,k}|^2 + \sum_{\ell=\tilde{\ell}_1}^{\infty} |\tilde{a}_\ell|^2 e^{2T \mathrm{Re}\, \lambda_\ell^p} + \sum_{|k|=\tilde{k}_1}^{\infty} |\tilde{b}_k|^2 \right]. \tag{4.30}
$$

On the other hand, from (2.100) and (2.101), we see that

$$
z(t,x) = \sum_{j=1}^{n_0} e^{\mu_j t} \sum_{k=0}^{m_j-1} \tilde{a}_{j,k} \sum_{s=0}^{k} \frac{t^{k-s}}{(k-s)!} q_{j,s}(x) + \sum_{\ell=\tilde{\ell}_1}^{\infty} \tilde{a}_\ell e^{\lambda_\ell^p t} q_\ell^p(x) + \sum_{|k|=\tilde{k}_1}^{\infty} \tilde{b}_k e^{\lambda_k^h t} q_k^h(x),
$$

where $(t,x) \in (0,T) \times (-1,0)$. Therefore

$$
\begin{aligned}
|z_x(\cdot, -1)|_{L^2(0,T)}^2 = \int_0^T & \left| \sum_{j=1}^{n_0} e^{\mu_j t} \sum_{k=0}^{m_j-1} \tilde{a}_{j,k} \sum_{s=0}^{k} \frac{t^{k-s}}{(k-s)!} \partial_x q_{j,s}(-1) \right. \\
& \left. + \sum_{\ell=\tilde{\ell}_1}^{\infty} \tilde{a}_\ell e^{\lambda_\ell^p t} \partial_x q_\ell^p(-1) + \sum_{|k|=\tilde{k}_1}^{\infty} \tilde{b}_k e^{\lambda_k^h t} \partial_x q_k^h(-1) \right|^2 dt.
\end{aligned} \tag{4.31}
$$

Now, by (2.43), (2.47) and (2.7), it is easy to check that

$$
\partial_x q_\ell^p(x) = -\frac{\lambda_\ell^p \left(e^{\lambda_\ell^p x} + e^{-\lambda_\ell^p (x+2)} \right)}{\ell^2 \pi^2 \left(1 + e^{2\ell^2 \pi^2} \right)}, \qquad \partial_x q_k^h(x) = \frac{e^{\lambda_k^h x} + e^{-\lambda_k^h (x+2)}}{2} i.
$$

Therefore,

$$\partial_x q_\ell^p(-1) = -\frac{2\lambda_\ell^p e^{-\lambda_\ell^p}}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)}, \qquad \partial_x q_k^h(-1) = ie^{-\lambda_k^h}. \tag{4.32}$$

Also, we have

$$\begin{aligned}
&\sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\tilde a_{j,k}\sum_{s=0}^{k}\frac{t^{k-s}}{(k-s)!}\partial_x q_{j,s}(-1) = \sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\tilde a_{j,k}\sum_{s=0}^{k}\frac{t^s}{s!}\partial_x q_{j,k-s}(-1)\\
&= \sum_{j=1}^{n_0} e^{\mu_j t}\sum_{s=0}^{m_j-1}\left(\frac{1}{s!}\sum_{k=s}^{m_j-1}\tilde a_{j,k}\partial_x q_{j,k-s}(-1)\right)t^s \\
&= \sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\left(\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde a_{j,s}\partial_x q_{j,s-k}(-1)\right)t^k.
\end{aligned} \tag{4.33}$$

Combining (4.31), (4.32) and (4.33), we get

$$\begin{aligned}
|z_x(\cdot,-1)|^2_{L^2(0,T)} = \int_0^T\Bigg|&\sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\left(\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde a_{j,s}\partial_x q_{j,s-k}(-1)\right)t^k \\
&-\sum_{\ell=\tilde\ell_1}^{\infty}\frac{2\tilde a_\ell\lambda_\ell^p e^{-\lambda_\ell^p}}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}e^{\lambda_\ell^p t} + i\sum_{|k|=\tilde k_1}^{\infty}\tilde b_k e^{-\lambda_k^h}e^{\lambda_k^h t}\Bigg|^2 dt.
\end{aligned} \tag{4.34}$$

Now, in view of (4.30), (4.34) and (1.11), we conclude that

$$\begin{aligned}
&\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|\tilde a_{j,k}|^2 + \sum_{\ell=\tilde\ell_1}^{\infty}|\tilde a_\ell|^2 e^{2T\,\mathrm{Re}\,\lambda_\ell^p} + \sum_{|k|=\tilde k_1}^{\infty}|\tilde b_k|^2 \\
&\le C\int_0^T\Bigg|\sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\left(\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde a_{j,s}\partial_x q_{j,s-k}(-1)\right)t^k \\
&\quad -\sum_{\ell=\tilde\ell_1}^{\infty}\frac{2\tilde a_\ell\lambda_\ell^p e^{-\lambda_\ell^p}}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}e^{\lambda_\ell^p t} + i\sum_{|k|=\tilde k_1}^{\infty}\tilde b_k e^{-\lambda_k^h}e^{\lambda_k^h t}\Bigg|^2 dt.
\end{aligned} \tag{4.35}$$

Now, for any fixed integer $N \ge \max(\tilde\ell_1, \tilde k_1)$ and any given sequences $\{a_\ell\}_{\ell=\tilde\ell_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde k_1}^{\infty}$ satisfying (4.27), we choose sequences $\{\tilde a_\ell\}_{\ell=\tilde\ell_1}^{\infty}$ and $\{\tilde b_k\}_{|k|=\tilde k_1}^{\infty}$ as follows

$$\begin{aligned}
\tilde a_\ell &= \begin{cases} -\dfrac{\ell^2\pi^2(1+e^{2\ell^2\pi^2})}{2\lambda_\ell^p e^{-\lambda_\ell^p}}a_\ell, & \ell = \tilde\ell_1, \tilde\ell_1+1,\cdots,N,\\[2mm] 0, & \ell \ge N+1,\end{cases}\\
\tilde b_k &= \begin{cases} -ie^{\lambda_k^h}b_k, & |k| = \tilde k_1, \tilde k_1+1,\cdots,N,\\ 0, & |k|\ge N+1.\end{cases}
\end{aligned} \tag{4.36}$$

Further, we claim that for any complex numbers $a_{j,k}$, the following finite dimensional systems of linear equations admits one and only one solution $\tilde{a}_{j,k}$:

$$\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde{a}_{j,s}\partial_x q_{j,s-k}(-1)=a_{j,k},\qquad k=0,1,\cdots,m_j-1;\ j=1,2,\cdots,n_0. \tag{4.37}$$

In fact, for any fixed $j=1,2,\cdots,n_0$, (4.37) can be re-written as follows:

$$\begin{cases}\dfrac{1}{(m_j-1)!}\tilde{a}_{j,m_j-1}\partial_x q_{j,0}(-1)=a_{j,m_j-1},\\ \dfrac{1}{(m_j-2)!}\Big(\tilde{a}_{j,m_j-1}\partial_x q_{j,1}(-1)+\tilde{a}_{j,m_j-2}\partial_x q_{j,0}(-1)\Big)=a_{j,m_j-2},\\ \cdots,\\ \tilde{a}_{j,m_j-1}\partial_x q_{j,m_j-1}(-1)+\tilde{a}_{j,m_j-2}\partial_x q_{j,m_j-2}(-1)+\cdots+\tilde{a}_{j,0}\partial_x q_{j,0}(-1)=a_{j,0}.\end{cases} \tag{4.38}$$

By Remark 2.5, and using (2.6) and (2.7), it is easy to deduce that

$$\partial_x q_{j,0}(-1)\neq 0. \tag{4.39}$$

By (4.39), we see that system (4.38) admits a unique solution $(\tilde{a}_{j,m_j-1},\tilde{a}_{j,m_j-2},\cdots,\tilde{a}_{j,0})$. Also, by (4.37), one concludes that there is a constant $C>0$, independent of $a_{j,k}$ and $\tilde{a}_{j,k}$, such that

$$\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2\le C\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|\tilde{a}_{j,k}|^2. \tag{4.40}$$

In view of (4.35), (4.36), (4.37) and (4.40), we get

$$\begin{aligned}&\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{N}\left|\frac{\ell^2(1+e^{2\ell^2\pi^2})}{\lambda_\ell^p e^{-\lambda_\ell^p}}a_\ell\right|^2 e^{2T\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{N}|e^{\lambda_k^h}b_k|^2\\ &\le C\int_0^T\left|\sum_{j=1}^{n_0}e^{\mu_j t}\sum_{k=0}^{m_j-1}a_{j,k}t^k+\sum_{\ell=\tilde{\ell}_1}^{N}a_\ell e^{\lambda_\ell^p t}+\sum_{|k|=\tilde{k}_1}^{N}b_k e^{\lambda_k^h t}\right|^2dt,\end{aligned} \tag{4.41}$$

where $C>0$ is a constant, independent of N.

From (2.29), we conclude that there exists a constant $c_1>0$ such that

$$\left|\frac{\ell^2(1+e^{2\ell^2\pi^2})}{\lambda_\ell^p e^{-\lambda_\ell^p}}\right|^2\ge c_1 e^{-2\mathrm{Re}\,\lambda_\ell^p},\qquad\forall\,\ell\in\mathbb{N}. \tag{4.42}$$

Similarly, by (2.34) in Proposition 2.2 and using (2.10), we conclude that there exists a constant $c_2>0$ such that

$$|e^{\lambda_k^h}|^2\ge c_2,\qquad\forall\,|k|=\tilde{k}_1,\tilde{k}_1+1,\cdots. \tag{4.43}$$

Now, combining (4.41), (4.42) and (4.43), we arrive at

$$\begin{aligned}&\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{N}|a_\ell|^2e^{2(T-1)\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{N}|b_k|^2\\&\leq C\int_0^T\left|\sum_{j=1}^{n_0}e^{\mu_jt}\sum_{k=0}^{m_j-1}a_{j,k}t^k+\sum_{\ell=\tilde{\ell}_1}^{N}a_\ell e^{\lambda_\ell^pt}+\sum_{|k|=\tilde{k}_1}^{N}b_ke^{\lambda_k^ht}\right|^2dt,\end{aligned}\tag{4.44}$$

where $C>0$ is a constant, independent of N. Finally, taking $N\to\infty$ in (4.44), we get (4.26) immediately. □

4.5 Proof of the observability result

In this subsection, we will give the proof of the observability result, Theorem 4.4.

For any $(y_0,z_0,z_1)\in V'$, one can find $\{\tilde{a}_{j,0},\cdots,\tilde{a}_{j,m_j-1}\}_{j=1}^{n_0}\subset\mathbb{C}$, and two sequences $\{\tilde{a}_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{\tilde{b}_k\}_{|k|=\tilde{k}_1}^{\infty}$ in $\mathbb{C}$ with

$$\sum_{\ell=\tilde{\ell}_1}^{\infty}|\tilde{a}_\ell|^2+\sum_{|k|=\tilde{k}_1}^{\infty}\frac{|\tilde{b}_k|^2}{|k|e^{\sqrt{2|k|\pi}}}<\infty\tag{4.45}$$

such that

$$(y_0,z_0,z_1)=\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}u_{j,k}+\sum_{\ell=\tilde{\ell}_1}^{\infty}\tilde{a}_\ell(p_\ell^p,q_\ell^p,r_\ell^p)+\sum_{|k|=\tilde{k}_1}^{\infty}\tilde{b}_k(p_k^h,q_k^h,r_k^h).$$

Therefore, the corresponding solution of equation (1.1) with this datum is given by

$$\begin{aligned}(y(t),z(t),z_t(t))=&\sum_{j=1}^{n_0}e^{\mu_jt}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}\sum_{s=0}^{k}\frac{t^{k-s}}{(k-s)!}u_{j,s}\\&+\sum_{\ell=\tilde{\ell}_1}^{\infty}\tilde{a}_\ell(p_\ell^p,q_\ell^p,r_\ell^p)e^{\lambda_\ell^pt}+\sum_{|k|=\tilde{k}_1}^{\infty}\tilde{b}_k(p_k^h,q_k^h,r_k^h)e^{\lambda_k^ht}.\end{aligned}\tag{4.46}$$

It is easy to see that $(y(t),z(t),z_t(t))\in V'$ for all $t\in[0,T]$. Therefore,

$$\begin{aligned}&|(y(T),z(T),z_t(T))|_{V'}^2\\&\leq C\left(\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|\tilde{a}_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|\tilde{a}_\ell|^2e^{2T\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{\infty}\frac{|\tilde{b}_k|^2}{|k|e^{\sqrt{2|k|\pi}}}\right).\end{aligned}\tag{4.47}$$

On the other hand, from (4.46) and (2.101), we see that

$$y(t,x)=\sum_{j=1}^{n_0}e^{\mu_jt}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}\sum_{s=0}^{k}\frac{t^{k-s}}{(k-s)!}p_{j,s}(x)+\sum_{\ell=\tilde{\ell}_1}^{\infty}\tilde{a}_\ell e^{\lambda_\ell^pt}p_\ell^p(x)+\sum_{|k|=\tilde{k}_1}^{\infty}\tilde{b}_ke^{\lambda_k^ht}p_k^h(x),$$

where $(t,x)\in(0,T)\times(0,1)$. Therefore, in view of (4.6), we get

$$
\begin{aligned}
|y_x(\cdot,1)|^2_{L^2(0,T)} = \int_0^T \Bigg| &\sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}\sum_{s=0}^{k}\frac{t^{k-s}}{(k-s)!}\partial_x p_{j,s}(1) \\
&+\sum_{\ell=\tilde{\ell}_1}^{\infty}\frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\Big(1+e^{-2\lambda_\ell^p}\Big)\tilde{a}_\ell}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)\Big(1+e^{2\sqrt{\lambda_\ell^p}}\Big)}e^{\lambda_\ell^p t} \\
&+\sum_{|k|=\tilde{k}_1}^{\infty}\frac{e^{\sqrt{\lambda_k^h}}\Big(1+e^{-2\lambda_k^h}\Big)\tilde{b}_k}{i\Big(1+e^{2\sqrt{\lambda_k^h}}\Big)}e^{\lambda_k^h t}\Bigg|^2 dt.
\end{aligned}
\tag{4.48}
$$

However, by (2.33) in Proposition 2.2 and (2.9), we see that there is a constant $C>0$ such that

$$
\frac{1}{C}\le |e^{\sqrt{\lambda_\ell^p}x}|\le C,\qquad \forall\, x\in[-2,2] \text{ and } \ell=\tilde{\ell}_1,\tilde{\ell}_1+1,\cdots.
$$

This fact, combined with (2.46), (2.9) and Remark 2.3, implies that

$$
\frac{1}{C}\le\left|\frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\Big(1+e^{-2\lambda_\ell^p}\Big)}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)\Big(1+e^{2\sqrt{\lambda_\ell^p}}\Big)}\right|\le C,\qquad \forall\,\ell\in\mathbb{N}
\tag{4.49}
$$

for some constant $C>0$.

On the other hand, arguing as in (4.33), we have

$$
\sum_{j=1}^{n_0} e^{\mu_j t}\sum_{k=0}^{m_j-1}\tilde{a}_{j,k}\sum_{s=0}^{k}\frac{t^{k-s}}{(k-s)!}\partial_x p_{j,s}(1)=\sum_{j=1}^{n_0}e^{\mu_j t}\sum_{k=0}^{m_j-1}\left(\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde{a}_{j,s}\partial_x p_{j,s-k}(1)\right)t^k.
\tag{4.50}
$$

We put

$$
\begin{aligned}
a_{j,k}&=\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde{a}_{j,s}\partial_x p_{j,s-k}(1),\qquad k=0,1,\cdots,m_j-1;\ j=1,2,\cdots,n_0,\\
a_\ell&\triangleq\frac{2\lambda_\ell^p e^{\sqrt{\lambda_\ell^p}}\Big(1+e^{-2\lambda_\ell^p}\Big)\tilde{a}_\ell}{\ell^2\pi^2\Big(1+e^{2\ell^2\pi^2}\Big)\Big(1+e^{2\sqrt{\lambda_\ell^p}}\Big)},\qquad \ell=\tilde{\ell}_1,\tilde{\ell}_1+1,\cdots\\
b_k&=\frac{e^{\sqrt{\lambda_k^h}}\Big(1+e^{-2\lambda_k^h}\Big)\tilde{b}_k}{i\Big(1+e^{2\sqrt{\lambda_k^h}}\Big)},\qquad k=\pm\tilde{k}_1,\pm(\tilde{k}_1+1),\cdots.
\end{aligned}
\tag{4.51}
$$

Then, by (4.45) and (4.49), and using Proposition 4.1, we see that $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty}$ satisfy (4.27). Hence, using Lemma 4.1, we get

$$
\begin{aligned}
&\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2 e^{2T\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{\infty}|b_k|^2\\
&\le C\int_0^T\left|\sum_{j=1}^{n_0}e^{\mu_j t}\sum_{k=0}^{m_j-1}a_{j,k}t^k+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell e^{\lambda_\ell^p t}+\sum_{|k|=\tilde{k}_1}^{\infty}b_k e^{\lambda_k^h t}\right|^2 dt.
\end{aligned}
\tag{4.52}
$$

Now, similar to the analysis of equations (4.37), from the equations

$$\frac{1}{k!}\sum_{s=k}^{m_j-1}\tilde{a}_{j,s}\partial_x p_{j,s-k}(1)=a_{j,k},\qquad k=0,1,\cdots,m_j-1;\ j=1,2,\cdots,n_0,$$

one deduces that for any fixed j, every $\tilde{a}_{j,s}$ can be expressed as a linear combination of $a_{j,0}$, $a_{j,1}$, $\cdots$, and a_{j,m_j-1}, the combination coefficients depending only on j, m_j and the generalized eigenvectors. Therefore, we conclude that there is a constant $C>0$ such that

$$\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|\tilde{a}_{j,k}|^2\le C\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2. \tag{4.53}$$

By (4.47), noting (4.49), (4.51) and (4.53), and using Proposition 4.1, we have

$$|(y(T),z(T),z_t(T))|_{V'}^2\le C\left[\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2e^{2T\mathrm{Re}\,\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{\infty}|b_k|^2\right]. \tag{4.54}$$

Now, combining (4.48), (4.50), (4.51), (4.52) and (4.54), we obtain the desired result (4.24).

Finally, by (4.48) and a direct computation, one gets (4.25) easily. This completes the proof of Theorem 4.4. □

4.6 Proof of the controllability result

This subsection is devoted to proving the controllability result, Theorem 4.3. The proof is almost standard. However, for the sake of completeness, we will give the details.

For any $(u_0,v_0,v_1)\in\mathcal{S}^{-1}V$, our aim is to find a control $g_1\in L^2(0,T)$ such that the solution of system (1.7) satisfies

$$u(T)=0\quad\text{in }(0,1)\qquad\text{and } v(T)=v_t(T)=0\quad\text{in }(-1,0). \tag{4.55}$$

For this purpose, we use the duality argument.

Thanks to Theorem 4.4, we see that, for any $(\phi_0,\psi_0,\psi_1)\in V'$, the following system

$$\begin{cases}-\phi_t-\phi_{xx}=0 & \text{in }(0,T)\times(0,1),\\ \psi_{tt}-\psi_{xx}=0 & \text{in }(0,T)\times(-1,0),\\ \phi(t,1)=0 & t\in(0,T),\\ \psi(t,-1)=0 & t\in(0,T),\\ \phi(t,0)=\psi(t,0),\quad \phi_x(t,0)=\psi_x(t,0) & t\in(0,T),\\ \phi(T)=\phi_0 & \text{in }(0,1),\\ \psi(T)=\psi_0,\quad \psi_t(T)=\psi_1 & \text{in }(-1,0)\end{cases} \tag{4.56}$$

admits a unique solution $(\phi,\psi,\psi_t)\in C([0,T];V')$.

Let us introduce the following linear subspace of $L^2(0,T)$ (recall (4.22) for W):

$$\mathcal{Y} \triangleq \Big\{\phi_x(\cdot,1) \;\Big|\; (\phi,\psi) \text{ solves system (4.56) with } (\phi_0,\psi_0,\psi_1) \in W\Big\} \tag{4.57}$$

and define a linear functional on $\mathcal{Y}$ as follows:

$$\mathcal{L}(\phi_x(\cdot,1)) = ((\phi(0),\psi(0),\psi_t(0)),\mathcal{S}(u_0,v_0,v_1))_H. \tag{4.58}$$

By means of the first conclusion in Theorem 4.4, from (4.58), we see that

$$|\mathcal{L}(\phi_x(\cdot,1))| \le |\mathcal{S}(u_0,v_0,v_1)|_V |(\phi(0),\psi(0),\psi_t(0))|_{V'} \le C|\mathcal{S}(u_0,v_0,v_1)|_V |\phi_x(\cdot,1)|_{L^2(0,T)}.$$

Hence, $\mathcal{L}$ defined above is a bounded linear functional on $\mathcal{Y}$. By Hahn-Banach Theorem, $\mathcal{L}$ can be extended to a bounded linear functional on $L^2(0,T)$. For simplicity, we use the same notion for the extension. Now, Riesz Representation Theorem allows us to find a function $\eta \in L^2(0,T)$ such that

$$\int_0^T \phi_x(t,1)\eta(t)dt = ((\phi(0),\psi(0),\psi_t(0)),\mathcal{S}(u_0,v_0,v_1))_H. \tag{4.59}$$

We claim that

$$g_1(t) = -\eta(t) \tag{4.60}$$

is exactly the control we need.

In fact, for any $(\phi_0,\psi_0,\psi_1) \in W$ and any smooth (y_0,z_0,z_1) and g_1, by equations (1.7) and (4.56), using integration by parts, and recalling the definition of $\mathcal{S}$ in (4.23), we get

$$\begin{aligned}
0 &= \int_0^T \int_{-1}^0 \psi(v_{tt}-v_{xx})dxdt + \int_0^T \int_0^1 \phi(u_t-u_{xx})dxdt \\
&= \int_0^T \phi_x(t,1)g_1(t)dt + \int_{-1}^0 \psi(T,x)v_t(T,x)dx - \int_{-1}^0 \psi(0,x)v_1(x)dx \\
&\quad - \int_{-1}^0 \psi_t(T,x)v(T,x)dx + \int_{-1}^0 \psi_t(0,x)v_0(x)dx \\
&\quad + \int_0^1 \phi(T,x)u(T,x)dx - \int_0^1 \phi(0,x)u_0(x)dx \\
&= \int_0^T \phi_x(t,1)g_1(t)dt \\
&\quad + \Big((\psi(T),\phi(T)),(v_t(T),u(T))\Big)_{L^2(-1,1)} + (\psi_t(T),-v(T))_{L^2(-1,0)} \\
&\quad - \Big((\psi(0),\phi(0)),(v_1,u_0)\Big)_{L^2(-1,1)} - (\psi_t(0),-v_0)_{L^2(-1,0)} \\
&= \int_0^T \phi_x(t,1)g_1(t)dt + ((\phi_0,\psi_0,\psi_1),\mathcal{S}(u(T),v(T),v_t(T)))_H \\
&\quad - ((\phi(0),\psi(0),\psi_t(0)),\mathcal{S}(u_0,v_0,v_1))_H.
\end{aligned} \tag{4.61}$$

Hence, by (4.59), (4.60) and (4.61), using the density argument, we conclude that

$$((\phi_0,\psi_0,\psi_1),\mathcal{S}(u(T),v(T),v_t(T)))_H = 0, \qquad \forall\ (\phi_0,\psi_0,\psi_1) \in W,$$

which implies (4.55) immediately. This completes the proof of Theorem 4.3. $\square$

5 Appendix A: Proof of Lemma 2.1

This appendix is devoted to prove Lemma 2.1.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and $(p, q, r) \in D(\mathcal{A}) \setminus \{0\}$ be the corresponding eigenvector. Then it is easy to check that

$$\begin{cases} p_{xx} = \lambda p, & \text{in } (0,1), \\ q_{xx} = \lambda^2 q, & \text{in } (-1,0), \\ r = \lambda q, & \text{in } (-1,0), \\ p(1) = q(-1) = 0, & \\ p(0) = q(0), \quad p_x(0) = q_x(0). & \end{cases} \tag{5.1}$$

By the first two equations in (5.1), we conclude that there exist four constants α, β, γ and δ such that

$$\begin{aligned} q(x,\lambda) &= \alpha e^{\lambda x} + \beta e^{-\lambda x}, \qquad && x \in (-1,0), \\ p(x,\lambda) &= \gamma e^{\sqrt{\lambda} x} + \delta e^{-\sqrt{\lambda} x}, && x \in (0,1). \end{aligned} \tag{5.2}$$

By the last two equations in (5.1), we get

$$\begin{cases} \alpha e^{-\lambda} + \beta e^{\lambda} = 0, \\ \gamma e^{\sqrt{\lambda}} + \delta e^{-\sqrt{\lambda}} = 0, \\ \alpha + \beta = \gamma + \delta, \\ (\alpha - \beta)\sqrt{\lambda} = \gamma - \delta. \end{cases} \tag{5.3}$$

We claim that $\lambda \neq 0$. Otherwise, assume $\lambda = 0$. Then, by the first two equations in (5.3), we get $\alpha + \beta = 0$ and $\gamma + \delta = 0$. Thus, by (5.2) and the third equation in (5.1), it is easy to see that $(p, q, r) = 0$. This is a contradiction.

Also, we claim that $\alpha \neq 0$. Otherwise, assume $\alpha = 0$. Then, by the first equation in (5.3), we get $\beta = 0$. Thus, from the last two equations in (5.3), one sees that $\gamma + \delta = 0$ and $\gamma - \delta = 0$, which implies that $\gamma = \delta = 0$. Therefore, by (5.2) and the third equation in (5.1), we get $(p, q, r) = 0$. This is a contradiction. Therefore, in the sequel, without loss of generality, we may choose $\alpha = 1$. Thus, by the first equation in (5.3), we get $\beta = -e^{-2\lambda}$.

Let us show that (2.5) holds. By the second and the fourth equation in (5.3), we have

$$\gamma\Big(1 + e^{2\sqrt{\lambda}}\Big) = \sqrt{\lambda}(1 + e^{-2\lambda}). \tag{5.4}$$

Now, if (2.5) does not hold, i.e.

$$1 + e^{2\sqrt{\lambda}} = 0, \tag{5.5}$$

by (5.4), one gets

$$1 + e^{-2\lambda} = 0. \tag{5.6}$$

Obviously, (5.5) and (5.6) can not hold simultaneously. Thus, (2.5) holds.

Now, by (5.4) and the second equation in (5.3), we get

$$\gamma = \frac{\sqrt{\lambda}(1 + e^{-2\lambda})}{1 + e^{2\sqrt{\lambda}}}, \qquad \delta = -\frac{\sqrt{\lambda}(1 + e^{-2\lambda})e^{2\sqrt{\lambda}}}{1 + e^{2\sqrt{\lambda}}}. \tag{5.7}$$

Combining (5.2), (5.7) and the third equation in (5.1), we see that (p, q, r) is given by (2.7).

On the other hand, by (5.7) and the third equation in (5.3), we get

$$1 - e^{-2\lambda} = \frac{\sqrt{\lambda}(1 + e^{-2\lambda})}{1 + e^{2\sqrt{\lambda}}}\left(1 - e^{2\sqrt{\lambda}}\right),$$

from which, we conclude that

$$\kappa(\lambda) = 0, \tag{5.8}$$

where $\kappa(\lambda)$ is as in (2.4). This means that every eigenvalue λ of $\mathcal{A}$ satisfies the algebraic equation (5.8).

It is easy to check that every non-zero root λ of $\kappa(\lambda)$ is an eigenvalue of $\mathcal{A}$, and $\mu(p, q, r)$ is a corresponding eigenvector, where (p, q, r) is given by (2.7), μ is any non-zero complex number. Thus, we have shown that $\sigma_p(\mathcal{A}) = \{\lambda \neq 0 \mid \kappa(\lambda) = 0\}$.

Finally, let us show that $\sigma_p(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$. Assume this is not correct. Then one can find a $\mu_0 \in \sigma_p(\mathcal{A})$ with $\operatorname{Re}\mu_0 \geq 0$. Let $(p_0, q_0, r_0) \neq 0$ be the corresponding eigenvector of $\mathcal{A}$. Then the solution of (1.1) with initial data $(y_0, z_0, z_1) = (p_0, q_0, r_0)$ is given by

$$(y(t), z(t), z_t(t)) = (p_0, q_0, r_0)e^{\mu_0 t}. \tag{5.9}$$

Now, by (5.9), (1.5) and the energy dissipation law (1.6), it is easy to conclude that

$$(\operatorname{Re}\mu_0)e^{2t\operatorname{Re}\mu_0}|(p_0, q_0, r_0)|_H^2 + \frac{|\mu_0|^2}{2}e^{2t\operatorname{Re}\mu_0}\int_{\Omega_1}|p_0|^2 dx = 0. \tag{5.10}$$

Note that $\mu_0 \neq 0$. Thus, (5.10) and the fact that $\operatorname{Re}\mu_0 \geq 0$ imply $\int_{\Omega_1}|p_0|^2 dx = 0$, which implies that $p_0 = 0$. Therefore, $y(t) \equiv 0$ for all $t \geq 0$. However, by the transmission condition in (1.1), this implies that $z(t, 0) = z_x(t, 0) = 0$ for all $t \geq 0$. Hence, the unique continuation property of wave equations yields that $z(t) = z_t(t) \equiv 0$. Thus $(p_0, q_0, r_0) = 0$. But this is a contradiction. This completes the proof of Lemma 2.1. $\square$

6 Appendix B: Proofs of Propositions 2.1-2.4 and 4.1

Proof of Proposition 2.1. By $\mu \in B_{|k|^{-1/2}}(0)$ and noting the definition of λ_k^1 (see (2.10)), one gets

$$\sqrt{\mu + \lambda_k^1} - \sqrt{\lambda_k^1} = \frac{\mu}{\sqrt{\mu + \lambda_k^1} + \sqrt{\lambda_k^1}} = O(k^{-1}).$$

Thus, (2.13) holds.

On the other hand, note that for any $j \in \mathbb{N}$, it holds

$$e^{-2\sqrt{\lambda_k^1}} = O(k^{-j}) \quad \text{as } k \to +\infty, \qquad e^{2\sqrt{\lambda_k^1}} = O(|k|^{-j}) \quad \text{as } k \to -\infty. \tag{6.1}$$

By (2.13) and the first formula in (6.1), for $k \geq 0$, we have

$$\begin{aligned} &\frac{e^{-2\sqrt{\mu+\lambda_k^1}}+1}{e^{-2\sqrt{\mu+\lambda_k^1}}-1} = \frac{e^{-2(\sqrt{\lambda_k^1}+O(k^{-1}))}+1}{e^{-2(\sqrt{\lambda_k^1}+O(k^{-1}))}-1} = \frac{e^{-2\sqrt{\lambda_k^1}}(1+O(k^{-1}))+1}{e^{-2\sqrt{\lambda_k^1}}(1+O(k^{-1}))-1} \\ &= \frac{O(k^{-2})(1+O(k^{-1}))+1}{O(k^{-2})(1+O(k^{-1}))-1} = -1+O(k^{-2}), \qquad \forall\, \mu \in B_{|k|^{-1/2}}(0). \end{aligned} \tag{6.2}$$

Similarly, by (2.13) and the second formula in (6.1), for $k < 0$, we have

$$\frac{e^{-2\sqrt{\mu+\lambda_k^1}}+1}{e^{-2\sqrt{\mu+\lambda_k^1}}-1} = \frac{1+e^{2\sqrt{\mu+\lambda_k^1}}}{1-e^{2\sqrt{\mu+\lambda_k^1}}} = 1+O(|k|^{-2}), \qquad \forall\, \mu \in B_{|k|^{-1/2}}(0). \tag{6.3}$$

Now, combining (6.2) and (6.3), we get (2.14). This completes the proof of Proposition 2.1. ☐

Proof of Proposition 2.2. By (2.9) and (2.11), we get

$$\sqrt{\lambda_\ell^p} = \sqrt{\lambda_\ell^0} + O(\ell^{-1}), \qquad \lambda_\ell^p = \lambda_\ell^0 + O(1) = -\ell^2\pi^2 + O(1).$$

Hence, for any $x \in (0,2]$ and any $j \in \mathbb{N}$, we have $e^{\lambda_\ell^p x} = e^{(-\ell^2\pi^2+O(1))x} = O(\ell^{-j})$. This gives (2.31).

Now, using the elementary identity $e^{t_1} - e^{t_2} = (t_1 - t_2)e^{t_2}\int_0^1 e^{s(t_1-t_2)}ds$, we get

$$\frac{e^{x\mathrm{Re}\,\lambda_\ell^p} - e^{x\mathrm{Re}\,\lambda_\ell^0}}{e^{x\mathrm{Re}\,\lambda_\ell^0}} = \mathrm{Re}\,(\lambda_\ell^p - \lambda_\ell^0)x\int_0^1 e^{s\mathrm{Re}\,(\lambda_\ell^p-\lambda_\ell^0)x}ds = O(1), \qquad x \in [-4,4].$$

This gives (2.32).

Similarly, for $x \in [-2,2]$, one has

$$e^{\sqrt{\lambda_\ell^p}x} - e^{\sqrt{\lambda_\ell^0}x} = \Big(\sqrt{\lambda_\ell^p} - \sqrt{\lambda_\ell^0}\Big)xe^{x\sqrt{\lambda_\ell^0}}\int_0^1 e^{sx(\sqrt{\lambda_\ell^p}-\sqrt{\lambda_\ell^0})}ds = O(\ell^{-1}).$$

This gives (2.33).

On the other hand, noting that $\lambda_k^h = \lambda_k^1 + O(|k|^{-1/2})$ and recalling $\lambda_k^1 = (1/2+k)\pi i$, we have

$$e^{\lambda_k^h x} - e^{\lambda_k^1 x} = (\lambda_k^h - \lambda_k^1)xe^{\lambda_k^1 x}\int_0^1 e^{sx(\lambda_k^h-\lambda_k^1)}ds = O(|k|^{-1/2}), \qquad x \in [-2,2].$$

This gives (2.34). This completes the proof of Proposition 2.2. ☐

Proof of Proposition 2.3. By $\lambda_\ell^0 = -\ell^2\pi^2$, we see that for any $x \in (0,1)$ and any $\ell \geq \ell_1$, it holds

$$-\frac{\sqrt{\lambda_\ell^0}(1+e^{-2\lambda_\ell^p})\Big(e^{\sqrt{\lambda_\ell^0}x} - e^{\sqrt{\lambda_\ell^0}(2-x)}\Big)}{\ell^2\pi^2(1+e^{2\ell^2\pi^2})(1+e^{2\sqrt{\lambda_\ell^0}})} = \frac{1+e^{-2\lambda_\ell^p}}{\ell\pi(1+e^{2\ell^2\pi^2})}\sin \ell\pi x = p_\ell^0(x). \tag{6.4}$$

By (2.43), (2.7), (2.9) and (6.4), we get (recall (1.2) for W_1 and W_2)

$$\begin{aligned}
&\sum_{\ell=\ell_1}^{\infty}\Big[|p_\ell^p-p_\ell^0|_{W_1}^2+|q_\ell^p|_{W_2}^2+|r_\ell^p|_{L^2(-1,0)}^2\Big]\\
&=\sum_{\ell=\ell_1}^{\infty}\frac{1}{\ell^4\pi^4(1+e^{2\ell^2\pi^2})^2}\Big[|1+e^{-2\lambda_\ell^p}|^2\int_0^1\Big|\frac{\lambda_\ell^p}{1+e^{2\sqrt{\lambda_\ell^p}}}\Big(e^{\sqrt{\lambda_\ell^p}x}+e^{\sqrt{\lambda_\ell^p}(2-x)}\Big)\\
&\quad-\frac{\lambda_\ell^0}{1+e^{2\sqrt{\lambda_\ell^0}}}\Big(e^{\sqrt{\lambda_\ell^0}x}+e^{\sqrt{\lambda_\ell^0}(2-x)}\Big)\Big|^2dx\\
&\quad+|\lambda_\ell^p|^2\int_{-1}^0\Big|e^{\lambda_\ell^p x}+e^{-\lambda_\ell^p(x+2)}\Big|^2dx+|\lambda_\ell^p|^2\int_{-1}^0\Big|e^{\lambda_\ell^p x}-e^{-\lambda_\ell^p(x+2)}\Big|^2dx\Big].
\end{aligned}\tag{6.5}$$

By (2.11) in Lemma 2.2, and noting (2.9), we get $\lambda_\ell^p=\lambda_\ell^0+O(1)$. By (2.33) in Proposition 2.2, for any $x\in[0,1]$, we have

$$e^{\sqrt{\lambda_\ell^p}x}=e^{\sqrt{\lambda_\ell^0}x}+O(\ell^{-1}),\quad e^{\sqrt{\lambda_\ell^p}(2-x)}=e^{\sqrt{\lambda_\ell^0}(2-x)}+O(\ell^{-1}),\quad e^{2\sqrt{\lambda_\ell^p}}=e^{2\sqrt{\lambda_\ell^0}}+O(\ell^{-1}).$$

Thus,

$$\begin{aligned}
&\frac{\lambda_\ell^p}{1+e^{2\sqrt{\lambda_\ell^p}}}\Big(e^{\sqrt{\lambda_\ell^p}x}+e^{\sqrt{\lambda_\ell^p}(2-x)}\Big)\\
&=\frac{(\lambda_\ell^0+O(1))}{1+e^{2\sqrt{\lambda_\ell^0}}+O(\ell^{-1})}\Big(e^{\sqrt{\lambda_\ell^0}x}+e^{\sqrt{\lambda_\ell^0}(2-x)}+O(\ell^{-1})\Big)\\
&=\Big(\frac{\lambda_\ell^0}{1+e^{2\sqrt{\lambda_\ell^0}}}+O(\ell)\Big)\Big(e^{\sqrt{\lambda_\ell^0}x}+e^{\sqrt{\lambda_\ell^0}(2-x)}+O(\ell^{-1})\Big)\\
&=\frac{\lambda_\ell^0}{1+e^{2\sqrt{\lambda_\ell^0}}}\Big(e^{\sqrt{\lambda_\ell^0}x}+e^{\sqrt{\lambda_\ell^0}(2-x)}\Big)+O(\ell).
\end{aligned}\tag{6.6}$$

Also, it is easy to see that there is a constant $C>0$, independent of ℓ, such that

$$|1+e^{-2\lambda_\ell^p}|\le C(1+e^{2\ell^2\pi^2}).\tag{6.7}$$

From (6.6) and (6.7), we see that

$$\begin{aligned}
&\sum_{\ell=\ell_1}^{\infty}\frac{|1+e^{-2\lambda_\ell^p}|^2}{\ell^4\pi^4(1+e^{2\ell^2\pi^2})^2}\int_0^1\Big|\frac{\lambda_\ell^p}{1+e^{2\sqrt{\lambda_\ell^p}}}\Big(e^{\sqrt{\lambda_\ell^p}x}+e^{\sqrt{\lambda_\ell^p}(2-x)}\Big)\\
&\quad-\frac{\lambda_\ell^0}{1+e^{2\sqrt{\lambda_\ell^0}}}\Big(e^{\sqrt{\lambda_\ell^0}x}+e^{\sqrt{\lambda_\ell^0}(2-x)}\Big)\Big|^2dx\le C\sum_{\ell=\ell_1}^{\infty}\frac{1}{\ell^2}<\infty.
\end{aligned}\tag{6.8}$$

On the other hand, by (2.32) in Proposition 2.2, we have $e^{x\mathrm{Re}\,\lambda_\ell^p}=e^{\lambda_\ell^0 x}(1+O(1))$ uniformly for all $x\in[-4,4]$. Thus, there is a constant $C>0$, independent of ℓ, such that

$$\begin{aligned}
&|\lambda_\ell^p|^2\int_{-1}^0\Big|e^{\lambda_\ell^p x}+e^{-\lambda_\ell^p(x+2)}\Big|^2dx+|\lambda_\ell^p|^2\int_{-1}^0\Big|e^{\lambda_\ell^p x}-e^{-\lambda_\ell^p(x+2)}\Big|^2\\
&\le 4|\lambda_\ell^p|^2\int_{-1}^0\Big(\Big|e^{\lambda_\ell^p x}\Big|^2+\Big|e^{-\lambda_\ell^p(x+2)}\Big|^2\Big)dx=4|\lambda_\ell^p|^2\int_{-1}^0\Big(e^{2\mathrm{Re}\,\lambda_\ell^p x}+e^{-2\mathrm{Re}\,\lambda_\ell^p(x+2)}\Big)dx\\
&\le C|\lambda_\ell^0|^2\int_{-1}^0\Big(e^{2\lambda_\ell^0 x}+e^{-2\lambda_\ell^0(x+2)}\Big)dx=C|\lambda_\ell^0|^2\frac{1-e^{-4\lambda_\ell^0}}{2\lambda_\ell^0}\le C\ell^2(1+e^{2\ell^2\pi^2})^2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\sum_{\ell=\ell_1}^{\infty} \frac{1}{\ell^4\pi^4(1+e^{2\ell^2\pi^2})^2}\Big[|\lambda_\ell^p|^2\int_{-1}^0 \Big|e^{\lambda_\ell^p x}+e^{-\lambda_\ell^p(x+2)}\Big|^2 dx + |\lambda_\ell^p|^2\int_{-1}^0 \Big|e^{\lambda_\ell^p x}-e^{-\lambda_\ell^p(x+2)}\Big|^2\Big] dx \\
&\le C\sum_{\ell=\ell_1}^{\infty}\frac{1}{\ell^2}<\infty.
\end{aligned} \tag{6.9}
$$

Finally, combining (6.5), (6.8) and (6.9), we arrive at (2.45). This completes the proof of Proposition 2.3. □

Proof of Proposition 2.4. Using (2.7), (2.47) and (2.50), we get

$$
\begin{aligned}
&\sum_{|k|=k_1}^{\infty}[|p_k^h|_{W_1}^2+|q_k^h-q_k^1|_{W_2}^2+|r_k^h-r_k^1|_{L^2(-1,0)}^2] \\
&=\frac{1}{4}\sum_{|k|=k_1}^{\infty}\Big[|1+e^{-2\lambda_k^h}|^2\int_0^1\left|\frac{e^{\sqrt{\lambda_k^h}x}+e^{\sqrt{\lambda_k^h}(2-x)}}{1+e^{2\sqrt{\lambda_k^h}}}\right|^2 dx \\
&\quad+\int_{-1}^0\Big|e^{\lambda_k^h x}+e^{-\lambda_k^h(x+2)}-e^{\tilde\lambda_k^h x}-e^{-\tilde\lambda_k^h(x+2)}\Big|^2 dx \\
&\quad+\int_{-1}^0\Big|e^{\lambda_k^h x}-e^{-\lambda_k^h(x+2)}-e^{\tilde\lambda_k^h x}+e^{-\tilde\lambda_k^h(x+2)}\Big|^2 dx\Big] \\
&\le\sum_{|k|=k_1}^{\infty}\Big[|1+e^{-2\lambda_k^h}|^2\int_0^1\frac{e^{2x\mathrm{Re}\sqrt{\lambda_k^h}}+e^{2(2-x)\mathrm{Re}\sqrt{\lambda_k^h}}}{|e^{2\mathrm{Re}\sqrt{\lambda_k^h}}-1|^2}dx \\
&\quad+\int_{-1}^0\Big(\Big|e^{\lambda_k^h x}-e^{\tilde\lambda_k^h x}\Big|^2+\Big|e^{-\lambda_k^h(x+2)}-e^{-\tilde\lambda_k^h(x+2)}\Big|^2\Big)dx\Big].
\end{aligned} \tag{6.10}
$$

By (2.13) in Proposition 2.1, we have $\sqrt{\lambda_k^h}=\sqrt{\lambda_k^1}+O(|k|^{-1})$. Recall that $\lambda_k^1=(1/2+k)\pi i$. Thus

$$
\mathrm{Re}\sqrt{\lambda_k^h}=\frac{\mathrm{sgn}\,(k)}{2}\sqrt{|1+2k|\pi}+O(|k|^{-1}). \tag{6.11}
$$

Hence,

$$
\lim_{k\to-\infty}\mathrm{Re}\sqrt{\lambda_k^h}=-\infty,\qquad \lim_{k\to+\infty}\mathrm{Re}\sqrt{\lambda_k^h}=+\infty.
$$

Therefore, there is a constant $C>0$, independent of k, such that

$$
\left|\frac{e^{2\mathrm{Re}\sqrt{\lambda_k^h}}+1}{e^{2\mathrm{Re}\sqrt{\lambda_k^h}}-1}\right|\le C,\qquad \forall\,|k|\ge k_1. \tag{6.12}
$$

By (6.11) and (6.12), we get

$$
\begin{aligned}
&\int_0^1\frac{e^{2x\mathrm{Re}\sqrt{\lambda_k^h}}+e^{2(2-x)\mathrm{Re}\sqrt{\lambda_k^h}}}{|e^{2\mathrm{Re}\sqrt{\lambda_k^h}}-1|^2}dx=\frac{e^{4\mathrm{Re}\sqrt{\lambda_k^h}}-1}{2|e^{2\mathrm{Re}\sqrt{\lambda_k^h}}-1|^2\mathrm{Re}\sqrt{\lambda_k^h}} \\
&=\frac{e^{2\mathrm{Re}\sqrt{\lambda_k^h}}+1}{2\Big|e^{2\mathrm{Re}\sqrt{\lambda_k^h}}-1\Big|\mathrm{Re}\sqrt{\lambda_k^h}}\le C|1+2k|^{-1/2}+O(|k|^{-1}).
\end{aligned} \tag{6.13}
$$

By (2.34) in Proposition 2.2, we have $e^{-2\lambda_k^h} = e^{-2\lambda_k^1} + O(|k|^{-1/2})$. Recall that $\lambda_k^h = \lambda_k^1 + O(|k|^{-1/2})$. Thus

$$\left|1 + e^{-2\lambda_k^h}\right|^2 = \left|1 + e^{-2\lambda_k^1} + O(|k|^{-1/2})\right|^2 = O(|k|^{-1}). \tag{6.14}$$

Now, from (6.13) and (6.14), we get

$$\sum_{|k|=k_1}^{\infty} |1 + e^{-2\lambda_k^h}|^2 \int_0^1 \frac{e^{2x\mathrm{Re}\sqrt{\lambda_k^h}} + e^{2(2-x)\mathrm{Re}\sqrt{\lambda_k^h}}}{|e^{2\mathrm{Re}\sqrt{\lambda_k^h}} - 1|^2} dx < \infty. \tag{6.15}$$

On the other hand, by (2.28) in Lemma 2.3, we have

$$\lambda_k^h = \tilde{\lambda}_k^h + O(|k|^{-1}), \qquad \forall\, |k| \geq k_1. \tag{6.16}$$

Thus for any $|k| \geq k_1$, by (6.16), we see that

$$e^{\lambda_k^h x} - e^{\tilde{\lambda}_k^h x} = x(\lambda_k^h - \tilde{\lambda}_k^h) e^{\tilde{\lambda}_k^h x} \int_0^1 e^{sx(\lambda_k^h - \tilde{\lambda}_k^h)} ds = O(|k|^{-1}) \tag{6.17}$$

holds uniformly for $x \in [-2, 2]$. Therefore

$$\sum_{|k|=k_1}^{\infty} \int_{-1}^0 \left(\left|e^{\lambda_k^h x} - e^{\tilde{\lambda}_k^h x}\right|^2 + \left|e^{-\lambda_k^h (x+2)} - e^{-\tilde{\lambda}_k^h (x+2)}\right|^2\right) dx \leq C \sum_{|k|=k_1}^{\infty} |k|^{-2} < \infty. \tag{6.18}$$

Finally, combining (6.10), (6.15) and (6.18), we get (2.51). This completes the proof of Proposition 2.4. $\square$

Proof of Proposition 4.1. First of all, by (2.42), noting that $e^{-2\lambda_k^1} = -1$, and using $(1-z)^{-1} = 1 + z + O(|z|^2)$ when $|z| < 1$, one gets

$$e^{-2\lambda_k^h} = -\left[1 - \left(\frac{2\mathrm{sgn}\,(k)}{\sqrt{\lambda_k^1}} + O(|k|^{-1})\right)\right]^{-1} = -1 - \frac{2\mathrm{sgn}\,(k)}{\sqrt{\lambda_k^1}} + O(|k|^{-1}).$$

Recall that $\lambda_k^1 = (1/2 + k)\pi i$. Therefore, there is a constant $C > 0$ such that

$$\frac{1}{C}\sqrt{|k|} \leq \left|1 + e^{-2\lambda_k^h}\right|^{-1} \leq C\sqrt{|k|}. \tag{6.19}$$

On the other hand, by (6.11), it is easy to see that

$$\begin{aligned} \frac{\left|1 + e^{2\sqrt{\lambda_k^h}}\right|^2}{\left|e^{\sqrt{\lambda_k^h}}\right|^2} &= \left|e^{\sqrt{\lambda_k^h}} + e^{-\sqrt{\lambda_k^h}}\right|^2 = e^{2\mathrm{Re}\sqrt{\lambda_k^h}} + e^{-2\mathrm{Re}\sqrt{\lambda_k^h}} + 2\mathrm{Re}\left(e^{2i\mathrm{Im}\sqrt{\lambda_k^h}}\right) \\ &= e^{\sqrt{|1+2k|\pi} + O(|k|^{-1})} + O(1) = e^{\sqrt{2|k|\pi} + O(|k|^{-1/2})} + O(1). \end{aligned}$$

Therefore, there is a constant $C > 0$ such that

$$\frac{1}{C} e^{\sqrt{2|k|\pi}} \leq \frac{\left|1 + e^{2\sqrt{\lambda_k^h}}\right|^2}{\left|e^{\sqrt{\lambda_k^h}}\right|^2} \leq C e^{\sqrt{2|k|\pi}}. \tag{6.20}$$

Finally, combining (6.19) and (6.20), we conclude (4.4) immediately. This completes the proof of Proposition 4.1. $\square$

7 Appendix C: Spectral analysis in the general case of intervals with different lengths

This section is devoted to the spectral analysis of the generator $\mathcal{A}$ of the underlying semigroup of system (1.1) when the wave and heat domain, $(-1,0)$ and $(0,1)$, are replaced respectively by $(-b,0)$ and $(0,a)$ for some positive numbers a and b.

Obviously, in this case we may choose the underlying energy space of system (1.1) as $H \triangleq \Big\{(f,g,h) \ \Big|\ (g,f) \in H_0^1(-a,b),\ h \in L^2(-b,0)\Big\}$. Also, $\mathcal{A}$ can be defined similar to (2.1) and (2.2).

Recall that, our spectral analysis for $\mathcal{A}$ in Section 2 is developed for the parabolic domain and the hyperbolic one, first separately, and then combined by considering the transmission condition on the interface. Clearly, by scaling, this first step is length free. On the other hand, we do not need to consider the length of the intervals in the second step. Consequently, there are no essential differences between the spectral analysis for the general case and the special case considered in Section 2. Hence, in what follows, we will only list the main spectral analysis results without proofs.

First, we put

$$\begin{aligned}
f &= f(x,r) \triangleq \frac{\sqrt{r}(1+e^{-2br})}{1+e^{2a\sqrt{r}}}\Big(e^{\sqrt{r}x} - e^{\sqrt{r}(2a-x)}\Big), && x \in (0,a),\\
g &= g(x,r) \triangleq e^{rx} - e^{-r(x+2b)}, && x \in (-b,0),\\
h &= h(x,r) \triangleq r\Big[e^{rx} - e^{-r(x+2b)}\Big], && x \in (-b,0).
\end{aligned} \tag{7.1}$$

Similar to Lemma 2.1, we see that the eigenvectors of $\mathcal{A}$, corresponding to every eigenvalue r, are of the form $\mu(f,g,h)$ for some $\mu \neq 0$.

Next, we have

Lemma 7.1 *There exist two positive integers ℓ_3 and k_3 such that $\mathcal{A}$ has two classes of eigenvalues $\{r_\ell^p\}_{\ell=\ell_3}^\infty$ and $\{r_k^h\}_{|k|=k_3}^\infty$. Furthermore, the following asymptotic estimates hold:*

$$\sqrt{r_\ell^p} = \frac{\ell\pi}{a}i + \frac{1}{\ell\pi i} + O(\ell^{-2}), \quad r_k^h = -\frac{1}{\sqrt{|1+2k|b\pi}} + \frac{(1/2+k)\pi}{b}i + \frac{\mathrm{sgn}\,(k)}{\sqrt{|1+2k|b\pi}}i + O(|k|^{-1}).$$

Now, we choose

$$\begin{aligned}
&f_\ell^p(x) \triangleq -\frac{a^2 f(x,r_\ell^p)}{\ell^2\pi^2(1+e^{2b\ell^2\pi^2/a^2})}, \qquad x \in (0,a),\\
&g_\ell^p(x) \triangleq -\frac{a^2 g(x,r_\ell^p)}{\ell^2\pi^2(1+e^{2b\ell^2\pi^2/a^2})}, \qquad h_\ell^p(x) \triangleq -\frac{a^2 h(x,r_\ell^p)}{\ell^2\pi^2(1+e^{2b\ell^2\pi^2/a^2})}, \qquad x \in (-b,0);
\end{aligned}$$

$$\begin{aligned}
&f_k^h(x) \triangleq -\frac{f(x,r_k^h)}{2ir_k^h}, \qquad x \in (0,a),\\
&g_k^h(x) \triangleq -\frac{g(x,r_k^h)}{2ir_k^h}, \qquad h_k^h(x) \triangleq -\frac{h(x,r_k^h)}{2ir_k^h}, \qquad x \in (-b,0).
\end{aligned}$$

Then, similar to Theorem 2.1, we get

Theorem 7.1 *There exist positive integers n_1, $\tilde{\ell}_3 \geq \ell_3$ and $\tilde{k}_3 \geq k_3$ such that*

$$\{u_{j,0}, \cdots, u_{j,m_j-1}\}_{j=1}^{n_1} \bigcup \{(f_\ell^p, g_\ell^p, h_\ell^p)\}_{\ell=\tilde{\ell}_3}^{\infty} \bigcup \{(f_k^h, g_k^h, g_k^h)\}_{|k|=\tilde{k}_3}^{\infty}$$

form a Riesz basis of H, where $u_{j,0}$ is an eigenvector of $\mathcal{A}$ with respect to some eigenvalue r_j of $\mathcal{A}$ ($j = 1, 2, \cdots, n_1$) with algebraic multiplicity m_j, $\{u_{j,1}, \cdots, u_{j,m_j-1}\}$ is the associated Jordan chain of the corresponding generalized eigenvectors of $\mathcal{A}$ with respect to r_j and $u_{j,0}$, i.e., $\mathcal{A}u_{j,0} = r_j u_{j,0}$ and $\mathcal{A}u_{j,k} = r_j u_{j,k} + u_{j,k-1}$ ($k = 1, \cdots m_j - 1$).

Once these spectral results are established, the main results of this paper on the polynomial decay of smooth solutions and the control from the parabolic extreme can be immediately extended to this more general case.

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