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IDENTIFICATION OF THE CLASS OF INITIAL DATA FOR THE INSENSITIZING CONTROL OF THE HEAT EQUATION

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A Philippe Ciarlet, avec gratitude et admiration

Abstract. This paper is devoted to analyze the class of initial data that can be insensitized for the heat equation. This issue has been extensively addressed in the literature both in the case of complete and approximate insensitization (see [19] and [1], respectively).

But in the context of pure insensitization there are very few results identifying the class of initial data that can be insensitized. This is a delicate issue which is related to the fact that insensitization turns out to be equivalent to suitable observability estimates for a coupled system of heat equations, one being forward and the other one backward in time. The existing Carleman inequalities techniques can be applied but they only give interior information of the solutions, which hardly allows identifying the initial data because of the strong irreversibility of the equations involved in the system, one of them being an obstruction at the initial time $t = 0$ and the other one at the final one $t = T$.

In this article we consider different geometric configurations in which the subdomains to be insensitized and the one in which the external control acts play a key role. We show that, under rather restrictive geometric restrictions, initial data in a class that can be characterized in terms of a summability condition of their Fourier coefficients with suitable weights, can be insensitized. But, the main result of the paper, which might seem surprising, shows that this fails to be true in general, so that even the first eigenfunction of the system can not be insensitized. This result is similar to those obtained in the context of the null controllability of the heat equation in unbounded domains in [14] where it is shown that smooth and compactly supported initial data may not be controlled.

Our proofs combine the existing observability results for heat equations obtained by means of Carleman inequalities, energy and gaussian estimates and Fourier expansions.

1. **Statement of the problem.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be an open and bounded set of class C^2. Let $T > 0$ and let ω and $\mathcal{O}$ be two open and non empty subsets of Ω.

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We consider the following parabolic system:

$$\begin{cases} v_t - \Delta v = h 1_\omega & \text{in } \ Q = \Omega \times (0,T) \\ v = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\ v(x,0) = y^0(x) + \tau v^0 & \text{in } \ \Omega \end{cases} \tag{1}$$

where 1_ω denotes the characteristic function of the set ω; y^0 is given in $L^2(\Omega)$, and $h = h(x,t)$ is a control term in $L^2(Q)$. The data of the state equation (1) are incomplete in the following sense:

- $v^0 \in L^2(\Omega)$ is unknown and $\|v^0\|_{L^2(\Omega)} = 1$; and represents some *uncertainty* on the initial data.
- $\tau \in I\!R$ is unknown and small enough.

The problem of insensitizing controls was introduced by J. L. Lions [13] and can be stated, roughly, as follows: We say that the control h *insensitizes* the differentiable functional $\Phi(v)$ defined for the solutions of (1) if

$$\left.\frac{\partial \Phi(v(x,t;h,\tau))}{\partial \tau}\right|_{\tau=0} = 0. \tag{2}$$

When (2) holds the functional Φ is locally insensitive to the perturbation τv^0. The problem consists precisely in identifying if a functional $\Phi(v)$ can be insensitized and, to build the controls that insensitize it.

There are of course many possible choices of Φ. The simplest one, to which this paper is devoted, is the square of the L^2 norm of the state in some observation subset $\mathcal{O} \subset \Omega$, i.e.,

$$\Phi(v) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} v^2(x,t)\,dx\,dt. \tag{3}$$

In this case the insensitivity condition (2) is equivalent to a null control problem. This equivalence is given in the following proposition.

Proposition 1. *([1]) Let us consider the following cascade system of heat equations:*

$$\begin{cases} y_t - \Delta y = h 1_\omega & \text{in } Q \\ y = 0 & \text{on } \ \Sigma \\ y(.,0) = y^0 & \text{in } \Omega, \end{cases} \tag{4}$$

$$\begin{cases} -q_t - \Delta q = y 1_{\mathcal{O}} & \text{in } \ Q \\ q = 0 & \text{on } \ \Sigma \\ q(.,T) = 0 & \text{in } \Omega. \end{cases} \tag{5}$$

Then, the insensitivity condition (2) is equivalent to

$$q(0) = 0. \tag{6}$$

More precisely, if h is such that (2) holds for Φ defined as in (3), the solution (y,q) of (4)-(5) is such that (6) holds. The reciprocal is also true. For completeness, the proof of Proposition 1 can be found in an appendix at the end of the paper.

Observe that (6) is precisely a null controllability property for the cascade system (4)-(5). Note however that this situation is more complex than the standard one since the control that acts on the equation of q is y, the solution of (4), restricted to $\mathcal{O}$, which is the solution of the heat equation in which the control h is applied. Thus, the control h acts on the equation (5) in a indirect way and this adds important difficulties with respect to the standard control problems.

It is by now well understood that the null controllability of a system is equivalent to an observability inequality of its adjoint (see e.g. [12, 6, 8]). Let us describe how this duality principle applies in this particular case.

Let (p, z) be the adjoint state solution to the adjoint system:

$$\begin{cases} p_t - \Delta p = 0 & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(0) = p^0 & \text{in } \Omega\,, \end{cases} \tag{7}$$

$$\begin{cases} -z_t - z_{xx} = p 1_{\mathcal{O}} & \text{in } Q\,, \\ z = 0 & \text{on } \Sigma \\ z(T) = 0 & \text{in } \Omega. \end{cases} \tag{8}$$

The following holds:

Proposition 2. ([19]) *Suppose that for every y^0 in $L^2(\Omega)$ there exists $h \in L^2(Q)$ and $C > 0$ such that the solution to (1)-(2) corresponding to h satisfies $q(0) = 0$ with*

$$\|h\|_{L^2(\omega\times(0,T))} \leq C\|y^0\|_{L^2(\Omega)}. \tag{9}$$

Then, for some positive constant $C > 0$ the following observability inequality holds for any solution of (7)-(8):

$$\int_\Omega z^2(0)dx \leq C\int_0^T\int_\omega z^2 dxdt. \tag{10}$$

Reciprocally, if the observability inequality (10) holds then for any data $y^0 \in L^2(\Omega)$ it is possible to obtain a control such that q, the corresponding solution to (5), satisfies (6), with h verifying (9).

In [19] the first author proved that when $\Omega\backslash\bar{\omega} \neq \emptyset$, there exist data $y^0 \in L^2(\Omega)$ such that for every $h \in L^2(\omega \times (0,T))$, $q(0) \neq 0$ and, consequently, inequality (10) is not true. It is important to emphasize that this negative result is due to the fact that the equation (7) is *forward* in time and equation (8) is *backward* in time. When both equations are simultaneously backward or forward, and $\mathcal{O} \cap \omega \neq \emptyset$, the corresponding observability inequality can be proved (see e.g. [9]). But in the present case, the problem is much more delicate. Indeed, the information that the right hand side term of (10) yields about z in ω, when $\omega \cap \mathcal{O} \neq \emptyset$, provides information on p in $\omega \cap \mathcal{O}$ for $0 < t < T$. Using the existing observability estimates for the heat equation (7) that p satisfies, this yields estimates on p everywhere in Ω and for $0 < t \leq T$, but not for $t = 0$. This is an obstacle to recover estimates on z at $t = 0$ as well.

There are weaker versions of the insensitivity property that are easier to be fulfilled. Bodart and Fabre [1] relaxed the notion of insensitizing controls in the following way: Given $\varepsilon > 0$, the control h is said to *ε-insensitize* Φ if

$$\Big| \left.\frac{\partial\Phi(v(x,t;h,\tau))}{\partial\tau}\right|_{\tau=0} \Big| \leq \varepsilon. \tag{11}$$

In this case, for the functional defined by (3), the corresponding control condition is also weaker than (2). More precisely, ε-insensitivity is equivalent to $\|q(.,0)\|_{L^2(\Omega)} \leq \varepsilon$ instead of (2). Thus, while *insensitivity* corresponds to a *null-controllability* property, the *ε-insensitivity* is equivalent to an *approximate controllability* problem.

Using this fact and the already existing tools for the approximate controllability of the heat equation, in [1], the existence of ε-insensitizing controls was proved in bounded domains for the functional (3) and globally Lipschitz nonlinearities of class C^1 provided $\omega \cap \mathcal{O} \neq \emptyset$. In [11] this results have being extended to the case in which $\mathcal{O} \cap \omega = \emptyset$. The technique used involves the Fourier expansion of the solutions and therefore is not of use for the linear case with potentials depending on x and t and, as a consequence, useless in the non linear framework.

Let us now return to the insensitivity problem under consideration, which, obviously, can also be be viewed as a limit of the ε-insensitivity property as ε tends to zero.

In [19], for $\mathcal{O} = \Omega$, a sufficient condition on the initial data y^0 to be insensitized was given. More specifically, the following was proved.

Proposition 3. *([19]) Let $S(\cdot)$ denote the heat semigroup with Dirichlet boundary conditions. Let $\tau > 0$. Then, there exists a constant $C = C(\alpha)$ such that for every solution (p, z) of the adjoint system (7)-(8), $\mathcal{O} = \Omega$ and initial data $p^0 \in L^2(\Omega)$ the following inequality holds true:*

$$\|S(\tau)z(0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega z^2 dx dt. \tag{12}$$

Note that this yields very weak information on the initial datum $z(0)$ because of the very smoothing effect of the semigroup $S(\cdot)$ generated by the heat equation. Indeed, taking into account that $S(t) = \exp(-t\Delta)$, the norm in the left hand side term of (12) is weaker than any norm in any negative order Sobolev space.

Another result in [19], based on classical Carleman estimates ([8]), local energy estimates, and energy estimates, (see (8), p. 42 and (50), p. 54) that is going to be used in this paper, is the following "weighted" observability inequality for solutions of (7)-(8):

Proposition 4. *There exist $M > 0$ large enough, and $C > 0$, such that for every solution of (7)-(8)*

$$\int_0^T\int_\Omega |p|^2 e^{\frac{-M}{t}} dx dt + \int_0^T\int_\Omega |z|^2 e^{\frac{-M}{t}} dx dt \leq C \int_0^T \int_\omega z^2 dx dt. \tag{13}$$

Note however that, from this global estimate, it is hard to get information about z at $t = 0$ because of the degeneracy of the exponential (in time) weight function at $t = 0$.

This paper is devoted to further analyze the space of initial data that can (or cannot) be insensitized. To do this we try to determine under which conditions on the observation and control regions $\mathcal{O}$ and ω it is possible to obtain an observability inequality of the form (10), possibly with a weaker norm on the left hand side term, for the solutions of the adjoint system.

More generally, we shall discuss inequalities of the form

$$\|z(0)\|_V \leq C\|z\chi_{(\omega\times(0,T))}\|_H \tag{14}$$

for all (p, z) solutions to the adjoint system (7)-(8), with norms $||\cdot||_V$ and $||\cdot||_H$ that we will try to identify.

We consider various geometric configurations and present both positive and negative results that can be summarized as follows:

- The first result concerns the most favorable case in which $\mathcal{O} \subset \omega$, i. e. the control region covers the whole set in which the solution is being insensitized.

In this case one recovers a Sobolev norm on $z(\cdot, 0)$. Thus, insensitization can be achieved for initial data in a Sobolev space.

- The second one concerns the case in which $\mathcal{O} = \Omega$. In this case we prove an improved version of (12) in which the norm in the left hand side term turns out to be the one in the domain of the operator $\exp(c\sqrt{-\Delta})$, for a suitable positive constant $c > 0$. This case is favorable for insensitivity since the norm to be insensitized is distributed everywhere in the domain Ω, and then, due to the energy dissipation law, it is less sensitive to perturbations than when localized in any other subset of Ω. This situation is similar to the one encountered in the context of the classical null control problem for the heat equation ([5]) in the sense that the observed norm that Proposition 3 predicts can be significantly improved. Indeed, note that the observed norm that Proposition 3 yields is the one in the domain of $\exp(-\tau\Delta)$ for all $\tau > 0$ rather than the one in $\exp(c\sqrt{-\Delta})$ for some $c > 0$.
- The last result is of negative nature and it is the most siginificant and surprising one of the paper. We show that, when the geometric conditions of both cases above fail, one can not even insensitize the first eigenfunction of the Laplacian. This lack of spectral insensitivity shows in particular that the space of data that can be insensitized can not be described in terms of the Fourier coefficients of the solutions, as happens to the standard norms we are used to deal with. In other words, this inequality (14) may not be achieved with a norm $||\cdot||_V$ that can be expressed in terms of the Fourier coefficients of z. To be precise, we show that, the first Fourier component of z at $t = 0$ can not be guaranteed to be finite even when z is in $L^2(\omega \times (0, T))$. This is so because we can build singular initial data p^0, with singularities in $\mathcal{O}$ but away from ω, that generate solutions that are smooth in ω, and such that z, because of the singularity of p in $\mathcal{O}$, may not have a finite projection over the first eigenspace as $t \to 0^+$.

This last result closes the issue under consideration, to some extent, since it shows that it is hopeless to have inequalities of the form (14) in which the norm in the left hand side might be identified in Fourier terms. This seems to be the minimal requirement to deal with more general problems (variable coefficients, equations with potentials, semilinear equations,...) and to implement existing methods, for instance based on Carleman inequalities, whose utility has to be excluded as well.

Our proofs combine the existing observability inequalities, and in particular Proposition 4, energy and gaussian estimates and Fourier expansions.

2. **Some positive results.**

2.1. **The case $\mathcal{O} \subset \omega$.** This is the simplest case since the domain $\mathcal{O}$ to be insensitized is fully contained in the control set ω. In this case one can improve inequality (12) to get estimates on Sobolev norms of $z(\cdot, 0)$:

Theorem 2.1. *If $\mathcal{O} \subset \omega$ then*

$$\int_\Omega |\nabla z(x, 0)|^2 dx \leq C \int_0^T \int_\omega (|z_t|^2 + |\Delta z|^2) dx dt. \tag{15}$$

Proof. From classical energy estimates, i.e. multiplying the equation satisfied by z by $-\Delta z$ and integrating by parts in Q, we have that

$$\int_\Omega |\nabla z(x,0)|^2 dx \leq \int_0^T \int_{\mathcal{O}} p^2 dxdt.$$

Now, (15) holds immediately in view of this inequality, the identity (8) and the fact that $\mathcal{O} \subset \omega$. □

Remark 1. In a similar way one can get the weaker version:

$$\int_\Omega z^2(x,0) \leq C\left[||z_t||^2_{L^2(0,T;H^{-1}(\omega))} + \int_0^T \int_\omega |\nabla z|^2 dxdt \right], \tag{16}$$

when the characteristic function of the set $\mathcal{O}$ in (8) is replaced by a smooth cut-off function. According to this result the insensibilization of all the data $y_0 \in L^2(\Omega)$ holds with controls h in $H^{-1}(0,T;H_0^1(\omega)) + L^2(0,T;H^{-1}(\omega))$.

Whether a similar result holds when the right hand side term in (15) is replaced by the $L^2(\omega \times (0,T))$-nom, obtaining weaker estimates on $z(0)$, is an open problem. In other words, the class of initial data that can be insensitized with L^2-controls is still to be identified.

Similar results hold for a large class of parabolic equations with variable coefficients depending both on space and time. Note that, here, essentially, we have only used the well-posedness of the system and the fact that $\mathcal{O} \subset \omega$. Thus, similar results hold for other models too.

2.2. **The case $\mathcal{O} = \Omega$.** In this section, let φ_j denote the eingenfuntions of the Dirichlet Laplacian corresponding to the eigenvalue λ_j. That is

$$\begin{gathered} -\Delta\varphi_j = \lambda_j \varphi_j \text{ in } \Omega \\ \varphi_j = 0 \text{ on } \partial\Omega. \end{gathered}$$

We have the following result.

Theorem 2.2. *Let $\mathcal{O} = \Omega$, then there exist $B, C > 0$ such that for every z solution of (8) the estimate*

$$\sum_{j=1}^\infty e^{-B\sqrt{\lambda_j}} |z_j(0)|^2 \leq C \int_0^T \int_\omega z^2 dxdt \tag{17}$$

holds, where $z_j(0)$ are the Fourier coefficients of the solution z at time $t = 0$, that is, $z_j(0) = \int_\Omega z(0)\varphi_j dx$.

Remark 2. This result significantly improves the estimate in Proposition 3. Indeed, the estimate in (12), when written in Fourier series, reads

$$\sum_{j=1}^\infty e^{-\tau\lambda_j} |z_j(0)|^2 \leq C_\tau \int_0^T \int_\omega z^2 dxdt$$

for all $\tau > 0$ and a suitable $C_\tau > 0$, which is significantly weaker than (17).

Proof. We write

$$p_0 = \sum_{j=1}^\infty p_0^j \varphi_j.$$

We have that

$$p(t)=\sum_{j=1}^{\infty} p_0^j e^{-\lambda_j t}\varphi_j. \tag{18}$$

According to the by now well known observability properties of the heat equation we know that, for some $B>0$, (see e.g. [5], p. 27, (6.8)),

$$\sum_{j=1}^{\infty} |p_0^j|^2 e^{-B\sqrt{\lambda_j}} \le C\int_0^T\int_\Omega e^{-M/t}p^2(x,t)dxdt, \tag{19}$$

with M as in (13).

On the other hand, from (18) we have that

$$z(t)=\sum_{j=1}^{\infty}(\frac{e^{-\lambda_j t}}{2\lambda_j}-\frac{e^{-\lambda_j(2T-t)}}{2\lambda_j})p_0^j\varphi_j$$

and

$$z_j(0)=(\frac{1}{2\lambda_j}-\frac{e^{-2\lambda_j T}}{2\lambda_j})p_0^j.$$

From that we obtain that

$$\begin{aligned}\sum_{j=1}^{\infty}|z_j(0)|^2e^{-B\sqrt{\lambda_j}} &= \sum_{j=1}^{\infty}\left|\frac{1}{2\lambda_j}-\frac{e^{-2\lambda_j T}}{2\lambda_j}\right|^2|p_0^j|^2e^{-B\sqrt{\lambda_j}}\\ &\le C\sum_{j=1}^{\infty}|p_0^j|^2e^{-B\sqrt{\lambda_j}}\end{aligned}$$

The last inequality combined with (19) and (13) implies that

$$\sum_{j=1}^{\infty}e^{-B\sqrt{\lambda_j}}|z_j(0)|^2\le C\int_0^T\int_\omega z^2dxdt.$$

□

Remark 3. The previous result shows that when $\mathcal{O}=\Omega$ it is possible to insensibilize initial data $y_0=\sum_{j=1}^{\infty}b_j\varphi_j$ with

$$\sum_{j=1}^{\infty}e^{B\sqrt{\lambda_j}}b_j^2<\infty,$$

with L^2-controls.

Remark 4. This result is optimal.

To begin with, using the same arguments and constructions as in [5] and [16], one can show that inequality (13) is optimal. This is particularly simple in this case in which $\mathcal{O}=\Omega$, since one can then easily write down explicitly the solution z in terms of p using the heat kernel and the variation of constants formula. This means, in particular, that one can show the existence of a constant $M^*(\Omega,\omega)>0$ such that this inequality (13) fails when $M<M^*(\Omega,\omega)$. On the other hand, as shown in [5], the two quantities in (19) are equivalent. This shows the optimality.

Recall that the argument in [5] is based on the use of the following very singular solution of the heat equation

$$p(x,t)=\cos\left(\frac{Ax_1}{2t}\right)e^{A^2/4t}G(x,t) \tag{20}$$

while that in [16] is based on using the solution of the heat equation with Dirichlet boundary conditions and a Dirac delta as initial datum located away from the control set. Both proofs can be adapted to the present setting. That in [16] yields a better estimate on the optimal value for M, whose sharp geometric characterization is unknown even in the classical context of the observability of the heat equation.

In the present setting the optimality of Theorem 2.2 has to be understood in the sense that the estimate it provides may only hold for some $B > 0$ sufficiently large, whose optimal value is by now unknown.

3. **The main negative result when $\mathcal{O} \neq \Omega$ and $\mathcal{O} \not\subset \omega$.** In the previous sections we have considered the cases in which either $\mathcal{O}$ coincides with the whole domain Ω or it is contained in the control set ω and we have been able to identify the space of data that can be insensitized in terms of the Fourier coefficients. In this section we show that this may not be done in all the other geometric configurations. In other words, we prove that there are not positive weights $\{\rho_j\}$ so that

$$\sum_{j=1}^{\infty} \rho_j |z_j(0)|^2 \leq C \int_0^T \int_\omega z^2 dx dt \tag{21}$$

holds.

To be more precise, we prove that the spectral observability property (21) fails, whatever the weights $\{\rho_j\}$ are. Actually, even the first Fourier component of the solution z at $t = 0$ can not be estimated.

This is, in a first view, a surprising negative result. It is similar to the one proved in [14] on the lack of null controllability of the heat equation on unbounded domains and also of the parabolic equation $u_t + (-\Delta)^{1/2} u = 0$ in [15].

To prove this negative result we consider the $1 - d$ case and combine the Fourier expansion of solutions with gaussian estimates:

Theorem 3.1. *Let Ω be a bounded open interval of $\mathbb{R}$. There exist non empty subdomains $\mathcal{O} \neq \Omega$, $\mathcal{O} \not\subset \omega$ such that the spectral inequality*

$$|\int_\Omega z(x,0)\varphi_1(x)dx|^2 \leq C \int_0^T \int_\omega z^2 \, dx \, dt,$$

fails. Here φ_1 stands for the first eigenfunction of the Dirichlet Laplacian in Ω. In other words,

$$\sup_{(z,p) \text{ solutions of } (7)-(8)} \frac{|\int_\Omega z(x,0)\varphi_1(x)dx|^2}{\int_0^T \int_\omega z^2 \, dx \, dt} = \infty. \tag{22}$$

Remark 5. In view of this result, in this geometric setting, we can not expect an inequality of the form (21) to hold.

As a consequence of this result and, by duality, we deduce, in particular, that the initial datum $y_0 = \varphi_1$ can not be insensitized with L^2-controls, which is, to some extent, the worst scenario we could think of. This situation is similar to the one encountered for the heat equation in an unbounded domain by means of controls localized in a bounded set. In the latter, even if approximate controllability holds, compactly supported smooth initial data can not be controlled ([14]).

Recently, Kavian and de Teresa [11] proved that system (4)-(5) is partially approximately controllable in the context of Theorem 3. That means that for any

$\varepsilon > 0$ there exists $h \in L^2(\omega \times (0,T))$ such that q, the solution to (5), satisfies $\|q(0)\|_{L^2} \leq \varepsilon$. Here again the case $\omega \cap \mathcal{O} \neq \emptyset$ was easy to solve but the case where $\mathcal{O}$ and ω are disjoint is much more intricate and there are still several open problems on the subject (see [11]).

In any case, our negative result shows also that the controls in [11] necessarily diverge as ε tends to zero.

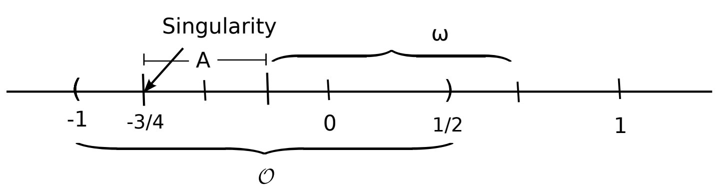
Proof. Since we are in $1-d$, without loss of generality, we assume that $\Omega = (-1,1)$. In the other cases the constants in our estimates may change but the proof is the same. We proceed in 3 steps.

First step. Let us assume that $-3/4 \notin \omega$ then, there exists $A > 0$ such that $|x + 3/4| \geq A$ for every $x \in \omega$. Take, $\mathcal{O} = (-1, 1/2)$. This choice of the domain $\mathcal{O}$ simplifies the Fourier series representation of the solution. Actually, we do not know if it is possible to construct a counterexample for every $\mathcal{O}$ satisfying $\mathcal{O} \not\subset \omega$, $\mathcal{O} \neq \Omega$.

Let us consider the following heat equation with singular initial datum at $x = -3/4$:

$$\begin{cases} p_t - p_{xx} = 0 & \text{in } \Omega \times (0,T) \\ p = 0 & \text{on } \partial\Omega \times (0,T) \\ p(0) = (-\partial_x^2)^k \delta_{x=-3/4} & \text{in } \Omega \end{cases} \tag{23}$$

for some $k \geq 1$ that we will made precise below.



Let $\lambda_j = j^2\pi^2/4$ denote the eigenvalues of the Dirichlet Laplacian, and φ_j the corresponding eigenfunctions, that is

$$\varphi_j(x) = \sin\left(\frac{j\pi(x+1)}{2}\right).$$

We have

$$p_{j,0} = <\varphi_j, (-\partial_x^2)^k \delta_{x=-3/4}> = \left(\frac{j\pi}{2}\right)^{2k} \sin\left(\frac{j\pi}{8}\right).$$

Clearly there exists a subsequence ($j = 4(2l+1)$) such that

$$\beta \geq \frac{|p_{j,0}|}{j^{2k}} \geq \alpha > 0. \tag{24}$$

Let now z be the solution to

$$\begin{cases} -z_t - z_{xx} = p\chi_{\mathcal{O}} & \text{in } \Omega \times (0,T) \\ z = 0 & \text{on } \partial\Omega \times (0,T) \\ z(T) = 0 & \text{in } \Omega. \end{cases} \tag{25}$$

We define $z_{0,1} = <\varphi_1, z(0)>$ the first Fourier coefficient of z at $t = 0$. Using the Fourier expansion of the solution z we have that

$$z_{0,1} = \sum_{j=1}^{\infty} c_{1,j}\left(\frac{1}{2\lambda_j} - \frac{e^{-2\lambda_j T}}{2\lambda_j}\right)p_0^j$$

with

$$c_{1,j} = \int_{-1}^{1/2} \sin\left(\frac{j\pi(x+1)}{2}\right)\sin\left(\frac{\pi(x+1)}{2}\right)dx.$$

Observe that for $j = 4m$

$$c_{1,j} = C(-1)^m\left(\frac{1}{4m-1} + \frac{1}{4m+1}\right),$$

with C independent of m.

Clearly

$$c_{1,4m} \sim \frac{1}{m}. \tag{26}$$

Let us see that $z_1(0)$ is not well defined, or, to be more precise, that

$$| < \varphi_1, z(t) > | \to \infty \text{ as } t \to 0^+.$$

We argue by contradiction. Suppose that the series

$$\sum_{j\geq 1} c_{1,j}\left(\frac{1}{2\lambda_j} - \frac{e^{-2\lambda_j T}}{2\lambda_j}\right)p_0^j$$

converges. Then, necessarily,

$$p_0^j \frac{c_{1,j}}{\lambda_j} \to 0,$$

and, in particular, this sequence is bounded. Thus, in view of (24), for $j = 4(2l+1)$ we have that $p_0^j \neq 0$ and

$$|c_{1,4(2l+1)}| \leq C\frac{(4(2l+1))^2}{(4(2l+1))^{2k}}.$$

For $k > 1$ this contradicts (26), proving the divergence of the series

$$\sum_{j\geq 1} p_0^j \frac{c_{1,j}}{\lambda_j}.$$

In the next two steps we prove that $z(x,t) \in L^2(\omega \times (0,T))$. This suffices to prove (22).

Second step. In order to prove that $z(x,t) \in L^2(\omega \times (0,T))$ we consider the solution to the cascade system in $\mathbb{R}\times(0,T)$ (instead of the bounded domain $(-1,1)\times(0,T)$) and later compare the solution in $\mathbb{R}$ with the solution in $(-1,1)$. In this aim we study the following auxiliary system:

$$\begin{cases} \tilde{p}_t - \tilde{p}_{xx} = 0 & \text{in } \mathbb{R}\times(0,T) \\ \tilde{p}(0) = (-\partial_x^2)^k \delta_{x=-3/4} & \text{in } \mathbb{R}. \end{cases} \tag{27}$$

$$\begin{cases} -\tilde{z}_t - \tilde{z}_{xx} = \tilde{p}\chi_{\mathcal{O}} & \text{in } \mathbb{R}\times(0,T) \\ \tilde{z}(T) = 0 & \text{in } \mathbb{R}. \end{cases} \tag{28}$$

Observe that

$$\tilde{p}(x,t) = (-\partial_x^2)^k G_{-3/4}(x,t)$$

where $G_{-3/4} = G(x+3/4,t)$ denotes the translation of G centered at $x=-3/4$ of the heat kernel

$$G(x,t) = (4\pi t)^{-1/2}\exp(\frac{-|x|^2}{4t})$$

and

$$(-\partial_x^2)^k(G(x,\sigma)) = G(x,\sigma)q(x,\frac{1}{\sigma})$$

where where q is a polynomial of degree $2k$ in both variables.

From the variation of constants formula we get that

$$\begin{aligned}
\tilde{z}(\cdot,t) &= \int_t^T G(\sigma-t)*(\chi_{\mathcal{O}}\tilde{p})(\sigma)d\sigma \\
&= \int_t^T G_{-3/4}(\sigma-t)*G_{-3/4}(\sigma)q(y,\frac{1}{\sigma})1_{\mathcal{O}}(y)d\sigma \\
&= \int_t^T\int_{\mathbb{R}} (4\pi(\sigma-t))^{-1/2}\exp(\frac{-|x-y|^2}{4(\sigma-t)}) \\
&\quad \cdot(4\pi\sigma)^{-1/2}\exp(\frac{-|y+3/4|^2}{4\sigma})q(y+3/4,\frac{1}{\sigma})1_{\mathcal{O}}(y)dyd\sigma.
\end{aligned}$$

Observe that for $y\in\mathcal{O}$

$$|q(y+3/4,\frac{1}{\sigma})| \le C(\mathcal{O})|F(\frac{1}{\sigma})|$$

where $C(\mathcal{O})$ is a positive constant and F a polynomial of degree $2k$. Thus

$$\begin{aligned}
|\tilde{z}(x,t)| &\le C\int_t^T G_{-3/4}(2\sigma-t)|F(\frac{1}{\sigma})|d\sigma \\
&= C\int_t^T (4\pi(2\sigma-t))^{-1/2}\exp(\frac{-|x+3/4|^2}{4(2\sigma-t)})|F(\frac{1}{\sigma})|d\sigma.
\end{aligned}$$

We also have, for every $x\in\omega$

$$\exp(\frac{-|x+3/4|^2}{4(2\sigma-t)}) \le \exp(\frac{-A^2}{4(2\sigma-t)})$$

and then

$$\begin{aligned}
|\tilde{z}(x,t)| &\le C\int_t^T (2\sigma-t)^{-1/2}\exp(\frac{-A^2}{4(2\sigma-t)})|F(\frac{1}{\sigma})|d\sigma \\
&\le C\int_t^T \exp(\frac{-A^2}{2(2\sigma-t)})|F(\frac{1}{\sigma})|d\sigma.
\end{aligned}$$

On the other hand, for every $t\le\sigma$,

$$\frac{-A^2}{4\sigma} \ge \frac{-A^2}{2(2\sigma-t)}$$

and then

$$|\tilde{z}(x,t)| \le C\int_0^T \exp(\frac{-A^2}{4\sigma})|F(\frac{1}{\sigma})|d\sigma.$$

Taking into account that F is a polynomial of degree $2k$ the largest term in the integral above corresponds precisely to the power $2k$. But, whatever β and $\gamma\ge 0$ are, the function $s^{-\gamma}\exp(-\beta/s)$ reaches its maximum in $s=\beta/\gamma$. Thus we obtain that for every $x\in\omega$ and $t\in(0,T)$,

$$|\tilde{z}(x,t)| \le Ce^{-2k}(\frac{A^2}{4(2k)})^{-2k}.$$

In particular $\tilde{z}(x,t) \in L^2(\omega \times (0,T))$. Observe that the same arguments are valid in order to show that $\tilde{z}(1,t)$ and $\tilde{z}(-1,t)$ belong to $L^\infty(0,T)$.

Third Step. Let $\phi = \tilde{p} - p$ and $\psi = \tilde{z} - z$ be the solutions of

$$\begin{cases} \phi_t - \phi_{xx} = 0 & \text{in } \Omega \times (0,T) \\ \phi = \tilde{p} & \text{on } \partial\Omega \times (0,T) \\ \phi(0) = 0 & \text{in } \Omega. \end{cases} \tag{29}$$

$$\begin{cases} -\psi_t - \psi_{xx} = \phi \chi_{\mathcal{O}} & \text{in } \Omega \times (0,T) \\ \psi = \tilde{z} & \text{on } \partial\Omega \times (0,T) \\ \psi(T) = 0 & \text{in } \Omega. \end{cases} \tag{30}$$

Clearly ϕ is smooth and bounded since $\tilde{p}$ is regular in the boundary of Ω. The estimates on $\tilde{z}$ above allow seeing that ψ is also smooth. We conclude that z belongs to $L^2(\omega \times (0,T))$.

Fourth Step. In view of the construction above it is easy to build a family of solutions showing that the supremum in (22) is unbounded. In fact it is sufficient to take as initial data for p the value of the solution p above at $t = \varepsilon$ with $\varepsilon \to 0$. We obtain that

$$\int_0^T \int_\omega z_\varepsilon^2 dx dt \leq C$$

uniformly and

$$\lim_{\varepsilon \to 0} \frac{| < \varphi_1, z_\varepsilon(0) > |^2}{\displaystyle\int_0^T \int_\omega z_\varepsilon^2 dx dt} = \infty.$$

□

Remark 6. This construction, leading to the lack of spectral observability inequalities can be generalized easily for hypercubes in $\mathbb{R}^n$ and control and observation sets of the same type. However, it seems difficult to obtain a counterexample valid for all possible domains Ω. Indeed, note that in this construction we have used rather explicitly the structure of the spectrum of the Laplacian, in particular when deriving (24) and (26).

In any case it is important to underline that, in view of this result, one has to exclude the possibility of getting weak observability estimates of the form (21) by, for instance, standard Carleman type inequalities since our counterexamples show that they fail dramatically at least for some choices of the subdomains $\mathcal{O}$ and ω.

4. **Further remarks.** As far as we know there are few papers on the problem of insensitizing other models. The following ones are worth mentionning.

Recently, Dager in [4] considered the one dimensional wave equation and showed that all initial data in an appropriate space can be insensitized from the boundary for $T > 0$ large enough. Using other techniques the same result was extended by Tebou [18] to the wave equation in several space dimensions. Of course, in this case, geometric conditions on the observability and on the controllability set are required to guarantee that the propagation along bicharacteristic rays is captured. These results show that the wave equation is behaved very differently than the heat one and this is due to the time reversibility of the wave equation.

On the other hand, as mentioned above, in [11] it is shown a unique continuation result, that implies an approximate insensitizing result for the heat equation, without any condition on the intersection of the observability set $\mathcal{O}$ and the controllability set ω as it is required in [4] and [18] for the ε-insensitinzing the wave equation with a control in the interior of the domain. This is in agreement on previous results on the control of wave and heat processes showing that no geometric requirements are needed for controlling the heat equation while they are necessary for the wave one.

As we mentioned above, the problem under consideration is particularly complex because one of the equations of the system under discussion is singular at $t = 0$ while the other one is it at $t = T$. The situation is different for a cascade system of two coupled one dimensional heat equations oriented in the same sense of time as below, considered in [7],

$$\begin{cases} y_t - \nu y_{xx} = 0 & \text{in } (0,T) \times (0,1)\,, \\ y(t,0) = v & \text{in } (0,T)\,, \\ y(t,1) = 0 & \text{in } (0,T) \\ y(.,0) = y^0 & \text{in } (0,1)\,, \end{cases}$$

$$\begin{cases} q_t - q_{xx} = y & \text{in } (0,T) \times (0,1), \\ q(t,0) = q(t,1) = 0 & \text{in } (0,T)\,, \\ q(.,0) = q^0 & \text{in } (0,1)\,. \end{cases}$$

In this case the approximate controllability property does not hold when $\sqrt{\nu} \in \mathbb{Q}$.

On the other hand, Bodart et all [2], [3] obtained insensitizing results for the heat equation with other boundary conditions and also for nonlinear heat equations with slightly superlinear growth. These issues have also been analyzed in Pérez-García [17].

Guerrero [10], has presented interesting results for the heat equation when the functional to be insensitized is

$$\Phi_{\mathcal{O}}(v(h,\tau)) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\nabla v(t,x)|^2 dxdt.$$

instead of (3).

Regarding other equations a lot of work is to be done. The results in [11] allow to prove the existence of ε-insensitizing controls for the Stokes equation. However the (null) insensitinzing property, as far as we know, has to be done. It is also an open question if the same kind of results apply to the Schrödinger equation.

Appendix. This appendix is devoted to the proof of Proposition 1.

Recall that

$$\Phi_{\mathcal{O}}(v(h,\tau)) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} v^2(t,x) dxdt.$$

Taking derivatives with respect to τ and evaluating at $\tau = 0$ we get that (2) is precisely

$$\int_0^T \int_{\mathcal{O}} y v_\tau = 0 \tag{31}$$

for every $v^0 \in L^2$, $\|v^0\| = 1$, where y is the solution corresponding to $\tau = 0$ and v_τ the derivative of v solution to (1) at $\tau = 0$.

That is y solves,

$$\begin{cases} y_t - \Delta y = h 1_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(.,0) = y^0, & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} v_{\tau,t} - \Delta v_\tau = 0 & \text{in } Q \\ v_\tau = 0 & \text{on } \Sigma \\ v_\tau(.,0) = v^0 & \text{in } \Omega. \end{cases} \tag{32}$$

Take q the solution of the adjoint system to (32) corresponding to a second member $y1_{\mathcal{O}}$, in other words, q solves

$$\begin{cases} q_t + \Delta q = y1_{\mathcal{O}} & \text{in } Q \\ q = 0 & \text{on } \Sigma \\ q(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{33}$$

Multiplying (32) by q and integrating by parts we get that

$$\int_\Omega q(0)v^0 = \int_0^T \int_{\mathcal{O}} y v_\tau$$

and then (31) is equivalent to ask

$$\int_\Omega q(0)v^0 = 0 \quad \forall v^0 \in L^2(\Omega), \quad \|v^0\| = 1,$$

that is

$$q(0) = 0.$$

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