Convergence of a multigrid method for the controllability of a 1-d wave equation

Convergence d'une méthode multigrille pour la contrôlabilité d'une équation d'ondes 1-d

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Abstract

We consider the problem of computing numerically the boundary control for the wave equation. It is by now well known that, due to high frequency spurious oscillations, numerical instabilities occur and may led to the failure of convergence of some apparently natural numerical algorithms. Several remedies have been proposed in the literature to compensate this fact: Tychonoff regularization, Fourier filtering, mixed finite elements,... In this Note we prove that the two-grid method proposed by Glowinski [G] does indeed provide a convergent algorithm. This is done in the context of the finite-difference semi-discrete approximation of the 1-d wave equation. *To cite this article: Mihaela Negreanu, Enrique Zuazua, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

On considère le problème de l'approximation numérique du contrôle frontière de l'équation des ondes. Il est maintenant bien connu que la plupart des méthodes de différences et éléments finis classiques ne donnent pas des approximations convergentes à cause des instabilités dues aux hautes fréquences. Plusieurs remèdes on été proposés dans la littérature pour compenser ce fait : régularisation de Tychonoff, filtrage en Fourier, éléments finis mixtes... Dans cette Note on démontre la convergence de la méthode de multi-grille proposée par Glowinski [G] dans le cas de l'approximation semi-discrète de l'équation des ondes par différences finies. *Pour citer cet article : Mihaela Negreanu, Enrique Zuazua, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Version française abrégée

On considère le problème de l'approximation numérique du contrôle frontière de l'équation des ondes. On traite le cas modèle de l'équation des ondes $1-d$ à coefficients constants dans l'intervalle $0<x<1$ avec des conditions aux limites de Dirichlet. Le contrôle agit sur l'extrême $x=1$. On sait que si le temps de contrôle T est supérieur ou égal à 2 alors le système est exactement contrôlé dans l'espace $L^2(0,1)\times H^{-1}(0,1)$ avec des contrôles $L^2(0,T)$.
Il est maintenant bien connu que le contrôle d'une approximation numérique discrète ou semi-discrète convergeante de ces équations peut ne pas converger (voir [G], [IZ], [Z],...). Ceci est dû au fait que la plupart des méthodes numériques classiques engendrent des ondes numériques artificielles de longueur d'onde et vitesse de propagation de l'ordre du pas du maillage, faisant que, pour tout temps de contrôle T fini donné, les contrôles puissent diverger. Par conséquent, "contrôler une approximation numérique du système n'est pas nécessairement une méthode efficace pour obtenir une bonne approximation du contrôle".
Plusieurs méthodes ont été proposées pour compenser ce manque de convergence : régularisation de Tychonoff, filtrage en Fourier, éléments finis mixtes,... Toutes ces méthodes ont en commun le fait qu'elles éliminent les contributions pathologiques des hautes fréquences.
Dans cette Note on analyse la méthode de bi-grille proposée par R. Glowinski dans [G] dans le cas des approximations semi-discrètes en différences finies de l'équation des ondes $1-d$ et on montre que la méthode converge. Le principe de la méthode est le suivant. La méthode HUM de J.L. Lions [L] caractérise le contrôle optimal comme une trace normale d'une solution du système adjoint que minimise une certaine fonctionnelle quadratique. À cause des hautes fréquences pathologiques cette fonctionnelle manque de coercivité uniforme par rapport au pas du maillage. La méthode proposée dans [G] consiste à minimiser cette fonctionnelle dans un sous-espace de données initiales lentes pour le problème adjoint provenant d'un maillage plus grossier (de double pas, par exemple). Il s'agit donc de données initiales qui oscillent "lentement". Cette condition sur les données initiales permet d'estimer les hautes fréquences et de prouver la coercivité uniforme de la fonctionnelle par une méthode de multiplicateurs discrets. On obtient ainsi des contrôles bornés. Il faut cependant signaler que :

- Le temps de contrôle obtenu est le double que celui de l'équation des ondes. Ce temps est optimal, comme l'indique l'analyse des courbes de dispersion.
- Les contrôles obtenus ne contrôlent pas exactement l'état discret mais uniquement sa projection sur le maillage grossier.

Les contrôles ainsi obtenus convergent vers le contrôle frontière de l'équation des ondes.

1. Introduction

Let us consider the $1-d$ wave equation:

$$\begin{cases} u_{tt}-u_{xx}=0, & 0<x<1, \quad 0<t<T,\\ u(0,t)=0, \quad u(1,t)=v(t), & 0<t<T,\\ u(x,0)=u^0(x), \qquad u_t(x,0)=u^1(x), & 0<x<1, \end{cases} \tag{1}$$

where (u^0,u^1) is the initial state and v is the control function that acts on the system through the extreme $x=1$ of the space interval $(0,1)$. System (1) describes the vibration of a controlled string.

When no control acts on the boundary $x = 1$, i.e., when $v = 0$, the energy of solutions is conserved:

$$E(t) = \frac{1}{2}\int_0^1 \left(| u_x(x,t) |^2 + | u_t(x,t) |^2\right) dx. \tag{2}$$

The starting point of our study is the following well known exact controllability result for (1) (see [L]): *Given $T \geq 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists a control function $v \in L^2(0,T)$ such that the solution $u = u(x,t)$ of (1) satisfies:*

$$u(T) = u_t(T) = 0, \qquad 0 < x < 1. \tag{3}$$

Moreover, there exists a unique control of minimal L^2-norm among the admissible controls with the property (3). Note that (3) makes sense since, system (1) admits a unique solution $u \in \mathcal{C}([0,T]; L^2(0,1)) \cap \mathcal{C}^1([0,T]; H^{-1}(0,1))$.
Let us now introduce the homogeneous system (the adjoint of (1))

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ \phi(0,t) = \phi(1,t) = 0, & 0 < t < T, \\ \phi(x,T) = \phi^0(x), \qquad \phi_t(x,T) = \phi^1(x), & 0 < x < 1. \end{cases} \tag{4}$$

When $(\phi^0, \phi^1) \in H_0^1(0,1) \times L^2(0,1)$ it admits a unique solution $\phi = \phi(x,t) \in \mathcal{C}^0\left([0,T], H_0^1(0,1)\right) \cap \mathcal{C}^1\left([0,T], L^2(0,1)\right)$.
With the aid of the *Hilbert Uniqueness Method (HUM)* introduced by J.L. Lions in [L] the exact controllability of (1) is shown to be equivalent to the following observability inequality of the adjoint system (4): *Given $T \geq 2$ there exists a positive constant $C(T) > 0$ such that*

$$E(0) \leq C(T) \int_0^T | \phi_x(1,t) |^2 \, dt \tag{5}$$

holds for every solution $\phi = \phi(x,t)$ of the adjoint problem (4).
Here we are concerned with the numerical approximation of the control of system (1). A possible procedure is to approximate the wave operator by some sequence of discrete or semi-discrete operators and to obtain this control $v(t)$ as the limit of the sequence of controls of the approximating equations: take $N \in \mathbb{N}$, set $h = 1/(N+1)$ and consider the finite-difference space semi-discretization of (1):

$$\begin{cases} u_j''(t) = \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2}, & t > 0, \quad j = 1, ..., N, \\ u_0(t) = 0, \ \ u_{N+1}(t) = v_h(t), & t > 0, \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, & j = 1, ..., N. \end{cases} \tag{6}$$

Following the HUM method, the existence of a control v_h for a given h is equivalent to the existence of a constant C_h such that

$$E_h(t) \leq C_h \int_0^T \left| \frac{\phi_N(t)}{h} \right|^2 dt \tag{7}$$

where $\{\phi_j(t)\}_j$ is the solution of the homogeneous adjoint semi-discrete system

$$\begin{cases} \phi_j''(t) = \dfrac{\phi_{j+1}(t) - 2\phi_j(t) + \phi_{j-1}(t)}{h^2}, & 0 < t < T, \quad j = 1, ..., N, \\ \phi_0(t) = \phi_{N+1}(t) = 0, & 0 < t < T, \\ \phi_j(T) = \phi_j^0, \quad \phi_j'(T) = \phi_j^1, & j = 1, ..., N \end{cases} \tag{8}$$

and E_h is the energy of this system

$$E_h(t) = \frac{h}{2}\sum_{j=1}^{N} |\phi_j(t)|^2 + \frac{h}{2}\sum_{j=0}^{N} \frac{|\phi_{j+1}(t) - \phi_j(t)|^2}{h^2}. \tag{9}$$

Note that the energy (9) is also conserved in time and represents a discretization of the continuous one. Inequality (7) holds for all $T > 0$ and $h > 0$ for a suitable $C_h = C_h(T)$. However, in [IZ] it was proved that, for all $T > 0$, the best constant C_h in (7) blows-up, i.e., $C_h \to \infty$, as $h \to 0$. As a consequence of this, there exist initial data for the wave equation (1) such that the controls v_h of (6) are unbounded in $L^2(0, T)$ as $h \to 0$. Thus, computing v_h is not an efficient way to approximate the control of (1).
A possible cure is to use the 2-grid algorithm introduced by R. Glowinski [G] to avoid the high frequency pathologies producing the blow-up phenomena described above. In this Note we prove the convergence of this algorithm.

2. Uniform observability of the 2-grid approximation scheme

Let $N \in \mathbb{N}$ be an odd number. With $h = 1/(N + 1)$ we introduce the coarse grid G_C^{2h}, an equidistant division of the interval $(0, 1)$, $x_0 = 0 < x_1 = 2h < ... < x_{(N+1)/2} = 1$, with $x_j = 2jh$, $j = 0, ..., (N + 1)/2$ and the fine grid G_F^h, $y_0 = 0 < y_1 = h < ... < y_N = Nh < y_{N+1} = 1$ with $y_j = jh$, $j = 0, ..., N + 1$.
Let us introduce the space V_h

$$V_h = \left\{ (\bar{\phi}_h^0, \bar{\phi}_h^1) \in \mathbb{R}^N \times \mathbb{R}^N : \phi_{2j+1}^l = \frac{\phi_{2j}^l + \phi_{2j+2}^l}{2}, \ j = 0, ..., \frac{N-1}{2}, \ l = 0, 1 \right\}. \tag{10}$$

Note that V_h is constituted by discrete functions defined on the fine grid obtained by interpolation of functions defined on the coarse one. Thus, V_h contains only slowly oscillatory functions.
Our first result guarantees that the observability inequality for (8) is uniform within the class V_h:
Theorem 2.1 *Let $h > 0$ and $T > 4$. Then, there exist constants $C_j(T) > 0$, $j = 1, 2$ independent of h, such that*

$$C_1(T)E_h(0) \leq \int_0^T \left|\frac{\phi_N(t)}{h}\right|^2 dt \leq C_2(T)E_h(0) \tag{11}$$

for all solution of (8) with initial data in the space V_h and all $h > 0$.
The proof of this theorem uses discrete multiplier techniques similar to those in [IZ]. We have the following identity for all the solutions of system (8) (see formula (2.40) in [IZ]):

$$TE_h(0) + X_h(t)|_0^T = \frac{1}{2}\int_0^T \left|\frac{\phi_N(t)}{h}\right|^2 dt + \frac{h}{4}\sum_{j=0}^{N}\int_0^T \left|\phi_{j+1}'(t) - \phi_j'(t)\right|^2 dt, \tag{12}$$

with $X_h(t) = h\sum_{j=1}^N \phi_j'(t)j(\phi_{j+1}(t) - \phi_{j-1}(t))$. Observe that, as indicated in [IZ], this identity is very close to the one one gets for the $1-d$ wave equation except for the last term. As indicated in [IZ], when estimating this term one losses completely the observability inequality if the class of solutions under consideration is not restricted. Here, using Fourier expansion of solutions, we are able to obtain the following sharp estimate for the solution of (8) in the class V_h of slowly varying initial data:

$$\sum_{j=0}^{N} \left|\phi_{j+1}'(t) - \phi_j'(t)\right|^2 \le 2\sum_{j=0}^{N} \left|\phi_j'(t)\right|^2, \tag{13}$$

Note that this estimate coincides with that one gets for the solutions of (8) involving only the eigenvalues $\sqrt{\lambda} \le 1/h$ in its Fourier expansion, i.e., for solutions in which the high frequency components corresponding to the upper half of the spectrum have been filtered. The remaining terms are estimated exactly as in [IZ]). ◇

Remark 1. It is important to underline that the inequality in (11) is uniform in h precisely because of considering slowly oscillating initial data in the fine grid.
The time $T > 4$ is twice the observability time for the continuous case. The analysis of the dispersion diagram of filtered solutions predicts a propagation velocity equal to $1/2$ in the class of solutions under consideration and confirms the observability time $T = 4$.

3. Construction and convergence of the controls

We now briefly describe the algorithm for building an efficient approximation of the control v of (1) which is inspired in Glowinski [G]. Let us consider the functional $J_h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$J_h\left((\bar{\phi}_h^0, \bar{\phi}_h^1)\right) = \frac{1}{2}\int_0^T \left|\frac{\phi_N(t)}{h}\right|^2 dt + h\sum_{j=1}^{N}\left[u_j^1\phi_j(0) - u_j^0\phi_j'(0)\right] \tag{14}$$

where $\bar{\phi}_h$ is the solution of the adjoint problem (8) with initial data $(\bar{\phi}_h^0, \bar{\phi}_h^1) \in V_h$ and $(\bar{u}_h^0, \bar{u}_h^1)$ are the initial data of system (6) to be controlled. This functional is convex and continuous. Moreover, in view of inequality (11), it is uniformly (with respect to $h > 0$) coercive in V_h provided $T > 4$. Then, there exists a minimizer $(\hat{\phi}_h^0, \hat{\phi}_h^1) \in V_h$. The corresponding Euler equation reeds

$$\int_0^T \hat{\phi}_N(t)\phi_N(t)/h^2 dt + h\sum_{j=1}^{N}[u_j^1\phi_j(0) - u_j^0\phi_j'(0)] = 0, \tag{15}$$

for every solution $\bar{\phi}_h$ of the adjoint equation (8). We define $v_h(t) = \hat{\phi}_N(t)/h$ and set

$$\int_0^T v_h(t)\phi_N(t)/h\, dt + h\sum_{j=1}^{N}[u_j^1\phi_j(0) - u_j^0\phi_j'(0)] = 0. \tag{16}$$

In view of the uniform inequality (11), the sequence of controls we have obtained v_h is bounded in $L^2(0,T)$. Observe that, in this method, the functional J_h is minimized on V_h, in contrast to the direct application of the minimization method of the functional J_h (that leads to divergent controls as $h \to 0$ for some pathological initial data, see [IZ] and [M]). The uniform observability inequality (11) guarantees the uniform coercivity of J_h when restricted to V_h and, consequently, the boundedness of the controls.

Note that (16) does not imply that the solution of (6) vanishes at time $t = T$. Rather a suitable projection of the state (u, u') on the coarse grid vanishes. More precisely, if we define the restriction $U_j(T) = (u_{2j}(T) + 1/2u_{2j+1}(T) + 1/2u_{2j-1}(T))/2$ and $U'_j(T) = (u'_{2j}(T) + 1/2u'_{2j+1}(T) + 1/2u'_{2j-1}(T))/2$, with $j = 1, ..., (N + 1)/2$, where $\bar{u}_h$ is the solution of (6), we have $(\bar{U}_h(T), \bar{U}'_h(T)) = (\bar{0}, \bar{0})$. ◇

Let us define the Hilbert spaces of square summable sequences ℓ^2 and $\hbar^{-1}$ as follows

$$\ell^2 = \{\{c_k\} \ : \ \|c_k\|^2_{\ell^2} = \sum_{k\in\mathbb{N}} | \ c_k \ |^2 < \infty\}, \quad \hbar^{-1} = \{\{c_k\} \in \ell^2 \ : \ \|c_k\|^2_{\hbar^{-1}} = \sum_{k\in\mathbb{N}} |c_k/(k\pi)|^2 < \infty\}. \tag{17}$$

Given an initial state $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ of the continuous system (1), we develop it in Fourier series $(u^0, u^1) = \sum_{k=1}^{\infty}(c^0_k, c^1_k)\varphi_k(x)$, with $(c^0_k, c^1_k) \in \ell^2 \times \hbar^{-1}$, $\varphi_k(x) = \sin(k\pi x)$. We now consider initial states: $(\bar{u}^0_h, \bar{u}^1_h) = \sum_{k=1}^{N}(c^0_k, c^1_k)\bar{\varphi}_k$, with $\bar{\varphi}_k = (\varphi_{k,1}, ..., \varphi_{k,N})$, $\varphi_{k,j} = \sin(k\pi j\Delta x)$, $j, k = 1, ..., N$. We set the Fourier series expansion of the solution u of (1): $u(x, t) = \sum_{k\in\mathbb{N}} c_k(t)\varphi_k(x)$ with $\sup_{t\in[0,T]}[\|c_k(t)\|_{\ell^2} + \|c'_k(t)\|_{\hbar^{-1}}] < \infty$. The solution of (6) may be written in a similar form as $\bar{u}_h(t) = \sum_k c_{k,h}(t)\bar{\varphi}_k$ by putting $c_{k,h}(t) = 0$ for $k > N$. The following holds:

Theorem 3.1 *Let $T > 4$. Let $\bar{u}_h$ and u be the solutions of (6) and (1) as above. Then*

$$\{c_{k,h}(\cdot)\}_{k\in\mathbb{N}} \to \{c_k(\cdot)\}_{k\in\mathbb{N}} \text{ strongly in } L^p(0, T; \ell^2) \cap W^{1,p}(0, T; \hbar^{-1}) \text{ as } h \to 0, \text{ with } 1 \le p < \infty \tag{18}$$

$$v_h(\cdot) \to v(\cdot) \quad \text{strongly in } L^2(0, T), \text{ as } h \to 0, \tag{19}$$

where $v_h = v_h(t)$ is the control of (6) constructed by the 2-grid algorithm above and $v = v(t)$ is the HUM control for the continuous wave equation which drives the initial data (u^0, u^1) to rest in time T.

Remark 2. Similar results can be obtained for the finite element semi-discretization of the $1 - d$ wave equation.

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