Stability results for the wave equation with indefinite damping

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December 19, 2002

*Partially supported by JNICT (Portugal), under projects PBIC/C/MAT/2139/95 and PPFUI-89/94.
†Partially supported by DGICYT(Spain) grant PB93-1023 and UE grant CHRX-CT94-0471.

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Abstract. The one-dimensional wave equation with damping of indefinite sign in a bounded interval with Dirichlet boundary conditions is considered. It is proved that solutions decay uniformly exponentially to zero provided the damping potential is in the BV-class, has positive average, is small enough and satisfies a finite number of further constraints guaranteeing that the derivative of the real part of the eigenvalues is negative when the damping vanishes. This sharp result completes a previous one by the first author showing that an indefinite sign damping always produces unstable solutions if it is large enough and it answers by the afirmative to a conjecture concerning small damping terms. The method of proof relies on the developments by S. Cox and the second author on the high frequency asymptotic expansion of the spectrum for damped wave equations and on the characterization of the decay rate in terms of the spectral abscissa.

1 Introduction

Consider the linear wave equation of the form

$$u_{tt} + 2a(x)u_t = u_{xx}, \tag{1}$$

on the interval $(0,1)$, together with Dirichlet boundary conditions and where $a \in L^\infty(0,1)$. In the case where the damping term a is nonnegative, and positive on an open subset, it is known that the energy of a solution will decay exponentially in time (see, for instance [CZ]). More precisely, let us introduce the energy of the system:

$$E(t) = \frac{1}{2}J\int_0^1 [|u_t(x,t)|^2 + |u_x(x,t)|^2]dx.$$

Then, under the conditions above, there exists $C, \omega > 0$ such that

$$E(t) \leq Ce^{-\omega t}E(0), \ \forall t > 0$$

for every finite energy solution of (1) with homogeneous Dirichlet boundary conditions.

In a similar way, if a is nonpositive, and negative on a nonempty open subset, all but the trivial solution will grow to infinity exponentially.

When the function a is allowed to change sign, the question of stability of the trivial solution of (1) becomes much more delicate. In particular, the techniques that are normally employed in the definite case, such as energy methods, will now fail.

In a sense, the problem becomes one of being able to measure *how positive* a given damping term is, and when it will cease to be enough to counteract the effect of the negative part, which will then take over, making the trivial solution unstable. In [CFNS], it was suggested that one way of measuring this might be via integrals of the form

$$I_k = \int_0^1 a(x)\sin^2(k\pi x)dx, \ \ k = 1, 2, \cdots.$$

The idea behind it is that these terms are related to the derivative of the eigenvalues of the associated eigenvalue problem, when it is considered as a perturbation of the undamped equation. In particular, if the k^{th} order term is positive, then the corresponding eigenvalue will be to the left of the imaginary axis for a small enough perturbation. It turns out that this is not enough to ensure stability, as eigenvalues which are to the left for small perturbations may move to the right as the pertubation increases, though not uniformly. Thus, it is also necessary to take into account the norm of the function a. This has been done in [F], where it was shown that for any damping term taking both positive and negative values on subsets of positive measure, and provided that its L^∞ norm is large enough, the trivial solution will be unstable (for a precise statement, see [F]).

All this suggests that a more adequate way of tackling the problem of the stability of the trivial solution of a linear wave equation of this type, is to consider a damping term of the form $\epsilon a(x)$, where ϵ is a positive parameter. In this way, it is possible to study what happens along a direction a, depending on the norm of the perturbation.

The techniques used in [F] are not appropriate to deal with the case of small ϵ, as in this case there are no real eigenvalues. A conjecture presented at the end of that paper was that, provided that the integrals I_k are uniformly positive, and that ϵ is small enough, then the trivial solution will be stable. The purpose of the present note is to show that this is indeed the case, when a is in BV.

The proof of this fact consists mainly of two steps. The first ensures that, for any positive values of the parameter ϵ, all existing eigenvalues can be followed to an eigenvalue of the undamped wave equation situated on the imaginary axis, as the parameter is decreased to zero. This shows that there can exist no eigenvalues coming from the point at infinity, and that, for small values of ϵ, it is possible to control the position of each eigenvalue with respect to the imaginary axis by looking at its first derivative. The second part of the proof then consists of showing that if all the integrals I_k are uniformly positive, for sufficiently small ϵ this control will be uniform in k and all eigenvalues will have negative real parts. Both aspects of the proof are basically variations on the methods used in [CZ] and rely on a shooting method that allows us to get high–frequency asymptotic expansions of the spectrum. As a consequence, our proof is restricted to the one–dimensional case.

The outline of the paper is as follows. In the next section we introduce some notation and state the main stability result for the linear problem. Sections 3 and 4 consist of the statement and proof of the several results needed to obtain the main result. Finally, in section 5 we discuss the results obtained and make some comments on, in particular, the n–dimensional problem.

2 Notation and main result

We consider the following wave equation on the interval $(0, 1)$

$$\begin{cases} u_{tt} + 2\epsilon a(x)u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x). \end{cases} \tag{2}$$

Here ϵ is a positive parameter, $a \in L^\infty(0,1)$ and the initial condition (ϕ, ψ) is taken to lie in the energy space $X = H_0^1(0,1) \times L^2(0,1)$, considered with the usual inner product defined by

$$< (f,g),(u,v) > = \int_0^1 f'\bar{u}' + g\bar{v}dx.$$

Our main result gives sufficient conditions for the trivial solution of (2) to be globally asymptotically stable in the space X for small values of ϵ, that is, $(u(\cdot,t), u_t(\cdot,t))$ converges to zero strongly in that space, as t goes to $+\infty$. In order to state these conditions we shall need some notation. We begin by writing equation (2) as a system of the form $U_t = L_\epsilon U$, where $U = (u, u_t)$, and $L_\epsilon : D(L_\epsilon) \to X$ with

$$L_\epsilon = \begin{bmatrix} 0 & I \\ A & -2\epsilon a \end{bmatrix}, \quad D(L_\epsilon) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \ ,$$

where $A = \partial^2/\partial x^2$. Denote by γ_k, $k = 0, 1, \cdots$ the eigenvalues of this last operator, with $\gamma_0 > \gamma_1 \cdots$, and by v_k the corresponding normalized eigenfunctions.

From [F] it is known that the following are necessary conditions for the trivial solution of (2) to be stable for small values of ϵ in a more general frame in which $A = \partial^2/\partial x^2 + b(x)$, where b is a bounded (possibly changing sign) potential:

(C1) $\gamma_k < 0$ for all $k = 0, 1, \cdots$.

(C2) The integral inequalities

$$I_k = \int_0^1 a(x) v_k^2(x) dx > 0,$$

hold for all $k = 0, 1, \cdots$.

The first condition has to do with the fact that each positive eigenvalue γ_k of the operator A gives rise to both a positive and a negative eigenvalue for the operator L_ϵ, which persist for all positive values of the parameter. On the other hand, a negative eigenvalue γ_k of A gives rise to a pair of purely imaginary eigenvalues of L_0. Designate these by $\lambda_k^\pm$. As ϵ increases they will change continuously with the parameter, and we can thus refer to the eigenvalues of L_ϵ by $\lambda_k^\pm(\epsilon)$. These functions are differentiable at $\epsilon = 0$, and the expression for their derivatives at this point can easily be seen to be

$$(\lambda_k^\pm)'(0) = -I_k.$$

Thus if any of the integrals I_k is negative, the corresponding pair of eigenvalues will be to the right of the imaginary axis for small values of ϵ and the trivial solution will be unstable.

It turns out that condition $(C1)$ together with a stronger version of condition $(C2)$, namely

(C3) The integral inequalities

$$I_k = \int_0^1 a(x) v_k^2(x) dx > c,$$

hold for all $k = 0, 1, \cdots$ and some positive constant c.

are enough to ensure stability of the trivial solution. More precisely, we have

Theorem 2.1 *Assume that $a \in BV(0,1)$ and that condition $(C3)$ holds. Then there exists a positive constant ϵ_0, depending on the function a, such that for all $0 < \epsilon < \epsilon_0$ the trivial solution of equation (2) is globally asymptotically stable in the space X.*

Furthermore, there exist constants C and ω depending only on a such that

$$E(t) \leq C e^{-\epsilon \omega t} E(0), \ \forall t > 0 \tag{3}$$

for every finite energy solution of (2) provided $0 < \epsilon < \epsilon_0$.

The proof of this Theorem will be done in the next sections, and consists mainly of the following steps. We study the eigenvalue problem associated with the operator L_ϵ and begin by showing that for any positive value of ϵ the high frequencies are asymptotically close to the eigenvalues of an associated eigenvalue problem whose spectrum is known. This will enable us to guarantee that the whole spectrum can be continued to the spectrum of the operator L_0 as ϵ approaches zero, and thus that there are no other eigenvalues – see [F]. Moreover, it will give us information on the location of the spectrum. In particular, the real part of the eigenvalues will be uniformly asymptotically close to minus ϵ times the average of a_0 of the function a. As, from the asymptotics for the eigenfunctions of the operator A, we have that $I_k \to a_0$ as $k \to \infty$, this, together with condition $(C3)$, shows that the high frequencies will be uniformly to the left of the imaginary axis. It also means that instead of condition $(C3)$ we could have considered condition $(C2)$ if it had been assumed that the damping coefficient had positive average.

We next prove that there exists $\epsilon_1 > 0$ such that, for all $\epsilon \in (0, \epsilon_1)$, there exists an integer N, independent of ϵ in this interval, with the property that for all $n > N$ the band where the eigenvalues $\lambda_n^\pm(\epsilon)$ are situated does not intersect the imaginary axis. These results are acomplished by adapting the estimates made in [CZ] to the present situation.

All this then enables us to consider only the remaining finite number of eigenvalues, and, by using the fact that the I_k's are positive, to show that there will exist a positive value ϵ_0 for which *all* eigenvalues are uniformly to the left of the imaginary axis for $\epsilon \in (0, \epsilon_0)$. This shows that the trivial solution is asymptotically stable for ϵ in this interval.

3 Basic estimates for the high frequencies

The eigenvalue problem associated with the operator L_ϵ can be written as

$$y_{xx} = \lambda^2 y + 2\epsilon\lambda a(x) y, \tag{4}$$

together with the boundary conditions $y(0) = y(1) = 0$. In this section we shall determine the asymptotic form of the eigenvalues of this problem. This is done by using a shooting method and follows closely the estimates made in section 5 of [CZ]. Consider the solution of (4) satisfying the initial conditions $y(0, \lambda) = 0$ and $y'(0, \lambda) = 1$. The idea is to use the following ansatz suggested by Horn [H]

$$y(x, \lambda) = \phi(x) e^{\lambda\xi(x)} \sum_{n=0}^{\infty} f_n(x)\lambda^{-n}, \quad f_0(x) \equiv 1, \tag{5}$$

in order to find a first approximation to the eigenvalue problem (4) that will enable us to obtain information on the asymptotic behaviour of the eigenvalues. By substituting this expression into equation (4) and equating like powers of λ yields as a first term ($n = 0$)

$$z_1(x, \lambda) = e^{\lambda x + \epsilon \int_0^x a(t)dt}.$$

This function will not, in general, satisfy equation (4), but

$$-z'' + \lambda^2 z + 2\epsilon\lambda a(x) z = -\left[\epsilon^2 a^2(x) + \epsilon a'(x)\right] z \tag{6}$$

(note that when a is of bounded variation the standard weak form of this equation is well–defined). We now use the solutions of this equation to obtain estimates for the zeros of the function $\lambda \to y(1, \lambda)$, which coincide with the eigenvalues of problem (4). Via reduction of order, we have that (6) also has

$$z_2(x, \lambda) = z_1(x, \lambda) \int_0^x z_1^{-2}(t, \lambda)dt = e^{\lambda x + \epsilon \int_0^x a(t)dt} \int_0^x e^{-2\lambda t - 2\epsilon \int_0^t a(s)ds} dt \tag{7}$$

as a solution which satisfies

$$z_2(0, \lambda) = 0 \text{ and } z_2'(0, \lambda) = 1.$$

In the very particular case where $a' = -a^2$, that is, $a(x) = (x + c)^{-1}$ for some positive constant c, the functions z_2 and y coincide, and the spectrum is given by the zeros of the function that takes λ to $\int_0^1 e^{-2\lambda x}(x + c)^{-2}dx$.

We now estimate the difference between z_2 and y in the case where a is a general function of bounded variation, and use those estimates to prove that in this case the roots of $y(1, \cdot)$ are asymptotically close to those of $\lambda^{-1}\sinh(\lambda + \epsilon a_0)$. In order to do this, we begin by obtaining some estimates for z_2. Multiplying equation (4) by $\bar{y}$ and separating into real and imaginary parts gives (from the imaginary part) that complex eigenvalues must satisfy

$$Re(\lambda) = -\epsilon \frac{\displaystyle\int_0^1 a(x)|y(x)|^2 dx}{\displaystyle\int_0^1 |y(x)|^2 dx},$$

and so $-\epsilon a^+ \leq Re(\lambda) \leq -\epsilon a^-$, where a^- and a^+ denote, respectively, the essential infimum and supremum of a on $(0, 1)$. This means that in our estimates we only have to consider values of λ such that $|Re(\lambda)| \leq \epsilon\bar{a}$, where $\bar{a} = ||a||_{L^\infty(0,1)}$ (note that for ϵ small enough no real eigenvalue exist).

Lemma 3.1 *Let $a \in BV(0,1)$ and T_a be its total variation on that interval. Then*

$$|z_2(x,\lambda)| \le \frac{e^{6\epsilon\bar{a}}(1+\epsilon\bar{a})}{|\lambda|}$$

and

$$\left| z_2(x,\lambda) - \frac{\sinh\left(\lambda x + \epsilon \int_0^x a(t)dt\right)}{\lambda} \right| \le \frac{\epsilon e^{6\epsilon\bar{a}}}{|\lambda|^2}\left(\bar{a} + \epsilon\bar{a}^2 + T_a\right).$$

Proof: Integrating (7) by parts gives

$$z_2(x,\lambda) = \frac{1}{\lambda}\sinh\left[\lambda x + \epsilon\int_0^x a(t)dt\right] - \frac{\epsilon}{\lambda} e^{\lambda x + \epsilon\int_0^x a(t)dt}\int_0^x a(t)e^{-2\lambda t - 2\epsilon\int_0^t a(s)ds}dt. \tag{8}$$

¿From this we have

$$|z_2(x,\lambda)| \le \frac{e^{6\epsilon\bar{a}}(1+\epsilon\bar{a})}{|\lambda|}.$$

Integrating again by parts in (8) we obtain

$$z_2(x,\lambda) = \frac{1}{\lambda}\sinh\left[\lambda x + \epsilon\int_0^x a(t)dt\right] - \frac{\epsilon}{2\lambda^2} e^{\lambda x + \epsilon\int_0^x a(t)dt}\left[a(0) - a(x)e^{-2\lambda x - 2\epsilon\int_0^x a(t)dt} + \right.$$
$$\left. + \int_0^x \left[a'(t) - 2\epsilon a^2(t)\right] e^{-2\lambda t - 2\epsilon\int_0^t a(s)ds}dt\right],$$

from which the result follows. □

Proposition 3.2 *If $a \in BV(0,1)$, then there exists a constant C_0 such that*

$$\left| y(x,\lambda) - \frac{\sinh\left(\lambda x + \epsilon \int_0^x a(t)dt\right)}{\lambda} \right| \le \frac{C_0(\epsilon,\bar{a},T_a)}{|\lambda|^2},$$

uniformly for $0 < x < 1$ and $-\epsilon\bar{a} \le Re(\lambda) \le \epsilon\bar{a}$. Furthermore, $C_0 = c\epsilon + o(\epsilon^2)$, for some positive constant c.

Remark 3.1 *We can choose $C_0(\epsilon)$ to be of the form $C_0(\epsilon) = \epsilon C_0$ with C_0 a positive constant depending only on a, provided ϵ is taken to be bounded. Since we are assuming ϵ to be small enough, in what follows we will assume that the constant $C_0(\epsilon,\bar{a},T_a)$ in Proposition 3.2 has this form.*

Proof: We begin by noting that y is a solution of

$$-z'' + \lambda^2 z + 2\epsilon\lambda a(x)z + \epsilon[\epsilon a^2(x) + a'(x)]z = \epsilon[\epsilon a^2(x) + a'(x)]z, \;\; z(0) = 0, \; z'(0) = 1.$$

Thus, if $K(x,t,\lambda) = z_1(x,\lambda)z_2(t,\lambda) - z_1(t,\lambda)z_2(x,\lambda)$, y is also a solution of the integral equation

$$y(x,\lambda) = z_2(x,\lambda) + \epsilon\int_0^x K(x,t,\lambda)\left[\epsilon a^2(t) + a'(t)\right] y(t,\lambda)dt.$$

As in [CZ], we solve this equation in series form, that is, we look for solutions of the form

$$y(x,\lambda) = \sum_{n=0}^{\infty} S_n(x,\lambda),$$

where $S_0 = z_2$ and

$$\begin{aligned} S_n(x,\lambda) &= \epsilon \int_0^x K(x,t,\lambda)\left[\epsilon a^2(t) + a'(t)\right] S_{n-1}(t,\lambda)dt \\ &= \epsilon \int \cdots \int_{0\le t_1\le\cdots\le t_{n+1}=x} z_2(t_1,\lambda)\Pi_{i=1}^n \left[K(t_{i+1},t_i,\lambda)\left[\epsilon a^2(t)+a'(t)\right]\right] dt_1\cdots dt_n. \end{aligned}$$

¿From the expression for z_2 in (7) it follows that K can be written in terms of z_1 only as

$$K(x,t,\lambda) = z_1(x,\lambda)z_1(t,\lambda)\int_x^t z_1^{-2}(s,\lambda)ds.$$

By substituting now z_1 and rearranging we have

$$\begin{aligned} K(x,t,\lambda) &= -\frac{1}{\lambda}\sinh\left[\lambda(x-t)+\epsilon\int_t^x a(s)ds\right] - \\ &\quad -\frac{\epsilon}{\lambda}e^{\lambda(x+t)+\epsilon\int_0^x a(s)ds+\epsilon\int_0^t a(s)ds}\int_x^t a(s)e^{-2\lambda s-2\epsilon\int_0^s a(r)dr}ds, \end{aligned}$$

from which it follows that

$$\begin{aligned} |K(x,t,\lambda)| &\le \frac{1}{|\lambda|}e^{\alpha(x-t)+\epsilon\int_t^x a(s)ds} + \frac{\epsilon\bar{a}}{|\lambda|}\int_t^x \left|e^{\lambda(x-s)+\lambda(t-s)+\epsilon\int_s^x a(r)dr+\epsilon\int_s^t a(r)dr}\right| ds \\ &\le \frac{e^{2\epsilon\bar{a}(x-t)}}{|\lambda|} + \frac{\epsilon\bar{a}}{|\lambda|}\int_0^1 e^{\epsilon\bar{a}((x-s)-(t-s)+(x-s)+(s-t))}ds \\ &\le \frac{(1+\epsilon\bar{a})e^{2\epsilon\bar{a}(x-t)}}{|\lambda|} \\ &\le \frac{(1+\epsilon\bar{a})e^{2\epsilon\bar{a}}}{|\lambda|}. \end{aligned}$$

Using this in the expression for S_n yields

$$\begin{aligned} |S_n(x,\lambda)| &\le \frac{e^{6\epsilon\bar{a}}(1+\epsilon\bar{a})^{n+1}}{|\lambda|^{n+1}}e^{2\epsilon\bar{a}}\int\cdots\int_{0\le t_1\le\cdots\le t_{n+1}=x}\Pi_{i=1}^n|\epsilon^2a^2(t_i)+\epsilon a'(t_i)|dt_1\cdots dt_n \\ &\le \frac{e^{8\epsilon\bar{a}}(1+\epsilon\bar{a})^{n+1}}{|\lambda|^{n+1}}\frac{(\epsilon^2\bar{a}^2+\epsilon T_a)^n}{n!}, \end{aligned}$$

and, for $|\lambda|\ge 1$,

$$|S_n(x,\lambda)| \le \frac{e^{8\epsilon\bar{a}}(1+\epsilon\bar{a})^{n+1}}{|\lambda|^2}\frac{(\epsilon^2\bar{a}^2+\epsilon T_a)^n}{n!}.$$

This proves that the S_n are summable and we have

$$\begin{aligned} &\left|y(x,\lambda) - \frac{\sinh\left(\lambda x + \int_0^x a(t)dt\right)}{\lambda}\right| \le \\ &\le |y(x,\lambda)-z_2(x,\lambda)| + \left|z_2(x,\lambda) - \frac{\sinh\left(\lambda x+\int_0^x a(t)dt\right)}{\lambda}\right| \\ &\le \sum_{n=1}^\infty |S_n(x,\lambda)| + \frac{\epsilon e^{6\epsilon\bar{a}}}{|\lambda|^2}(\bar{a}+\epsilon\bar{a}^2+T_a) \\ &\le \frac{e^{8\epsilon\bar{a}}(1+\epsilon\bar{a})}{|\lambda|^2}\left[e^{(1+\epsilon\bar{a})(\epsilon^2\bar{a}^2+\epsilon T_a)}-1\right] + \frac{\epsilon e^{6\epsilon\bar{a}}}{|\lambda|^2}(\bar{a}+\epsilon\bar{a}^2+T_a), \end{aligned}$$

as desired. □

As in [CZ], this result together with Rouché's Theorem can now be used to obtain that the zeros of the function $y(1,\lambda)$ will have to lie in a neighbourhood of the roots of $\lambda^{-1}\sinh(\lambda+\epsilon a_0)$, that is

$$-\epsilon a_0 \pm in\pi, \quad n = \pm 1, \pm 2, \cdots.$$

In order to do this, define

$$\Gamma_{\pm n} \equiv \{z : |z+\epsilon a_0 \mp in\pi| = 2\epsilon C_0/(n\pi)\}, \quad n \in \mathbb{N}$$

where, recall, C_0 is assumed to be independent of ϵ.

Lemma 3.3 *If $z \in \Gamma_{\pm n}$, then $|\sinh(z+\epsilon a_0)| > \epsilon C_0/|z|$ for small enough ϵ and every $n \in \mathbb{N}$.*

Proof: It is of course enough to show the result for the curves Γ_n, as the eigenvalues appear in pairs of complex conjugates. If $z \in \Gamma_n$, then it can be written as $-\epsilon a_0 + in\pi + \epsilon\rho_n e^{i\theta}$, with $\rho_n = 2\epsilon C_0/(n\pi)$ and $\theta \in [0, 2\pi)$. Hence, for $z \in \Gamma_n$,

$$|\sinh(z+\epsilon a_0)|^2 = \sinh^2[\epsilon\rho_n\cos(\theta)] + \sin^2[\epsilon\rho_n\sin(\theta)].$$

This function achieves its minimum at $\theta = \pi/2$ and thus $|\sinh(z+\epsilon a_0)| \geq |\sin(\epsilon\rho_n)|$. For small enough ϵ, to prove the inequality of the lemma it is thus sufficient to show that $\sin(\epsilon\rho_n) > \epsilon C_0/|z|$. The right–hand side of this inequality satisfies

$$\frac{\epsilon C_0}{|z|} = \frac{\epsilon C_0}{|-\epsilon a_0 + in\pi + \epsilon\rho_n e^{i\theta}|} \leq \frac{n\pi\epsilon C_0}{n^2\pi^2 - 2\epsilon C_0},$$

for small enough ϵ. Furthermore, as ρ_n is also small, we only have to prove that

$$\epsilon\rho_n + o(\epsilon^3\rho_n^3) > \frac{n\pi\epsilon C_0}{n^2\pi^2 - 2\epsilon C_0}.$$

But this is equivalent to

$$n\pi C_0 - \frac{4\epsilon C_0^2}{n\pi} + (n^2\pi^2 - 2\epsilon C_0)o(\epsilon^2\rho_n^3) > 0,$$

which is clearly true for every $n \in \mathbb{N}$ if ϵ small enough.□

Theorem 3.4 *If $a \in BV(0,1)$ then there exists a positive number ϵ_1 such that the operator L_ϵ has one and only one simple eigenvalue in the region Γ_n, and another in the region Γ_{-n} for each $n \in \mathbb{N}$ and each $\epsilon \in (0, \epsilon_1)$. These eigenvalues exhaust the spectrum of L_ϵ.*

Proof: From Proposition 3.2 and the previous Lemma we have that

$$\left|y(1,\lambda) - \frac{\sinh(\lambda+\epsilon a_0)}{\lambda}\right| \leq \frac{C_0}{|\lambda|^2} < \left|\frac{\sinh(\lambda+\epsilon a_0)}{\lambda}\right|.$$

Hence, by Rouché's Theorem, $y(1,\lambda)$ has the same number of zeros as the function $\sinh(\lambda+\epsilon a_0)$ inside each $\Gamma_{\pm n}$ and in the complement of their union. □

This result shows that the spectrum of the operator L_ϵ consists only of the eigenvalues $\lambda_n^\pm$ of this operator which can be continued, as ϵ goes to zero, to an eigenvalue of L_0 which is situated on the imaginary axis.

4 Proof of Theorem 2.1

We are now ready to prove Theorem 2.1.
Proof of Theorem 2.1:

First of all we prove the asymptotic stability of the trivial solution. Then we obtain the uniform decay estimate (3).

¿From Theorem 3.4 we have that if $2C_0/(n\pi) < a_0$, that is $n > 2C_0/(a_0\pi)$, then the curves $\Gamma_{\pm n}$ are totally to the left of the imaginary axis. More precisely, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$sup_{|n|\geq n_0} Re(\lambda_n^{\pm}) \leq -C\epsilon$$

for all $0 < \epsilon < \epsilon_1$. As $C_0 = c\epsilon + o(\epsilon^2)$, we have that for small enough ϵ it is enough to have $n \geq n_0$, where n_0 is the smallest integer which is greater than $2c/(a_0\pi)$.

¿From $(C3)$ we now have that there exists ϵ_2 such that the remaining $2(n_0 - 1)$ eigenvalues will also be to the left of the imaginary axis for all $\epsilon \in (0, \epsilon_2)$. Thus if we now take ϵ_0 to be the minimum of the ϵ_j's, $j = 1, 2$ (where ϵ_1 was defined in Theorem 3.4), it follows that the spectrum of the operator L_ϵ is uniformly to the left of the imaginary axis for $\epsilon \in (0, \epsilon_0)$ and more precisely that $Re(\lambda_n^{\pm}) \leq -C\epsilon$ for all $n \in \mathbb{N}$. This proves that the trivial solution of (2) is globally asymptotically stable for these values of the parameter ϵ.

Let us discuss now the uniform decay rate (3). In [CZ] it was proved that the decay rate coincides with the spectral abscissa. More precisely, it was shown that the eigenfunctions of L_ϵ constitute a Riesz basis of the energy space. This implies the existence of a constant $C(\epsilon) > 0$ such that

$$E(t) \leq C(\epsilon) e^{\omega(\epsilon)t} E(0), \ \forall t > 0 \tag{9}$$

for all finite energy solutions where $\omega = sup_{n \in \mathbb{N}} Re(\lambda_n^{\pm})$. We have proved that $\omega \leq -C\epsilon$. Thus it is sufficient to show that the constant $C(\epsilon)$ remains bounded as $\epsilon \to 0$.

To do this we observe that $C(\epsilon)$ only depends on the constants appearing on the Riesz basis condition. Since we have a good convergence in X of the eigenfunctions of L_ϵ to the eigenfunctions of the limit conservative problem as $\epsilon \to 0$, by virtue of the bounds of the form of those used in the proof of Theorem 3.4, it is natural to expect this constant to be bounded and, actually, to converge to $C(0) = 1$.

To do this rigorously it is sufficient to reproduce the arguments of [CZ, section 6] based on the use of Bari's Theorem. Note that in this case, as we are dealing with the one–dimensional problem and ϵ is small, all the eigenvalues will be simple and thus the associated Jordain Chains used to construct the biorthogonal sequence needed to apply Bari's Theorem will be of length one. □

5 Discussion

In this paper we have shown that it is possible for the trivial solution of (2) to be stable when the damping coefficient changes sign, provided that it is in BV, satisfies certain integral inequalities (in particular is of positive average) and its norm is small enough. One way of interpreting this is to say that, in this case, the coupling induced by the diffusion term is sufficiently strong to ensure that all modes decay to zero as time goes to infinity. From this and the instability results in [F], we might expect it to be possible to show that there existed a positive value of ϵ, say $\bar{\epsilon}$, such that the trivial solution would be stable for $\epsilon < \bar{\epsilon}$ and unstable for $\epsilon > \bar{\epsilon}$. However, the problem seems to be more complicated than this, as suggested by some of the (numerical) results presented in [FGK] for a finite–dimensional system consisting of an m^{th}–order discretization of equation (2). These show that it is possible to have alternate stability and instability intervals (in ϵ), that is, the trivial solution can be, for instance, unstable, then become stable, and then unstable again as ϵ is increased. It would be interesting to see if these stability switches can also occur in the infinite dimensional problem, and to understand what the mechanisms responsible for this type of behaviour are.

We have not treated here the more general case of a damped wave equation with a zero order potential b in $L^2(0,1)$, i. e.

$$u_{tt} - u_{xx} + 2\epsilon a(x) u_t + b(x) u = 0.$$

Indeed, the developments of section 3 do not apply directly in this case. With the same ansatz one is lead to consider

$$-z'' + \lambda^2 z + 2\epsilon\lambda a(x) z + \epsilon[\epsilon a^2(x) + a'(x)]z = \epsilon[\epsilon a^2(x) + a'(x)]z - b(x)z, \ \ z(0) = 0, \ z'(0) = 1.$$

We are now unable to obtain estimates of the form of those of Proposition 3.2 with a constant C_0 of the order of ϵ since b is now of the order of unity. In order to handle this problem one should get a suitable modification of the particular solution (7) by means of an ansatz similar to (5) (inspired in[H]) but taking into account the zero order potential b.

If we were able to solve this problem, then the variable coefficient case

$$\rho(x) u_{tt} - (\sigma(x) u_x)_x + 2\epsilon a(x) u_t + b(x) u = 0$$

would be treatable since, as it is well known, this equation can be reduced to have constant coefficients in the principle part by a suitable change of the x-variable and unknown.

We have not considered here the n–dimensional problem. In view of the results in [L] the conjecture has to be modified when $n \geq 2$. Indeed, in [L] it was shown that when the damping is non-negative, the decay rate coincides with the supremum of the spectral abscissa and the infimum of the ammount of damping concentrated on all rays of geometric optics. Taking this into account, it is natural to expect Theorem 2.1 to hold in several dimensions if, in addition to (C3), the damping a is assumed to be such that there exists some $T > 0$ such that the infimum of the averages of the damping along all the rays of geometric optics of lenght T lying in the domain and reflected at the boundary is strictly positive.

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