FINITE DIFFERENCE APPROXIMATION OF HOMOGENIZATION PROBLEMS FOR ELLIPTIC EQUATIONS*

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Abstract. In this paper, the problem of the approximation by finite differences of solutions to elliptic problems with rapidly oscillating coefficients and periodic boundary conditions is considered. The mesh-size is denoted by h while ε denotes the period of the rapidly oscillating coefficient. Using Bloch wave decompositions, we analyze the case where the ratio h/ε is rational. We show that if h/ε is kept fixed, being a rational number, even when $h, \varepsilon \to 0$, the limit of the numerical solution does not coincide with the homogenized one obtained when passing to the limit as $\varepsilon \to 0$ in the continuous problem. Explicit error estimates are given showing that, as the ratio h/ε approximates an irrational number, solutions of the finite difference approximation converge to the solutions of the homogenized elliptic equation. We consider both the 1-d and the multi-dimensional case. Our analysis yields a quantitative version of previous results on numerical homogenization by M. Avellaneda, Th.Y. Hou and G. Papanicolaou [1].

Key words. finite differences, homogenization, multiscale, Bloch waves

AMS subject classifications. 35B27, 65N06, 65N12

1. Introduction. Homogenization of elliptic equations with rapidly oscillating periodic coefficients is by now a well understood problem. Roughly speaking, the limit equation turns out to be elliptic, with constant coefficients, and the effective coefficients may be computed by solving an auxiliary problem on the unit periodic cell. The interested reader may find a fairly complete study of this problem in the book by A. Bensoussan, J.L. Lions and G. Papanicolaou [2].

One of the main applications of homogenization theory is related to the numerical resolution of elliptic problems with rapidly oscillating coefficients. More precisely, in agreement with the homogenization result mentioned above, instead of approximating the problem with rapidly oscillating coefficients one can solve numerically the homogenized one. The later is much easier to handle since its coefficients are constants. For a long time this has been the only feasible numerical approach to problems with rapidly oscillating coefficients since the direct application of classical finite difference or finite element methods required meshes of a size h asymptotically smaller than the period of the rapidly oscillating coefficient. This made computations unfeasible in practice.

A new approach to the problem was proposed by B. Engquist in [15] who introduced the notion of convergence essentially independent of the wave of the oscillation in the approximation of oscillatory solutions of hyperbolic problems. The proof of this type of convergence relies on fundamental results in Ergodic Theory and convergence of random numbers in Monte Carlo methods (see H. Niederreiter [28]). This approach has been successfully applied, in particular, in the context of numerical methods of hyperbolic problems (B. Engquist and Th.Y. Hou [16], B. Engquist and J.-G. Liu [17]),

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finite difference approximations of elliptic equations (M. Avellaneda, Th.Y. Hou and G. Papanicolaou [1]), multigrid methods of elliptic equations (B. Engquist and E. Luo [18]).

In recent years important progresses have been made also in the context of finite elements. The multiscale finite element method (MsFEM) was introduced by Th.Y. Hou, X.H. Wu and Z. Cai in [20] and [21] for solving elliptic problems with oscillatory coefficients. The idea of MsFEM is to capture small scale information through the base functions (which in general are of oscillatory nature) constructed in the elements whose sizes are much larger than the small scales of the problem. For a detailed analysis of the MsFEM we refer to [12], [13], [14], and [21].

Another contribution is due to A.M. Matache, I. Babuska and C. Schwab [25] and A.M. Matache and C. Schwab [26] who, using Fourier-Bochner representation of solutions of the oscillatory problem, proved that the numerical approximation problem may be attacked directly, without using the homogenization theory, by constructing special Galerkin bases adapted to the coefficients of the equation. This approach has also been successfully applied to equations in rapidly oscillating perforated domains and, in principle, does not require periodicity assumptions.

In the present work we present a complete study of the homogenization of the finite difference schemes approximating a family of elliptic equations with rapidly oscillating coefficients.

This problem was previously investigated by S.M. Kozlov in [23]. He approximated the homogenized coefficients of second order elliptic equations using linear polynomial interpolates of the coefficients of the finite difference system. Later, A. Piatnitski and E. Remy in [31] studied the asymptotic behavior as $\varepsilon \to 0$ of the effective coefficients for a family of random finite difference schemes. They proved the discrete analogue of the compensated compactness lemma, adapted to difference operators, originally introduced by F. Murat and L. Tartar in [27] in the continuous case. With it, they defined the H-convergence of difference operators.

In this work we study how the homogenization of the solutions of finite difference approximations of elliptic equations with rapidly oscillating coefficients is related to the homogenized solutions of the corresponding differential operators. A first result by Avellaneda, Hou and Papanicolaou in [1] shows that if the mesh-size h is of the order of ε, the period of the oscillating coefficients, and the ratio h/ε is irrational so that the numerical mesh correctly samples the spatial domain (with respect to the coefficients of the equation under consideration), then solutions of the numerical approximation do converge to the solution of the continuous homogenized problem. Moreover, in the multidimensional case, the finite difference approximation scheme does not provide the right homogenized coefficients unless the components of the ratio h/ε are close enough to an integer number.

In this paper, we further pursue the approach in [1]. We analyze the case where the ratio h/ε is rational, using the Bloch wave decomposition at the discrete level. We obtain explicit error estimates that depend, in particular, on the denominator of the rational number h/ε. It is shown that the error tends to zero as the denominator tends to infinity and therefore, roughly speaking, the error tends to zero as h/ε approaches an irrational number. Thus, our results are in agreement with those of [1] mentioned above and provide a constructive way of recovering them by means of classical results on diophantine approximation and explicit error bounds.

To do this we use the Bloch wave decomposition both at continuous and discrete level. We refer to the following works and the references therein for an introduction to the theory of Bloch waves: F. Bloch [3], A. Bensoussan, J.L. Lions and G. Papanicolaou [2], C. Conca, J. Planchard and M. Vanninathan [9].

Our work is inspired in that by C. Conca and M. Vanninathan [10] where a new proof of the convergence of solutions of elliptic problems with rapidly oscillating periodic coefficients towards the homogenized solution was obtained using Bloch waves decomposition. In fact, in [10], it was shown that the problem of homogenization may be reduced to the analysis of the first Bloch mode. It was then established that the Bloch waves representing the periodic medium approach Fourier waves representing the homogenized one and this fact may be easily interpreted as an homogenization result in the Fourier space.

We follow the same approach to analyze the behavior of u_h^ε (this stands for the numerical approximation with numerical mesh-size h of the elliptic problem with rapidly oscillating coefficients of period ε), as $\varepsilon \to 0$ and $h \to 0$. In particular, our approach allows analyzing whether u_h^ε converges to the solution u^* of the homogenized equation. As we shall see, this can indeed be proved provided h/ε approximates an irrational number.

But there is a difference on the way h/ε has to be chosen between the one-dimensional and the multi-dimensional case. There are two reasons. First, we need to guarantee that the discrete operator is positive semi-definite and also to estimate the difference between the homogenized coefficients of the continuous problem and the homogenized coefficients associated to the discrete system. According to this, in the multi-dimensional case, further restrictions on h/ε are needed, other then requiring that h/ε approaches an irrational number.

We restrict ourselves to the simplest case of periodic boundary conditions with ε tending to zero along a sequence such that the space-domain contains exactly an integer number of periodic cells of the rapidly oscillating coefficients. This simplifies significantly the Bloch representation of the solution both at the continuous and discrete level. Important further developments should be done to handle general domains. We refer to C. Conca, R. Orive and M. Vanninathan [7] and [8] for the analysis of the problem of homogenization of continuous elliptic problems in general bounded domains with Dirichlet boundary conditions. Finally, we note that numerical experiments using Bloch waves were performed by C. Conca and S. Natesan in [6] providing a better approximation to the exact homogenized solution than the classical first order corrector in the case of smooth coefficients.

Organization of the paper. In Section 2 we present our main results and discuss its significance. In Section 3 we define the Bloch waves for discrete spaces, and present several properties of Bloch eigenvalues and eigenvectors. The proof of these properties are given in Appendix C. Section 4 is devoted to the study of the numerical problem using Bloch waves. We prove the error estimates in numerical homogenization in several space dimensions. In Section 5, we recover the results of [1] in the 1-d case when h/ε is irrational using our analysis and classical results in diophantine Number Theory. In Appendix A we discuss the continuous problem using the Bloch wave decomposition and in Appendix B we give a more explicit and simpler proof of the numerical homogenization result in the 1-dimensional case based on the explicit representation of solutions.

Notation. All along this article the following notations will be used:

$$\text{For any } p = (p_1, \dots, p_d) \in \mathbb{R}^d,\ p_i \neq 0,\ \text{ we denote } \frac{1}{p} = \left(\frac{1}{p_1}, \cdots, \frac{1}{p_d} \right).$$

For $x, z \in \mathbb{R}^d$, we write $xz = (x_1z_1, \ldots, x_dz_d) \in \mathbb{R}^d$.
For $x, z \in \mathbb{R}^d$, we denote $x \cdot z = x_1z_1 + \cdots + x_dz_d$.
For $x \in \mathbb{R}^d$, its euclidean norm is denoted by $|x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$.
For any $p \in \mathbb{N}^d$, we write $\dot{p} = p_1 \cdot p_2 \cdot \cdots \cdot p_d$, the product of all the components p_i.

The space of Y-periodic functions in $H^s_{loc}(\mathbb{R}^d)$ will be denoted by $H^s_{\#}(Y)$. The norm in the space $H^s(Y)$ will be denoted as $\|\cdot\|_s$. In particular, the norm in $L^2(Y)$ (which coincides with $L^2_{\#}(Y)$) is $\|\cdot\|_0$.

2. Presentation of the problems and main results.

2.1. Homogenization problem with periodic boundary conditions. Let us introduce the continuous problem to be discretized later on. We consider the elliptic operator

$$A = -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right), \quad y \in \mathbb{R}^d \tag{2.1}$$

where the coefficients (a_{ij}) satisfy:

$$\begin{cases} a_{ij} \in L^\infty_{\#}(Y) \quad \text{where} \quad Y = [0, 2\pi[^d, \text{ i.e., each coefficient } a_{ij} \text{ is a} \\ Y\text{-periodic, bounded measurable function defined in } \mathbb{R}^d, \text{ such that} \\ \exists \alpha > 0 : \quad a_{ij}(y)\eta_i\eta_j \geq \alpha|\eta|^2 \quad \forall \eta \in \mathbb{R}^d, \\ a_{ij} = a_{ji} \quad \forall i, j = 1, \ldots, d. \end{cases} \tag{2.2}$$

For each $\varepsilon > 0$, we consider the operator A^ε:

$$A^\varepsilon = -\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}(x)\frac{\partial}{\partial x_j}\right) \quad \text{with} \quad a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad x \in \mathbb{R}^d.$$

Associated with A^ε, we consider the following periodic boundary-value problem

$$\begin{cases} A^\varepsilon u^\varepsilon = f \qquad \text{in} \quad Y, \\ u^\varepsilon \in H^1_{\#}(Y), \quad m(u^\varepsilon) = \dfrac{1}{|Y|}\displaystyle\int_Y u^\varepsilon dx = 0. \end{cases} \tag{2.3}$$

If f is in $L^2_{\#}(Y)$ with $m(f) = 0$, then the equation (2.3) is well-posed: it has a unique solution. In this paper, we consider the case where the space-domain contains exactly an integer number of periodic cells of the coefficients $\{a^\varepsilon_{ij}\}$, i.e.,

$$\frac{1}{\varepsilon} = s \in \mathbb{N}. \tag{2.4}$$

The limit of the solutions of (2.3) as $\varepsilon \to 0$ solves an elliptic equation related to the following constant coefficient homogenized operator A^*:

$$A^* = -a^*_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.5}$$

The homogenized coefficients a^*_{ij} are defined as follows

$$2a^*_{ij} = \frac{1}{|Y|}\int_Y \left(2a_{ij} - \frac{\partial a_{j\ell}}{\partial y_\ell}\chi^i - \frac{\partial a_{i\ell}}{\partial y_\ell}\chi^j\right) dy, \tag{2.6}$$

where, for any $k = 1, \dots, d$, χ^k is the unique solution of the cell problem

$$\begin{cases} A\chi^k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} & \text{in} \quad Y, \\ \chi^k \in H^1_\#(Y), & m(\chi^k) = 0. \end{cases} \tag{2.7}$$

The classical theory of homogenization provides the following result (see [2]):

THEOREM 2.1. *Let the coefficients $a_{k\ell}$ satisfy assumptions (2.2) and u^ε and u^* be the solutions of (2.3) and (2.8), respectively. Then, if f belongs to $L^2_\#(Y)$ with $m(f) = 0$, the sequence of solutions u^ε of (2.3) converges weakly in $H^1(Y)$, as $\varepsilon \to 0$, to the so-called homogenized solution u^* characterized by*

$$\begin{cases} A^* u^* = f & \textit{in} \quad Y, \\ u^* \in H^1_\#(Y), & m(u^*) = 0. \end{cases} \tag{2.8}$$

Furthermore, we have

$$\|u^\varepsilon - u^*\|_0 \le c\varepsilon \|f\|_0. \tag{2.9}$$

In Appendix A we give a new proof of (2.9) using Bloch waves as in [10], where the Dirichlet problem was studied both in bounded and unbounded domains. Here, we adapt the Bloch waves approach to the case of periodic boundary conditions.

2.2. The finite-difference scheme for elliptic PDEs. We now introduce a finite difference approximation of (2.3).

Let $h = (h_1, \dots, h_d)$ be a vector with positive components

$$h_i = \frac{2\pi}{n_i} \quad \text{with } n_i \in \mathbb{N}, \tag{2.10}$$

and denote by Γ_h the following subgroup of $\mathbb{R}^d$:

$$\Gamma_h = \{y \in Y \mid y = (z_1 h_1, \dots, z_d h_d),\ z_j \in \mathbb{Z},\ 1 \le j \le d\}. \tag{2.11}$$

$\|\cdot\|_h$ denotes the discrete L^2-norm in a mesh Γ_h

$$\|f\|_h^2 = h_1 \cdots h_d \sum_{x \in \Gamma_h} |f(x)|^2,$$

and $(\cdot,\cdot)_h$ its discrete inner product

$$(f,g)_h \overset{\text{def}}{=} \sum_{x \in \Gamma_h} h_1 h_2 \cdots h_d f(x) \bar{g}(x).$$

Let e_j, $j = 1, \dots, d$ be the unit vectors in the coordinate directions. Define

$$\nabla_i^{\pm h} g(x) = \frac{1}{\pm h_i}[g(x \pm h_i e_i) - g(x)],$$

with $i = 1, \dots, d$. Note that for any $x \in \Gamma_h$, $\nabla_i^{\pm h} g(x)$ is an approximation (of order 2) of $\partial_i g$ at $(x \pm h_i\, e_i/2)$. For numerical purposes it is therefore natural to replace the

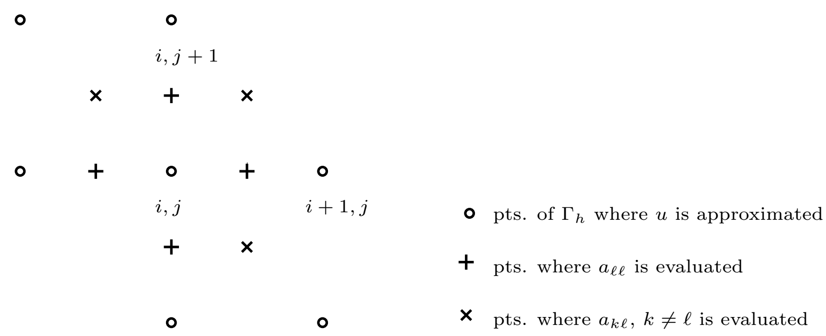

FIG. 2.1. *Points involved in the finite difference approximation of* (Au) *at* $x = (ih_1, jh_2)$

boundary value problem (2.3) by the discrete one

$$\sum_{i,j=1}^{d} -\nabla_i^{-h}\left[a_{ij}^{\varepsilon}(x(i,j))\nabla_j^{+h}u_h^{\varepsilon}(x)\right] = f(x), x \in \Gamma_h, \tag{2.12}$$

$$u_h^{\varepsilon}(x + 2\pi m) = u_h^{\varepsilon}(x) \quad \forall m \in \mathbb{Z}^d (2\pi\text{-periodicity}), \tag{2.13}$$

$$\sum_{x \in \Gamma_h} u_h^{\varepsilon}(x) = 0, \tag{2.14}$$

where

$$x(i,j) = x + \frac{1}{2}h_i e_i + (1 - \delta_{ij})\frac{1}{2}h_j e_j, \qquad (\text{Kronecker } \delta_{ij}). \tag{2.15}$$

This discrete method is convergent of order 1.

When f is continuous, $f(x)$ are the values of f for $x \in \Gamma_h$. When f is not continuous, the value $f(x)$ may be replaced by the average of f on a cell around the point x of the mesh. We assume that

$$\sum_{x \in \Gamma_h} f(x) = 0, \tag{2.16}$$

to guarantee that (2.12)–(2.14) is well-posed. This is consistent with the assumption $m(f) = 0$ made in the continuous problem.

Note that the numerical scheme provides approximations u_h^{ε} of u^{ε} at the points of the mesh Γ_h (points denoted by $\circ$ in the Figure 2.1 above). However, the coefficients a_{ij}^{ε} of the discrete system (2.12)–(2.14) are evaluated at the points $x(i,j)$ defined in (2.15). Observe that these points do not belong to the mesh Γ_h. In fact, for $i = j$, these points are located symmetrically between x and $x + h_i e_i$ (points denoted by $+$ in Figure 2.1). Finally, when $i \neq j$, $x(i,j)$ is located in the diagonal.

From (2.12), for u, v satisfying (2.13), the following identity holds

$$\sum_{x \in \Gamma_h} \sum_{i,j=1}^{d} -\nabla_i^{-h}\left[a_{ij}^{\varepsilon}(x(i,j))\nabla_j^{+h}u(x)\right]v(x) = \sum_{x \in \Gamma_h} \sum_{i,j=1}^{d} a_{ij}^{\varepsilon}(x(i,j))\nabla_j^{+h}u(x)\nabla_i^{+h}v(x). \tag{2.17}$$

Therefore, as the coefficients $\{a_{ij}^{\varepsilon}(x(i,j))\}$ are coercive (this will be shown in Section 3.2), the bilinear form (2.17) associated with the linear system (2.12)–(2.13) is non negative. On the other hand, the constant discrete functions, $u_h^\varepsilon(x) = c$ for any $x \in \Gamma_h$, solve (2.12)–(2.13) and are its unique solutions when $f \equiv 0$. Thus, when the compatibility condition (2.14) holds, there exists an unique u_h^ε satisfying (2.12)–(2.14).

One expects the solution u_h^ε of (2.12) - (2.14) to be an approximation to the exact solution u^ε. This is indeed true but the error estimates depend on the rapidly oscillating character of the coefficients a_{ij}^ε.

Let us mention the following classical result on the convergence of finite-difference approximations for elliptic problems (see [4] and [29]).

THEOREM 2.2. *Let u be the solution of the periodic boundary problem*

$$\begin{cases} Au = f & in \quad Y, \\ u \in H^1_\#(Y), & m(u) = 0. \end{cases}$$

Let u_h be the solution of the finite difference approximation. Then, if the order of accuracy of the scheme is ν and f belongs to $C^{0,\lambda}$, we have

$$\sup_{x\in\Gamma_h} |u(x) - u_h(x)| \le c(h^{\min(\nu,\lambda)} + \varepsilon_h(a))\|f\|_{C^\lambda}$$

where $\varepsilon_h(a) = \max[a_{k\ell}(x) - a_{k\ell}(x')]$, with $|x - x'| \le h$ and for $k, \ell = 1, \dots, d$.

Applying this theorem to (2.3) and (2.12)-(2.14) with $\varepsilon > 0$ fixed, it can be shown that, when the coefficients $a_{k\ell}$ are continuous, the solution $u_h^\varepsilon(x)$ of (2.12)-(2.14) converges to $u^\varepsilon(x)$ when $h \to 0$. In particular, we have the following estimate, when f is Lipschitz:

$$\sup_{x\in\Gamma_h} |u_h^\varepsilon(x) - u^\varepsilon(x)| \le c\left(h + \frac{h}{\varepsilon}\right). \tag{2.18}$$

where $c > 0$ denotes a positive constant which depends on the Lipschitz constant of the coefficients and the ellipticity constant but it is independent of h and ε.

2.3. Main results. In this section, we analyze how well u_h^ε approximates u^*. From the estimate (2.9) provided by the theory of homogenization and interpolation inequalities, we get for h sufficiently small (see Remark 4.3)

$$\|u^\varepsilon - u^*\|_h \le ch + c'\varepsilon. \tag{2.19}$$

On the other hand, by the error estimate (2.18), we obtain

$$\|u_h^\varepsilon - u^*\|_h \le \|u_h^\varepsilon - u^\varepsilon\|_h + \|u^\varepsilon - u^*\|_h \le c_1\frac{h}{\varepsilon} + ch + c'\varepsilon. \tag{2.20}$$

Therefore, if $h << \varepsilon$, u_h^ε converges to u^*. In this case, the rapid oscillations of the coefficients a_{ij}^ε are captured by the numerical approximation since the numerical mesh is much finer. But this requires h to be asymptotically smaller than ε and makes computations infeasible in practice.

Existing results in 1-d. Estimate (2.20) does not provide a complete information on the behavior of numerical solutions since, as we mentioned above in the introduction, the convergence of u_h^ε to u^* may hold with h and ε of the same order, and more precisely, when $h/\varepsilon \to 2\pi r \neq 0$ as $h, \varepsilon \to 0$ provided r is a suitable irrational number (see [1]).

To see this, we consider the following 1-d problem:

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(a^\varepsilon(x)\dfrac{\partial u^\varepsilon}{\partial x}(x)\right) = f(x), \qquad 0 < x < 2\pi, \\ u^\varepsilon(0) = b, \quad u^\varepsilon(2\pi) = c. \end{cases} \tag{2.21}$$

The homogenized solution in the 1-d case satisfies

$$\begin{cases} -a^*\dfrac{\partial^2 u^*}{\partial x^2} = f(x), \qquad 0 < x < 2\pi, \\ u^*(0) = b, \quad u^*(2\pi) = c, \end{cases} \tag{2.22}$$

where a^* is defined by

$$a^* = \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dy}{a(y)}\right)^{-1}. \tag{2.23}$$

Theorems 1 and 1a of [1] show that the finite difference approximation associated to (2.21) satisfy:

THEOREM A *For any continuous and bounded function $f(x)$, we have*

$$\lim_{\varepsilon,h\to 0}\sup_{x\in\Gamma_h}|u_h^\varepsilon(x) - u^*(x)| \to 0, \tag{2.24}$$

provided $h/\varepsilon = 2\pi r$ with r any irrational number with Γ_h defined in (2.11).

THEOREM B *Under the assumptions of Theorem A, given $\tau > 0$ there exists $h_0 > 0$ and a set $S(\varepsilon, h_0) \subset [0, h_0]$ defined by*

$$S(\varepsilon,h_0) = \Big\{0 < h \le h_0 \mid \quad \left|\frac{kh}{\varepsilon} - 2\pi j\right| \ge \frac{\tau}{|k|^{3/2}} \quad j = 1,\dots,\left[\frac{kh_0}{\varepsilon}\right] + 1, \quad \text{for } 0 \ne k \in \mathbb{Z}, \quad 0 < \varepsilon \le 1\Big\},$$

with Lebesgue measure $|S(\varepsilon,h_0)| \ge h_0(1-3\tau)$, such that for $0 < \varepsilon \le 1$

$$\sup_{x\in\Gamma_h}|u_h^\varepsilon(x) - u^*(x)| \le \tau \tag{2.25}$$

These results guarantee that the numerical homogenization, i.e. the convergence of the finite-difference solution towards the solution of the continuous homogenized problem, occurs even when h and ε are of the same order, provided the ratio h/ε is kept irrational.

Main results in 1-d. The following theorem concerns the more natural case where $h/2\pi\varepsilon$ is a rational number.

THEOREM 2.3. *Assume u_h^ε is the finite difference approximation of the solution of (2.21), u^* satisfies (2.22) and f is a continuous function. Assume that h and ε are such that*

$$\frac{h}{\varepsilon} = 2\pi\frac{q}{p}, \qquad \text{with } H.C.F.(p,q) = 1, \quad q,p \in \mathbb{N}. \tag{2.26}$$

Then, there exist $c_1, c_2 > 0$ independent of h, ε and f such that

$$\sup_{x\in\Gamma_h}|u_h^\varepsilon(x) - u^*(x)| \le c_1 ph + \frac{c_2}{p}. \tag{2.27}$$

Here and the sequel *H.C.F.* stands for the *highest common factor*. This result will be proved in Appendix B by means of the explicit formulas of solutions.

Theorem 2.3 concerns the Dirichlet problem (2.21). We have chosen to state our main 1-d result for the Dirichlet problem in order to facilitate the comparison with Theorems A and B of [1] stated above.

The error estimate (2.27) contains two different terms. The first one tends to zero as $h \to 0$ when h/ε is kept fixed. However, in order for the second one to tend to zero we need to take $p \to \infty$, which, in practice, requires $h/2\pi\varepsilon$ to approximate an irrational number.

Using discrete Bloch waves we get the following result in 1-d:

THEOREM 2.4. *Let u_h^ε be the solution of (2.12)-(2.14) in dimension $d = 1$. Let ε be as in (2.4) and consider $|h| < h_0$ satisfying (2.26). Then, for any continuous function f with $m(f) = 0$, there exists a discrete function $u^*_{q/p}$ such that*

$$\|u_h^\varepsilon - u^*_{q/p}\|_h \le c\,|ph|\,\|f\|_h, \tag{2.28}$$

with c independent of h, ε. Moreover, $u^*_{q/p}$ is a discrete Fourier approximation with mesh-size h of the solution of

$$\begin{cases} -a_p^* \dfrac{\partial^2 v}{\partial x^2}(x) = f(x), & 0 < x < 2\pi, \\ v \quad \text{is } 2\pi\text{-periodic}, & m(v) = 0, \end{cases} \tag{2.29}$$

where a_p^* is defined by

$$a_p^* = \left(\frac{1}{p} \sum_{j=1}^{p} \frac{1}{a(2\pi(j+1/2)/p)} \right)^{-1}. \tag{2.30}$$

Furthermore,

$$\|u^*_{q/p} - u^*\|_h \le c\,\frac{1}{p}, \tag{2.31}$$

where u^* is the homogenized solution (2.22) and $c > 0$ is independent of h, ε.

It is important to note that the discrete homogenized problem (2.29) differs from the continuous one. Consequently $u^*_{q/p}$ does not coincide with the continuous homogenized solution u^*. In fact, in (2.31) we give an explicit estimate of the distance between $u^*_{q/p}$ and u^*. We see that it is of order $O(1/p)$. This is in agreement with the fact that convergence towards the continuous homogenized solution requires the ratio $h/2\pi\varepsilon$ to be irrational, or, in other words, the denominator p of the rational ratio $h/2\pi\varepsilon = q/p$ to tend to infinity.

REMARK 2.5. *Here, we compare Theorems 2.3 and 2.4. Both hold under the same hypotheses. In Theorem 2.3 we get the error estimate in the ℓ^∞-norm (2.27). Estimates (2.28) and (2.31) in Theorem 2.4 yield the same error estimate as in (2.27) but in the norm $\|\cdot\|_h$, i.e.*

$$\|u_h^\varepsilon - u^*\|_h \le c_1 ph + c_2 \frac{1}{p}. \tag{2.32}$$

We state separately the numerical homogenization result (2.28) (with ratio rational) and the necessity of the ratio being irrational in (2.31) to converge towards the homogenized solution.

As a consequence of these results with rational ratio we recover the statements of Theorems A and B using classical results in approximation of irrational numbers by rational ones in Section 5. Theorems A and B above concern the case of Dirichlet boundary conditions but similar results apply also with periodic ones.

Main results in several space dimensions. . In several space dimensions we get a weaker version of Theorems 2.3 and 2.4. According to (2.4) and (2.10), we have

$$\frac{h_i}{\varepsilon} = 2\pi \frac{s}{n_i}, \quad i = 1, \ldots, d. \tag{2.33}$$

In particular, taking into account that $s \in \mathbb{N}$, we have

$$\left\{ \begin{array}{l} \exists\, p_i, q_i \in \mathbb{N} \quad \text{such that} \quad n_i = p_i\, r_i, \quad s = q_i\, r_i, \\ \text{where} \quad r_i{=}\text{H.C.F.}(s, n_i) \quad \text{with } i = 1, \ldots, d. \end{array} \right. \tag{2.34}$$

Thus, h and ε satisfy

$$\frac{h_i}{\varepsilon} = 2\pi \frac{q_i}{p_i}, \quad \text{with H.C.F.}(p_i, q_i) = 1, \quad i = 1, \ldots, d. \tag{2.35}$$

Let us now state how u_h^ε approximates u^* in several space dimensions.

THEOREM 2.6. *Assume that $d \geq 2$ and h_0 to be sufficiently small. Let ε and h be as in (2.4) and (2.35) with $|h| < h_0$. Furthermore, assume that*

$$\frac{q}{p} - \left[\frac{q}{p}\right] = \frac{\rho}{p}, \quad \textit{with } \left|\frac{\rho}{p}\right| < c_a, \tag{2.36}$$

where c_a sufficiently small and depending only on the lower and upper bounds of the coefficients. Let u_h^ε be the solution of (2.12)–(2.14). Then, for any continuous function f with $m(f) = 0$, there exists a discrete function $u^*_{q/p}$ such that

$$\|u_h^\varepsilon - u^*_{q/p}\|_h \leq c\, |ph|\, \|f\|_h, \tag{2.37}$$

for all $h, \varepsilon > 0$ as above with $c > 0$ independent of h, ε, f. The function $u^*_{q/p}$ is the discrete Fourier approximation with mesh-size h of the solution of

$$\left\{ \begin{array}{l} -a_{ij}^{*,q/p} \dfrac{\partial^2 v}{\partial x_i \partial x_j} = f \quad \textit{in} \quad Y, \\ v \in H^1_\#(Y), \qquad m(v) = 0. \end{array} \right. \tag{2.38}$$

In general, this solution does not coincide with the homogenized solution (2.8). We have the following explicit error estimate

$$\|u^*_{q/p} - u^*\|_h \leq c\, \delta\, \|f\|_h, \tag{2.39}$$

where $\delta > 0$ is given by

$$\delta = \max\left(\left|\frac{\rho}{p}\right|, 1 - \frac{\sigma_m}{\sigma_M}, \frac{\sigma_M}{\sigma_m} - 1 \right) \tag{2.40}$$

with $\sigma_M = \max(\sigma_i)$ and $\sigma_m = \min(\sigma_i)$, where $\sigma = q/\rho$.

REMARK 2.7. *As we said above the discrete function $u^*_{q/p}$ is the discrete Fourier approximation of the solution of (2.38) where the coefficients $\{a_{ij}^{*,q/p}\}$ are constants* *and depend on q and p. In general these coefficients do not coincide with the homogenized coefficients and, consequently, the solution of (2.38) does not coincide with the homogenized solution either. We will give an estimate of $|a^*_{ij} - a^{*,q/p}_{ij}|$ depending on suitable conditions of p, q (see Proposition 3.11). As in the context of homogenization of continuous problems (see [10]), the discrete homogenized coefficients can be derived from the Hessian of the first discrete Bloch eigenvalue in (3.21). An explicit formula of this Hessian is given in Lemma 3.7.*

REMARK 2.8. *In the one-dimensional case, the numerical homogenized solution $u^*_{q/p}$ exists whatever the choice of q and p is. In several space dimensions, we need the extra hypothesis (2.36) to identify the numerical homogenized solution. Condition (2.36) guarantees the ellipticity of the discrete system (2.12) (see Section 3.2). This condition is needed since the ellipticity of $\{a^\varepsilon_{ij}(x(i,j))\}$ is not automatic in several space dimensions.*

REMARK 2.9. *According to Theorem 2.4 u^ε_h converges to $u^*_{q/p}$, with q, p fixed. Moreover, $u^*_{q/p}$ differs in general from u^* with error estimates (2.31) or (2.39) in one and several space dimensions, respectively, and these error estimates are independent of the sequence h, ε provided (2.35) is fulfilled. On the other hand, as we shall in Section 5, there exist sequences q, p such that $u^*_{q/p}$ converges to u^*. In particular, in one space dimension that happens when q/p tends to an irrational number.*

In several space dimensions, according to (2.39), we get the same result provided δ tends to zero. For this to be true, in view of (2.40) we need two conditions. The first one requires that the residue $\rho/p \to 0$. This condition is similar to that in Theorem 3 in [1] where the residue converges to zero through a sequence of irrational numbers. This is significantly more restrictive than in the one-dimensional case. In one space dimension, the homogenized coefficient is explicit and is given by (2.23). But, in several space dimensions, the homogenized coefficients (2.6) depend on χ_k, solution of (2.7) that, in general, may not be computed explicitly and need to be approximated numerically. The function χ_k is approximated with the mesh-size ρ/p that needs to be sufficiently small (see Appendix C.3).

The other condition which is needed for $\delta \to 0$ is that $\sigma_M/\sigma_m \to 1$, which indicates that the ratio between all components of q/ρ tends to one. This condition is automatically satisfied when h_i/ε is independent of i. When the mesh-size is different in several space directions, the finite difference approximation scheme for χ_k solution of (2.7) with the mesh-size ρ/p does not coincide with the discrete analogue obtained when applying the discrete Bloch approximation. The later is given in (3.24). In Appendix C.3 we give an estimate on the difference of these two quantities in terms of $|\sigma_m/\sigma_M - \sigma_M/\sigma_m|$. This explains why this quantity enters in the definition of δ in (2.40) appearing in the estimate (2.39).

2.4. Numerical examples. First, we consider an example of numerical homogenization in one space dimension. We consider the 2π-periodic coefficient

$$a(y) = \frac{1}{1 + 0.5|\sin(y/2)|} \qquad y \in (0, 2\pi).$$

The corresponding homogenized coefficient is $a^* = \pi/(1+\pi)$. The numerical homogenized coefficient a^*_p defined in (2.30) with $q = 5$, $p = 19$, as predicted by the theory differs with a^* by an error of the order of 10^{-3}. Considering the constant function $f = 1$, we calculate u^ε_h using the formula (B.5) and we compare with the continuous functions u^ε, u^* and $u^*_{q/p}$ solutions of the equation (2.21), (2.22) and (2.29), respectively, with homogeneous Dirichlet boundary conditions. As seen in Table 2.1, as h

TABLE 2.1
One dimension. Errors of the solutions with $q = 5$, $p = 19$

h	ε	$\|u^\varepsilon - u_h^\varepsilon\|_h$	$\|u^\varepsilon - u_h^\varepsilon\|_\infty$	$\|u^* - u_h^\varepsilon\|_h$	$\|u^* - u_h^\varepsilon\|_\infty$	$\|u^*_{q/p} - u_h^\varepsilon\|_h$
$\frac{2\pi}{38}$	$\frac{1}{10}$	0.038	0.0217	0.078	0.069	0.0778
$\frac{2\pi}{380}$	$\frac{1}{100}$	0.0039	0.0027	0.0086	0.0085	0.0079
$\frac{2\pi}{1900}$	$\frac{1}{500}$	0.0033	0.0018	0.0036	0.0022	0.0016
$\frac{2\pi}{19000}$	$\frac{1}{5000}$	0.0033	0.0018	0.0033	0.0018	$1.5 \cdot 10^{-4}$
$\frac{2\pi}{190000}$	$\frac{1}{50000}$	0.0033	0.0018	0.0033	0.0018	$1.58 \cdot 10^{-5}$

TABLE 2.2
One dimension. Errors of the solutions with different q, p

h	ε	q	p	$\|u^\varepsilon - u_h^\varepsilon\|_h$	$\|u^* - u_h^\varepsilon\|_h$	$\|u^*_{q/p} - u_h^\varepsilon\|_h$	$\|a^* - a^*_p\|$
$\frac{2\pi}{19000}$	$\frac{1}{5000}$	5	19	0.0033	0.0033	$1.5 \cdot 10^{-4}$	$2.08 \cdot 10^{-4}$
$\frac{2\pi}{19100}$	$\frac{1}{5100}$	51	191	$5.37 \cdot 10^{-5}$	$1.61 \cdot 10^{-4}$	$1.57 \cdot 10^{-4}$	$2.06 \cdot 10^{-6}$
$\frac{2\pi}{19100}$	$\frac{1}{5110}$	511	1910	$3.72 \cdot 10^{-5}$	$1.56 \cdot 10^{-4}$	$1.55 \cdot 10^{-4}$	$2.06 \cdot 10^{-8}$

and ε go to zero with $h/\varepsilon = 2\pi q/p$ and $q = 5$, $p = 19$, the approximation u_h^ε does not converge to u^*, but rather to $u^*_{q/p}$.

In Table 2.2 we exhibit the results for various choices of p and q. More precisely, we consider the pairs (p, q)=(19,5), (191,51) and (1910,511). The theory predicts (see (2.32) for the error estimate) that the accuracy of the approximation is improved when q/p approximates an irrational number, as $p \to \infty$. This is clearly seen in the column of Table 2.2 devoted to the estimates of $\|u^* - u_h^\varepsilon\|_h$, that shows that, by keeping h and ε essentially fixed, but increasing p from 19 to 1910 this error decreases from $3.3 \cdot 10^{-3}$ to $1.56 \cdot 10^{-4}$.

In two space dimensions we consider the 2π-periodic coefficients

$$a_{11}(y_1, y_2) = a_{22}(y_1, y_2) = 1 + |\sin(y_1/2)\sin(y_2/2)|, \qquad a_{12}(y_1, y_2) = 1/2. \tag{2.41}$$

Using the formula (C.26) we obtain the numerical homogenized coefficients of Table 2.3 with different values of p and q. The column corresponding to (q_1, q_2, p_1, p_2) =(1,1,71,71) presents the values of the approximation in finite differences of the homogenized coefficients with mesh sizes $2\pi/71$ in both space directions. We give the explicit value of $a_{kj}^{*,q/p}$ for $k, j = 1, 2$. According to the data in Table 2.3 we see that the results are significantly different in the first three cases corresponding to (q_1, q_2, p_1, p_2)=(1,1,71,71), (72,71,71,70) and (72,72,71,71) that in the last two in which (q_1, q_2, p_1, p_2)=(31,103,70,72) and (1031,121,71,71). In the first three cases the values of the numerical homogenized coefficients are rather similar but not in the last two. This is in agreement with the error estimate in Proposition 3.11 with δ defined in (2.40). Indeed, while in the first three cases ρ =(1,1), $\sigma = (q_1, q_2)$, and accordingly, δ is small, that is not the case in the last two cases. We recall that the error estimate on the homogenized coefficients (3.31) affects automatically the error estimate on the homogenized solutions (see (2.39)).

3. Discrete Bloch waves.

3.1. Bloch decomposition. The solutions of the elliptic continuous problem associated with the operator A^ε may be decomposed in Bloch waves and this allows

TABLE 2.3
Numerical homogenized coefficients with different values p and q

q_1,q_2,p_1,p_2	1,1,71,71	72,71,71,70	72,72,71,71	31,103,70,72	1031,121,70,72
$a_{11}^{*,q/p}$	1.3728	1.3727	1.3727	1,3684	1.3656
$a_{22}^{*,q/p}$	1.3728	1.3727	1.3727	1.3679	1.3672
$a_{12}^{*,q/p}$	0.5010	0.5009	0.5010	0.4939	0.4896

describing the homogenization process as $\varepsilon \to 0$. This will be done in Appendix A.1. We need to perform a similar Bloch decomposition for the numerical problem.

In the finite difference system (2.12) the coefficients a^{ε}_{ij} are hp-periodic in the mesh Γ_h defined in (2.11). Indeed, taking into account that h and ε satisfy (2.35), for any $x \in \Gamma_h$ and $z \in \Gamma_{hp} \subset \Gamma_h$ we obtain that

$$a^{\varepsilon}_{ij}\left(x+z\right) = a_{ij}\left(\frac{x+z}{\varepsilon}\right) = a_{ij}\left(\frac{x}{\varepsilon}\right) = a^{\varepsilon}_{ij}(x). \tag{3.1}$$

Here, we have used the fact that $z/\varepsilon \in 2\pi\mathbb{Z}^d$. Indeed, $z \in \Gamma_{hp}$ and therefore $z = nhp$ with $n \in \mathbb{Z}^d$. Thus, according to (2.35), $z/\varepsilon = 2\pi nq \in 2\pi\mathbb{Z}^d$. We conclude that the coefficients $\{a^{\varepsilon}_{ij}\}$ are hp-periodic (or $q\varepsilon Y-$periodic in the mesh Γ_h by the relation (2.35)).

Now, we define the discrete Bloch waves associated with the linear system (2.12), from the following family of spectral problems: To find $\mu = \mu(\xi) \in \mathbb{R}$ and $\varphi^{\varepsilon}_h = \varphi^{\varepsilon}_h(x;\xi)$ (non identically zero) such that

$$\begin{aligned} &\sum_{i,j=1}^{d} -\nabla^{-h}_i \left[a^{\varepsilon}_{ij}\left(x(i,j)\right)\nabla^{+h}_j(e^{ix\cdot\xi}\varphi^{\varepsilon}_h(x;\xi))\right] = \mu(\xi)e^{ix\cdot\xi}\varphi^{\varepsilon}_h(x;\xi), \quad x \in \Gamma^p_h \\ &\qquad \varphi^{\varepsilon}_h(x;\xi) \text{ is } ph\text{-periodic in } x, \text{ i.e.,} \quad \varphi^{\varepsilon}_h(x + p_k h_k e_k;\xi) = \varphi^{\varepsilon}_h(x;\xi), \end{aligned} \tag{3.2}$$

where, using that the coefficients are hp-periodic in Γ_h, we use the mesh Γ^p_h defined by

$$\Gamma^p_h = \{x = (n_1h_1,\dots,n_dh_d): \quad 0 \le n_i < p_i, \quad n_i \in \mathbb{Z}, \forall i = 1,\dots,d\}.$$

The discrete Bloch waves are then given by $\psi^{\varepsilon}_h(x;\xi) = e^{ix\cdot\xi}\varphi^{\varepsilon}_h(x;\xi)$. By (2.35) we get:

$$\psi^{\varepsilon}_h(x + mhp;\xi) = e^{imhp\cdot\xi}\psi^{\varepsilon}_h(x;\xi), \quad m \in \mathbb{Z}^d.$$

It is clear that this property remains the same if ξ is replaced by $\xi + n(q\varepsilon)^{-1}$, $n \in \mathbb{Z}^d$, since $(q\varepsilon)^{-1}$ is a multiple of 2π. So, there is no loss of generality in confining ξ to the cell $[0,(q\varepsilon)^{-1}[$ or in the translated one $[-(2q\varepsilon)^{-1},(2q\varepsilon)^{-1}[$.

We consider the following bilinear form associated with (3.2):

$$a^{\varepsilon}_h(\xi)(u,v) = \sum_{x\in\Gamma^p_h}\sum_{i,j=1}^{d} a^{\varepsilon}_{ij}(x(i,j))\nabla^{+h}_j(e^{ix\cdot\xi}u(x))\overline{\nabla^{+h}_i(e^{ix\cdot\xi}v(x))}\,, \tag{3.3}$$

for ph-periodic functions u, v. We note that the bilinear form $a^{\varepsilon}_h(\xi)$ is Hermitian, i.e.,

$$a^{\varepsilon}_h(\xi)(u,v) = \overline{a^{\varepsilon}_h(\xi)(v,u)}.$$

Indeed, by (3.3), we get

$$\overline{a_h^\varepsilon(\xi)(v,u)} = \sum_{x\in\Gamma_h^p}\sum_{i,j=1}^{d} a_{ij}^\varepsilon(x(i,j))\overline{\nabla_i^{+h}(e^{ix\cdot\xi}u(x))}\nabla_j^{+h}(e^{ix\cdot\xi}v(x)).$$

The coefficients (a_{ij}) and the mesh-points (2.15) being symmetric we deduce that $a_h^\varepsilon(\xi)$ is Hermitian. Therefore, the eigenvalues of (3.2), that we shall refer to as discrete Bloch eigenvalues, are real (see [34], p. 25). We denote them by $\{\mu_{h,m}^\varepsilon(\xi)\}_{m=1}^{\dot p}$, where $\dot p = p_1\cdots p_d = \#(\Gamma_h^p)$, i.e, the number of points in the mesh Γ_h^p, and $\{\varphi_{h,m}^\varepsilon(x;\xi)\}_{m=1}^{\dot p}$ their corresponding orthonormal eigenvectors.

Given a discrete function f with values $f(x)$ for $x\in\Gamma_h$, we define the mth Bloch coefficient of f at the ε scale as follows:

$$\widehat{f}_{h,m}^\varepsilon(k) = (2\pi)^{-\frac{d}{2}}\dot h(\dot p)^{\frac12}\sum_{x\in\Gamma_h} f(x)e^{-ik\cdot x}\overline{\varphi_{h,m}^\varepsilon}(x;k) \quad \forall m\ge 1,\ k\in\Lambda_{q\varepsilon}, \tag{3.4}$$

when h and ε satisfy (2.35), and with

$$\Lambda_{q\varepsilon} = \left\{k=(k_1,\dots,k_d)\in\mathbb{Z}^d,\ \text{such that}\ \left[\frac{-1}{2q_i\varepsilon}\right]+1\le k_i\le\left[\frac{1}{2q_i\varepsilon}\right]\right\}. \tag{3.5}$$

The following holds:

THEOREM 3.1. *Let $\varepsilon, h>0$ satisfy (2.35) and Γ_h be defined as in (2.11). For any Y-periodic discrete function f the following representation formula holds:*

$$f(x) = (2\pi)^{-\frac{d}{2}}(\dot p)^{\frac12}\sum_{k\in\Lambda_{q\varepsilon}}\sum_{m=1}^{\dot p}\widehat{f}_{h,m}^\varepsilon(k)e^{ik\cdot x}\varphi_{h,m}^\varepsilon(x;k).$$

Further, we have Parseval's identity,

$$\|f\|_h^2 = \sum_{m=1}^{\dot p}\sum_{k\in\Lambda_{q\varepsilon}}|\widehat{f}_m^\varepsilon(k)|^2,$$

and Plancherel's identity:

$$(f,g)_h = \sum_{m=1}^{\dot p}\sum_{k\in\Lambda_{q\varepsilon}}\widehat{f}_{h,m}^\varepsilon(k)\overline{\widehat{g}_{h,m}^\varepsilon}(k).$$

Proof. We use the same idea as in the case $f\in L^2(\mathbb{R}^d)$ whose detailed proof can be found in [2], p. 616, and also in [30]. First, we consider the function

$$f(x;k) = (2\pi)^{-\frac{d}{2}}\dot h(\dot p)^{\frac12}\sum_{z\in\Gamma_{hp}} f(x+z)e^{-ik\cdot(x+z)}, \qquad k\in\Lambda_{q\varepsilon}. \tag{3.6}$$

Γ_{hp} is a mesh in Y defined as Γ_h but with hp mesh-size. Note that $f(x;k)$ is hp-periodic in x. Indeed,

$$\frac{(2\pi)^{\frac{d}{2}}}{\dot h(\dot p)^{\frac12}}f(x+h_jp_je_j;k) = \sum_{\substack{z\in\Gamma_{hp}\\ z_j\ne 0}} f(x+z)e^{-ik\cdot(x+z)} + \sum_{\substack{z\in\Gamma_{hp}\\ z_j=2\pi}} f(x+z)e^{-ik\cdot(x+z)}.$$

Thanks to the fact that f is Y-periodic and $k \in \mathbb{Z}^d$, we see

$$\sum_{\substack{z\in\Gamma_{hp}\\ z_1=2\pi}} f(x+z)e^{-ik\cdot(x+z)} = \sum_{\substack{z\in\Gamma_{hp}\\ z_1=0}} f(x+z)e^{-ik\cdot(x+z)}.$$

The function $f(x;k)$ being hp-periodic in x we have $\{f(x;k)| \quad x \in \Gamma_h^p\} \in \mathbb{C}^{\dot{p}}$. Since $\{\varphi^\varepsilon_{h,m}(x;k)| \quad x \in \Gamma_h^p\}_{m=1}^{\dot{p}}$ is an orthonormal basis in $\mathbb{C}^{\dot{p}}$, we get for $x \in \Gamma_h^p$

$$\begin{aligned} f(x;k) &= \sum_{m=1}^{\dot{p}} \varphi^\varepsilon_{h,m}(x;k) \sum_{x'\in\Gamma_h^p} f(x';k)\overline{\varphi^\varepsilon_{h,m}}(x';k) \\ &= \sum_{m=1}^{\dot{p}} \varphi^\varepsilon_{h,m}(x;k)\,(2\pi)^{-\frac{d}{2}}\dot{h}(\dot{p})^{\frac{1}{2}} \sum_{x'\in\Gamma_h^p}\sum_{z\in\Gamma_{hp}} f(x'+z)e^{-ik\cdot(x'+z)}\overline{\varphi^\varepsilon_{h,m}}(x'+z;k). \end{aligned}$$

Note that $\Gamma_h^p \oplus \Gamma_{hp} = \Gamma_h$, i.e., for any $x \in \Gamma_h$ there exist an unique $x' \in \Gamma_h^p$ and $z \in \Gamma_{hp}$ such that $x = x' + z$, and $\varphi^\varepsilon_{h,m}$ defined in (3.2) is hp-periodic in x. Then, by (3.4), we get

$$f(x;k) = \sum_{m=1}^{\dot{p}} \varphi^\varepsilon_{h,m}(x;k)\,\widehat{f}^\varepsilon_{h,m}(k). \tag{3.7}$$

Moreover, using that $\{\psi^\varepsilon_{h,m}(x;k)| \quad x \in \Gamma_h^p\}_{m=1}^{\dot{p}}$ are orthonormal, we get for any $k \in \Lambda_{q\varepsilon}$:

$$\sum_{x\in\Gamma_h^p} |f\,(x;k)|^2 = \sum_{m=1}^{\dot{p}} |\widehat{f}^\varepsilon_{h,m}(k)|^2. \tag{3.8}$$

Now, we proceed to show the inverse formula for f. For $z, z' \in \Gamma_{ph}$, it follows that

$$\sum_{k\in\Lambda_{q\varepsilon}} e^{ik\cdot(z-z')} = \begin{cases} 0 & \text{if } z \neq z', \\ \#(\Lambda_{q\varepsilon}) & \text{if } z = z', \end{cases} \tag{3.9}$$

and, for $k \in \Lambda_{q\varepsilon}$,

$$\sum_{z\in\Gamma_{ph}} e^{ik\cdot z} = \begin{cases} 0 & \text{if } k \neq 0, \\ \#(\Gamma_{ph}) & \text{if } k = 0. \end{cases} \tag{3.10}$$

The proof of these formulas is immediate (note that $\#(\Lambda_{q\varepsilon}) = \#(\Gamma_{ph})$). We get by (3.10)

$$f(x) = (2\pi)^{-\frac{d}{2}}(\dot{p})^{\frac{1}{2}} \sum_{k\in\Lambda_{q\varepsilon}} f\,(x;k)\,e^{ik\cdot x}, \qquad x \in \Gamma_h,$$

and, by (3.7), we obtain the representation formula of the theorem. On the other hand, using the definition of $f(x;k)$ and the formula (3.9), we obtain

$$\sum_{k\in\Lambda_{q\varepsilon}}\sum_{x\in\Gamma_h^p} |f\,(x;k)|^2 = \frac{(\dot{h})^2\dot{p}}{(2\pi)^d} \sum_{x\in\Gamma_h^p}\sum_{z\in\Gamma_{ph}} |f(x+z)|^2\#(\Lambda_{q\varepsilon}),$$

and, by (2.35) and $\Gamma_h^p \oplus \Gamma_{hp} = \Gamma_h$, we get

$$\sum_{k\in\Lambda_{q\varepsilon}} \sum_{x\in\Gamma_h^p} = \sum_{x\in\Gamma_h} h_1\cdots h_d |f(x)|^2 \left|f\left(x;k\right)\right|^2.$$

Parseval's identity is a consequence of (3.8). Analogously, Plancherel's identity follows. This concludes the proof of Theorem 3.1. □

Using Theorem 3.1, we have the following result.

THEOREM 3.2. *Let $\{u_h^\varepsilon(x)|\, x \in \Gamma_h\}$ be the unique solution of (2.12)–(2.14). Let $\widehat{u}_{h,m}^\varepsilon(k)$ and $\widehat{f}_{h,m}^\varepsilon(k)$ be the m^{th} Bloch coefficients of u_h^ε and f, respectively, with $m = 1,\ldots,\dot{p}$ and $k \in \Lambda_{q\varepsilon}$. Then, we have*

$$\mu_{h,m}^\varepsilon(k)\widehat{u}_{h,m}^\varepsilon(k) \;=\; \widehat{f}_{h,m}^\varepsilon(k), \quad m = 1,\ldots,\dot{p}, \quad \forall k \in \Lambda_{q\varepsilon}. \tag{3.11}$$

Proof. According to Theorem 3.1

$$u_h^\varepsilon(x) = (2\pi)^{-\frac{d}{2}}\dot{h}(\dot{p})^{\frac12} \sum_{m=1}^{\dot{p}} \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{h,m}^\varepsilon(k) e^{ik\cdot x}\varphi_{h,m}^\varepsilon\left(x;k\right), \quad x \in \Gamma_h. \tag{3.12}$$

Using (3.12) in the discrete problem (2.12), we have, for any $x \in \Gamma_h$:

$$f(x) = \frac{\dot{h}(\dot{p})^{\frac12}}{(2\pi)^{\frac{d}{2}}} \sum_{m=1}^{\dot{p}} \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{h,m}^\varepsilon(k) \left(- \sum_{i,j=1}^{d} \nabla_i^{-h}\left[a_{ij}^\varepsilon(x(i,j))\nabla_j^{+h}(e^{ik\cdot x}\varphi_{h,m}^\varepsilon(x;k))\right]\right).$$

Since $\{\varphi_{h,m}^\varepsilon(x;k)|\, x \in [0, ph]\cap\Gamma_h\}$ satisfies (3.2) we obtain

$$f(x) = (2\pi)^{-\frac{d}{2}}\dot{h}(\dot{p})^{\frac12} \sum_{m=1}^{\dot{p}} \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{h,m}^\varepsilon(k)\mu_{h,m}^\varepsilon(k)e^{ik\cdot x}\varphi_{h,m}^\varepsilon(x;k), \qquad x \in \Gamma_h.$$

Then, using the representation formula for f given in Theorem 3.1, (2.12) can be written as

$$\sum_{m=1}^{\dot{p}} \sum_{k\in\Lambda_{q\varepsilon}} \left(\mu_{h,m}^\varepsilon(k)\widehat{u}_{h,m}^\varepsilon(k) - \widehat{f}_{h,m}^\varepsilon(k)\right) e^{ik\cdot x}\varphi_{h,m}^\varepsilon(x;k) = 0, \quad x \in \Gamma_h.$$

Using the completeness of the Bloch eigenvectors in $\mathbb{C}^{\dot{p}}$ and their ph-periodicity, we conclude the proof. □

3.2. Properties of Bloch waves. In this section we present some properties of the discrete Bloch eigenvalues and eigenvectors that we will use to prove the numerical homogenization results. The proof of these properties will be given in Appendix C.

We have defined the discrete Bloch waves for $x \in \Gamma_h^p$ in (3.2). Now, making a change of variable, we use the discrete Bloch waves in the following mesh

$$\Gamma_{\frac{2\pi}{p}} = \left\{ y \in Y |\, y = \left(\frac{2\pi}{p_1}n_1,\ldots,\frac{2\pi}{p_d}n_d\right),\, 0 \le n_i < p_i,\, n_i \in \mathbb{Z},\, i = 1,\ldots,d \right\}. \tag{3.13}$$

Note that $\#(\Gamma_{\frac{2\pi}{p}}) = \#(\Gamma_h^p) = \dot{p}$ and that $y \in \Gamma_{\frac{2\pi}{p}}$, whenever $q\varepsilon y \in \Gamma_h^p$ by (2.35). Now, for $x \in \Gamma_h^p$, we introduce the following relation

$$\varphi_h^\varepsilon(x;\xi) = \varphi_p(y;\eta), \tag{3.14}$$

where (x,ξ) and (y,η) are related by $y = x/(q\varepsilon)$, $\eta = q\varepsilon\xi$. Recall that ξ belongs to $[-1/(2q\varepsilon), 1/(2q\varepsilon)[$. Hence, $\eta \in [-1/2, 1/2[^d = Y'$. By the relations (3.14) and (2.35),

$$\nabla_j^{+h}(e^{ix\cdot\xi}\varphi_h^\varepsilon(x;\xi)) = \frac{1}{q_j\varepsilon}\nabla_j^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_p(y;\eta)).$$

Therefore, we get

$$\nabla_i^{-h}\left[a_{ij}^\varepsilon\left(x(i,j)\right)\nabla_j^{+h}(e^{ix\cdot\xi}\varphi_h^\varepsilon(x;\xi))\right] = \nabla_i^{-\frac{2\pi}{p}}\left[\frac{a_{ij}\left(qy(i,j)\right)}{\varepsilon^2 q_i q_j}\nabla_j^{+\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_p(y;\eta))\right],$$

where, for $i,j = 1,\dots d$,

$$y(i,j) = y + \frac{\pi}{p_i}e_i + (1-\delta_{ij})\frac{\pi}{p_j}e_j. \tag{3.15}$$

Thus, we consider the following family of spectral problems: To find $\mu = \mu(\eta) \in \mathbb{R}$ and $\varphi = \varphi_p(y;\eta)$ (non identically zero) such that

$$\begin{gathered} -\sum_{i,j=1}^{d}\nabla_i^{-\frac{2\pi}{p}}\left[\frac{1}{q_iq_j}a_{ij}\left(qy(i,j)\right)\nabla_j^{+\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_p(y;\eta))\right] = \mu(\eta)e^{iy\cdot\eta}\varphi_p(y;\eta), \\ \varphi_p(y,\eta) \text{ is } Y\text{-periodic in } y. \end{gathered} \tag{3.16}$$

Its eigenvalues $\{\mu_m(\eta)\}_{m=1}^{\hat{p}}$ and eigenvectors $\{\varphi_{p,m}(y;\eta) |\, y \in \Gamma_{\frac{2\pi}{p}}\}$ verify

$$\begin{aligned} \mu_{h,m}^\varepsilon(\xi) &= \varepsilon^{-2}\mu_m(q\varepsilon\xi), \qquad \xi \in \left[-\frac{1}{2q\varepsilon}, \frac{1}{2q\varepsilon}\right[, \\ \varphi_{h,m}^\varepsilon(x;\xi) &= \varphi_{p,m}(\frac{x}{q\varepsilon}; q\varepsilon\xi), \qquad x \in [0, ph[\cap\Gamma_h. \end{aligned} \tag{3.17}$$

We consider the following Hermitian bilinear form associated with (3.16):

$$a(\eta)(u,v) = \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i,j=1}^{d}\frac{1}{q_iq_j}a_{ij}\left(qy(i,j)\right)\nabla_j^{\frac{2\pi}{p}}(e^{iy\cdot\eta}u(y))\overline{\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}v(y))}, \tag{3.18}$$

for $\eta \in Y'$ and Y-periodic discrete functions u, v. Note that the coefficients $\{a_{ij}\}$ defined in (2.2) are bounded and coercive. However, the values $\{a_{ij}(qy(i,j))\}$ considered in (3.16) depend on $\{ij\}$, and are taken in different points of the mesh (see Fig. 2.1). The main consequence of this fact is the loss of the ellipticity in (3.18) in several space dimensions. In the following lemma, we get the ellipticity of $\{a_{ij}(qy(i,j))\}$ under the key assumption (2.36).

LEMMA 3.3. *Assume that the coefficients $\{a_{ij}\}$ defined in (2.2) are Lipschitz continuous, α and β are respectively their boundedness and ellipticity constants and $q,p \in \mathbb{N}^d$ satisfy (2.36). Then, for $|\rho/p| < \alpha/(2cd\pi)$ where c is the Lipschitz constant of the coefficients, we have*

$$\begin{aligned} &\sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\eta_j \le 2\beta|\xi||\eta|, \qquad \textit{(boundedness)}, \\ &\sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\xi_j \ge \frac{\alpha}{2}|\xi|^2, \qquad \textit{(coercivity)}. \end{aligned}$$

Its proof will be given in Appendix C. As an immediate consequence we get that the Hermitian bilinear form $a(\eta)$ in (3.18) is non negative.

LEMMA 3.4. *Let $a(\eta)$ be the Hermitian bilinear form in (3.18). Under the hypotheses of Lemma 3.3, $a(\eta)$ is positive semi-definite.*

Proof. Using formula (3.18) and the coercivity property of Lemma 3.3, we conclude

$$a(\eta)(v,v) \geq \frac{\alpha}{2} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \left|\nabla_i^{\frac{2\pi}{p}} (e^{iy\cdot\eta} v(y))\right|^2, \qquad \forall \eta \in Y'. \qquad \square \tag{3.19}$$

The eigenvalues μ_m of $a(\eta)$ are non-negative and can be ordered in an increasing way:

$$0 \leq \mu_1(\eta) \leq \cdots \leq \mu_{\dot{p}}(\eta).$$

Moreover, thanks to (3.19), we have that

$$\mu_1 \text{ is simple, } \mu_1(0) = 0, \quad \varphi_{p,1}(y;0) = \frac{1}{\sqrt{p_1\cdots p_d}}, \quad \forall y \in \Gamma_{\frac{2\pi}{p}}. \tag{3.20}$$

LEMMA 3.5. *Assume that the hypotheses of Lemma 3.3 hold. There exists a positive constant $c > 0$, independent of η, p and q such that for any $m \geq 2$*

$$\mu_m(\eta) \geq c \min_{i=1,..,d} \frac{1}{q_i^2}, \quad \forall\, \eta \in Y'.$$

PROPOSITION 3.6. *There exists $\delta > 0$ sufficiently small such that the first eigenvalue $\mu_1(\eta)$ and the corresponding eigenvector $\varphi_{p,1}(y;\eta)$ are analytic with respect to η in B_δ.*

The proof of the previous results will be given in Appendix C.

As in the context of homogenization of continuous elliptic problems, the homogenized coefficients for the discrete problem are given by the Hessian of the first discrete Bloch eigenvalue μ_1 at $\eta = 0$. In particular, we define the discrete homogenized coefficients as:

$$a_{k\ell}^{*,q/p} = \frac{q_k q_\ell}{2} \frac{\partial^2 \mu_1}{\partial \eta_k \eta_\ell}(0) \tag{3.21}$$

The following proposition gives the values of the derivatives of μ_1 at $\eta = 0$.

LEMMA 3.7. *Let μ_1 and $\varphi_{p,1}$ be the first eigenvalue and eigenvector of the spectral problem (3.16). Then, the origin $\eta = 0$ is a critical point of the first Bloch eigenvalue:*

$$\frac{\partial \mu_1}{\partial \eta_k}(0) = 0 \qquad \forall k = 1,\ldots,d. \tag{3.22}$$

Moreover, the second derivatives at $\eta = 0$ are

$$\begin{aligned}\frac{\partial^2 \mu_1}{\partial \eta_k \eta_\ell}(0) = \frac{1}{\dot{p}} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} & \frac{2a_{k\ell}(qy(k,\ell))}{q_k q_\ell} \\ & - \frac{1}{\dot{p}} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{j=1}^{d} \left[\frac{\Theta_q^k(y)}{q_\ell q_j} \nabla_j^{\frac{2\pi}{p}} a_{j\ell}(qy(\ell,j)) + \frac{\Theta_q^\ell(y)}{q_k q_j} \nabla_j^{\frac{2\pi}{p}} a_{jk}(qy(k,j)) \right]\end{aligned} \tag{3.23}$$

where $\{\Theta_q^k(y)\,|\,y\in\Gamma_{\frac{2\pi}{p}}\}$ *is the unique solution of*

$$\left\{\begin{array}{l} -\sum\limits_{i,j=1}^{d}\nabla_i^{-\frac{2\pi}{p}}\left[\dfrac{a_{ij}\,(qy(i,j))}{q_iq_j}\nabla_j^{+\frac{2\pi}{p}}(\Theta_q^k(y))\right]=\sum\limits_{j=1}^{d}\dfrac{1}{q_kq_j}\nabla_j^{-\frac{2\pi}{p}}a_{jk}(qy(k,j)),\\ \qquad\Theta_q^k(y)\quad Y\text{-periodic},\qquad\qquad \sum\limits_{y\in\Gamma_{\frac{2\pi}{p}}}\Theta_q^k(y)=0.\end{array}\right. \tag{3.24}$$

The derivatives of the first Bloch eigenvector are as follows:

$$\frac{\partial\varphi_{p,1}}{\partial\eta_k}(y;0)=\frac{i}{(\dot{p})^{\frac{1}{2}}}\Theta_q^k(y),\qquad k=1,\ldots,d. \tag{3.25}$$

As a consequence of the analyticity of μ_1 and Taylor's formula we get the following bounds on μ_1 and $\varphi_{p,1}$.

LEMMA 3.8. *Under the hypotheses of Lemma 3.3, the map* $\eta\in Y'\to\mu_1(\eta)\in\mathbb{R}$ *has a strict global minimum at* $\eta=0$ *where* $\mu_1(0)=0$. *Furthermore, there exist* $c_1,c_2,c>0$ *independent of* p *and* q*, such that*

$$c_1\left|\frac{\eta}{q}\right|^2\le\mu_1(\eta)\le c_2\left|\frac{\eta}{q}\right|^2,\qquad\forall\eta\in Y', \tag{3.26}$$

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^{d}\frac{1}{q_i^2}\left|\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_{p,1}(y;\eta))\right|^2\le c\left|\frac{\eta}{q}\right|^2. \tag{3.27}$$

LEMMA 3.9. *Under the hypotheses of Lemma 3.3, there exists* $c>0$ *independent of* p,q *such that for any* $\eta\in B_\delta$*:*

$$\left|\mu_1(\eta)-\frac{1}{2}\frac{\partial^2\mu_1}{\partial\eta_i\eta_j}(0)\eta_i\eta_j\right|\le c\left|\frac{\eta}{q}\right|^2|\eta|. \tag{3.28}$$

Finally, we estimate the difference of the discrete homogenized coefficients (3.21) with the homogenized coefficients $a^*_{k\ell}$ distinguishing one-dimensional and multidimensional case.

PROPOSITION 3.10. *(One space dimension) We assume that the coefficient* a *is Lipschitz continuous.* a^* *and* a^*_p *are defined by (2.23) and (2.30), respectively. Then, we get*

$$\left|a^*-a_p^*\right|\le\frac{\pi\beta^2}{2\alpha^2}\frac{c}{p}, \tag{3.29}$$

for all $p\ge1$*, where* β*,* c *and* α *are the upper bound, the Lipschitz and the coercivity constants of the coefficient* a*, respectively. Moreover, the second derivative of* μ_1 *at* $\eta=0$ *satisfies*

$$\frac{\partial^2\mu_1}{\partial\eta^2}(0)\;=\;2\,\frac{a_p^*}{q^2}. \tag{3.30}$$

PROPOSITION 3.11. *(Several space dimensions) We assume that the coefficients* $\{a_{k\ell}\}$*, their first derivatives and the functions* $\{\chi^k\}$ *defined by (2.7) are Lipschitz* *continuous. Let $a^*_{k\ell}$ and $a^{*,q/p}_{k\ell}$ be defined in (2.6) and (3.21), respectively. Then, we get*

$$\left| a^*_{k\ell} - a^{*,q/p}_{k\ell} \right| \leq c\,\delta, \tag{3.31}$$

where δ is defined in (2.40) and c is independent of δ, q, p.

REMARK 3.12. *The hypotheses in the multi-dimensional case are stronger than in 1-d. In 1-d, the homogenized coefficient is explicitly given by (2.23). While, in several dimensions, the homogenized coefficients (2.6) depend on χ_k, solution of (2.7). In general, the auxiliary functions χ_k may not be computed explicitly and need to be approximated numerically. This is the reason of the difference between the 1-d and the several space dimensions cases.*

4. Homogenization via discrete Bloch waves. In Theorem 3.2 we have seen that problem (2.12)–(2.14), under the assumption (2.35), is equivalent to

$$\mu^\varepsilon_{h,m}(k)\widehat{u}^\varepsilon_{h,m}(k) = \widehat{f}^\varepsilon_{h,m}(k), \qquad \forall m = 1, \dots, \dot{p}, \quad k \in \Lambda_{q\varepsilon}, \tag{4.1}$$

with $\Lambda_{q\varepsilon}$ defined in (3.5). Our goal in this section is to pass to the limit in (4.1) as $\varepsilon \to 0$. In Section 4.1 we show that $\widehat{u}^\varepsilon_m(k)$ with $(m \geq 2)$ are negligible for any $k \in \Lambda_{q\varepsilon}$. As a consequence of this result these Bloch components of u^ε_h do not play any role in the limit as $\varepsilon \to 0$. It is then sufficient to analyze the limit behavior of the equation

$$\mu^\varepsilon_{h,1}(k)\widehat{u}^\varepsilon_{h,1}(k) = \widehat{f}^\varepsilon_{h,1}(k), \qquad k \in \Lambda_{q\varepsilon}. \tag{4.2}$$

The limit will be given thanks to the fact that the discrete first Bloch transform representing the periodic medium tends to the discrete Fourier transform representing the homogeneous one. It is a consequence of the fact that the first Bloch coefficient defined in Theorem 3.1 tends to the usual discrete Fourier coefficient (see Section 4.2). Consequently, the limit of equation (4.2) is the Fourier transform of the homogenized equation (see Section 4.3).

4.1. Estimates on the higher order Bloch modes. Here we are going to prove that $\widehat{u}^\varepsilon_{h,m}(k)$ with $m \geq 2$ are negligible as $\varepsilon \to 0$. We set

$$v^\varepsilon_h(x) = (2\pi)^{-\frac{d}{2}}(\dot{p})^{\frac{1}{2}} \sum_{k\in\Lambda_{q\varepsilon}} \sum_{m=2}^{\dot{p}} \widehat{u}^\varepsilon_{h,m}(k) e^{ik\cdot x} \varphi^\varepsilon_{h,m}(x;k), \qquad x \in \Gamma_h, \tag{4.3}$$

the projection of the solution u^ε_h of (2.12)–(2.14) over the orthogonal subspace to the first Bloch component.

PROPOSITION 4.1. *Let v^ε_h defined in (4.3). Then $\|v^\varepsilon_h\|_h \leq c|q\varepsilon|^2\,\|f\|_h$.*

Proof. By Parseval's identity and (4.1), we deduce that

$$\|v^\varepsilon_h\|^2_h = \sum_{k\in\Lambda_{q\varepsilon}} \sum_{m=2}^{\dot{p}} |\widehat{u}^\varepsilon_{h,m}(k)|^2 = \sum_{k\in\Lambda_{q\varepsilon}} \sum_{m=2}^{\dot{p}} \frac{|\widehat{f}^\varepsilon_{h,m}(k)|^2}{(\mu^\varepsilon_{h,m}(k))^2}.$$

Recalling that $\mu^\varepsilon_{h,m}(k) = \varepsilon^{-2}\mu_m(k/q\varepsilon)$ by (3.17), we arrive at

$$\|v^\varepsilon_h\|^2_h = \varepsilon^4 \sum_{k\in\Lambda_{q\varepsilon}} \sum_{m=2}^{\dot{p}} \frac{|\widehat{f}^\varepsilon_{h,m}(k)|^2}{(\mu_m(\frac{k}{q\varepsilon}))^2}.$$

At this point, we use the lower bound of the eigenvalues μ_m with $m \geq 2$ proved in Lemma 3.5 and we obtain that

$$\|v_h^\varepsilon\|_h^2 \leq c\varepsilon^4 \max_{i=1,\dots,d}(q_i)^4 \sum_{k\in\Lambda_{q\varepsilon}} \sum_{m=2}^{\dot{p}} |\widehat{f}_{h,m}^\varepsilon(k)|^2 \leq c|q\varepsilon|^4 \|f\|_h^2. \qquad \square$$

As a consequence of this proposition, the problem of passing of the limit as $\varepsilon \to 0$ in (4.1) is reduced to analyze the equation (4.2) characterizing the first Bloch coefficient.

4.2. First Bloch mode and the discrete Fourier decomposition. Here, we prove that the first Bloch transform defined in (3.4) tends to the usual discrete Fourier transform.

First, we present a brief introduction of discrete Fourier waves (see [11], p 59). Suppose that $f \in L^2_{loc}(\mathbb{R}^d)$ is Y-periodic. Its development in a Fourier series is given by

$$f(x) = (2\pi)^{-\frac{d}{2}} \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ik\cdot x}, \tag{4.4}$$

where $\widehat{f}_k$ is the k^{th} Fourier coefficient of f defined by

$$\widehat{f}_k = (2\pi)^{-\frac{d}{2}} \int_Y f(x) e^{-ik\cdot x} dx, \quad \forall k \in \mathbb{Z}^d. \tag{4.5}$$

Identity (4.5) follows from the orthonormality of the plane waves $e^{ik\cdot y}$ in $L^2(Y)$. The norm in $H^s_\#(Y)$ is defined by

$$\|f\|_s = \left(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s |\widehat{f}_k|^2 \right)^{\frac{1}{2}}.$$

For $n \in \mathbb{N}^d$ we consider the finite-dimensional space S_n generated by the functions $\{e^{ik\cdot x}\}$ with $k \in \Delta_n$, a finite subset of $\mathbb{Z}^d$ defined by

$$\Delta_n = \left\{ (k_1, \cdots, k_d) \in \mathbb{Z}^d : -\left[\frac{n_i+1}{2}\right] + 1 \leq k_i \leq n_i - \left[\frac{n_i+1}{2}\right] \; i = 1, \dots, d \right\}.$$

Note that S_n is of dimension $\dot{n}$ since $\#(\Delta_n) = \dot{n}$. The best approximation of f in S_n (in the L^2-sense) is the function $f_n \in S_n$, obtained by truncating the development (4.4) for $k \in \Delta_n$

$$f_n(x) = (2\pi)^{-\frac{d}{2}} \sum_{k\in\Delta_n} \widehat{f}_k e^{ik\cdot x}. \tag{4.6}$$

Consider also the function $f^{(n)} \in S_n$ *interpolating* f at the points $x \in \Gamma_h$, $h = 2\pi/n$ and Γ_h defined in (2.11), such that

$$f^{(n)}(x) = f(x), \qquad x \in \Gamma_h,$$

and its Fourier decomposition

$$f^{(n)}(x) = \sum_{\ell\in\Delta_n} \widetilde{f}_\ell e^{i\ell\cdot x}, \tag{4.7}$$

where

$$\widetilde{f}_\ell = \frac{1}{n_1 \cdots n_d} \sum_{x \in \Gamma_h} f(x) e^{-i\ell \cdot x}. \tag{4.8}$$

We observe that $\#(\Gamma_h) = \#(\Lambda_n) = \dot{n}$. We define the discrete Fourier transform:

DEFINITION 4.2. *The mapping* $\mathbb{C}^{\dot{n}} \longrightarrow \mathbb{C}^{\dot{n}}$ *such that*

$$\{f(x)\}_{x\in\Gamma_h} \longrightarrow \{\widetilde{f}_\ell\}_{\ell\in\Delta_n}$$

defined according to the formula (4.8) is the discrete Fourier transform.

From [5] we have that for $0 \le m \le s$ the following estimates hold

$$\|f - f_n\|_m \le c_1 |h|^{s-m} \|f\|_s, \qquad \|f - f^{(n)}\|_m \le c_2 |h|^{s-m} \|f\|_s, \tag{4.9}$$

where c_1, c_2 are positive constants independent of n and f.

REMARK 4.3. *The interpolating functions allow us to estimate the discrete norm* $\|\cdot\|_h$ *of continuous functions. Indeed, we have for* $h = 2\pi/n$ *that* $\|f\|_h = \|f^{(n)}\|_0$. *Now, using this property we prove (2.19). Since* u^ε *and* u^* *are continuous*

$$\|u^\varepsilon - u^*\|_h = \|u^{\varepsilon(n)} - u^{*(n)}\|_0 \le \|u^{\varepsilon(n)} - u^\varepsilon\|_0 + \|u^* - u^{*(n)}\|_0 + \|u^\varepsilon - u^*\|_0,$$

and applying (4.9) (note that $u^\varepsilon \in H^1(Y)$ *and* $u^* \in H^2(Y)$*) and (2.9), we obtain (2.19).*

We prove the convergence of the first Bloch coefficient to the discrete Fourier transform.

PROPOSITION 4.4. *Let* f *be a continuous function. Let* $\widehat{f}^\varepsilon_{h,1}(k)$ *be the first Bloch coefficient of* f *defined by (3.4). Let* $h, \varepsilon > 0$ *be satisfying (2.35). Let* B_δ *be the analyticity region in Proposition 3.6. Then, for* $(kq\varepsilon) \in B_\delta$ *we get*

$$|\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}} \widetilde{f}_k| \le c\,|k|\,|q\varepsilon|\,\|f\|_h, \tag{4.10}$$

where $\widetilde{f}_k$ *is the discrete Fourier coefficient defined in (4.8). Furthermore, if* f *is a Lipschitz function, there exists c depending on the Lipschitz constant of* f *such that*

$$\sum_{(kq\varepsilon)\in B_\delta} |\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}} \widetilde{f}_k|^2 \le c\,|q\varepsilon|^2. \tag{4.11}$$

Proof. By the definitions of the discrete Fourier coefficients $\widetilde{f}_k$ and the first Bloch coefficients $\widehat{f}^\varepsilon_{h,1}(k)$ we can write for $k \in \Lambda_{\varepsilon q}$

$$\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}} \widetilde{f}_k = (2\pi)^{\frac{-d}{2}} \dot{h} (\dot{p})^{\frac{1}{2}} \sum_{x\in\Gamma_h} f(x) e^{-ik\cdot x} [\overline{\varphi^\varepsilon_{h,1}}(x;k) - \overline{\varphi^\varepsilon_{h,1}}(x;0)]$$

since $\varphi^\varepsilon_{h,1}(x;0) = \dot{p}^{-1/2}$ for any $x \in \Gamma_h$ (see (3.20)). Since $\Gamma_h = \Gamma_{ph} \oplus \Gamma^p_h$ and $\varphi^\varepsilon_{h,1}$ is ph-periodic (see (3.2)), we get

$$\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}} \widetilde{f}_k = \sum_{x\in\Gamma^p_h} f(x;k) e^{-ix\cdot k} [\overline{\varphi^\varepsilon_{h,1}}(x;k) - \overline{\varphi^\varepsilon_{h,1}}(x;0)],$$

with $f(x;k)$ defined in (3.6). Applying the Cauchy-Schwartz inequality, we obtain

$$|\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}} \widetilde{f}_k|^2 \le \sum_{x\in\Gamma^p_h} |f(x;k)|^2 \sum_{x\in\Gamma^p_h} |\varphi^\varepsilon_{h,1}(x;k) - \varphi^\varepsilon_{h,1}(x;0)|^2. \tag{4.12}$$

Note that by (3.17), it follows that

$$\sum_{x\in\Gamma_h^p} |\varphi_{h,1}^\varepsilon(x;k) - \varphi_{h,1}^\varepsilon(x;0)|^2 = \sum_{y\in\Gamma_{\frac{2\pi}{p}}} |\varphi_1(y;kq\varepsilon) - \varphi_1(y;0)|^2.$$

In Proposition 3.6, we established the analyticity of the map $\eta \to \varphi_1(\cdot;\eta)$. Here we only need its Lipschitz property to get

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} |\varphi_1(y;kq\varepsilon) - \varphi_1(y;0)|^2 \le c\,|q\varepsilon|^2\ |k|^2. \tag{4.13}$$

In view of (3.8), by Parseval's identity get

$$\sum_{x\in\Gamma_h^p} |f\,(x;k)|^2 = \sum_{m=1}^{\dot{p}} |\widehat{f}_{h,m}^\varepsilon(k)|^2 \le \|f\|_h^2.$$

Then, applying (4.13) in (4.12), we prove (4.10). On the other hand, by (3.8), (4.12) and (4.13), we obtain

$$\sum_{(kq\varepsilon)\in B_\delta} |\widehat{f}_{h,1}^\varepsilon(k) - (2\pi)^{\frac{d}{2}}\widetilde{f}_k|^2 \le c\,|q\varepsilon|^2 \sum_{(kq\varepsilon)\in B_\delta}\sum_{m=1}^{\dot{p}} |k|^2|\widehat{f}_{h,m}^\varepsilon(k)|^2.$$

This concludes the proof of (4.11). ◻

4.3. Numerical homogenization. It is convenient first to pass heuristically to the limit in (4.2) to motivate the discrete homogenized limit this produces. The rigorous proof of the main results are in Section 4.4.

First, we study the homogenized problem (2.8) using the discrete Fourier transform of Section 4.2. We consider the interpolation $f^{(n)}$ of f in S_n as in (4.7). Note that $\widetilde{f}_0 = 0$ since $m(f) = 0$. We now define

$$u^{(n)}(x) = \sum_{\ell\in\Lambda_n} \widetilde{u}_\ell e^{i\ell\cdot x}, \quad x \in Y,$$

with

$$\widetilde{u}_k = \frac{\widetilde{f}_k}{a_{ij}^* k_i k_j} \quad \text{and} \quad \widetilde{u}_0 = 0. \tag{4.14}$$

It is immediate to see that

$$\begin{cases} A^* u^{(n)} = f^{(n)} & \text{in} \quad Y, \\ u^{(n)} \in H^1_\#(Y), & m(u^{(n)}) = 0. \end{cases}$$

Furthermore, if $f \in H^s_\#(Y)$, $s \ge 0$, and u^* is the solution of (2.8), we have from (4.9) that

$$\|u^* - u^{(n)}\|_2 \le \|f - f^{(n)}\|_0 \le c|h|^s\|f\|_s,$$

and then

$$\|u^* - u^{(n)}\|_0 \le c|h|^{s+2}\|f\|_s. \tag{4.15}$$

We consider the equation (4.2), multiply both sides by $(2\pi)^{-d/2}$ and using (3.17), we get

$$\varepsilon^{-2}\mu_1(q\varepsilon k)(2\pi)^{-\frac{d}{2}}\widehat{u}^{\varepsilon}_{h,1}(k) = (2\pi)^{-\frac{d}{2}}\widehat{f}^{\varepsilon}_{h,1}(k), \qquad k \in \Lambda_{\varepsilon q}.$$

Expanding μ_1 by Taylor's formula (see Lemma 3.7) and applying (4.10) (concerning the convergence of the first Bloch coefficient $f^{\varepsilon}_1(k)$), we have

$$\frac{1}{2}\partial^2_{ij}\mu_1(0)q_ik_iq_jk_j(2\pi)^{-\frac{d}{2}}\widehat{u}^{\varepsilon}_{h,1}(k) = \widetilde{f}_k + |k|O(|q\varepsilon|) + |qk|^3O(\varepsilon)\widehat{u}^{\varepsilon}_{h,1}(k).$$

Here and the sequel we use the notation $\partial_i = \frac{\partial}{\partial\eta_i}$. We define $\widehat{u}^*_{q/p}(k)$ by

$$a^{*,q/p}_{ij}k_ik_j\widehat{u}^*_{q/p}(k) = \widetilde{f}_k, \quad \text{and} \quad \widehat{u}^*_{q/p}(0) = 0. \tag{4.16}$$

We note that $\widehat{u}^*_{q/p}(k)$ are the discrete Fourier coefficients of the solution of (2.38). Further, since the discrete homogenized coefficients satisfy (3.21), then we can write

$$(2\pi)^{-d/2}\widehat{u}^{\varepsilon}_{h,1}(k) = \widehat{u}^*_{q/p}(k) + \epsilon_k,$$

where ϵ_k is the difference of $(2\pi)^{-d/2}\widehat{u}^{\varepsilon}_{h,1}(k)$ with $\widehat{u}^*_{q/p}(k)$. In the following section we give estimates on the error ϵ_k.

Now, we calculate the difference between the coefficients $\widehat{u}^*_{q/p}(k)$ and the discrete Fourier coefficients of the homogenized solution $\widetilde{u}_k$ defined in (4.14).

PROPOSITION 4.5. *(Several space dimensions) Let u^* be the solution of the homogenized problem (2.8). We define the discrete function $u^*_{q/p}$ as*

$$u^*_{q/p}(x) = \sum_{k\in\Lambda_n}\widehat{u}^*_{q/p}(k)e^{ix\cdot k}, \qquad x \in \Gamma_h, \tag{4.17}$$

where $\widehat{u}^*_{q/p}(k)$ is defined by (4.16). Under the hypotheses of Proposition 3.11, we have

$$\|u^* - u^*_{q/p}\|_h \le c\,(|h| + \delta).$$

PROPOSITION 4.6. *(One space dimension) Let u^* be the homogenized solution. We define the discrete function $u^*_{q/p}$ as in (4.17). Under the hypotheses of Proposition 3.10, we have*

$$\|u^* - u^*_{q/p}\|_h \le c(h + p^{-1}). \tag{4.18}$$

Proof of Proposition 4.5. First, thanks to Remark 4.3, we get

$$\|u^* - u^*_{q/p}\|_h = \|u^{*(n)} - u^*_{q/p}\|_0,$$

where $u^{*(n)}$ is defined interpolating u^* at the points of Γ_h. On the other hand, thanks to the definition of $\widehat{u}^*_{q/p}(k)$ and $\widetilde{u}_k$ defined in (4.16) and (4.14), we get

$$\widehat{u}^*_{q/p}(k) - \widetilde{u}_k = \widetilde{f}_k\left(\frac{1}{a^{*,q/p}_{ij}k_ik_j} - \frac{1}{a^*_{ij}k_ik_j}\right), \qquad k \in \Lambda_n,\ k \ne 0.$$

Now, using the coercivity of $a_{ij}^{*,q/p}$ (see Lemma C.7) and of the coefficients $\{a_{ij}^*\}$, by Proposition 3.11 we obtain

$$|\widehat{u}_{q/p}^*(k) - \widetilde{u}_k| \le c\delta \frac{|\widetilde{u}_k|}{\alpha^2}.$$

Thus, by Parseval's identity,

$$\|u_{q/p}^* - u^{(n)}\|_0 \le c\delta \frac{\|u^*\|_0}{\alpha^2}.$$

Then, by (4.9) and (4.15), we get

$$\|u_{q/p}^* - u^{*(n)}\|_0 \le \|u^{(n)} - u^{*(n)}\|_0 + \|u_{q/p}^* - u^{(n)}\|_0^2 \le c|h| + \|u_{q/p}^* - u^{(n)}\|_0,$$

and this concludes the proof. □

The proof of Proposition 4.6 is the same but under the weaker assumptions of Proposition 3.10 required to compare $\partial^2\mu_1(0)$ with a^*.

4.4. Convergence estimates. Applying the techniques and results above, we can now prove the main estimate in several space dimensions stated in Theorem 2.6. The proof consists of several propositions which correspond to different estimates on the discrete Bloch space and the discrete Fourier space. To do it, we decompose u_h^ε as follows

$$u_h^\varepsilon(x) = u_1^\varepsilon(x) + v_h^\varepsilon(x), \qquad x \in \Gamma_h,$$

where v_h^ε is defined in Proposition 4.1 and u_1^ε is defined by

$$u_1^\varepsilon(x) = (2\pi)^{-\frac{d}{2}} (\dot{p})^{\frac{1}{2}} \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{h,1}^\varepsilon(k) e^{ix\cdot k} \varphi_{h,1}^\varepsilon(x;k). \tag{4.19}$$

Thanks to Proposition 4.1, we can neglect the higher Bloch modes of u_h^ε. In particular,

$$\|u_h^\varepsilon - u_1^\varepsilon\|_h = \|v_h^\varepsilon\| \le c\,|q\varepsilon|^2 \|f\|_h. \tag{4.20}$$

Now, we decompose u_1^ε using the following discrete functions:

$$u_1^{*,\varepsilon}(x) = (\dot{p})^{\frac{1}{2}} \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{q/p}^*(k) e^{ix\cdot k} \varphi_{h,1}^\varepsilon(x;k), \tag{4.21}$$

$$u_2^{*,\varepsilon}(x) = \sum_{k\in\Lambda_{q\varepsilon}} \widehat{u}_{q/p}^*(k) e^{ix\cdot k}, \tag{4.22}$$

where $\widehat{u}_{q/p}^*(k)$ is defined in (4.16). We are going to prove the following results:

Proposition 4.7. *Let u_1^ε and $u_1^{*,\varepsilon}$ be the discrete functions defined in (4.19) and (4.21), respectively. Under the assumptions of Proposition 4.4, we have*

$$\|u_1^\varepsilon - u_1^{*,\varepsilon}\|_h \le c|q\varepsilon|\,\|f\|_h.$$

Proposition 4.8. *Let $u_1^{*,\varepsilon}$ and $u_2^{*,\varepsilon}$ be the discrete functions defined in (4.21) and (4.22), respectively. Then, we have*

$$\|u_1^{*,\varepsilon} - u_2^{*,\varepsilon}\|_h \le c\,|q\varepsilon|\,\|f\|_h.$$

To prove these propositions we only need the analyticity of μ_1 and φ_1 and the convergence of the first discrete Bloch mode to the discrete Fourier transform. We note that $\widehat{u}_1^\varepsilon(0) = \widehat{u}^*_{q/p}(0) = 0$. Therefore, the contribution of the index $k = 0$ in the estimates is negligible and we do not need to take it into account in the following analysis.

Proof of Proposition 4.7. First, we are going to neglect the non analytic components. We use Parseval's identity and obtain

$$\|u_1^\varepsilon - u_1^{*,\varepsilon}\|_h^2 = \sum_{k\in\Lambda_{q\varepsilon}} |\widehat{u}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widehat{u}^*_{q/p}(k)|^2.$$

Taking into account that $\widehat{u}^\varepsilon_{h,1}(k)$ and $\widehat{u}^*_{q/p}(k)$ satisfy (4.2) and (4.16), respectively, and thanks to the existence of $c_1, c_2 > 0$ such that

$$\mu^\varepsilon_{h,1}(k) \geq c_1|k|^2 \qquad \text{and} \qquad a^{*,q/p}_{ij} k_i k_j \geq c_2|k|^2, \tag{4.23}$$

we can reduce our analysis to the points $k \in B_{\delta/q\varepsilon}$. In fact, for $k \in \Lambda_{q\varepsilon}$ we get

$$\begin{aligned} \sum_{\varepsilon qk\notin B_\delta} |\widehat{u}^\varepsilon_{h,1}(k)|^2 &\leq \sum_{\varepsilon qk\notin B_\delta} \frac{|\widehat{f}^\varepsilon_{h,1}(k)|^2}{(c_1|k|^2)^2} \leq c_\delta\,|hp|^4\,\|f\|_h^2, \\ \sum_{\varepsilon qk\notin B_\delta} |\widehat{u}^*_{q/p}(k)|^2 &\leq \sum_{\varepsilon qk\notin B_\delta} \frac{|\widetilde{f}_k|^2}{(c_2|k|^2)^2} \leq c_\delta\,|hp|^4\,\|f\|_h^2. \end{aligned} \tag{4.24}$$

Thus, we have

$$\|u_1^\varepsilon - u_1^{*,\varepsilon}\|_h^2 \leq \sum_{k\in\Lambda_{\varepsilon q}\cap B_{\delta/q\varepsilon}} |\widehat{u}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widehat{u}^*_{q/p}(k)|^2 \,+\, c_\delta\,|hp|^4\|f\|_h^2. \tag{4.25}$$

Using the definitions of $\widehat{u}^\varepsilon_{h,1}(k)$ and $\widehat{u}^*_{q/p}(k)$ we get for $k \neq 0$

$$\widehat{u}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widehat{u}^*_{q/p}(k) = \frac{\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widetilde{f}(k)}{\mu^\varepsilon_{h,1}(k)} + \widetilde{f}(k)[\frac{1}{\mu^\varepsilon_{h,1}(k)} - \frac{2}{\partial_{ij}\mu_1(0)q_ik_iq_jk_j}],$$

where $\mu^\varepsilon_{h,1}(k)$ is defined by (3.17). For $\varepsilon qk \in B_\delta$ we apply Taylor's expansion of μ_1 and obtain

$$\mu^\varepsilon_{h,1}(k) = \varepsilon^{-2}\mu_1(\varepsilon qk) = \frac{1}{2}\partial^2_{ij}\mu_1(0)q_ik_iq_jk_j + c\,|q\varepsilon|\,|k|^3. \tag{4.26}$$

Therefore, we get the following estimate

$$|\widehat{u}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widehat{u}^*_{q/p}(k)| \leq \frac{|\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widetilde{f}(k)|}{\mu^\varepsilon_{h,1}(k)} + \frac{c|\widetilde{f}(k)||qk|^3\varepsilon}{\mu^\varepsilon_{h,1}(k)\partial^2_{ij}\mu_1(0)q_ik_iq_jk_j},$$

and using (4.23), we obtain

$$|\widehat{u}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widehat{u}^*_{h/\varepsilon}(k)| \leq \frac{c}{|k|^2}|\widehat{f}^\varepsilon_{h,1}(k) - (2\pi)^{\frac{d}{2}}\widetilde{f}(k)| + c|\widetilde{f}(k)|\frac{|q|\varepsilon}{|k|}.$$

Finally, using the results of Proposition 4.4, we obtain

$$\sum_{k\in\Lambda_{\varepsilon q}\cap\varepsilon q B_\delta} \frac{c}{|k|^4}|\widehat{f}^{\varepsilon}_{h,1}(k)-(2\pi)^{\frac{d}{2}}\widetilde{f}(k)|^2 \le c\,|q\varepsilon|^2 \sum_{(kq\varepsilon)\in B_\delta}\sum_{m=1}^{\dot p}\frac{1}{|k|^2}|\widehat{f}^{\varepsilon}_{h,m}(k)|^2.$$

Therefore, we get

$$\sum_{k\in\Lambda_{\varepsilon q}\cap\varepsilon q B_\delta}|\widehat{u}^{\varepsilon}_{h,1}(k)-(2\pi)^{\frac{d}{2}}\widehat{u}^{*}_{q/p}(k)|^2 \le c\sum_{(kq\varepsilon)\in B_\delta}\sum_{m=1}^{\dot p}\frac{|\widehat{f}^{\varepsilon}_{h,m}(k)|^2}{|k|^2}+c'|q\varepsilon|^2\sum_{(kq\varepsilon)\in B_\delta}\frac{|\widetilde{f}(k)|^2}{|k|^2}.$$

Then, since $|k|\ge 1$, we conclude the proof thanks to (4.25). □

Proof of Proposition 4.8. We observe that for any $x\in\Gamma_h$ $\varphi^{\varepsilon}_{h,1}(x;0)=1/\sqrt{\dot p}$. Therefore

$$u_1^{*,\varepsilon}(x)-u_2^{*,\varepsilon}(x)=\sqrt{\dot p}\sum_{k\in\Lambda_{q\varepsilon}}\widehat{u}^{*}_{q/p}(k)e^{ix\cdot k}[\varphi^{\varepsilon}_{h,1}(x;k)-\varphi^{\varepsilon}_{h,1}(x;0)].$$

By (3.17), an easy computation shows that

$$\|u_1^{*,\varepsilon}-u_2^{*,\varepsilon}\|_h^2=(2\pi)^d\sum_{k\in\Lambda_{q\varepsilon}}|\widehat{u}^{*}_{q/p}(k)|^2\sum_{y\in\Gamma_{\frac{2\pi}{p}}}|\varphi_1(y;q\varepsilon k)-\varphi_1(y;0)|^2.$$

Now, by (4.24) we reduce our analysis to the points $q\varepsilon k\in B_\delta$ as in the previous proof

$$\|u_1^{*,\varepsilon}-u_2^{*,\varepsilon}\|_h^2\le(2\pi)^d\sum_{(q\varepsilon k)\in\Lambda_{q\varepsilon}\cap B_\delta}|\widehat{u}^{*}_{q/p}(k)|^2\sum_{y\in\Gamma_{\frac{2\pi}{p}}}|\varphi_1(y;q\varepsilon k)-\varphi(y;0)|^2+c_\delta|q\varepsilon|^4\|f\|_h^2.$$

By the analyticity of $\varphi_1(\cdot;\eta)$ in the variable η (see Proposition 3.6), we obtain

$$\begin{aligned}\|u_1^{*,\varepsilon}-u_2^{*,\varepsilon}\|_h^2&\le c\sum_{(q\varepsilon k)\in\Lambda_{q\varepsilon}\cap B_\delta}|\widehat{u}^{*}_{q/p}(k)|^2|\varepsilon qk|^2+c_\delta|q\varepsilon|^4\|f\|_h^2\\&\le c|q\varepsilon|^2\|f\|_h^2+c_\delta|q\varepsilon|^4\|f\|_h^2.\end{aligned}$$

This concludes the proof of Proposition 4.8. □

Now, combining (4.20) and Propositions 4.5, 4.7 and 4.8, we prove Theorem 2.6.

Proof of Theorem 2.6. We consider u^{ε}_h, the finite difference approximation of (2.3), and u^*, the homogenized solution of (2.8). For any $x\in\Gamma_h$ we introduce the decomposition

$$\begin{aligned}u^{\varepsilon}_h(x)-u^{*}_{q/p}(x)&=(u^{\varepsilon}_h(x)-u^{\varepsilon}_1(x))+(u^{\varepsilon}_1(x)-u_1^{*,\varepsilon}(x))+(u_1^{*,\varepsilon}(x)-u_2^{*,\varepsilon}(x))\\&\quad+(u_2^{*,\varepsilon}(x)-u^{*}_{q/p}(x)),\end{aligned}\tag{4.27}$$

where u^{ε}_1, $u_1^{*,\varepsilon}$, $u_2^{*,\varepsilon}$ and $u^{*}_{q/p}$ are defined in (4.19), (4.21), (4.22) and (4.16), respectively. Since higher Bloch modes can be neglected (see Section 4.1), we get (4.20). Then, using the previous Propositions 4.7 and 4.8, we get

$$\|u^{\varepsilon}_h-u^{*}_{q/p}\|_h\le c(|q\varepsilon|+|q\varepsilon|^2)\|f\|_h+\|u_2^{*,\varepsilon}-u^{*}_{q/p}\|_h.\tag{4.28}$$

Now, by (4.17) and (4.22) we observe that

$$u^*_{q/p}(x) - u^{*,\varepsilon}_2(x) = \sum_{k\in\Lambda_n/\Lambda_{q\varepsilon}} \widehat{u}^*_{q/p}(k)e^{ix\cdot k}.$$

Using the same analysis as in (4.24), we have

$$\|u^{*,\varepsilon}_2 - u^*_{q/p}\|^2_h \leq \sum_{k\in\Lambda_n/\Lambda_{q\varepsilon}} |\widehat{u}^*_{q/p}(k)|^2 \leq c|q\varepsilon|^4\|f\|^2_h \tag{4.29}$$

and we prove (2.37). Using this estimate and Proposition 4.5, we conclude the proof. □

We conclude this section with the proof of Theorem 2.4, which is specific to the 1-d case.

Proof of Theorem 2.4. We consider the decomposition (4.27). First, we apply the estimates of Propositions 4.7 and 4.8. The assumptions of these propositions are not needed because (4.23) and (4.26) are satisfied in 1-d with p, q defined as in (2.20). Then, we have (4.28).

On the other hand, (4.29) is satisfied in 1-d and, under the hypotheses of Proposition 3.10, we obtain (4.18). Then, applying (4.29) and (4.18) in (4.28), we conclude the proof. □

5. Diophantine approximations. In this section, using our results, we recover a quantitative version of previous results on numerical homogenization by Avellaneda, Hou and Papanicolaou (see Theorem A and Theorem B in Section 2.3).

Proof of Theorem A. Given an irrational number r we are going to prove (2.24) using Theorem 2.3 and a classical results of approximation of irrational numbers by rational ones. We know (see [19], p. 189-190) that there exists a sequence of natural numbers (p_n, q_n) such that

$$\left| r - \frac{q_n}{p_n} \right| \leq \frac{1}{\sqrt{5}p_n^2} \to 0 \qquad \text{when } n \to \infty.$$

Now, we consider another sequence $\{a_n\} \subset \mathbb{N}$ such that $a_n \to \infty$. Then, taking $\varepsilon = 1/(a_n q_n)$ and $h = 2\pi/(a_n p_n)$, we get by Theorem 2.3 that

$$\sup_{x\in\Gamma_h} |u^\varepsilon_h(x) - u^*(x)| \leq c\left(\frac{1}{a_n} + \frac{1}{p_n}\right),$$

and, in particular, when $n \to \infty$ the error converges to 0 while $h/2\pi\varepsilon$ goes to r. Thus, we conclude the proof of Theroem A. □

Now, we prove Theorem B: Given $\varepsilon \in (0, 1)$, if $h \in S(\varepsilon, h_0)$, we have to prove that (2.25) holds using the estimate of Theorem 2.3.

Proof of Theorem B. Given $h \in S(\varepsilon, h_0)$, when $h/2\pi\varepsilon \in \mathbb{Q}$ we obtain immediately (2.25) by the estimate of Theorem 2.3. Now, we study the case $h/2\pi\varepsilon \notin \mathbb{Q}$. We slightly perturb ε by $\epsilon \neq \varepsilon$ such that

$$\frac{h}{\epsilon} = 2\pi\frac{q}{p} \in \mathbb{Q} \qquad \text{with} \qquad \text{M.C.D.}(q, p) = 1.$$

First, we calculate the difference between u_h^ε and u_h^ϵ, the approximations of the discrete problems with scale ε and ϵ respectively. Using that u_h^ϵ and u_h^ε are solutions of (2.12)–(2.14) with coefficient a_h^ϵ and a_h^ε respectively, it is immediate that

$$\sum_{x\in\Gamma_h} a_h^\epsilon(x)|\nabla(u_h^\epsilon(x)-u_h^\varepsilon(x))|^2 = \sum_{x\in\Gamma_h}[a_h^\varepsilon(x)-a_h^\epsilon(x)]\nabla u_h^\varepsilon(x)\nabla(u_h^\epsilon(x)-u_h^\varepsilon(x)).$$

Since a is Lipschitz, we get

$$|a_h^\varepsilon(x)-a_h^\epsilon(x)| \le c\frac{1}{h}\left|\frac{h}{\varepsilon}-\frac{q}{p}\right|, \qquad \forall x\in\Gamma_h.$$

Therefore, by the coercivity of a, we get

$$\left(\sum_{x\in\Gamma_h}|\nabla(u_h^\epsilon(x)-u_h^\varepsilon(x))|^2\right)^{\frac{1}{2}} \le ch^{-1}\left|\frac{h}{\varepsilon}-\frac{q}{p}\right|, \tag{5.1}$$

where c depends on a. On the other hand, applying Theorem 2.3, we obtain that

$$\sup_{x\in\Gamma_h}|u_h^\epsilon(x)-u^*(x)| \le c\,hp + c'\frac{1}{p}, \tag{5.2}$$

where c and c' only depend on α, β, a and $\|f\|_\infty$.

Now, we check that there exist q and p that provide the result we are looking for. In fact, using (5.1) and (5.2) we get (2.25) when $p,q\in\mathbb{N}$ satisfy:

$$\left|\frac{h}{\varepsilon}-2\pi\frac{q}{p}\right| \le \tau h \qquad \text{and} \qquad \tau^{-1}\le p\le \tau h^{-1}. \tag{5.3}$$

By the Dirichlet's theorem (a classical result of approximation of irrational numbers, see [33] p. 34), there exist $p,q\in\mathbb{N}$ such that $p\le\tau h^{-1}$ and

$$\left|\frac{h}{\varepsilon}-2\pi\frac{q}{p}\right| < \frac{h}{p\tau}. \tag{5.4}$$

To obtain (5.3) it is also necessary that $h/p\tau\le\tau h$. Therefore, p satisfying (5.4) has to be larger than τ^{-2}, i.e., p must belong to the interval $[\tau^{-2},\tau h^{-1}]$. In fact, using that $h\in S(\varepsilon,h_0)$, we have that

$$\left|\frac{h}{\varepsilon}-2\pi\frac{q}{p}\right| \ge \frac{\tau}{p^{\frac{5}{2}}}.$$

From this inequality, we get $\tau p^{-5/2}\le h/p\tau$, i.e., $p\ge\tau^{4/3}h^{-2/3}$.

In short, by Dirichlet's result we know that there exist p,q with $p\le\tau h^{-1}$ satisfying (5.4) and, since $h\in S(\varepsilon,h_0)$, then $p\in[\tau^{4/3}h^{-2/3},\tau h^{-1}]$.

Thus, we conclude the proof if $\tau^{4/3}h^{-2/3}\ge\tau^{-2}$, i.e., if $h\le\tau^5$. Later, if $h_0=\tau^5$, for any $h\in S(\varepsilon,h_0)$, there exists p,q satisfying (5.3) and, therefore, (2.25) is proved using the estimates of Theorem 2.3. ∎

Appendix A. Homogenization of the continuous problem.

A.1. Bloch wave decomposition. In this section we recall some basic results on Bloch wave decomposition. We refer to [7], [9] and [10] for more details.

Let us consider the following spectral problem family by the parameter $\eta \in \mathbb{R}^d$: To find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\psi = \psi(x;\eta)$ (not identically zero) such that

$$\begin{cases} A\psi(\cdot;\eta) = \lambda(\eta)\psi(\cdot;\eta) \quad \text{in} \quad \mathbb{R}^d, \\ \psi(\cdot;\eta) \text{ is } (\eta, Y)\text{-periodic, i.e.,} \\ \psi(y+2\pi m;\eta) = e^{2\pi i m\cdot\eta}\psi(y;\eta) \quad \forall m \in \mathbb{Z}^d,\ y \in \mathbb{R}^d, \end{cases} \tag{A.1}$$

where A is the elliptic operator in divergence form defined in (2.1). We can write $\psi(y;\eta) = e^{iy\cdot\eta}\phi(y;\eta)$, ϕ being Y-periodic in the variable y. It is clear from (A.1) that η can be confined to the dual cell $\eta \in Y' = [-1/2, 1/2[^d$. Under these conditions, it is known (see [10]) that the spectral problem above admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \le \lambda_1(\eta) \le \cdots \le \lambda_n(\eta) \le \cdots \to \infty, \\ \lambda_m(\eta) \text{ is a Lipschitz function of } \eta \in Y',\ \forall m \ge 1. \end{cases} \tag{A.2}$$

Besides, the corresponding eigenfunctions denoted by $\psi_m(\cdot;\eta)$ and $\phi_m(\cdot;\eta)$, form orthonormal bases in the subspaces of $L^2_{loc}(\mathbb{R}^d)$ of (η, Y)-periodic and Y-periodic functions, respectively. Moreover, as a consequence of the min-max principle, it follows that (see [10])

$$\lambda_2(\eta) \ge \lambda_2^{(N)} > 0, \quad \forall \eta \in Y', \tag{A.3}$$

where $\lambda_2^{(N)} > 0$ is the second eigenvalue of A in the cell Y with Neumann boundary conditions.

To express the equation $A^\varepsilon u^\varepsilon = f$ in an equivalent way in the Bloch space, we introduce the Bloch eigenvalues $\{\lambda_m^\varepsilon(\xi)\}_{m\ge1}$ and eigenvectors $\{\phi_m^\varepsilon(x;\xi)\}$ in the ε-scale:

$$\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\varepsilon\xi), \qquad \phi_m^\varepsilon(x;\xi) = \phi_m(\frac{x}{\varepsilon};\varepsilon\xi). \tag{A.4}$$

Then, given $f \in L^2_\#(Y)$, the mth Bloch coefficient of f at the ε scale is defined as follows:

$$\widehat{f}_m^\varepsilon(k) = \varepsilon^{-\frac{d}{2}} \int_Y f(x) e^{-ik\cdot x}\overline{\phi_m^\varepsilon}(x;k)dx \quad \forall m \ge 1,\ k \in \Lambda_\varepsilon, \tag{A.5}$$

where

$$\Lambda_\varepsilon = \{k = (k_1, \ldots, k_d) \in \mathbb{Z}^d, \text{ such that } [-1/2\varepsilon] + 1 \le k_i \le [1/2\varepsilon]\}. \tag{A.6}$$

THEOREM A.1. *Let $\varepsilon > 0$ be defined by (2.4). For any $f \in L^2(Y)$ the following representation formula holds:*

$$f(x) = \varepsilon^{\frac{d}{2}} \sum_{k\in\Lambda_\varepsilon} \sum_{m\ge1} \widehat{f}_m^\varepsilon(k) e^{ik\cdot x}\phi_m^\varepsilon(x;k).$$

Further, we have the Parseval's identity:

$$\int_Y |f(x)|^2 dx = \varepsilon^d \sum_{k\in\Lambda_\varepsilon} \sum_{m\ge1} |\widehat{f}_m^\varepsilon(k)|^2.$$

More generally, the following Plancherel identity is also valid:

$$\int_Y f(x)g(x)dy = \varepsilon^d \sum_{k\in\Lambda_\varepsilon} \sum_{m\geq 1} \widehat{f}_m^\varepsilon(k)\overline{\widehat{g}_m^\varepsilon(k)}, \quad \forall f,g \in L^2_\#(Y).$$

The proof of this theorem follows the same steps of the proof of Theorem 3.1.

We also have the following result on the dependence of λ_1 and ϕ_1 with respect to the parameter η (see [7] and [10]).

PROPOSITION A.2. *Assume that the coefficients $a_{k\ell}$ satisfy (2.2). Then there exists $\delta > 0$ such that the first eigenvalue λ_1 is an analytic function on $B_\delta = \{\eta : |\eta| < \delta\}$ and satisfies*

$$c_1|\eta|^2 \leq \lambda_1(\eta) \leq c_2|\eta|^2, \quad \forall \eta \in Y', \tag{A.7}$$

and

$$\begin{aligned} \lambda_1(0) = \partial_k\lambda_1(0) = 0, & k = 1,\ldots,N, \\ \partial^2_{k\ell}\lambda_1(0) = 2a^*_{k\ell}, & k,\ell = 1,\ldots,N, \\ \partial^\alpha\lambda_1(0) = 0, & \forall\alpha \text{ such that } |\alpha| \text{ is odd}, \end{aligned} \tag{A.8}$$

where $a^*_{k\ell}$ are the homogenized coefficients defined in (2.6). Furthermore, there is a choice of the first eigenfunction $\phi_1(y;\eta)$ satisfying

$$\begin{cases} \eta \in B_\delta \to \phi_1(y;\eta) \in L^\infty \cap L^2_\#(Y) \text{ is analytic}, \\ \phi_1(y;0) = (2\pi)^{-\frac{d}{2}}. \end{cases}$$

A.2. Homogenization results. As in [10], we deduce some classical homogenization results with periodic boundary conditions as a consequence of the properties of the first Bloch eigenvalue and eigenvector.

First, we observe that the equation (2.3) can be easily transformed to a set of algebraic equations for the Bloch coefficients. We show next that the energy of the solution u^ε of (2.3) contained in all Bloch modes $m \geq 2$ goes to zero (Proposition A.3). Thus it is sufficient to analyze the first Bloch mode corresponding to $m = 1$ since all higher modes can be neglected in the homogenization process. When passing to the limit in the first Bloch component ($m = 1$) we obtain the Fourier series corresponding to the limit homogeneous medium.

Let us now develop these ideas. Thanks to the following relation

$$A^\varepsilon(e^{ik\cdot x}\phi_m^\varepsilon(x;k)) = \lambda_m^\varepsilon(\xi)e^{ik\cdot x}\phi_m^\varepsilon(x;k),$$

which is satisfied for $k \in \Lambda_\varepsilon$, and, according to Theorem A.1, equation (2.3) is equivalent to

$$\lambda_m^\varepsilon(k)\widehat{u}_m^\varepsilon(k) = \widehat{f}_m^\varepsilon(k), \quad \forall\, m \geq 1,\ k \in \Lambda_\varepsilon. \tag{A.9}$$

Our goal is to pass to the limit in these equations as $\varepsilon \to 0$. First, we claim that one can neglect all the terms corresponding to $m \geq 2$.

PROPOSITION A.3. *Let*

$$v^\varepsilon(x) = \varepsilon^{\frac{d}{2}} \sum_{k\in\Lambda_\varepsilon} \sum_{m=2}^{\infty} \widehat{u}_m^\varepsilon(k)e^{ik\cdot x}\phi_m^\varepsilon(x;k),$$

where $\widehat{u}^\varepsilon_m(k)$ *are the Bloch coefficients of the solution of (2.3) with* $f \in L^2_\#(Y)$. *Then,*

$$\|v^\varepsilon\|_0 \le c\varepsilon^2 \|f\|_0, \qquad \|\nabla v^\varepsilon\|_0 \le c\varepsilon \|f\|_0.$$

The proof can be carried out as in Proposition 3.5 of [10], using the Parseval and Plancherel identities in Theorem A.1 above and the characterization of the eigenvalues of (A.1) by the min-max principle, together with the inequalities (A.3).

Let us now recall the classical Fourier series decomposition that will arise naturally when analyzing the limit behavior of the first Bloch component as $\varepsilon \to 0$. Suppose that $f \in L^2_\#(Y)$ and $k \in \mathbb{Z}^d$. The k^{th} Fourier coefficient of f is defined by (4.5) and the inverse formula is given by (4.4). Furthermore, Plancherel's identity is also valid:

$$\int_Y f(x)\overline{g(x)}dx = \sum_{k\in\mathbb{Z}} \widehat{f}_k\, \overline{\widehat{g}_k}, \quad \forall f, g \in L^2_\#(Y).$$

In the following proposition we give some convergence results of the first Bloch component towards the Fourier series:

Proposition A.4. *Under the assumptions of Proposition A.2, there exist* $c, c' > 0$ *such that for all* $f \in L^2_\#(Y)$ *they follow*

$$|\varepsilon^{\frac{d}{2}} \widehat{f}^\varepsilon_1(k) - \widehat{f}_k| \le c\varepsilon |k| \|f\|_0, \qquad \forall k \in \varepsilon \in B_\delta \cap \Lambda_\varepsilon, \tag{A.10}$$

$$\sum_{k\in\Lambda_\varepsilon\cap B_{\frac{\delta}{\varepsilon}}} \frac{2}{\lambda^\varepsilon_1(k)} |\varepsilon^{\frac{d}{2}} \widehat{f}^\varepsilon_1(k) - \widehat{f}_k|^2 \le c'\varepsilon^2 \|f\|^2_0, \tag{A.11}$$

with $c, c' > 0$ *independent of* ε, k *and* f.

The proof is a consequence of the analyticity of the first Bloch eigenvalue $\phi_1(y;\eta)$ with respect to η in B_δ. The proof can be carried out as in Proposition 3.6 of [10]. Our next aim is to pass to the limit in equation (A.9) corresponding to the first Bloch mode.

Proposition A.5. *Under the assumptions of Proposition A.2, for* $k\varepsilon \in B_\delta$ *it follows*

$$\varepsilon^{\frac{d}{2}} \widehat{u}^\varepsilon_1(k) \longrightarrow \widehat{u}^*_k \qquad as\ \varepsilon \to 0,$$

where $\widehat{u}^*_k$ *is the* k^{th} *Fourier coefficient of the homogenized solution* u^*. *In particular, the following estimate is satisfied*

$$|\varepsilon^{\frac{d}{2}} \widehat{u}^\varepsilon_1(k) - \widehat{u}^*_k| \le c\varepsilon \|f\|_0. \tag{A.12}$$

Proof. We get

$$|\varepsilon^{\frac{d}{2}} \widehat{u}^\varepsilon_1(k) - \widehat{u}^*_k| \le \frac{2}{(\lambda^\varepsilon_1(k))^2} |\varepsilon^{\frac{d}{2}} \widehat{f}^\varepsilon_1(k) - \widehat{f}_k|^2 + \frac{2|\widehat{f}_k|^2}{\lambda^\varepsilon_1(k) a^*_{ij} k_i k_j} |\lambda^\varepsilon_1(k) - a^*_{ij} k_i k_j|^2, \qquad k \ne 0$$

Using (A.10) and the Taylor expansion of the first Bloch eigenvalue up to second order (see in Proposition A.2), we obtain

$$|\varepsilon^{\frac{d}{2}} \widehat{u}^\varepsilon_1(k) - \widehat{u}^*_k| \le \varepsilon^2 \left(\frac{|k|^2 \|f\|^2_0}{(\lambda^\varepsilon_1(k))^2} + c \frac{|k|^4 |f_k|^2}{\lambda^\varepsilon_1(k) a^*_{ij} k_i k_j} \right),$$

for any $k\varepsilon \in B_\delta$, and by the inequalities (A.7), (A.12) is proved. On the other hand, since $m(u^\varepsilon) = 0$ by (2.3), we have $\widehat{u}_1^\varepsilon(0) = 0$ which, obviously, also converges to $\widehat{u}_0^* = 0$. ◻

We now proceed with the proof of the estimate (2.9) in Theorem 2.1.

Proof of Theorem 2.1. We use the same steps as in the proof of Theorem 1.8 in [7]. The Bloch decomposition of u^ε and the Fourier decomposition of u^* allow us to write

$$\begin{aligned} u^\varepsilon(x) - u^*(x) \\ &= v^\varepsilon(x) + \varepsilon^{\frac{d}{2}} \sum_{k\in\Lambda_\varepsilon\cap B_{\frac{\delta}{\varepsilon}}} \widehat{u}_1^\varepsilon(k) e^{ikx}[\phi_1(\frac{x}{\varepsilon};\varepsilon k) - \phi_1(\frac{x}{\varepsilon};0)] - (2\pi)^{-\frac{d}{2}} \sum_{k\in U_\varepsilon} \widehat{u^*}_k e^{ikx} \\ &\quad + \varepsilon^{\frac{d}{2}} \sum_{k\in\Lambda_\varepsilon\cap U_\varepsilon} \widehat{u}_1^\varepsilon(k) e^{ikx}\phi_1(\frac{x}{\varepsilon};\varepsilon k) + (2\pi)^{-\frac{d}{2}} \sum_{k\in\Lambda_\varepsilon\cap B_{\frac{\delta}{\varepsilon}}} [\varepsilon^{\frac{d}{2}}\widehat{u}_1^\varepsilon(k) - \widehat{u^*}_k] e^{ikx} \\ &= v^\varepsilon(x) + v_1^\varepsilon(x) + v_2^\varepsilon(x) + v_3^\varepsilon(x) + v_4^\varepsilon(x), \end{aligned}$$

where $U_\varepsilon = \mathbb{Z}^N - (\Lambda_\varepsilon \cap \varepsilon^{-1}B_\delta)$ and $v^\varepsilon(x)$ is defined in Proposition A.3. According to the second inequality in Proposition A.3 it is sufficient to estimate v_j^ε, $j = 1, \cdots, 4$.

Secondly, we can neglect the term v_2^ε thanks to the coercity of the homogenized coefficients and the term v_3^ε by (A.7). Now, using the analyticity of ϕ_1, we get

$$\|v_1^\varepsilon\|_0^2 \le c\varepsilon^2\varepsilon^d \sum_{k\in\Lambda_\varepsilon\cap B_{\frac{\delta}{\varepsilon}}} |k|^2 |\widehat{u}_1^\varepsilon(k)|^2 \le c\varepsilon^2\|f\|_0^2.$$

Finally, v_4^ε satisfies

$$\|v_3^\varepsilon\|_0^2 \le \sum_{k\in\Lambda_\varepsilon\cap B_{\frac{\delta}{\varepsilon}}} \frac{2}{(\lambda_1^\varepsilon(k))^2} |\varepsilon^{\frac{d}{2}}\widehat{f}_1^\varepsilon(k) - \widehat{f}_k|^2 + \frac{2|\widehat{f}_k|^2}{\lambda_1^\varepsilon(k) a_{ij}^* k_i k_j} |\lambda_1^\varepsilon(k) - a_{ij}^* k_i k_j|^2.$$

Thus, by (A.7), the coercivity of the homogenized coefficients and the analyticity of λ_1, we conclude the proof of (2.9). ◻

Appendix B. The one-dimensional case.

In this appendix we study the approximation in finite differences for the solutions of the 1-d problem (2.21). In particular, we prove Theorem 2.3.

Taking $h = 2\pi/n$, we denote for, $i \in \mathbb{N}$,

$$f_i = f(hi), \qquad a_i^\varepsilon = a\left(\frac{h}{\varepsilon}\left(i + \frac{1}{2}\right)\right).$$

We introduce the following system of linear equations with unknown $\{u_i^\varepsilon\}_{i=1}^{n-1}$:

$$\begin{cases} -a_i^\varepsilon u_{i+1}^\varepsilon + (a_i^\varepsilon + a_{i-1}^\varepsilon) u_i^\varepsilon - a_{i-1}^\varepsilon u_{i-1}^\varepsilon = h^2 f_i, & 1 \le i \le n-1, \\ u_0^\varepsilon = b, \quad u_n^\varepsilon = c. \end{cases} \tag{B.1}$$

Solutions of (2.21) are bounded in $H_0^1(0, 2\pi)$ independently of ε and converge weakly in $H_0^1(0, 2\pi)$ as $\varepsilon \to 0$ to $u^* \in H_0^1(0, 2\pi)$ solution of (2.22). Associated to (2.22), we have the following finite difference system for $h = 2\pi/n$:

$$\begin{cases} a^*(-u_{i-1}^* + 2u_i^* - u_{i+1}^*) = h^2 f_i, & 1 \le i \le n-1, \\ u_0^* = b, \quad u_n^* = c. \end{cases} \tag{B.2}$$

Thanks to Theorem 2.2, to prove (2.27) we only need to estimate $\{u_i^\varepsilon\}_{i=1}^n$ and $\{u_i^*\}_{i=1}^n$. To do it, we write explicitly these vectors. We define

$$U_i^{\varepsilon,h} = \frac{1}{h} a_i^\varepsilon (u_{i+1}^\varepsilon - u_i^\varepsilon), \quad \text{with} \quad 0 \le i \le n-1. \tag{B.3}$$

Then, (B.1) can be written as

$$-(U_i^{\varepsilon,h} - U_{i-1}^{\varepsilon,h}) = hf_i, \qquad 1 \le i \le n-1,$$

and consequently

$$U_i^{\varepsilon,h} = U_0^{\varepsilon,h} - \sum_{j=1}^{i} h\, f_j, \qquad 1 \le i \le n-1. \tag{B.4}$$

Now, by the definition (B.3) of $U_i^{\varepsilon,h}$, we get

$$u_{i+1}^\varepsilon = b + U_0^{\varepsilon,h} \sum_{j=0}^{i} \frac{h}{a_j^\varepsilon} - \sum_{j=1}^{i} \frac{h}{a_j^\varepsilon} \sum_{k=1}^{j} hf_k, \qquad 1 \le i \le n-1, \tag{B.5}$$

with $U_0^{\varepsilon,h}$ a constant that can be determined by the boundary conditions and f. In particular, we have

$$U_0^{\varepsilon,h} = \frac{a_h^{\varepsilon,*}}{2\pi}(c-b) + \frac{a_h^{\varepsilon,*}}{2\pi} \sum_{j=1}^{n-1} \left(\frac{1}{a_j^\varepsilon} \sum_{k=1}^{j} h^2 f_k \right), \tag{B.6}$$

with

$$a_h^{\varepsilon,*} = \left(\frac{1}{2\pi} \sum_{j=0}^{n-1} \frac{h}{a_j^\varepsilon} \right)^{-1}. \tag{B.7}$$

We observe that (B.5) is an approximation of the explicit formula of the solution of (2.21):

$$u^\varepsilon(x) = b + U_0^\varepsilon \int_0^x \frac{ds}{a^\varepsilon(s)} - \int_0^x \int_0^s \frac{f(t)}{a^\varepsilon(s)} dtds,$$

where U_0^ε is given by

$$U_0^\varepsilon = \left(c - b + \int_0^{2\pi} \int_0^s \frac{f(t)}{a^\varepsilon(s)} dtds \right) \left(\int_0^{2\pi} \frac{ds}{a^\varepsilon(s)} \right)^{-1}.$$

Analogously, the vector $\{u_i^*\}_{i=1}^n$ solution of (B.2) satisfies

$$u_{i+1}^* = b + \frac{(i+1)h}{a^*} U_0^{*,h} - \sum_{j=1}^{i} \sum_{k=1}^{j} \frac{h^2}{a^*} f_k, \tag{B.8}$$

where $U_0^{*,h}$ is defined by

$$U_0^{*,h} = a^*(c-b) + \sum_{j=0}^{n-1} \sum_{k=1}^{j} h^2 f_k. \tag{B.9}$$

Now, using (B.5) and (B.8), we write:

$$u_{i+1}^{\varepsilon} - u_{i+1}^{*} = U_0^{\varepsilon,h} \sum_{j=0}^{i} \frac{h}{a_j^{\varepsilon}} - U_0^{*,h} \frac{(i+1)h}{a^*} + \sum_{j=1}^{i} \sum_{k=1}^{j} \left(\frac{1}{a^*} - \frac{1}{a_j^{\varepsilon}} \right) h^2 f_k$$

$$= U_0^{\varepsilon,h} \sum_{j=0}^{i} \left(\frac{h}{a_j^{\varepsilon}} - \frac{h}{a^*} \right) + (U_0^{\varepsilon,h} - U_0^{*,h}) \frac{(i+1)h}{a^*} + \sum_{j=0}^{i} \left(\frac{h}{a^*} - \frac{h}{a_j^{\varepsilon}} \right) \sum_{k=1}^{j} h f_k. \quad \text{(B.10)}$$

B Using this property,

Now,we establish the following connections between $a_h^{\varepsilon,*}$ and a^*, necessary to estimate (B.10).

Lemma B.1. *Let a be a 2π-periodic Lipschitz function such that $0 < \alpha \le a(x) \le \beta$ for any $x \in (0, 2\pi)$. Let $h = 2\pi/n$ and $\varepsilon > 0$ be satisfying (2.26). Then,*

$$\left| \int_0^{2\pi} \frac{dy}{a(y)} - \frac{2\pi}{p} \sum_{j=0}^{p-1} \frac{1}{a_j^{\varepsilon}} \right| \le \frac{c}{2\alpha^2} \frac{1}{p}. \quad \text{(B.11)}$$

Moreover, if $a_h^{\varepsilon,*}$ is defined as in (B.7), we have:

$$\begin{aligned} &(i) \qquad \left| a^* - a_h^{\varepsilon,*} \right| \le \frac{c\beta^2}{2\alpha^2} \frac{1}{p}, \qquad \text{if } \frac{n}{p} \in \mathbb{N}. \\ &(ii) \qquad \left| a^* - a_h^{\varepsilon,*} \right| \le \frac{\beta^2}{2\alpha^2} \frac{c}{p} + \frac{\beta^2}{\alpha} hp, \qquad \text{if } \frac{n}{p} \notin \mathbb{N}. \end{aligned} \quad \text{(B.12)}$$

Proof. First, by the relation (2.26), we only consider p values of the coefficient a, since

$$a_{p+i}^{\varepsilon} = a \left(2\pi q + 2\pi \frac{q}{p} \left(i + \frac{1}{2} \right) \right) = a \left(2\pi \frac{q}{p} \left(i + \frac{1}{2} \right) \right) = a_i^{\varepsilon}. \quad \text{(B.13)}$$

Thus, the only distinct values $\{a_i^{\varepsilon}\}$ are $\{a_0^{\varepsilon}, \cdots, a_{p-1}^{\varepsilon}\}$. We denote

$$a_i = a \left(\frac{2\pi}{p} \left(i + \frac{1}{2} \right) \right).$$

Since h and ε satisfy (2.26), then $\{a_0^{\varepsilon}, \cdots, a_{p-1}^{\varepsilon}\} \equiv \{a_0, \cdots, a_{p-1}\}$. As a consequence, we get

$$\sum_{j=0}^{p-1} \frac{1}{a_j^{\varepsilon}} = \sum_{j=0}^{p-1} \frac{1}{a_j}. \quad \text{(B.14)}$$

Therefore, we obtain (B.11) thanks to the fact that a is Lipschitz continuous. In fact,

$$\left| \int_0^{2\pi} \frac{dy}{a(y)} - \frac{2\pi}{p} \sum_{j=0}^{p-1} \frac{1}{a_j^{\varepsilon}} \right| \le \sum_{j=0}^{p-1} \int_{\frac{2\pi j}{p}}^{2\pi \frac{j+1}{p}} \left| \frac{1}{a(y)} - \frac{1}{a_j} \right| dy \le \frac{c}{2\alpha^2} \frac{1}{p}.$$

Now, we prove (B.12). By the definition of a^* and $a^{\varepsilon,h}$, we get

$$|a^* - a_h^{\varepsilon,*}| \le \beta^2 \left| \int_0^{2\pi} \frac{dy}{a(y)} - \sum_{j=0}^{n-1} \frac{h}{a_j^{\varepsilon}} \right|. \quad \text{(B.15)}$$

Recall that $h = 2\pi/n$ and write $n = b_n p + c_n$ with $c_n \in \{0, 1, \cdots, p-1\}$ and $b_n = [n/p]$. Therefore, when $c_n = 0$

$$\sum_{j=0}^{n-1} \frac{h}{a_j^\varepsilon} = \frac{2\pi}{p} \sum_{j=0}^{p-1} \frac{1}{a_j^\varepsilon}.$$

Coming back to (B.15) and using (B.11), we obtain (i). On the other hand, when $c_n \in \{1, \cdots, p-1\}$, thanks to (B.14),

$$\sum_{j=0}^{n-1} \frac{h}{a_j^\varepsilon} = \sum_{j=0}^{p-1} \frac{hb_n}{a_j^\varepsilon} + \sum_{j=0}^{c_n-1} \frac{h}{a_j^\varepsilon} = \sum_{j=0}^{p-1} \frac{hb_n}{a_j} + \sum_{j=0}^{c_n-1} \frac{h}{a_j^\varepsilon},$$

and, then,

$$\begin{aligned} \left|a^* - a_h^{\varepsilon,*}\right| &\leq \beta^2 \left| \int_0^{2\pi} \frac{dy}{a(y)} - \frac{2\pi}{p} \sum_{j=0}^{p-1} \frac{1}{a_j} \right| + \beta^2 \left| \frac{2\pi}{p} - hb_n \right| \sum_{j=0}^{p-1} \frac{1}{a_j} + \beta^2 \sum_{j=0}^{c_n-1} \frac{h}{a_j^\varepsilon} \\ &\leq \frac{\beta^2}{2\alpha^2} \frac{c}{p} + 2\pi\beta^2 \left| \frac{1}{p} - \frac{b_n}{n} \right| \frac{p}{\alpha} + \beta^2 \frac{c_n h}{\alpha}, \end{aligned} \tag{B.16}$$

thanks to (B.11). Now, since $n = b_n + c_n$, we get

$$\frac{1}{p} - \frac{b_n}{n} = \frac{n - pb_n}{np} = \frac{c_n}{np}.$$

Returning to (B.16), since $c_n \leq p$, we prove (ii). ◻

As a consequence of Lemma B.1, we have:

LEMMA B.2. *For any $1 \leq i \leq n$, we get*

$$\left| \sum_{j=0}^{i} h \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) \right| \leq \frac{c}{2\alpha^2}\, b_i h \, + \frac{\beta - \alpha}{\alpha^2}\, hc_i, \tag{B.17}$$

where b_i is defined by $b_i = [i/p]$ and c_i satisfies that $c_i = i - pb_i$, $c_i \in \{0, \cdots, p-1\}$.

Proof. This proof is analogous to the case $n/p \notin \mathbb{N}$ in (B.12). First, by (B.13) and since a^* is constant,

$$\sum_{j=0}^{i} h \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) = \sum_{j=0}^{p-1} hb_i \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) + \sum_{j=0}^{c_i-1} h \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right). \tag{B.18}$$

Using that $\alpha \leq a^\varepsilon(x), a^* \leq \beta$, we have

$$\left| \sum_{j=0}^{c_i-1} h \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) \right| \leq \frac{\beta - \alpha}{\alpha^2} hc_i.$$

On the other hand, by the definition of a^*,

$$\sum_{j=0}^{p-1} hb_i \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) = hb_i p \sum_{j=0}^{p-1} \frac{1}{p} \left(\frac{1}{a_j^\varepsilon} - \frac{1}{a^*} \right) = hb_i p \left(\sum_{j=0}^{p-1} \frac{1}{p} \frac{1}{a_j^\varepsilon} - \frac{|}{2\pi} \int_0^{2\pi} \frac{dy}{a(y)} \right).$$

Coming back to (B.18) and by (B.11), we prove (B.17):

$$\left|\sum_{j=0}^{i} h\left(\frac{1}{a_j^\varepsilon}-\frac{1}{a^*}\right)\right| \le \frac{hb_ip}{\alpha^2}\frac{c}{p}+\frac{\beta-\alpha}{\alpha^2}hc_i \;\le\; \frac{c}{2\alpha^2}hb_i+\frac{\beta-\alpha}{\alpha^2}hc_i. \qquad \square$$

Proof of Theorem 2.3. We only need to estimate (B.10). We define

$$\begin{gathered} E_1(i)=U_0^{\varepsilon,h}\sum_{j=0}^{i}\left(\frac{h}{a_j^\varepsilon}-\frac{h}{a^*}\right), \quad E_2(i)=(U_0^{\varepsilon,h}-U_0^{*,h})\frac{h(i+1)}{a^*}, \\ E_3(i)=\sum_{j=0}^{i}\left(\frac{h}{a^*}-\frac{h}{a_j^\varepsilon}\right)F_j, \quad F_j=\sum_{k=1}^{j}hf_k. \end{gathered} \tag{B.19}$$

Then, $u_{i+1}^\varepsilon-u_{i+1}^*=E_1(i)+E_2(i)+E_3(i)$. Thus, we need to estimate E_j, $j=1,2,3$, to prove Theorem 2.3. $U_0^{\varepsilon,h}$, defined in (B.6), is bounded by

$$|U_0^{\varepsilon,h}|\le\beta|c-b|\,+\,\frac{\beta}{\alpha}\|f\|_\infty.$$

Then, thanks to (B.17), we get

$$|E_1(i)|\le\left(\beta|c-b|\,+\,\frac{\beta}{\alpha}\|f\|_\infty\right)\left(\frac{c}{2\alpha^2}\,b_ih\,+\,\frac{\beta-\alpha}{\alpha^2}\,hc_i\right). \tag{B.20}$$

Now, we study $E_3(i)$. To do it, we use Abel's formula:

$$\sum_{j=1}^{m}v_ju_j=u_{m+1}V_m-\sum_{j=1}^{m}V_j(u_{j+1}-u_j), \quad \text{with } V_j=\sum_{\ell=1}^{j}v_\ell. \tag{B.21}$$

Applying it to $E_3(i)$, we obtain

$$E_3(i)=F_{i+1}\sum_{j=0}^{i}\left(\frac{h}{a_j^\varepsilon}-\frac{h}{a^*}\right)-\sum_{j=0}^{i}hf_{j+1}\sum_{k=0}^{j}\left(\frac{h}{a_k^\varepsilon}-\frac{h}{a^*}\right).$$

By (B.19), $|F_j|\le\|f\|_\infty$. Then, thanks to (B.17), we have

$$|E_3(i)|\le 2\|f\|_\infty\left(\frac{c}{\alpha^2}\,\frac{1}{p}\,+\frac{\beta-\alpha}{\alpha^2}\,hp\right). \tag{B.22}$$

Finally, we estimate $E_2(i)$. We observe that

$$|E_2(i)|\le\frac{1}{\alpha}|U_0^{\varepsilon,h}-U_0^{*,h}|.$$

By (B.6) and (B.9), we get

$$U_0^{\varepsilon,h}-U_0^{*,h}=(c-b)(a_h^{\varepsilon,*}-a^*)+(a_h^{\varepsilon,*}-a^*)\sum_{j=0}^{n-1}F_j\frac{h}{a_j^\varepsilon}+a^*\sum_{j=0}^{n-1}\left(\frac{h}{a_j^\varepsilon}-\frac{h}{a^*}\right)F_j,$$

where F_j is defined by (B.19). Using (B.21) in the last term, we obtain

$$U_0^{\varepsilon,h} - U_0^{*,h} = (c-b)(\bar{a}_h^\varepsilon - a^*) + (\bar{a}_h^\varepsilon - a^*)\sum_{j=0}^{n-1} F_j \frac{h}{a_j^\varepsilon} + a^* F_n \sum_{j=0}^{n-1}\left(\frac{h}{a_j^\varepsilon} - \frac{h}{a^*}\right) - a^* \sum_{j=0}^{n-1} h f_{j+1} \sum_{k=0}^{j}\left(\frac{h}{a_k^\varepsilon} - \frac{h}{a^*}\right).$$

Then, applying (B.12) and (B.17),

$$|U_0^{\varepsilon,h} - U_0^{*,h}| \le \left(|c-b| + \frac{1}{\alpha}\|f\|_\infty\right)\left(\frac{\beta^2}{2\alpha^2}\frac{c}{p} + \frac{\beta^2}{\alpha}hp\right) + a^*\|f\|_\infty\left(\frac{c}{2\alpha^2}\frac{1}{p} + \frac{\beta-\alpha}{\alpha^2}hp\right). \tag{B.23}$$

By the estimates (B.20), (B.22) and (B.23), we conclude the proof. □

Appendix C. Properties of Bloch waves.

C.1. Properties of Bloch eigenvalues. In this section, we prove some results presented in Section 3.2. First, we see that the matrix $\{a_{ij}(qy(i,j))\}$ is elliptic thanks to the regularity of the coefficients a_{ij}.

LEMMA C.1. *We assume that the coefficients $\{a_{ij}\}$ defined in (2.2) are Lipschitz, and q, p satisfy (2.35) and*

$$\frac{q}{p} - \left[\frac{q}{p}\right] = \frac{\rho}{p}, \qquad \text{with} \qquad \rho_i \in \mathbb{N}. \tag{C.1}$$

Then, for any $y \in \Gamma_{\frac{2\pi}{p}}$ defined in (3.13), we have:

$$\begin{array}{rcl} \sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\eta_j & \le & \beta|\xi||\eta| + c\pi\left|\frac{\rho}{p}\right| |\xi|\,|\eta|, \\ \sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\xi_j & \ge & \alpha|\xi|^2 - c\pi\, d\left|\frac{\rho}{p}\right| |\xi|^2. \end{array} \tag{C.2}$$

Proof. We note that by the properties of the coefficients $\{a_{ij}\}$, we have that

$$\begin{array}{rcl} \sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\eta_j & = & \beta|\xi||\eta| + \Delta_1, \\ \sum_{i,j=1}^{d} a_{ij}(qy(i,j))\xi_i\xi_j & = & \alpha|\xi|^2 + \Delta_2, \end{array}$$

with

$$\Delta_1 = \sum_{i,j=1}^{d} [a_{ij}(qy(i,j)) - a_{ij}(qy)]\xi_i\eta_j, \quad \text{and} \quad \Delta_2 = \sum_{i,j=1}^{d} [a_{ij}(qy(i,j)) - a_{ij}(qy)]\xi_i\xi_j.$$

Using (C.1) and the periodicity of the coefficients, we obtain, for any $y \in \Gamma_{\frac{2\pi}{p}}$:

$$a_{ij}(qy) = a_{ij}(\rho y), \quad \text{and} \quad a_{ij}(qy(i,j)) = a_{ij}\left(\rho y + \frac{\pi\rho_i}{p_i}e_i + (1-\delta_{ij})\frac{\pi\rho_j}{p_j}e_j\right).$$

Moreover, since the coefficients $\{a_{ij}\}$ are Lipschitz continuous with constant $c > 0$, then

$$|a_{ij}(qy(i,j)) - a_{ij}(qy)| \leq c\pi \left(\frac{\rho_i^2}{p_i^2} + \frac{\rho_j^2}{p_j^2}(1-\delta_{ij}) \right)^{\frac{1}{2}}.$$

Thus, we get

$$\Delta_1 \leq c\pi \sum_{i,j=1}^{d} \left(\frac{\rho_i^2}{p_i^2} + \frac{\rho_j^2}{p_j^2}(1-\delta_{ij}) \right)^{\frac{1}{2}} |\xi_i||\eta_j| \leq c\pi \left| \frac{\rho}{p} \right| |\xi|\,|\eta|$$

$$\Delta_2 \leq c\pi \sum_{i,j=1}^{d} \left(\frac{\rho_i^2}{p_i^2} + \frac{\rho_j^2}{p_j^2}(1-\delta_{ij}) \right)^{\frac{1}{2}} |\xi_i||\xi_j| \leq c\pi\, d \left| \frac{\rho}{p} \right| |\xi|^2. \qquad \square$$

As a consequence of this lemma, we prove the ellipticity of $\{a_{ij}(qy(i,j))\}$ under assumptions (2.35).

Proof of Lemma 3.3. Thanks to Lemma C.1, the proof is immediate since $\{a_{ij}(qy(i,j))\}$ are bounded and coercive for sufficiently small $|\rho/p|$. In particular, $|\rho/p| \leq \alpha/(2cd\pi)$ where c is the Lipschitz constant of the coefficients. $\square$

With the coercivity of the coefficients we obtain that the discrete problem is positive semi-definite (see Lemma 3.4) and the second eigenvalue is bounded below by a positive constant. Let us prove it.

Proof of Lemma 3.5. We are going to use the min-max principle (see in [34], p. 99). We consider $\mathcal{V}_{\mathbb{C}}$, the set of complex subsets $W \subset \mathbb{C}^{\dot{p}}$ with dimension 2, and $\mathcal{V}_{\mathbb{R}}$, the set of subsets $W \subset \mathbb{R}^{\dot{p}}$ whose dimension is 2. Then, we have

$$\mu_2(\eta) = \min_{W \in \mathcal{V}_{\mathbb{C}}} \max_{v \in W-(0)} \frac{a(\eta)(v,v)}{|v|^2}. \tag{C.3}$$

Using the coercivity of the coefficients (see Lemma 3.3), we have

$$a(\eta)(v,v) \geq \alpha \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \left(\frac{p_k}{2\pi q_k} \right)^2 \left| e^{i\frac{2\pi}{p_k}\eta_k} v(y + \frac{2\pi}{p_k}e_k) - v(y) \right|^2. \tag{C.4}$$

We write $v \in W$ as $v(y) = v_r(y) + iv_i(y)$, $y \in \Gamma_{\frac{2\pi}{p}}$. We define for any $y \in \Gamma_{\frac{2\pi}{p}}$:

$$w(y) = \cos(\frac{2\pi}{p_k}\eta_k)v_r(y) + \sin(\frac{2\pi}{p_k}\eta_k)v_i(y) + i[\cos(\frac{2\pi}{p_k}\eta_k)v_i(y) - \sin(\frac{2\pi}{p_k}\eta_k)v_r(y)].$$

Using this function and the scalar product in $\mathbb{C}$, we have

$$\sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \left(\frac{p_k}{q_k} \right)^2 \left| e^{i\frac{2\pi}{p_k}\eta_k} v(y + \frac{2\pi}{p_k}e_k) - v(y) \right|^2$$

$$= \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \frac{(p_k)^2}{(q_k)^2} [|v(y + \frac{2\pi}{p_k}e_k)|^2 + |v(y)|^2 - 2\mathcal{R}e\langle v(y + \frac{2\pi}{p_k}e_k), w(y)\rangle].$$

By the Cauchy-Schwartz inequality and since $|w(y)| = |v(y)|$, we get

$$\left|\mathcal{R}e\langle v(y+\frac{2\pi}{p_k}e_k), w(y)\rangle\right| \leq |v(y+\frac{2\pi}{p_k}e_k)|\,|v(y)|.$$

Therefore, in view of (C.4), in (C.3) we have

$$\begin{aligned}\mu_2(\eta) &\geq \alpha \min_{W\in\mathcal{V}_{\mathbb{C}}} \max_{v\in W-(0)} \frac{1}{|v|^2} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \left(\frac{p_k}{2\pi q_k}\right)^2 \left(|v(y+\frac{2\pi}{p_k}e_k)| - |v(y)|\right)^2 \\ &\geq \alpha \min_{W\in\mathcal{V}_{\mathbb{R}}} \max_{v\in W-(0)} \frac{1}{|v|^2} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \left(\frac{p_k}{2\pi q_k}\right)^2 \left(v(y+\frac{2\pi}{p_k}e_k) - v(y)\right)^2. \quad \text{(C.5)}\end{aligned}$$

Note that this lower bound is independent of η. Moreover, the discrete eigenvalue problem associated to the bilinear form (C.5) is the finite difference approximation on $\Gamma_{\frac{2\pi}{p}}$ of

$$\begin{cases} \dfrac{1}{q_k^2}\dfrac{\partial^2 u}{\partial y_k^2} = \mu u, \\ u \text{ is } Y\text{-periodic.}\end{cases}$$

Now, the eigenvalues of the discrete system associated with the bilinear form (C.5) may be computed explicitly (see [22], p. 459). In particular, we have

$$\min_{W\in\mathcal{V}_{\mathbb{R}}} \max_{v\in W-(0)} \frac{1}{|v|^2} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{k=1}^{d} \left(\frac{p_k}{2\pi q_k}\right)^2 \left(v(y+\frac{2\pi}{p_k}e_k) - v(y)\right)^2 \geq \min(\frac{c}{q_i^2}),$$

and we conclude the proof. ◻

C.2. Regularity of the first Bloch eigenvalue and eigenvector. In spite of the fact that the eigenvalue problem defined in (3.16) depends exponentially in η, it is well known that the eigenvalues $\mu_m(\eta)$ are not, in general, smooth functions of $\eta \in Y'$ because of the possible change in the multiplicity of eigenvalues (see [32], p. 60). Now, using the min-max principle we prove that all the eigenvalues are Lipschitz continuous.

PROPOSITION C.2. *For all $m \geq 1$, $\mu_m(\eta)$ is a Lipschitz continuous function of η.*

Proof. Recall that the Hermitian bilinear form associated with the eigenvalue problem (3.16) is defined in (3.18). We notice that it can be decomposed as follows

$$a(\eta)(v,v) \;=\; a(\xi)(v,v) + R(v;\eta,\xi),$$

where, using the notations,

$$\begin{cases} y^k = y + \dfrac{2\pi}{p_k}e_k, \quad (e_k \text{ being the } k\text{-th canonical vector}), \\ b_{k\ell}(y) = a_{k\ell}(qy(k,\ell))\dfrac{p_k p_\ell}{q_k q_\ell (2\pi)^2}, \end{cases} \tag{C.6}$$

we can write

$$R(v;\eta,\xi)=\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{k,\ell=1}^{d}b_{k\ell}(y)[e^{i2\pi(\frac{\eta_k}{p_k}-\frac{\eta_\ell}{p_\ell})}-e^{i2\pi(\frac{\xi_k}{p_k}-\frac{\xi_\ell}{p_\ell})}]v(y^k)\overline{v(y^\ell)}$$
$$+\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{k,\ell=1}^{d}b_{k\ell}(y)[e^{i2\pi\frac{\xi_k}{p_k}}-e^{i2\pi\frac{\eta_k}{p_k}}]v(y^k)\overline{v(y)}$$
$$+\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{k,\ell=1}^{d}b_{k\ell}(y)[e^{-i2\pi\frac{\xi_\ell}{p_\ell}}-e^{-i2\pi\frac{\eta_k}{p_k}}]v(y)\overline{v(y^\ell)}.$$

Taking into account that the coefficients in (C.6) are bounded (see Lemma 3.3) and the function $e^{i\eta\cdot y}$ is Lipschitz on the variable η, we have

$$|R(v;\eta,\xi)|\le c(d,\beta)\,|\eta-\xi|\sum_{y\in\Gamma_{\frac{2\pi}{p}}}|v(y)|^2.$$

Then, using the min-max characterization of eigenvalues, we deduce that

$$\mu_m(\eta)\le\mu_m(\xi)+c(d,\beta)|\eta-\xi|$$

and, interchanging η and ξ, we conclude the proof. ∎

Unfortunately the Lipschitz character of eigenvalues is not enough to derive homogenization results. We are going to prove that $\mu_1(\eta)$ is analytic using the fact that the eigenvalue $\mu_1(0)$ (which is equal to 0) is simple. To prove it we use classical results of perturbation theory for linear operators in finite-dimensional spaces (see [24]).

Proof of Proposition 3.6. As remarked already, when $\eta=0$, the first eigenvalue $\mu_1(0)=0$ is simple. Furthermore, in Lemma 3.5 we prove that all eigenvalues $\{\mu_m(\eta)\}_{m\ge 2}$ are bounded below for any $\eta\in Y'$ by a positive constant. Hence, using Ostrowski's theorem on continuity of the eigenvalues (see [34], p. 63) there exists $\delta>0$ such that $\mu_1(\eta)$ is simple in $\eta\in B_\delta=B(0;\delta)$.

Now, applying a consequence of the Rellich's theorem on finite-dimensional spaces (see [32]), we obtain that $\mu_1(\eta)$ is analytic in $\eta\in B_\delta$. In fact, note that our finite eigenvalue problem (3.16) is perturbed by a d-dimensional variable $\eta\in Y'\subset\mathbb{R}^d$. Thus, by Theorem 5.16 in [24] p. 116, μ_1 is an analytic map in B_δ, since μ_1 is a simple eigenvalue in this region.

Since the eigenvalue problem (3.16) depends analytically on η, and the first eigenvalue does not change its multiplicity and remains analytic with respect to η in B_δ for δ sufficiently small, by Rellich's Theorem the choice of the first eigenvector can be made so that it depends analytically on $\eta\in B_\delta$. ∎

C.3. Derivatives of the first Bloch eigenvalue and eigenvector. The aim of this section is to obtain an expression of the homogenized coefficients in the finite difference analysis. We have divided the proof into a sequence of lemmas. First, we compute the derivatives of μ_1 and $\varphi_{p,1}$.

Proof of Lemma 3.7. Given that $\eta\to\mu_1(\eta)$ and $\eta\to\varphi_{p,1}(y;\eta)$ are smooth (see Proposition 3.6), it is straightforward to compute their derivatives at $\eta=0$. To do it, it is enough to differentiate the eigenvalue problem

$$-\sum_{i,j=1}^{d}e^{-iy\cdot\eta}\nabla_i^{-\frac{2\pi}{p}}\left[\frac{1}{q_iq_j}a_{ij}\left(qy(i,j)\right)\nabla_j^{\frac{2\pi}{p}}\left(e^{iy\cdot\eta}\varphi_{p,1}(y;\eta)\right)\right]=\mu_1(\eta)\varphi_{p,1}(y;\eta),\quad\text{(C.7)}$$

with respect to η twice and evaluate it at $\eta = 0$. Since the computations are classical, we present only the essential steps. Define

$$\mathcal{A}(y,\eta)v(y) = -\sum_{i,j=1}^{d} e^{-iy\cdot\eta}\nabla_i^{-\frac{2\pi}{p}}\left[\frac{1}{q_i q_j} a_{ij}\left(qy(i,j)\right)\nabla_j^{+\frac{2\pi}{p}} e^{iy\cdot\eta}v(y)\right].$$

With the notation (C.6), we get

$$\begin{aligned}\mathcal{A}(y,\eta)v(y) = &-\sum_{i,j=1}^{d} b_{ij}(y(i,j))[e^{i\frac{2\pi}{p_j}\eta_j}v(y^j) - v(y)]\\ &+\sum_{i,j=1}^{d} b_{ij}(y^{-i}(i,j))[e^{i\frac{2\pi}{p_j}\eta_j}e^{-i\frac{2\pi}{p_i}\eta_i}v(y^{j-i}) - e^{-i\frac{2\pi}{p_i}\eta_i}v(y^{-i})].\end{aligned}$$

Thus, we compute the first order derivatives in $\eta = 0$

$$\begin{aligned}\partial_k\mathcal{A}(y,0)v(y) = &-i\frac{2\pi}{p_k}\sum_{i=1}^{d} b_{ik}(y(i,k))v(y^k) + i\frac{2\pi}{p_k}\sum_{i=1}^{d} b_{ik}(y^{-i}(i,k))v(y^{k-i})\\ &- i\frac{2\pi}{p_k}\sum_{i=1}^{d} b_{ki}(y^{-k}(k,i))v(y^{i-k}) + i\frac{2\pi}{p_k}\sum_{i=1}^{d} b_{ki}(y^{-k}(k,i))v(y^{-k}).\end{aligned} \quad \text{(C.8)}$$

Also the second order derivatives are

$$\partial^2_{k\ell}\mathcal{A}(y,0)v(y) = \frac{(2\pi)^2}{p_k p_\ell}[b_{\ell k}(y^{-\ell}(\ell,k))v(y^{k-\ell}) + b_{k\ell}(y^{-k}(k,\ell))v(y^{\ell-k})], \quad \text{(C.9)}$$

if $k \neq \ell$, and

$$\begin{aligned}\partial^2_{kk}\mathcal{A}(y,0)v(y) = &\frac{(2\pi)^2}{(p_k)^2}\left[\sum_{i=1}^{d} b_{ik}(y(i,k))v(y^k) - \sum_{k\neq i=1}^{d} b_{ik}(y^{-i}(i,k))v(y^{k-i})\right]\\ &+\frac{(2\pi)^2}{(p_k)^2}\left[\sum_{i=1}^{d} b_{ki}(y^{-k}(k,i))v(y^{-k}) - \sum_{k\neq i=1}^{d} b_{ki}(y^{-k}(k,i))v(y^{i-k})\right].\end{aligned} \quad \text{(C.10)}$$

Differentiating in (C.7), evaluating at $\eta = 0$ and taking the scalar product in $\mathbb{C}^{\dot{p}}$, we obtain

$$\partial_k\mu_1(0) = \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \overline{\varphi_{p,1}(y;0)}\cdot\partial_k\mathcal{A}(y,0)\varphi_{p,1}(y;0).$$

Considering (3.20) and (C.9), we have

$$\partial_k\mu_1(0) = i\frac{2\pi}{p_k}\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^{d}[-b_{ik}(y(i,k)) + b_{ik}(y^{-i}(i,k))] = 0,$$

since

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^{d} b_{ik}(y(i,k)) = \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^{d} b_{ik}(y^{-i}(i,k)).$$

On the other hand, the first derivatives of $\varphi_{p,1}$ satisfy

$$\begin{cases} \mathcal{A}(y,0)\partial_k\varphi_{p,1}(y;0) = \partial_k\mathcal{A}(y,0)\varphi_{p,1}(y;0), \quad y \in \Gamma_{\frac{2\pi}{p}} \\ \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \partial_k\varphi_{p,1}(y;0)\overline{\varphi_{p,1}(y;0)} = 0, \end{cases}$$

for $k = 1, \ldots, d$. By (3.20) and (C.9), we write

$$\begin{aligned} \partial_k\mathcal{A}(y,0)\varphi_{p,1}(y;0) &= \frac{2\pi i}{p_k(\dot{p})^{\frac{1}{2}}} \sum_{j=1}^{d} \left[b_{kj}(y(k,j)) - b_{kj}\left(y(k,j) - \frac{2\pi}{p_j} e_j \right) \right] \\ &= \frac{i}{(\dot{p})^{\frac{1}{2}}} \sum_{j=1}^{d} \nabla_j^{-\frac{2\pi}{p}} \frac{a_{jk}}{q_k q_j}(qy(k,j)). \end{aligned}$$

Then, the first order derivatives of $\varphi_{p,1}$ are written as in (3.25). We differentiate again in (C.7), evaluate in $\eta = 0$ and take the scalar product to get

$$\begin{aligned} \partial^2_{k\ell}\mu_1(0) = &\sum_{y\in\Gamma_{\frac{2\pi}{p}}} \overline{\varphi_{p,1}(y;0)} \partial^2_{k\ell}\mathcal{A}(y,0)\varphi_{p,1}(y;0) \\ &+ \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \overline{\varphi_{p,1}(y;0)} \left[\partial_k\mathcal{A}(y,0)\partial_\ell\varphi_{p,1}(y;0) + \partial_\ell\mathcal{A}(y,0)\partial_k\varphi_{p,1}(y;0) \right]. \end{aligned}$$

Using (C.9) or (C.10) and with the notation (C.6), we have

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} \overline{\varphi_{p,1}(y;0)} \cdot \partial^2_{k\ell}\mathcal{A}(y,0)\varphi_{p,1}(y;0) = \frac{2}{\dot{p}} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \frac{1}{q_k q_\ell} a_{\ell k}(qy(\ell,k)).$$

On the other hand, in view of (3.25) and (C.8), we get

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} \overline{\varphi_{p,1}(y;0)} \cdot \partial_k\mathcal{A}(y,0)\partial_\ell\varphi_{p,1}(y;0) = -\frac{1}{\dot{p}} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \Theta_q^k(y) \sum_{j=1}^{d} \frac{1}{q_\ell q_j} \nabla_j^{-\frac{2\pi}{p}} a_{j\ell}(qy(\ell,j)).$$

Then, we obtain the second order derivatives of μ_1 and conclude the proof of Lemma 3.7. □

The one dimensional case. First, we observe that thanks to (3.23) the second order derivative of μ_1 is:

$$\frac{1}{2}\partial^2\mu_1(0) = \frac{1}{p}\frac{1}{q^2} \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \left[a(qy(1,1)) - \Theta_q^k(y)\nabla^{-\frac{2\pi}{p}} a(qy(1,1)) \right], \tag{C.11}$$

where $\{\Theta_q(y) \,|\, y \in \Gamma_{\frac{2\pi}{p}}\}$ satisfies

$$\begin{cases} -\nabla^{-\frac{2\pi}{p}} \left[a\left(qy(1,1)\right) \nabla^{\frac{2\pi}{p}} \left(\Theta_q(y)\right) \right] = \nabla^{-\frac{2\pi}{p}} a(qy(1,1)), \quad y \in \Gamma_{\frac{2\pi}{p}} \\ \Theta_q(y) \quad 2\pi\text{-periodic} \quad \text{and} \quad \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \Theta_q(y) = 0. \end{cases} \tag{C.12}$$

Then, since $y(1,1)$ is defined by (3.15), we compute the components of Θ_q:

$$\Theta_q(2\pi j/p) = \frac{2\pi}{p\sqrt{p}} \left\{ \frac{p-2j+1}{2} - a_p^* \left(b_p^* - \sum_{k=1}^{j} \frac{1}{a(\pi(2k+1)q/p)} \right) \right\}, \qquad \text{(C.13)}$$

for $j = 1, \ldots, p$, where a_p^* is defined in (2.30) and

$$b_p^* = \frac{1}{p} \sum_{k=1}^{p} \frac{p+1-k}{a(\pi(2k+1)/p)}. \qquad \text{(C.14)}$$

Then, replacing (C.13) and (C.14) in (C.11), the second derivative of μ_1 satisfies (3.30). Furthermore, a^* and a_p^* are defined in (2.23) and (2.30), respectively. Therefore, using that the coefficient a is Lipschitz, we conclude the proof of Proposition 3.10.

The case of several space dimensions. Now, we are going to prove that $\partial^2_{k\ell}\mu_1(0)$ is an approximation of the homogenized coefficient $a^*_{k\ell}$ defined in (2.6).

To do it, we need to find a relation between χ^k, solution of (2.7), and Θ_q^k that satisfies (3.24). Since $q, p \in \mathbb{N}$ we write

$$\frac{q}{p} - \left[\frac{q}{p}\right] = \frac{\rho}{p} \quad \text{and} \quad q_i = \rho_i \sigma_i, \quad \text{where } \sigma_i \in \mathbb{R}_+, \quad \sigma_i \geq 1, \quad \forall i = 1, \ldots, n. \qquad \text{(C.15)}$$

Then, thanks to the fact that $\{a_{ij}\}$ are Y-periodic, (3.24) coincides with

$$\begin{cases} -\sum\limits_{i,j=1}^{d} \nabla_i^{-\frac{2\pi\rho}{p}} \left[\dfrac{a_{ij}\left(y(i,j)\right)}{\sigma_i\sigma_j} \nabla_j^{\frac{2\pi\rho}{p}} \left(\Theta_\rho^k(y)\right) \right] = \sum\limits_{j=1}^{d} \dfrac{1}{\sigma_k\sigma_j} \nabla_j^{\frac{-2\pi\rho}{p}} a_{jk}(y(k,j)), \\ y \in \Gamma_{\frac{2\pi\rho}{p}}, \quad \Theta_\rho^k(y) \quad \rho Y\text{-periodic} \quad \text{and} \quad \sum\limits_{y \in \Gamma_{\frac{2\pi\rho}{p}}} \Theta_\rho^k(y) = 0, \end{cases} \qquad \text{(C.16)}$$

with

$$\Theta_q^k(y) = \frac{1}{\rho_k} \Theta_\rho^k(\rho y), \qquad \forall y \in \Gamma_{\frac{2\pi}{p}}. \qquad \text{(C.17)}$$

We note that Θ_ρ^k is the approximation in finite differences of the solution of the periodic boundary problem:

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(b_{ij}(y) \dfrac{\partial \chi_\rho^k}{\partial y_j} \right) = \dfrac{\partial b_{ik}(y)}{\partial y_i} \quad \text{in} \quad \rho Y, \\ \chi_\rho^k \in H^1_\#(\rho Y), \quad m_\rho(\chi_\rho^k) = \dfrac{1}{\rho Y} \displaystyle\int_{\rho Y} \chi_\rho^k(y)dy, \end{cases} \qquad \text{(C.18)}$$

where

$$b_{ij}(y) = \frac{a_{ij}(y)}{\sigma_i\sigma_j}, \qquad \forall i, j = 1, \ldots, n.$$

In particular, using classical error estimates (see Theorem 2.2), we obtain

$$\sup_{y \in \Gamma_{\frac{2\pi\rho}{p}}} \{|\chi_\rho^k(y) - \Theta_\rho^k(y)\} \leq c \frac{\beta}{\sigma_m} \left| \frac{\rho}{p} \right|, \qquad \text{(C.19)}$$

where $\sigma_m = \min\{\sigma_i\}$. Now, we study the relation between χ_ρ^k and χ^k, solution of (2.7). We have the following result:

LEMMA C.3. *Let χ^k and χ_ρ^k, $k = 1, \ldots, d$, be the test functions solving (2.7) and (C.18), respectively. Let $\delta > 0$ be defined in (2.40). Then, there exists c independent of δ, k, ρ and q such that*

$$\sup_{y \in \Gamma_{\frac{2\pi}{p}}} \{|\chi_\rho^k(y) - \chi^k(\rho y)|\} \leq c\delta. \tag{C.20}$$

Proof. Since $\rho \in \mathbb{N}^d$ and χ^k is Y-periodic, then χ^k is also the unique solution of

$$\begin{cases} A\chi^k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} & \text{in} \quad \rho Y, \\ \chi^k \in H^1_\#(\rho Y), & m_\rho(\chi^k) = 0. \end{cases} \tag{C.21}$$

We need another function to compare χ_ρ^k to χ^k. We consider the solution of

$$\begin{cases} A\chi_b^k = \sigma_k \nu \dfrac{\partial b_{k\ell}}{\partial y_\ell} & \text{in} \quad \rho Y, \\ \chi_b^k \in H^1_\#(\rho Y), & m_\rho(\chi_b^k) = 0, \end{cases} \tag{C.22}$$

with $\nu \in \mathbb{R}$ to be chosen later. Thus, we consider

$$\|\nabla(\chi^k - \chi_\rho^k)\|_2 \leq \|\nabla(\chi^k - \chi_b^k)\|_2 + \|\nabla(\chi_b^k - \chi_\rho^k)\|_2.$$

Since χ^k and χ_b^k satisfy (C.21) and (C.22), respectively, using the variational formulation

$$\|\nabla(\chi^k - \chi_b^k)\|_2 \leq \frac{c\beta}{\alpha} \sup_{i=1,\ldots,n} \{|(1 - \frac{\nu}{\sigma_i})|\}.$$

On the other hand, since χ_b^k satisfies (C.22), we have

$$\int_{\rho Y} A(\chi_b^k - \chi_\rho^k)\ (\chi_b^k - \chi_\rho^k) dy = -\int_{\rho Y} \left(\nu\sigma_k b_{jk} + a_{ij} \frac{\partial \chi_\rho^k}{\partial y_i} \right) \frac{\partial(\chi_b^k - \chi_\rho^k)}{\partial y_j}. \tag{C.23}$$

Now, since χ_ρ^k verifies (C.18),

$$\int_{\rho Y} \left(b_{jk} + b_{ji} \frac{\partial \chi_\rho^k}{\partial y_i} \right) \frac{\partial \varphi}{\partial y_j} dy = 0 \qquad \forall \varphi \in H^1_\#(\rho Y), \quad k = 1, \ldots, d.$$

Using this result in (C.23) and since $b_{ij} = a_{ij}/\sigma_i\sigma_j$, we get

$$\int_{\rho Y} A(\chi_b^k - \chi_\rho^k)\ (\chi_b^k - \chi_\rho^k) dy = \int_{\rho Y} a_{ij} \left(\frac{\sigma_k \nu}{\sigma_i \sigma_j} - 1 \right) \frac{\partial \chi_\rho^k}{\partial y_i} \frac{\partial(\chi_b^k - \chi_\rho^k)}{\partial y_j}.$$

Therefore, we obtain

$$\|\nabla(\chi_b^k - \chi_\rho^k)\|_2 \leq \|\nabla\chi_\rho^k/\sigma\|_2 \sup_{i,j=1,\ldots,n} \{\|a_{ij}(y)(\sigma_j - \frac{\nu\sigma_k}{\sigma_i})\|_\infty\}.$$

Since χ_ρ^k solves (C.18), we have

$$\|\nabla\chi_\rho^k/\sigma\|_2 = \left(\sum_{j=1}^n \frac{1}{\sigma_j^2}\left\|\frac{\partial\chi_\rho^k}{\partial y_j}\right\|^2\right)^{\frac{1}{2}} \leq c\frac{\beta}{\sigma_k\alpha}.$$

Since $a_{ij} \in L^\infty(Y)$ and considering $\nu \in [\sigma_m, \sigma_M]$, we prove that

$$\|\nabla(\chi^k - \chi_\rho^k)\|_2 \leq c\|a\|_\infty \sup_{i,j=1,\dots,n}\left\{\left|1-\frac{\sigma_j}{\sigma_i}\right|\right\}.$$

We conclude taking the definition of δ (2.40) into account. □

Thus, thanks to (C.20) and Lemma C.3 we have the following lemma:

LEMMA C.4. *Let χ^k , $k = 1,\dots,d$, be the test functions solution of (2.7). Let $\delta > 0$ be defined in (2.40). Then, we get that*

$$\sup_{y\in\Gamma_{\frac{2\pi}{p}}}\{|\rho_k\Theta_q^k(y) - \chi^k(\rho y)|\} \leq c\delta, \tag{C.24}$$

with c independent of k, ρ and q.

On the other hand, since the coefficients $a_{k\ell}^*$ are symmetric and $a_{k\ell}$ and χ^k are Y-periodic, the homogenized coefficients can be written as

$$a_{k\ell}^* = \frac{1}{|Y|}\int_Y \left(a_{k\ell}(\rho y) - \frac{\chi^k(\rho y)}{2\rho_j}\frac{\partial a_{j\ell}}{\partial y_j}(\rho y) - \frac{\chi^\ell(\rho y)}{2\rho_j}\frac{\partial a_{kj}}{\partial y_j}(\rho y)\right)dy. \tag{C.25}$$

Furthermore, since $a_{k\ell}^{*,q/p}$ is defined by (3.21) and q and p satisfy (2.27), we have thanks to (3.23) that

$$a_{k\ell}^{*,q/p} = \frac{1}{\dot{p}}\sum_{y\in\Gamma_{\frac{2\pi}{p}}} a_{k\ell}(\rho y(k,\ell))$$
$$-\frac{1}{\dot{p}}\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{j=1}^d\left[\frac{q_k\Theta_q^k(y)}{2q_j}\nabla_j^{-\frac{2\pi}{p}}a_{j\ell}(\rho y(\ell,j)) + \frac{q_\ell\Theta_q^\ell(y)}{2q_j}\nabla_j^{-\frac{2\pi}{p}}a_{jk}(\rho y(k,j))\right]. \tag{C.26}$$

Now, we are going to compare (C.25) with (C.26) in the following lemmas:

LEMMA C.5. *We assume that the coefficients $\{a_{k\ell}\}$ are Lipschitz continuous. Then, for any $k,\ell = 1,\dots,d$ and $|\rho/p| \leq \delta$, we get*

$$\left|\frac{1}{|Y|}\int_Y a_{k\ell}(\rho y)dy - \frac{1}{\dot{p}}\sum_{y\in\Gamma_{\frac{2\pi}{p}}} a_{k\ell}(\rho y(k,\ell))\right| \leq c\delta \tag{C.27}$$

where c is the Lipschitz constant of the coefficients.

Proof. It is immediate using that the coefficients are Lipschitz continuous. □

LEMMA C.6. *We assume that $\{a_{k\ell}\}$ and their derivatives are Lipschitz continuous and $\{\chi^k\}$ belong to $L^\infty(Y)$. Let $\delta > 0$ be defined in (2.40). Then,*

$$\left|\frac{1}{|Y|}\int_Y \frac{\chi^k(\rho y)}{\rho_j}\frac{\partial a_{j\ell}}{\partial y_j}(\rho y)dy - \frac{1}{\dot{p}}\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{j=1}^d\frac{q_k\Theta_q^k(y)}{q_j}\nabla_j^{-\frac{2\pi}{p}}a_{j\ell}(\rho y(\ell,j))\right| \leq c\,\delta. \tag{C.28}$$

Proof. Using (C.20) we get

$$\frac{1}{\dot{p}} \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \left| [q_k \Theta_q^k(y) - \sigma_k \chi^k(\rho y)] \sum_{j=1}^{d} \frac{1}{q_j} \nabla_j^{-\frac{2\pi}{p}} a_{j\ell}(\rho y(\ell, j)) \right| \leq c\,\delta, \tag{C.29}$$

where c depends of the Lipschitz constants of the coefficients. On the other hand,

$$\left| \sum_{j=1}^{d} \chi^k(\rho y) \left(\frac{\sigma_k}{q_j} - \frac{1}{\rho_j} \right) \nabla_j^{-\frac{2\pi}{p}} a_{j\ell}(\rho y(\ell, j)) \right| \leq c \sup_{k,j=1,\ldots,d} \left\{ \left| \frac{\sigma_k}{\sigma_j} - 1 \right| \right\}$$

where c depends of the $W^{1,\infty}$-norm of the coefficients and the L^∞-norm of the test function. Then, by (2.40), we get

$$\left| \sum_{j=1}^{d} \chi^k(\rho y) \left(\frac{\sigma_k}{q_j} - \frac{1}{\rho_j} \right) \nabla_j^{-\frac{2\pi}{p}} a_{j\ell}(\rho y(\ell, j)) \right| \leq c\,\delta.$$

Thus, by this estimate and (C.29), we only have to prove that

$$\left| \frac{1}{|Y|} \int_Y \frac{\chi^k(\rho y)}{\rho_j} \frac{\partial a_{j\ell}}{\partial y_j}(\rho y) dy - \frac{1}{\dot{p}} \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{j=1}^{d} \frac{\chi^k(\rho y)}{\rho_j} \nabla_j^{-\frac{2\pi}{p}} a_{j\ell}(\rho y(\ell, j)) \right| \leq c\delta.$$

But this property is immediate using that the coefficients are $C^{1,1}$. □

Finally, applying Lemmas C.3–C.6, we conclude the proof of Proposition 3.11.

C.4. Further properties of the first Bloch eigenvalue. First, we are going to see that the Hessian of μ_1 is coercive. In fact, we get:

LEMMA C.7. *Under the hypotheses of Lemma 3.3, there exist two constants $0 < \alpha \leq \beta$ satisfying:*

$$\sum_{i,j=1}^{d} q_k q_\ell \partial_{k\ell}^2 \mu_1(0) \xi_i \eta_j \leq \beta |\xi||\eta|, \qquad \textit{(boundedness)}, \tag{C.30}$$

$$\sum_{k,\ell=1}^{d} q_k q_\ell \partial_{k\ell}^2 \mu_1(0) \xi_k \xi_\ell \geq \alpha |\xi|^2, \qquad \textit{(coercivity)}. \tag{C.31}$$

Proof. We give the proof of (C.31); the other one being very similar. Note that

$$\sum_{k,\ell=1}^{d} q_k q_\ell \partial_{k\ell}^2 \mu_1(0) \xi_k \xi_\ell \geq \sum_{k,\ell=1}^{d} a_{k\ell}^* \xi_k \xi_\ell + \sum_{k,\ell=1}^{d} [\frac{q_k q_\ell}{2} \partial_{k\ell}^2 \mu_1(0) - a_{k\ell}^*] \xi_k \xi_\ell.$$

Then, since the homogenized coefficients are coercive and the error estimate (3.31) holds, the proof follows the arguments of Lemma 3.3 and C.1. □

As consequence of these results and the analyticity of μ_1, we prove Lemmas 3.8 and 3.9.

Proof of Lemma 3.8. First, for any $\eta \neq 0$, $\mu_1(\eta) > 0$. Using the bilinear form (3.18), we write

$$\mu_1(\eta) = \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{i,j=1}^{d} \frac{1}{q_i q_j} a_{ij}\,(qy(i,j))\,\nabla_j^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_{p,1}(y;\eta))\overline{\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\varphi_{p,1}(y;\eta))}.$$

Thanks to Lemma 3.3 and denoting $\psi_{p,1}(y;\eta) = e^{iy\cdot\eta}\varphi_{p,1}(y;\eta)$, we obtain

$$\alpha \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \frac{1}{q_i^2}\left|\nabla_i^{\frac{2\pi}{p}}\psi_{p,1}(y;\eta)\right|^2 \leq \mu_1(\eta) \leq \beta \sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \frac{1}{q_i^2}\left|\nabla_i^{\frac{2\pi}{p}}\psi_{p,1}(y;\eta)\right|^2. \quad \text{(C.32)}$$

If $\mu_1(\eta) = 0$ for some $\eta \neq 0$, we conclude that $\varphi_{p,1}(y;\eta) = ce^{-iy\cdot\eta}$. But $\varphi_{p,1}$ is Y-periodic in the variable y only for $\eta = 0$ in Y'. This is in contradiction with the fact $\eta \neq 0$. On the other hand, applying Taylor's formula, we have

$$\mu_1(\eta) = \frac{1}{2}\partial^2_{k\ell}\mu_1(0)\eta_k\eta_\ell + O(|\eta|^3).$$

Thus, thanks to (C.31),

$$\alpha\left|\frac{\eta}{q}\right|^2 + O(|\eta|^3) \leq \mu_1(\eta) \leq \beta\left|\frac{\eta}{q}\right|^2 + O(|\eta|^3), \qquad \forall\eta\in B_\delta.$$

Therefore, there exist $c, c' > 0$ such that

$$c'\left|\frac{\eta}{q}\right|^2 \leq \mu_1(\eta) \leq c\left|\frac{\eta}{q}\right|^2, \qquad \forall\eta\in B_\delta.$$

Finally, taking into account that $\mu_1(\eta) > 0$ for all $\eta \neq 0$, we deduce that (3.26) holds. Furthermore, applying (3.26) in (C.32) we obtain (3.27) and conclude the proof. □

Proof of Lemma 3.9. Using Taylor's formula, we get

$$\mu_1(\eta) - \frac{1}{2}\partial^2_{ij}\mu_1(0)\eta_i\eta_j = \sum_{ijk} c_{ijk}\partial^3_{ijk}\mu_1(\theta)\eta_i\eta_j\eta_k, \quad \text{where} \quad \theta\in B_{|\eta|}.$$

We need to see that

$$\left|\sum_{ijk} c_{ijk}\partial^3_{ijk}\mu_1(\theta)\eta_i\eta_j\eta_k\right| \leq c\left|\frac{\eta}{q}\right|^2 |\eta|,$$

where c is independent of the constant q. We note that the eigenvalue problem defined in (3.17) depends of q. First, we denote $\psi_{p,1}(y;\eta) = e^{iy\cdot\eta}\varphi_{p,1}(y;\eta)$ and obtain by (3.27) that

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \frac{1}{q_i^2}\left|\nabla_i^{\frac{2\pi}{p}}\psi_{p,1}(y;\eta)\right|^2 \leq c\left|\frac{\eta}{q}\right|^2.$$

Now, using similar computations to those in the proof of Lemma 3.7, we have

$$\partial_k \mu_1(\eta) = \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \frac{1}{q_k q_i} a_{ik}\left(qy(i,k)\right) \psi_{p,1}(y^k;\eta) \overline{\nabla_i^{\frac{2\pi}{p}}\left(\psi_{p,1}(y;\eta)\right)}$$
$$+ \sum_{y \in \Gamma_{\frac{2\pi}{p}}} \sum_{i=1}^{d} \frac{1}{q_k q_i} a_{ki}\left(qy(i,k)\right) \overline{\psi_{p,1}(y^k;\eta)} \nabla_i^{\frac{2\pi}{p}}\left(\psi_{p,1}(y;\eta)\right).$$

Then, by Lemma 3.3 we get

$$|q_k \partial_k \mu_1(\eta)| \le c \left|\frac{\eta}{q}\right|, \qquad \forall \eta \in B_\delta.$$

On the other hand, $\partial_k \varphi_{p,1}(\eta)$ satisfies

$$e^{iy\cdot\eta}[\mathcal{A}(\eta) - \mu_1(\eta)]\partial_k \varphi_{p,1}(\eta) = F_k(\eta) \quad \text{ and } \quad \langle \partial_k \varphi_{p,1}(\eta), \varphi_{p,1}(\eta)\rangle = 0,$$

where

$$F_k(y;\eta) = \sum_{i=1}^{d} \frac{1}{q_k q_i} a_{ik}\left(qy(i,k)\right) \nabla_i^{-\frac{2\pi}{p}}\left(\psi_{p,1}(y^k;\eta)\right)$$
$$+ \sum_{i=1}^{d} \frac{1}{q_k q_i} \nabla_i^{-\frac{2\pi}{p}}[a_{ik}\left(qy(i,k)\right)]\psi_{p,1}(y^{k-i};\eta)$$
$$+ \sum_{i=1}^{d} \frac{1}{q_k q_i} a_{ik}(qy^{-k}(i,k)) \nabla_i^{\frac{2\pi}{p}}\left(\psi_{p,1}(y^{-k};\eta)\right) + \partial_k \mu_1(\eta)\psi_{p,1}(y;\eta).$$

Then, there exists $C > 0$ such that

$$\sum_{y \in \Gamma_{\frac{2\pi}{p}}} |q_k F_k(y;\eta)|^2 \le c[1 + \left|\frac{\eta}{q}\right|^2] \le C,$$

where C depends on the dimension and the coefficients $a_{k\ell}$. Therefore,

$$\sum_{y \in \Gamma_{\frac{2\pi}{p}}} q_k^2 \sum_{i=1}^{d} \frac{1}{q_i^2} \left|\nabla_i^{\frac{2\pi}{p}}\left(e^{iy\cdot\eta}\partial_k \varphi_{p,1}(y;\eta)\right)\right|^2 \le C,$$

and, using Lemma 3.5,

$$\sum_{y \in \Gamma_{\frac{2\pi}{p}}} q_k^2 \left|\partial_k \varphi_{p,1}(y;\eta)\right)|^2 \le C|q|^2.$$

On the other hand, when $k \neq \ell$, $\partial^2_{k\ell}\mu_1(\eta)$ can be written as

$$\begin{aligned}\partial^2_{k\ell}\mu_1(\eta) = &\sum_{y\in\Gamma_{\frac{2\pi}{p}}} \frac{1}{q_k q_\ell} a_{\ell k}(qy(\ell,k))[\psi_{p,1}(y^k;\eta)\overline{\psi_{p,1}(y^\ell;\eta)} + \psi_{p,1}(y^\ell;\eta)\overline{\psi_{p,1}(y^k;\eta)}]\\
&- \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d \frac{1}{q_k q_i} a_{ik}(qy(i,k))\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\partial_\ell\varphi_{p,1}(y;\eta))\overline{\psi_{p,1}(y^k;\eta)}\\
&- \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d \frac{1}{q_k q_i} a_{ik}(qy(i,k))e^{iy^k\cdot\eta}\partial_\ell\varphi_{p,1}(y^k;\eta)\overline{\nabla_i^{\frac{2\pi}{p}}(\psi_{p,1}(y;\eta))}\\
&- \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d \frac{1}{q_\ell q_i} a_{i\ell}(qy(i,\ell))\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\partial_k\varphi_{p,1}(y;\eta))\overline{\psi_{p,1}(y^\ell;\eta)}\\
&- \sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d \frac{1}{q_\ell q_i} a_{i\ell}(qy(i,\ell))e^{iy^\ell\cdot\eta}\partial_k\varphi_{p,1}(y^\ell;\eta)\overline{\nabla_i^{\frac{2\pi}{p}}(\psi_{p,1}(y;\eta))}.\end{aligned}$$

Thus, we get

$$\begin{aligned}|q_k q_\ell \partial^2_{k\ell}\mu_1(\eta)| \le c&\left[1 + \left(\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d |q_\ell\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\partial_\ell\varphi_{p,1}(y;\eta))|^2\right)^{\frac12}\right]\\
&+c\left(\sum_{y\in\Gamma_{\frac{2\pi}{p}}}\sum_{i=1}^d |\nabla_i^{\frac{2\pi}{p}}(\psi_{p,1}(y;\eta))|^2\right)^{\frac12}\left(\sum_{y\in\Gamma_{\frac{2\pi}{p}}} |q_k\partial_k\varphi_{p,1}(y^\ell;\eta)|^2\right)^{\frac12}.\end{aligned}$$

Thus, applying the previous estimates, we obtain

$$|q_k q_\ell \partial^2_{k\ell}\mu_1(\eta)| \le \beta[1 + c(d) + C(d)|q|\left|\frac{\eta}{q}\right|] \le c(\beta, d), \qquad \forall \eta \in B_\delta.$$

We have similar results for the case $k = \ell$.

Now, we study $\partial^2_{k\ell}\varphi_{p,1}(\eta)$. Note that

$$\begin{gathered}e^{iy\cdot\eta}[\mathcal{A}(\eta) - \mu_1(\eta)]\partial^2_{k\ell}\varphi_{p,1}(\eta) = F_{k\ell}(\eta),\\
\langle\partial^2_{k\ell}\varphi_{p,1}(\eta), \varphi_{p,1}(\eta)\rangle = -\langle\partial_k\varphi_{p,1}(\eta), \partial_\ell\varphi_{p,1}(\eta)\rangle.\end{gathered}$$

Thus, we have that

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} q_k^2 q_\ell^2 \sum_{i=1}^d \frac{1}{q_i^2}\left|\nabla_i^{\frac{2\pi}{p}}(e^{iy\cdot\eta}\partial^2_{k\ell}\varphi_{p,1}(y;\eta))\right|^2 \le C|q\eta|^2 + \sum_{y\in\Gamma_{\frac{2\pi}{p}}} |F_{k\ell}(y;\eta)|^2.$$

We estimate $F_{k\ell}(\eta)$ arguing as for $F_k(\eta)$ and we see that

$$\sum_{y\in\Gamma_{\frac{2\pi}{p}}} |F_{k\ell}(y;\eta)|^2 \le c|q\eta|^2.$$

Finally, according to the previous estimates we proceed analogously to obtain

$$|q_i q_j q_k \partial^3_{ijk} \mu_1(\eta)| \leq c(d,\beta)|q|, \qquad \forall i,j,k,$$

and, by Taylor's formula, we derive (3.28). □

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