Finite Element Approximation of 2D Elliptic Optimal Design

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December 12, 2004

Abstract

We consider a problem of elliptic optimal design in two space dimensions. The control is the shape of the domain on which the Dirichlet problem for the Laplace equation is posed. In dimension $n = 2$, Šveràk [36] proved that there exists an optimal domain in the class of all open subsets of a given bounded open set, whose complementary sets have a uniformly bounded number of connected components. The proof in [36] is based on the compactness of this class of domains with respect to the complementary-Hausdorff topology H^c and the continuous dependence of the solutions of the Dirichlet laplacian in H^1 with respect to it. In this article we introduce a finite-element discrete version of this problem in which the domains under consideration are polygons

*This author has been partially supported by the Grant BFM2002-03345 of the Spanish MCYT, and the EU TMR Project "Smart Systems".

defined on the numerical mesh. The discrete optimal design problem admits at least one solution since it is a finite optimization problem. We prove that any limit in H^c of discrete optimal shapes, when the mesh-size tends to zero, is an optimal domain for the continuous optimal design problem. The proof relies on the following two key facts: a) any open bounded set of $\mathbf{R}^2$ can be approximated in H^c by a sequence of triangulated domains, b) finite-element approximations of the Dirichlet laplacian in the triangulated domains converge in H^1 to the solutions of the continuous Dirichlet problem whenever the triangulated domains converge in H^c.

Key words: Elliptic equation, Dirichlet problem, two space dimensions, shape optimization, optimal control, finite elements, complementary-Hausdorff topology, γ-convergence.

Mathematics Subject Classification: 35J05, 49Q10, 49M25, 65M60.

1 Introduction

We consider a problem of optimal control in which the control variable Ω is the domain on which a partial differential equation (PDE) is posed. The function we want to minimize depends on Ω through the solution of the PDE.

This subject has been widely studied in the last decades and there is an extensive literature.

We focus on the Dirichlet laplacian in $2D$ and, more precisely, on the problem of the numerical approximation of optimal shapes. We work in the functional and geometric setting introduced by Šveràk [36]. We then build a finite element approximation of the optimal design problem and prove that, in the complementary-Hausdorff topology H^c, every limit of discrete optimal shapes is an optimal shape for the Dirichlet problem for the continuous laplacian.

The geometric and functional setting in Šverák [36] seems to be the appropriate one to address this issue of numerical approximation of optimal shapes. Indeed, in dimension $d = 2$, according to [36], the solution of a Laplace-Dirichlet problem depends continuously on the domain on which it is posed *provided one works in the set* $\mathcal{O}^N$ *of all open subsets of a given open bounded set* D*, which have at most* N *holes* (N is a given number). This result is the key ingredient to prove the existence of optimal shapes for a number of optimal design problems. For the discrete/numerical optimal design problem, we shall work in the same geometric setting, by imposing the complement of the discrete shapes to have a finite, a priori fixed, number of connected components.

As we shall see, roughly speaking, this suffices to prove the convergence in H^c of the finite-element discrete optimal shapes to the continuous ones as the mesh-size tends to zero.

Let us describe more precisely the problem under consideration.

- D is a non-empty bounded lipschitz open set in $\mathbf{R}^2$.
- $\mathcal{O}$ is the set of all open subsets of D.
- For all $\Omega \in \mathcal{O}$, we consider a partial differential equation posed on Ω

$$y^{\Omega}: \quad A^{\Omega}\, y^{\Omega} = f^{\Omega}. \tag{1}$$

  For the sake of simplicity we shall focus on the Dirichlet problem for the Laplace operator. Any second order symmetric operator could be addressed with the same techniques. But considering the Dirichlet boundary conditions is essential to apply the arguments we shall develop in this article.

- For all $\Omega \in \mathcal{O}$, we define $j(\Omega) = J^{\Omega}(y^{\Omega})$, where J^{Ω} is a given functional.

The continuous optimal design problem we consider is as follows:

$$\textit{to find} \quad \Omega^* \quad \textit{such that} \quad j(\Omega^*) = \min_{\Omega} j(\Omega). \tag{2}$$

As we have mentioned above, the results by Šveràk [36] guarantee that the problem above achieves the minimum in an optimal shape Ω^* for a wide class of functionals, under the additional constraint that the domains under consideration have complementary sets with at most a finite prescribed number of connected components.

The problem we address is that of the numerical approximation of the optimal shapes solving (2). In particular we address the issue of whether the discrete optimal shapes for a suitable discretization of the problem above converge in H^c to an optimal shape for the continuous problem. As we shall see, the answer to this question is positive if the discrete optimization problem is conveniently built in the context of finite-element approximations.

We now introduce a discretization of this problem as follows.

- For any mesh size $h > 0$, we consider a triangulation $\mathcal{T}_h$ of the domain D. The triangulations are assumed to satisfy the classical requirements for finite elements.

- $\mathcal{O}_h$ is a set of open subsets of D constituted by unions of triangles T of the triangulation $\mathcal{T}_h$.

- For all $\Omega_h \in \mathcal{O}_h$, we consider the $P1$ finite element approximation of the PDE posed on Ω_h. Thus, its solution y_h solves a Galerkin variational approximation of the equation (1).

- We approximate J^Ω by a well-chosen functional $J_h^{\Omega_h}$, and we define

$$j_h(\Omega_h) = J_h^{\Omega_h}(y_h).$$

A classical family of functionals comes from one, say J, defined on $H_0^1(D)$. Extending any function of $H_0^1(\Omega)$ by 0, one can view it as a function of $H_0^1(D)$. Also the finite element solution in Ω_h belongs to $H_0^1(\Omega_h)$. In this case, we can take J for J^Ω and $J_h^{\Omega_h}$.

The discrete problem we consider is:

$$\textit{to find} \quad \Omega_h^* \quad \textit{such that} \quad j_h(\Omega_h^*) = \min_{\Omega_h} j_h(\Omega_h). \tag{3}$$

The triangulation $\mathcal{T}_h$ being fixed, the number of triangular domains under consideration for the discrete optimal design problem is finite. Thus, the existence of discrete optimal shapes is obvious.

The goal of this article is to describe a setting in which the following two properties are true:

- *convergence of the minima*:

$$\min_{\Omega_h} j_h(\Omega_h) \longrightarrow \min_{\Omega} j(\Omega) \quad \text{when} \quad h \longrightarrow 0,$$

- *convergence of the optimal shapes*: the limit in the topology H^c of discrete optimal shapes Ω_h^* solving (3) is an optimal shape for the continuous problem (2).

For this to be true, obviously, the mesh-size h of the triangulation $\mathcal{T}_h$ has to tend to zero.

The interest of this kind of convergence result is that it provides a rigorous justification to the most common engineering approach for computing optimal shapes that consists in solving a discrete finite-element version of the optimal design problem in order to compute an approximation of the continuous one.

In this article we discuss this problem in the context of the Dirichlet laplacian and describe the geometric, functional and finite element setting in which these convergence results hold.

Our results apply to a variety of functionals to be minimized. In particular, they apply to the most common example of minimizing the work of external loads, which is nothing but the internal energy of the system. They also apply to classical shape identification problems.

The proof we shall develop fits in the frame of Γ-convergence. Consequently, it relies essentially on the following two related but independent facts.

1. The first one is that *any* open subset Ω of D can be approximated in the complementary Hausdorff topology (H^c-topology) by domains Ω_h which are unions of triangles in the triangulation $\mathcal{T}_h$ as h tends to zero.

2. The second one is that $P1$ finite-element approximations in Ω_h converge to the solution of the Dirichlet problem in Ω, provided the triangulated domains Ω_h converge in H^c to Ω.

The second property may be viewed as a discrete version of the main result by Sveràk [36] guaranteeing the convergence of the solutions of the Dirichlet problem when the domains converge in the sense of H^c, under the additional assumption that their complementary set have an a priori bounded number of connected components.

To our knowledge the results in this paper are the very first ones in what concerns the convergence of discrete optimal shapes to continuous ones in the present geometric and functional setting, where optimal shapes may be very singular. For an introduction to this topic the interested reader is referred to the monographs [26] and [31].

Most of the ideas developed in this article may be of use for many other optimal design problems related with PDE's. But a complete developement at the level of convergence of numerical discretizations will certainly require the use of fine properties of the underlying continuous problem. In this article we fully rely on the results in [36] and therefore our results are restricted to the Dirichlet problem in $2D$. Note however that, although we work with the Laplace opertor, similar results could be obtained with the same techniques to many other Dirichlet problems: elliptic Stokes system, the wave and the heat equation, etc. The restriction to 2 dimensional space also refers to the use we do of the result of Šverák [36].

This paper is divided in five sections after this introduction. In Section 2, we recall some definitions and properties concerning Hausdorff topology, γ-convergence, Mosco-convergence, and how they are related. In Section 3, we present in detail a class of optimal design problems and their numerical approximations in which the techniques developed in this paper apply. Convergence is rigorously proved under some minimal requirements on the classes of admissible domains. In Section 4, we show that the general results of the previous section apply to the Dirichlet problem for the laplacian in $2D$ in the class of domains $\mathcal{O}^N$ in which the number of holes is a priori bounded by a finite number N. Section 5 is devoted to summarize the main results of the paper and to comment on some open problems and directions of future research. Section 6 is an Appendix, in which we give some examples showing that some of the most "intuitive" properties of Hausdorff convergence may fail.

As we mentioned above, there is an extensive literature on optimal design for PDE's both in the context of elasticity and fluid flows. The interested reader is referred to the monographies at the bibliography in the end of the article. This bibliography is by no means complete. However, we have tried to collect some representative works that we briefly comment now to close this introduction.

In the seventies and eighties the work in this field was done mainly in the context of smooth (Lipschitz) domains (see for instance [10, 11]). The method, originally introduced by Hadamard, consisting on considering only domains which are homeomorphic to a given reference domain was also intensively investigated (see [17, 18, 25, 29, 33, 35, 38] for instance). In both approaches the restrictions on the admissible domains are quite strong.

This method has been extensively used in engineering. Actually, in quite a lot of situations (building, car, aircraft, aerospace industries...), engineers know a priori the topology of the piece of material they have to build. They use shape optimization to improve its strength. This is usually done with gradient type methods. We can mention that this can be done using a discrete method: the gradient of the discrete functional is computed. It can also be done using a continuous point of vue: the differential of the continuous functional is computed, and then discretized. Quite often, these two methods give the same result. If not, they are asymptotic when the mesh size tends to zero, provided the discrete method is convergent. This is discussed in [22, 23].

More recently, techniques allowing to handle topological changes of the shapes have been developed, in a theoretical way as well as in a numerical one. Most of them are based on relaxation and homogenization ([1, 2, 3, 8, 28, 37]) and other with topological derivatives (see [25]).

For numerical results and experiments, we can refer for instance to [2, 3, 21, 25, 34] for representative engineering techniques.

In the last ten or twenty years new results of existence of continuous optimal shapes came out, requiring very little regularity on the admissible domains. We refer to [4, 5, 6, 7, 9, 19, 20, 24, 36] for up to date results in this direction. The key point in this approach is the use of the Hausdorff topology on sets of parts of $\mathbf{R}^d$. As we mentioned above the present paper relies heavily in the setting developed in [36] for the elliptic optimal design of the Dirichlet problem in $2D$.

This article is an extended version of [13] where the main results presented and fully developed here were announced.

2 Preliminaries

In what follows, we recall well-known results that can be found, in particular, in [19].

In Section 2.1, we recall properties concerning the Hausdorff topology. In Section 2.2, we consider the Laplace equation with Dirichlet boundary conditions, and we recall properties concerning the dependence of the solution with respect to the domain on which it is posed. In particular, we recall the definitions of γ-convergence, Mosco-convergence, and the relations between these convergence notions.

In what follows, D denotes an open bounded regular subset of $\mathbf{R}^d$. $\mathcal{O}$ denotes the set of all open subsets of D.

2.1 Hausdorff and complementary-Hausdorff topology

We first recall the definition of the Hausdorff and complementary-Hausdorff topologies. One has

Definition 2.1 *[5, 16, 19, 31]*

1. *The Hausdorff distance between two compact sets* K_1 *and* K_2 *of* $\mathbf{R}^2$ *is defined by*

$$d_H(K_1,\ K_2) = \max\{\max_{x_1\in K_1} d(x_1,K_2),\ \max_{x_2\in K_2} d(x_2,K_1)\},$$

where $d(x,K) = \min_{y\in K} \|\, x-y \,\|$, *and* $\|\, . \,\|$ *is the euclidian distance in* $\mathbf{R}^2$.

2. *The complementary-Hausdorff distance between two open subsets* Ω_1 *and* Ω_2 *of* D *is defined by*

$$d_{H^c}(\Omega_1,\Omega_2) = d_H(\ \overline{D}\setminus\Omega_1,\ \overline{D}\setminus\Omega_2)$$

□

Each of these distances defines a metric topology on the set of compact subsets of $\mathbf{R}^2$ and open subsets of D respectively . We denote

$$K_n \xrightarrow{H} K \quad\Longleftrightarrow\quad d_H(K_n,K)\longrightarrow 0,$$

$$\Omega_n \xrightarrow{H^c} \Omega \quad\Longleftrightarrow\quad d_{H^c}(\Omega_n,\Omega)\longrightarrow 0.$$

Remark 2.1 *(see [5, 19, 20, 31]) The following results hold:*

1. *The set of compact subsets* K *of* $\overline{D}$ *is* H*-compact, so* $\mathcal{O}$ *is* H^c*-compact.*

2. *Let* $(K_n)_n$ *be a sequence of compact subsets of* $\overline{D}$, H*-converging to* K. *Then*
$$K = \{x\in\overline{D};\ \exists\ x_n\in K_n \ s.t.\ x_n\longrightarrow x\}.$$

3. *Let* $(\Omega_n)_n$ *be a sequence of open subsets of* D, H^c*-converging to* Ω. *Then*

$$\forall K \ compact,\ K\subset\Omega,\quad \exists n_K\ \ s.t.\ \ n>n_K \Rightarrow\ K\subset\Omega_n.$$

□

For any $\Omega\in\mathcal{O}$, we denote by $\sharp_c\Omega$ the number of connected components of $\overline{D}\setminus\Omega$.

Definition 2.2 *For a given positive integer* N *and a small regular open subset* ω *of* D, *we define*

1. $\mathcal{O}^N = \{\Omega \in \mathcal{O};\ \sharp_c \Omega \leq N\}$.

2. $\mathcal{O}^N_\omega = \{\Omega \in \mathcal{O}^N;\ \omega \subset \Omega\}$.

□

One has

Lemma 2.1 *(see [20, 36]) The sets $\mathcal{O}^N$ and $\mathcal{O}^N_\omega$ are H^c-compact.*

□

Remark 2.2 *Some of the properties related to Hausdorff convergence that might seem "natural" and/or in agreement with intuition may fail. Here we present some of them. In Appendix 1 we give examples ([20]) showing that these properties fail in general:*

1. *For any K_1 and K_2 compact sets of $\mathbf{R}^2$, there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $d_H(K_1, K_2) = \| x_1 - x_2 \|$ but they are not necessarily on the boundary of K_1 and K_2. (see example 6.1)*

2. *The property that $\overline{\omega} \subset \Omega$ is not H^c-closed. (see example 6.2)*

3. *If a sequence of open subsets $(\Omega_n)_n$ of D H^c-converges to Ω, then the sequence $(\overline{\Omega_n})_n$ does not necessarily H-converge to $\overline{\Omega}$. (see example 6.2)*

4. *$\Omega_n \xrightarrow{H^c} \Omega$ does not imply that $\mu(\Omega_n) \longrightarrow \mu(\Omega)$, where $\mu(\Omega)$ denotes the Lebesgue measure of Ω. (see example 6.3)*

   In general one can only guarantee that

   $$lim\ inf\ \mu(\Omega_n) \geq \mu(\Omega).$$

5. *The same happens with the perimeter:*

   $$lim\ inf\ P(\Omega_n) \geq P(\Omega),$$

   where $P(\Omega)$ denotes the perimeter of Ω. (see example 6.3)

□

2.2 Dependence of the Dirichlet problem with respect to the domain

We remind that $H_0^1(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ for the H_0^1 topology, where $\mathcal{D}(\Omega)$ is the set of $\mathcal{C}^\infty$ functions with compact support in Ω. Accordingly, $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ and any function of $H_0^1(\Omega)$ can be extended by 0 to give a function of $H_0^1(\mathbf{R}^d)$. Note that these properties do not hold in $H^1(\Omega)$ without further restrictions on the regularity of Ω. This makes the Dirichlet problem much easier to treat than the Neumann one.

For any function $z \in H_0^1(\Omega)$, we denote by $\widetilde{z}$ its extension by 0 to D.

For any $f \in H^{-1}(D)$ and any $\Omega \in \mathcal{O}$, $\Omega \neq \emptyset$, one defines $y_f^\Omega \in H_0^1(\Omega)$ as the solution of the Dirichlet problem for the laplacian

$$\begin{cases} -\Delta y_f^\Omega = f & \text{in } \Omega \\ y_f^\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The variational formulation of (4) is as follows:

$$y_f^\Omega \in H_0^1(\Omega), \quad \int_\Omega \nabla y_f^\Omega \cdot \nabla z dx =< f,\ z >_{H^{-1}, H_0^1(\Omega)}, \quad \forall z \in H_0^1(\Omega). \tag{5}$$

Here and in the sequel $\cdot$ denotes the inner product in $\mathbf{R}^2$, and ∇z the gradient of z.

When $\Omega = \emptyset$ we use the notation $\widetilde{y_f^\emptyset} := 0$.

Given a sequence $(\Omega_n)_n \subset \mathcal{O}$ and a domain $\Omega \in \mathcal{O}$, we recall that Ω_n *γ-converges* to Ω if (see [19, 20])

$$\forall f \in H^{-1}(D), \qquad \widetilde{y_f^{\Omega_n}} \longrightarrow \widetilde{y_f^\Omega} \ \textit{strongly in } H_0^1(D).$$

On the other hand, Ω_n *Mosco-converges* to Ω and we denote it as $\Omega_n \stackrel{Mosco}{\longrightarrow} \Omega$ if (see [19, 27])

1. $\forall z \in H_0^1(\Omega), \ \ \exists\, z_n \in H_0^1(\Omega_n) \ \ s.t. \ \ \widetilde{z_n} \longrightarrow \widetilde{z} \ \ \textit{strongly in } H_0^1(D),$
2. $\forall (\Omega_{n_k})_k \subset (\Omega_n)_n,\ \forall z_{n_k} \in H_0^1(\Omega_{n_k})$, one has

$$\{\ \widetilde{z_{n_k}} \rightharpoonup w \quad \textit{weakly in } H_0^1(D)\ \} \implies \{\ \exists\, z \in H_0^1(\Omega) \quad s.t. \quad w = \widetilde{z}\ \}.$$

It is by now well known that these two notions coincide (see [19]), i. e.

$$\Omega_n \stackrel{\gamma}{\longrightarrow} \Omega \quad \Longleftrightarrow \quad \Omega_n \stackrel{Mosco}{\longrightarrow} \Omega.$$

Now, let us recall some relations between H^c-convergence and γ-convergence.

Lemma 2.2 *(see [5])*

If a sequence H^c-converges, then the first point of the definition of the Mosco convergence is satisfied.

In other words, if Ω_n converges to Ω in H^c, then, for all $z \in H_0^1(\Omega)$ there exists $z_n \in H_0^1(\Omega_n)$ such that $\widetilde{z_n} \longrightarrow \widetilde{z}$ strongly in $H_0^1(D)$.

□

It is well-known that, in general, H^c-convergence does not imply γ-convergence. Indeed, many situations are known where homogenization phenomena occur at the limit when the sequence of domains is allowed to develop an increasing number of holes. In those cases the limit of the solutions of the Dirichlet laplacian may be the solution of a different elliptic problem (see [15] and [1, 28, 31, 37]). Nevertheless, several situations are known where this does not happen. In [5], a list of subsets $\mathcal{U}$ of $\mathcal{O}$ on which H^c-convergence implies γ-convergence is given. The following one is due to V. Šveràk [36]:

Theorem 2.1 *([36])*

In two space dimensions, for any finite N, H^c-convergence and γ-convergence are equivalent properties on $\mathcal{O}^N$.

Notice that the properties that the dimension is 2 and that $\sharp_c\Omega \leq N$ for all sets in $\mathcal{O}^N$ are fundamental here.

3 Convergence of discrete optimal shapes towards continuous ones

In this section, we study the optimization problem described in the Introduction in a quite general setting.

For a matter a simplicity, we work here in dimension 2. Though, we emphasize the fact that it will become necessary only when we will use the result of Šverák.

3.1 Notations and definitions

As before, $D \subset \mathbf{R}^2$ is an open bounded regular set, and $\mathcal{O}$ the set of all open subsets of D. To fix ideas one can assume that D is for instance a rectangle in $\mathbf{R}^2$.

For any h, we consider a discretization or triangulation $\mathcal{T}_h$ of D made of finite elements T. Any finite element T is a closed triangle (see [32]) so that

$$D = \overbrace{\bigcup_{T \in \mathcal{T}_h} T}^{\circ}$$

where $\overbrace{A}^{\circ}$ denotes the interior of $A \subset \mathbf{R}^2$.

We assume that the mesh-size is $h > 0$. More precisely, any $T \in \mathcal{T}_h$ has a diameter at most equal to h.

Moreover, as usual in finite elements theory, we suppose that the triangulations are uniformly regular, that is

$$\exists \sigma > 0 \quad s.t. \quad \forall\ h > 0,\ \forall\ T \in \mathcal{T}_h, \quad 0 < \frac{h}{\rho(T)} \leq \sigma,$$

where $\rho(T)$ is the radius of the biggest ball which is contained in T.

We define $\mathcal{O}_h$ as the class of subdomains of D constituted by triangles T of the triangulation $\mathcal{T}_h$. More precisely, we say that $\Omega_h \in \mathcal{O}_h$ if and only if there exists $\mathcal{T}_h(\Omega_h) \subset \mathcal{T}_h$ such that

$$\Omega_h = \overbrace{\bigcup_{T \in \mathcal{T}_h(\Omega_h)} T}^{\circ}.$$

Obviously, the set $\mathcal{O}_h$ is finite.

We consider a functional

$$\widetilde{J} : \mathcal{O} \times H_0^1(D) \longrightarrow \mathbf{R} : (\Omega, \widetilde{z}) \mapsto \widetilde{J}(\Omega, \widetilde{z})$$

which is supposed to be continuous, $\mathcal{O}$ being equipped with the H^c-topology and $H_0^1(D)$ with its strong topology.

We consider the solution of the Laplace equation with Dirichlet boundary conditions.

Let $f \in H^{-1}(D)$ be given and for any $\Omega \in \mathcal{O}$, $\Omega \neq \emptyset$ consider the Dirichlet problem (4) or its weak version (5) in Ω. The right hand side term f being fixed in the sequel, the solution is denoted by y^Ω. We also set $\widetilde{y^\emptyset} := 0$.

For any $h > 0$ and any $\Omega_h \in \mathcal{O}_h$, we consider the $P1$ finite element space $V_h(\Omega_h)$. Obviously, $V_h(\Omega_h) \subset H_0^1(\Omega_h)$. We denote by y_h the finite element Galerkin approximation in Ω_h, namely

$$y_h \in V_h(\Omega_h), \quad \int_{\Omega_h} \nabla y_h \cdot \nabla z_h dx =< f,\ z_h >_{H^{-1}, H_0^1(\Omega_h)}, \ \ \forall z \in V_h(\Omega_h). \quad (6)$$

Remark 3.1 *It is important to distinguish* y_h*, which is the discrete finite-element solution in* Ω_h*, and* y^{Ω_h} *which is the solution of the continuous Dirichlet problem in* Ω_h*.*

We define the continuous and discrete functional to be optimized as follows:

$$j : \mathcal{O} \longrightarrow \mathbf{R} : \Omega \mapsto \widetilde{J}(\Omega, \widetilde{y^{\Omega}}),$$

$$j_h : \mathcal{O}_h \longrightarrow \mathbf{R} : \Omega_h \mapsto \widetilde{J}(\Omega_h, \widetilde{y_h}),$$

and for given subclasses of domains $\mathcal{U}_{ad} \subset \mathcal{O}$ and $\mathcal{U}_{ad,h} \subset \mathcal{O}_h$, we consider the optimization problems:

$$\Omega^* \in \mathcal{U}_{ad} : \quad j(\Omega^*) = \min_{\Omega \in \mathcal{U}_{ad}} j(\Omega), \tag{7}$$

$$\Omega_h^* \in \mathcal{U}_{ad,h} : \quad j_h(\Omega_h^*) = \min_{\Omega_h \in \mathcal{U}_{ad,h}} j_h(\Omega_h). \tag{8}$$

As we mentioned in the introduction, the goal of this paper is to give sufficient conditions on $\mathcal{U}_{ad}$ and $\mathcal{U}_{ad,h}$ insuring that the discrete minimization problems are good approximations of the continuous one.

Remark 3.2 *Note that, since* $\mathcal{O}_h$ *is a finite set, for any choice of* $\mathcal{U}_{ad,h}$ *the discrete minimization problem has at least one solution, say* $\Omega_h^\star$*. Obviously, this does not mean that efficiently computing that discrete optimal shape is an easy task at all. As we mentioned in the Introduction, this is a whole field of research in engineering (see [2, 3, 25, 34]).*

As we shall see, the existence of optimal shapes for the continuous optimization problems is quite subtle.

$\square$

Let us give some examples of functionals $\widetilde{J}$ which often arise in applications. The theory we shall develop in this article applies to all of them.

1. The first one is very standard and concerns the *compliance* of the system. It is defined by

   $$\widetilde{J}(\Omega, \widetilde{z}) =< f,\ \widetilde{z} >_{H^{-1}, H_0^1(D)},$$

   which gives

   $$j(\Omega) =< f,\ y^{\Omega} >_{H^{-1}, H_0^1(\Omega)} .$$

   Remark that, in this case, $j(\Omega) = \int_\Omega \mid \nabla y^{\Omega} \mid^2$, which coincides with the energy of solutions, and, in particular, is non-negative. Without any further constraint, the optimum is reached for the empty set $\Omega = \emptyset$, and the trivial solution $\widetilde{y^\Omega} = 0$. But this is often an irrelevant solution in applications. It is much more natural to impose some condition avoiding the possibility that the optimal shape degenerates to the empty set. This is done, for instance, imposing to Ω to contain a given non-empty set ω.

2. A second important example concerns shape identification problems.

   Let us consider a subdomain $\omega \in \mathcal{O}$, $\omega \neq \emptyset$. We suppose that a function y_g has been measured or observed on ω, which is a known or accesible part of the set Ω which is unknown and has to be identified.

   One then wants to minimize $|| \, y^\Omega - y_g \, ||_V$, where V is a suitable space, well-adapted to the problem under consideration. We can choose for instance $V = L^2(\omega)$ or $H^1(\omega)$. In this case, the functionals to be minimized are, for example, of the form

   $$\widetilde{J}(\Omega, \widetilde{z}) = \frac{1}{2} \, || \, \widetilde{z}_{|\omega} - y_g \, ||_V^2,$$

   which gives

   $$j(\Omega) = \frac{1}{2} \, || \, y^\Omega_{|\omega} - y_g \, ||_V^2 \; .$$

   We refer to [12] for a discrete formulation of this problem in the spirit of the theory of controllability of PDE's.

$\square$

3.2 The main result

The aim of this section is to prove the following

Theorem 3.1 *Suppose that we are given a set* $\mathcal{U} \subset \mathcal{O}$ *on which* H^c*-convergence implies* γ*-convergence. Suppose further that*

$$\mathcal{U}_{ad} \subset \mathcal{U}, \quad \mathcal{U}_{ad,h} \subset \mathcal{U}.$$

Morever, suppose that

1. $\forall \Omega \in \mathcal{U}_{ad}, \quad \exists \; \Omega_h \in \mathcal{U}_{ad,h} \quad s.t. \quad \Omega_h \xrightarrow{H^c} \Omega,$
2. *if a sequence* $\Omega_h \in \mathcal{U}_{ad,h}$ H^c*-converges to some* $\Omega \in \mathcal{O}$*, then* $\Omega \in \mathcal{U}_{ad}$*.*

Then the discrete optimal design problems converge to the continuous one in the sense that

1. *j reaches its minimum on $\mathcal{U}_{ad}$,*

2. *any accumulation point of any sequence $(\Omega_h^\star)_h$ of discrete minimizers (which is H^c-compact) is a continuous minimizer,*

3. *the whole sequence $(j_h(\Omega_h^\star))_h$ converges to $\min_{\Omega\in\mathcal{U}_{ad}} j(\Omega)$.*

$\square$

The proof of this theorem is a direct consequence of the following technical results.

Proposition 3.1 *Suppose that $\mathcal{U}_{ad}$ and $\mathcal{U}_{ad,h}$ are such that*

1. *(a)* $\forall\Omega\in\mathcal{U}_{ad}, \quad \exists\, \Omega_h\in\mathcal{U}_{ad,h} \quad s.t. \quad \Omega_h \xrightarrow{H^c} \Omega,$
   (b) if a sequence $\Omega_h \in \mathcal{U}_{ad,h}$ H^c-converges to some $\Omega \in \mathcal{O}$, then $\Omega \in \mathcal{U}_{ad}$,

2. *if $\Omega_h \in \mathcal{U}_{ad,h}$ and $\Omega \in \mathcal{O}$ are such that $\Omega_h \xrightarrow{H^c} \Omega$, then $\widetilde{y_h} \longrightarrow \widetilde{y^\Omega}$ strongly in $H_0^1(D)$.*

Then

1. *j reaches its minimum on $\mathcal{U}_{ad}$,*

2. *any accumulation point of any sequence $(\Omega_h^\star)_h$ of discrete minimizers (which is H^c-compact) is a continuous minimizer,*

3. *the whole sequence $(j_h(\Omega_h^\star))_h$ converges to $\min_{\Omega\in\mathcal{U}_{ad}} j(\Omega)$.*

Remark 3.3 *Hypothesis # 2 is a discrete finite-element version of the γ-convergence property.*

Remark 3.4 *In this proposition, the existence of continuous minimizers is obtained as limit of the discrete ones. No a priori assumptions on $\mathcal{U}_{ad}$ are made, other than Hypothesis # 1.*

□

Proof of Proposition 3.1

Let $(\Omega_h^\star)_h$ be a sequence of discrete minimizers. Any $\Omega_h^\star$ belongs to $\mathcal{O}$ which is H^c-compact. Let U be an accumulation point of this sequence.

From Hypothesis # 1.(b), we know that $U \in \mathcal{U}_{ad}$.

In view of Hypothesis # 2, we have

$$\widetilde{y_h} \longrightarrow \widetilde{y^U} \;\text{ strongly in } H_0^1(D),$$

and because of the continuity of $\widetilde{J}$, we obtain

$$j_h(\Omega_h^\star) \longrightarrow j(U).$$

Let us now check that U is a minimizer for j.

Let $\Omega \in \mathcal{U}_{ad}$ be given. From Hypothesis # 1.(a), we know that there exists $\Omega_h \in \mathcal{U}_{ad,h}$ such that $\Omega_h \xrightarrow{H^c} \Omega$, which implies, as before, that

$$j_h(\Omega_h) \longrightarrow j(\Omega).$$

Now, for each h, we have

$$j_h(\Omega_h^\star) \le j_h(\Omega_h).$$

Passing to the limit, we obtain

$$j(U) \le j(\Omega), \quad \forall \Omega \in \mathcal{U}_{ad}.$$

This proves points 1 and 2 of the proposition.

Also, we have seen that the only accumulation point of the sequence $(j_h(\Omega_h^\star))_h$ is nothing but $\min_{\Omega \in \mathcal{U}_{ad}} j(\Omega)$.

This ends the proof of the Proposition.

□

Let us now discuss hypothesis # 2 of Proposition 3.1.As we mentionned before, it is a discrete version of the property

$$\Omega_h \xrightarrow{H^c} \Omega \quad \Rightarrow \Omega_h \xrightarrow{\gamma} \Omega,$$

which allows passing to the limit on the solutions of the Dirichlet problem.

But, hypothesis # 2 concerns the convergence of the finite-element approximations. Obviously, this is related with the way $V_h(\Omega_h)$ approximates $H_0^1(\Omega)$ when $\Omega_h \longrightarrow \Omega$. We are going to give sufficient conditions for it to hold.

First, we prove the following lemma, which concerns test functions.

Lemma 3.1 *Let $\Omega \in \mathcal{U}_{ad}$ be given. Let $\Omega_h \in \mathcal{U}_{ad,h}$ be such that $\Omega_h \xrightarrow{H^c} \Omega$. Then*

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \exists \varphi_h \in V_h(\Omega_h) \quad s.t.$$

$$\widetilde{\varphi_h} \longrightarrow \ \widetilde{\varphi} \ \textit{ strongly in } H_0^1(D) \textit{ when } h \to 0.$$

Proof

Let $\varphi \in \mathcal{D}(\Omega)$ be given. We know that $\widetilde{\varphi} \in H_0^1(D)$. Moreover $supp\ \widetilde{\varphi} = supp\ \varphi = K \subset \Omega$, K being compact.

As φ is regular, we can use a standard finite element error bound (see [14, 32]).

Let us consider the interpolation operator $\pi_h : H_0^1(D) \longrightarrow V_h(D)$. As D is regular, one has

$$|| \ \widetilde{\varphi} - \pi_h \widetilde{\varphi} \ ||_{H_0^1(D)} \leq C(\sigma, D) h \ || \ \widetilde{\varphi} \ ||_{H^2(D)} = C(\sigma, D) h \ || \ \varphi \ ||_{H^2(\Omega)} \ .$$

Moreover, as we have $\Omega_h \xrightarrow{H^c} \Omega$, we know that

$$\exists \ h_o \ \ s.t. \ \ h < h_o \ \ \Rightarrow K \subset \Omega_h.$$

Therefore, the Lemma is proved if we choose φ_h as the restriction to Ω_h of $\pi_h \widetilde{\varphi}$.

□

Now, we can prove the following precise result.

Proposition 3.2 *Let $\Omega \in \mathcal{U}_{ad}$ be given, and $\Omega_h \in \mathcal{U}_{ad,h}$ be a sequence such that*

$$\Omega_h \xrightarrow{\gamma} \Omega \ \ \textit{and} \ \ \ \Omega_h \xrightarrow{H^c} \Omega.$$

Then

$$\widetilde{y_h} \longrightarrow \widetilde{y^\Omega} \ \ \textit{strongly} \ \ \textit{in} \ \ H_0^1(D).$$

Proof

Let us denote by $\widetilde{V_h(\Omega_h)}$ the vector space of all functions of $V_h(\Omega_h)$ extended by 0 to D.

Equation (6) can be rewritten

$$\widetilde{y_h} \in \widetilde{V_h(\Omega_h)}, \quad \int_D \nabla \widetilde{y_h} \cdot \nabla \widetilde{z_h} dx \ = < f, \ \widetilde{z_h} >_{H^{-1}, H_0^1(D)} \quad \forall \widetilde{z_h} \in \ \widetilde{V_h(\Omega_h)}.$$

For any h we have

$$\| \widetilde{y_h} \|_{H_0^1(D)} \leq \| f \|_{H^{-1}(D)} .$$

We first prove that $\widetilde{y^\Omega}$ is the only $weak - H_0^1(D)$ accumulation point of $(\widetilde{y_h})_h$. Then we prove the strong convergence.

Let w be a $weak - H_0^1(D)$ accumulation point of $(\widetilde{y_h})_h$. It is a weak limit in $H_0^1(D)$ of a subsequence $(\widetilde{y_{h_n}})_{h_n}$ of $(\widetilde{y_h})_h$.

Let us denote $\widetilde{y_n}$ for $\widetilde{y_{h_n}}$, Ω_n for Ω_{h_n}, V_n for $V_{h_n}(\Omega_{h_n})$, and $\widetilde{V_n}$ for $\widetilde{V_{h_n}(\Omega_{h_n})}$.

First, as $\Omega_n \xrightarrow{\gamma} \Omega$, we know that $\Omega_n \xrightarrow{Mosco} \Omega$. So, from point # 2 of the definition of the Mosco convergence, we know that there exists $u \in H_0^1(\Omega)$ such that $w = \widetilde{u}$.

Let us prove that $u = y^\Omega$ and that the convergence holds in the strong topology.

1. $\mathbf{u = y^\Omega}$

   The function y^Ω is caracterized by $y^\Omega \in H_0^1(\Omega)$ and

   $$\int_\Omega \nabla y^\Omega \cdot \nabla \varphi dx \ =< f,\ \varphi >_{H^{-1},H_0^1(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

   So, we have to prove that

   $$\int_\Omega \nabla u \cdot \nabla \varphi dx \ =< f,\ \varphi >_{H^{-1},H_0^1(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

   Let $\varphi \in \mathcal{D}(\Omega)$ be given. We know that $\Omega_n \xrightarrow{H^c} \Omega$ and we can apply Lemma 3.1. Then, there exists $\varphi_n \in V_n$ such that

   $$\widetilde{\varphi_n} \longrightarrow \widetilde{\varphi} \ \text{ strongly in } H_0^1(D) \ \text{ when } \ n \ \longrightarrow \infty.$$

   We have

   $$\int_{\Omega_n} \nabla y_n \cdot \nabla \varphi_n dx \ =< f,\ \varphi_n >_{H^{-1},H_0^1(\Omega)},$$

   or equivalently

   $$\int_D \nabla \widetilde{y_n} \cdot \nabla \widetilde{\varphi_n} dx \ =< f,\ \widetilde{\varphi_n} >_{H^{-1},H_0^1(D)} .$$

As $\widetilde{y_n} \rightharpoonup \widetilde{u}$ weakly in $H_0^1(D)$ and $\widetilde{\varphi_n} \longrightarrow \widetilde{\varphi}$ strongly in $H_0^1(D)$, we can pass to the limit and get

$$\int_D \nabla\widetilde{u} \cdot \nabla\widetilde{\varphi} dx \ =< f,\ \widetilde{\varphi} >_{H^{-1},H_0^1(D)},$$

or

$$\int_\Omega \ \nabla u \cdot \nabla\varphi dx \ =< f,\ \varphi >_{H^{-1},H_0^1(\Omega)}, \qquad \forall\varphi \in \mathcal{D}(\Omega).$$

So $u = y^\Omega$.

2. **Strong convergence in** $H_0^1(D)$

   One has

$$\int_D \mid \nabla\widetilde{y_h} - \nabla\widetilde{y^\Omega} \mid^2 dx \ = \int_D \mid \nabla\widetilde{y_h} \mid^2 dx - 2\int_D \nabla\widetilde{y_h}\cdot\nabla\widetilde{y^\Omega} dx + \int_D \mid \nabla\widetilde{y^\Omega} \mid^2 dx$$

$$=< f,\ \widetilde{y_h} >_{H^{-1},H_0^1(D)} - 2\int_D \nabla\widetilde{y_h} \cdot \nabla\widetilde{y^\Omega} dx + \int_D \mid \nabla\widetilde{y^\Omega} \mid^2 dx$$

$$\stackrel{h\to 0}{\longrightarrow} 0.$$

   Indeed, the limit of the first term $< f,\ \widetilde{y_h} >_{H^{-1},H_0^1(D)}$ is

$$< f,\ \widetilde{y^\Omega} >_{H^{-1},H_0^1(D)} = \int_D \mid \nabla\widetilde{y^\Omega} \mid^2 dx,$$

   because of the weak convergence in $H_0^1(D)$ of $\widetilde{y_h}$ to $\widetilde{y^\Omega}$. The limit of $\int_D \nabla\widetilde{y_h} \cdot \nabla\widetilde{y^\Omega} dx$ is $\int_D \mid \nabla\widetilde{y^\Omega} \mid^2 dx$ for the same reason.

This concludes the proof of Proposition 3.2. □

Now we can give the proof of Theorem 3.1.

Proof of Theorem 3.1.

This theorem is just a corollary of Propositions 3.1 and 3.2.

First, if a sequence $\Omega_h \in \mathcal{U}_{ad,h}$ H^c-converges to some Ω, we know that $\Omega \in \mathcal{U}_{ad}$. Then, as $\mathcal{U}_{ad,h} \subset \mathcal{U}$ and $\mathcal{U}_{ad} \subset \mathcal{U}$, by assumption on $\mathcal{U}$, it also γ-converges. So, we can apply Proposition 3.2 to get Hypothesis #2 of Proposition 3.1. Then the conclusion of Proposition 3.1 holds. □

Remark 3.5 *The results of this section apply in any space dimension. As we shall see in the next section, the assumption that the dimension d is equal to 2 will arise naturally because we will apply this result to the setting of Šverárk [36] which is restricted to $d = 2$.*

□

4 Application to optimal design in the 2-dimensional case

In this section, we check that in the 2-dimensional setting developed by Šverák [36] the conditions of the previous section are fulfilled.

4.1 The geometric setting

Recall that

$$\mathcal{O}^N = \{\Omega \in \mathcal{O};\ \sharp_c\Omega \leq N\}, \qquad \mathcal{O}^N_\omega = \{\Omega \in \mathcal{O}^N;\ \omega \subset \Omega\},$$

where N is a given integer and ω is a small regular subset of D. Note that both are H^c-compact.

In the sequel we choose $\mathcal{O}^N_\omega$ as the set $\mathcal{U}_{ad}$.

Remark 4.1 *The restriction $\omega \subset \Omega$ is imposed to avoid the optimal design to be the empty set. Often, one imposes a lower bound to the measure or perimeter of the domain. But, as indicated in Remark 2.2 points 4 and 5, the two later constraints do not suffice to work in the H^c topology.More precisely, it may happen that a converging sequence of domains with constant non-zero measure, has an empty limit (see Example 6.3 in Appendix 1) and therefore the class of domains $\{\Omega \subset D; \mu(\Omega) \geq m\}$ is not H^c-closed. The same happens with the perimeter. So, it is better to impose the constraint that all admissible domains contain a given subdomain ω. This is meaningful in applications in which a part of the structure to be designed is given a priori (its foundations, for instance), or in identification problems.*

Now, we define the set of discrete admissible domains.

Definition 4.1 *For each $h > 0$, we consider the set $\mathcal{O}_h$ of subdomains of D constituted by elements of the triangulation $\mathcal{T}_h$, as it has been defined in Section 3.1.*

Then we set

1. $\mathcal{O}^N_h = \{\ \Omega_h \in \mathcal{O}_h;\ \sharp_c\Omega_h \leq N\ \}$,

2. $\omega_h = \overbrace{\bigcup_{T\in\mathcal{T}_h,\ T\subset\omega} T}^{\circ}$, *and*

$$\mathcal{O}^N_{\omega,h} = \{\ \Omega_h \in \mathcal{O}^N_h;\ \omega_h \subset \Omega_h\}.$$

The set $\mathcal{O}^N_{\omega,h}$ is taken as $\mathcal{U}_{ad,h}$, the set of all discrete admissible domains.

$\square$

Remark 4.2 *Of course, one has*

$$\mathcal{O}^N_\omega \subset \mathcal{O}^N \subset \mathcal{O},$$

but, as ω_h is smaller than ω

$$\mathcal{O}^N_{\omega,h} \text{ is not a subset of } \mathcal{O}^N_\omega.$$

We only have

$$\mathcal{O}^N_{\omega,h} \subset \mathcal{O}^N.$$

$\square$

Now, we show that the hypothesis of Theorem 3.1 are satisfied so that the discrete problems do approach the continuous one.

4.2 Application of Theorem 3.1

We take $\mathcal{O}^N$ as $\mathcal{U}$. As we saw in Theorem 2.1, H^c-convergence and γ-convergence are equivalent on $\mathcal{O}^N$. Moreover $\mathcal{O}^N_\omega$ and $\mathcal{O}^N_{\omega,h}$ are subsets of $\mathcal{O}^N$.

We now check that $\mathcal{O}^N_\omega$ and $\mathcal{O}^N_{\omega,h}$ satisfy Hypothesis #1 and #2 of Theorem 3.1.

4.2.1 Hypothesis # 1.

The aim of this section is to prove that any $\Omega \in \mathcal{O}^N_\omega$ is the H^c-limit of a sequence $(\Omega_h)_h$ where each $\Omega_h \in \mathcal{O}^N_{\omega,h}$.

Let us first consider any $\Omega \in \mathcal{O}$. We set $F = \overline{D} \setminus \Omega$ and then

$$\overline{D} \; = \; \overset{\circ}{F} \cup \partial F \cup \Omega = F \cup \Omega.$$

For any $T \in \mathcal{T}_h$, one of the following properties holds:

$$T \subset \Omega, \quad \text{or} \quad T \cap \partial F \neq \emptyset, \quad \text{or} \quad T \subset \overset{\circ}{F}.$$

Now, we define

$$\Omega_h = \overbrace{\bigcup_{T \in \mathcal{T}_h,\ T \subset \Omega} T}^{\circ}\,.$$

Note that Ω_h has been built from Ω just as ω_h from ω.

We are going to show that this sequence $(\Omega_h)_h$ H^c-converges to Ω, and that if $\Omega \in \mathcal{O}_\omega^N$ then $\Omega_h \in \mathcal{O}_{\omega,h}^N$.

We also set

$$F_h = \bigcup_{T \in \mathcal{T}_h,\ T \bigcap F \neq \emptyset} T.$$

Then F_h is closed, Ω_h is open, $F_h \cap \Omega_h = \emptyset$, and clearly we have

$$\Omega_h \subset \Omega, \qquad F_h \cup \Omega_h = \overline{D}.$$

Remark 4.3 *Suppose that Ω has a hole H with empty interior (for instance a crack). Then any triangle T which meets H intersects F. Thus we cannot have $T \subset \Omega_h$. So, $H_h = \bigcup_{T \cap H \neq \emptyset} T$ is a hole in Ω_h and it has a non-empty interior. Nevertheless, H_h is not necessarily a neighbourhood of H. For example, if*

$$H \;=\; \{\ (x,y);\ x \ \in\ [0,1],\ y = x\},$$

and $h = \frac{1}{p}$, p being an integer, and considering the standard regular mesh of size h, one has

$$H_h \;=\; \bigcup_{i=-1}^{i=p} [i\ h,\ (i+1)\ h]^2.$$

□

Now, let us check that the Hypothesis # 1 of Theorem 3.1 is satisfied.

Proposition 4.1 *We have*

1. $\Omega_h \xrightarrow{H^c} \Omega$ *when* $h \longrightarrow 0$,
2. *for all h, $\sharp_c \Omega_h \leq \sharp_c \Omega$, so if $\Omega \in \mathcal{O}^N$, then for any h, $\Omega_h \in \mathcal{O}_h^N$.*
3. *if $\Omega \in \mathcal{O}_\omega^N$, then $\Omega_h \in \mathcal{O}_{\omega,h}^N$ for all h.*

Proof

1. By definition, we have to prove that $F_h = \overline{D} \setminus \Omega_h \xrightarrow{H} F = \overline{D} \setminus \Omega$.

   We have $F \subset F_h$, so that

   $$d_H(F_h, F) = \max_{x_h \in F_h} d(x_h, F).$$

Let x_h be in $F_h = \bigcup_{\{T \in \mathcal{T}_h,\ T \bigcap F \neq \emptyset\}} T$.

If $x_h \in F$, we have $d(x_h, F) = 0$.

If not, there exists $T \in \mathcal{T}_h$ such that $x_h \in T$ and $\partial F \cap T \neq \emptyset$. So there exists $y \in F$ such that

$$d(x_h, F) \leq \| x_h - y \| \leq h.$$

Therefore $d_H(F_h, F) \longrightarrow 0$.

2. Recall that if A and B are connected parts of $\mathbf{R}^2$, and $A \cap B \neq \emptyset$, then $A \cup B$ is a connected set.

   Let F_i be one connected component of F. We consider

$$F_h^i = \bigcup_{T \cap F^i \neq \emptyset} T = F^i \cup \Big(\bigcup_{T \cap \partial F^i \neq \emptyset} T \Big).$$

   Any T is of course a closed and connected set. If it intersects ∂F^i, then $T \cup F^i$ is also a connected set. This says that F_h^i is a connected set.

   Now, we have $F_h = \bigcup_i F_h^i$. So,

$$\sharp F_h \leq \sharp F,$$

   or

$$\sharp_c \Omega_h \leq \sharp_c \Omega.$$

   Observe that it may happen that $\sharp F_h < \sharp F$, since for two connected components F^i and F^j of F, the associated F_h^i and F_h^j may have a non empty intersection.

3. We have to prove that

$$\omega \subset \Omega \qquad \Rightarrow \qquad \omega_h \subset \Omega_h.$$

   This is clear because

$$\omega \subset \Omega, \quad \omega_h = \overbrace{\bigcup_{T \subset \omega} T}^{\circ} \subset \overbrace{\bigcup_{T \subset \Omega} T}^{\circ} = \Omega_h.$$

$\square$

Remark 4.4 *Notice that we have proved that any open subset Ω of D can be approximated in the sense of the H^c-topology by a sequence $(\Omega_h)_h$, where each Ω_h is a union of triangles. In particular Ω_h is lipschitz regular, even if Ω is very singular. For instance, it may happen that Ω has a boundary which has a non-zero measure.*

4.2.2 Hypothesis # 2

It suffices to prove the following result:

Proposition 4.2 *If a sequence $\Omega_h \in \mathcal{O}^N_{\omega,h}$ H^c-converges to some Ω, then $\Omega \in \mathcal{O}^N_\omega$.*

Proof

First, note that $\mathcal{O}^N_{\omega,h} \subset \mathcal{O}^N$, $\mathcal{O}^N_\omega \subset \mathcal{O}^N$, and $\mathcal{O}^N$ is H^c-closed. So, it is clear that $\Omega \in \mathcal{O}^N$. All we need to prove is that

$$\omega_h \subset \Omega_h \Longrightarrow \omega \subset \Omega.$$

Let us consider

$$F_h = \overline{D} \setminus \Omega_h, \qquad F = \overline{D} \setminus \Omega,$$
$$G_h = \overline{D} \setminus \omega_h, \qquad G = \overline{D} \setminus \omega.$$

By definition of ω_h, from Proposition 4.1 we know that $\omega_h \xrightarrow{H^c} \omega$, which is equivalent to the fact that $G_h \xrightarrow{H} G$.

We know that $F_h \cap \omega_h = \emptyset$ and we want to check that $F \cap \omega = \emptyset$.

Let $x \in F$ be given. We know (Remark 2.1) that

$$\exists\ x_h \in F_h \qquad s.t. \qquad x = \lim_{h \longrightarrow 0} x_h.$$

As $F_h \subset G_h$, we know that $x_h \in G_h$. As $G_h \xrightarrow{H} G$, also from Remark 2.1, we deduce that $x \in G = D \setminus \omega$. $\square$

Remark 4.5 *Observe that the results of this section 4.2.2 and those of section 4.2.1 apply in any finite dimension d. Also, the triangle shape of the finite elements does not matter. The result holds because the mesh size tends to* 0*, which is the case for any family of finite elements.*

4.2.3 The optimal design problem

In this section we apply Theorem 3.1 in the 2-dimensional case correponding to the framework of Šverák. We address the optimal design problems (7) and (8) of Section 3.1.

Note that the continuous optimal design problem (7) has a minimizer. This is so because $\mathcal{O}^N_\omega = \mathcal{U}_{ad}$ is H^c-closed and within this class, H^c convergence and γ-convergence are equivalent properties.

Taking $\mathcal{O}^N$ as $\mathcal{U}$, $\mathcal{O}_\omega^N$ as $\mathcal{U}_{ad}$, and $\mathcal{O}_{\omega,h}^N$ as $\mathcal{U}_{ad,h}$, all the hypotheses of Theorem 3.1 are satisfied.

So, we have proved that the discrete optimal design problems converge to the continuous ones. This means that the minima of the discrete functionals (8) converge to the minimum of the continuous one (7). Also, any accumulation point of sequences of discrete optimal shapes Ω_h^* is a continuous optimal one.

These results apply in particular to the two functionals introduced in section 3.1: the one related to the compliance and that corresponding to the identification of a partially known shape.

5 Conclusion and open problems

We have considered the problem of numerically approximating optimal shapes. More precisely, we have addressed the issue of whether discrete optimal shapes for a suitable discretization of the original continuous optimal design problem provide an approximation of the continuous optimal shapes.

The problem has been addressed in the context of minimizing functionals which depend continuously on the solution to the Dirichlet problem for the laplacian.

We have developed a $P1$ finite-element approximation for which this convergence result holds in the 2-dimensional case, working in the geometric setting introduced by V. Šveràk [36], namely domains with an a priori bounded number of holes. According to our results convergence holds in the complementary-Hausdorff topology.

This legitimates the usual engineering approach for computing numerically optimal shapes.

This article has been fully devoted to the Dirichlet problem for the laplacian. But the techniques we have developed could be used to solve similar optimal design problems for Dirichlet problems in $2D$ to many other equations including the elliptic Stokes system, the Lamé system in elasticity, the wave and heat equation, etc.

Our results have come out from a more general framework guaranteeing the convergence of discrete optimal shapes for a class of optimal design problems. The main properties required in this general framework are the continuous dependance of the solution of the PDE with respect to the domain on which it is posed, and the H^c-closedness of the set of admissible continuous domains. From these continuous properties, we have derived associated discrete ones from which we have deduced the results.

Changing the type of discretization is certainly just a technical issue, provided the approximation is conforming.

If the framework of Šverák could be generalized to higher dimension, and other families of admissible continuous shapes, it is likely that our result could follow. Though, such generalizations do not seem easy to obtain, and this has to be investigated.

It is clear that our technique only holds for the Dirichlet problem. How to deal with the Neumann problem is, to our knowledge, completely open.

The obtention of convergence rates would be of first interest. As far as we know, this subject is also completely open. Very likely, it will require some further information on the continuous optimal shapes. The regularity results obtained in [9] could be of some help for doing that. But this issue also remains to be investigated.

Finally, we have mentioned that the computation of discrete optimal shapes is not easy. A lot of work is being done in engineering research. Considering the experiments which can be seen by now, one can expect that this is not too far from reach.

6 Appendix 1

In this section we give some examples related to the properties mentioned in Section 2.1 that fail to hold under the assumption of H^c-convergence. They can be found in [20].

6.1 Example 1

This refers to the property we have mentionned in Remark 2.2, point 1. Consider

$$K_1 = \overline{B(0,1)}, \quad K_2 = \overline{B(0,2)} \setminus B(0, \frac{3}{2}).$$

Then, $d_H(K_1,\ K_2) = \| 0 - x_2 \|$ where x_2 is any point of norm $\frac{3}{2}$. The point 0 is not on the boundary of K_1.

6.2 Example 2

This refers to the properties we have mentionned in Remark 2.2, points 2 and 3.

Consider in one space dimension

$$\Omega_n =]-\frac{1}{n},\ \frac{1}{n}[\ \subset\ ]-1,+1[.$$

One has

$$F_n = [-1,-\frac{1}{n}]\ \cup\ [\frac{1}{n},1] \xrightarrow{H} [-1,+1],$$

so that $\Omega_n \xrightarrow{H^c} \Omega$ with $\Omega = \emptyset$. Though, $\overline{\Omega}_n \xrightarrow{H} \{0\}$.
Moreover, one has $\{0\} \subset \Omega_n$ for all n, though $\{0\}$ is not a subset of Ω.

6.3 Example 3

This refers to the property we have mentionned in Remark 2.2, points 4 and 5.

Assume we are in dimension 2.

We consider the function $\phi :]0,1[\longrightarrow \mathbf{R}$ defined by

$$\phi(x) = \begin{cases} 2x & \forall x \in]0, \frac{1}{2}[\\ 2-2x & \forall x \in]\frac{1}{2}, 1[, \end{cases}$$

and for any $n \in \mathbf{N},\ n \neq 0$

$$\begin{cases} \phi_n(x) = \phi(2^n x) & \forall x \in]0, \frac{1}{2^n}] \\ \phi_n(x) \text{ is } \frac{1}{2^n} \text{ periodic.} \end{cases}$$

Consider $D =]0,1[\times]-1,2[$ and

$$\Omega_n \ = \ \{\ (x,y);\ 0 < x < 1,\ -1 < y < \phi_n(x)\ \}.$$

The sequence $F_n = \overline{D} \setminus \Omega_n$ H-converges to $[0,1]\ \times\ [0,2]$, so that

$$\Omega_n \xrightarrow{H^c}\ \Omega =]0,1[\ \times\ ]-1,0[.$$

Though, denoting by μ the Lebesgue measure in $\mathbf{R}^2$, and P the perimeter, we have

$$\forall n, \quad \mu(\Omega_n) = 1 + \frac{1}{2}, \quad \mu(\Omega) = 1,$$

$$P(\Omega_n) \longrightarrow \infty, \quad P(\Omega) = 4.$$

So μ and P are not continuous with respect to the H^c-topology.

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