Parabolic singular limit of a wave equation with localized interior damping

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1 Introduction

In this paper we will consider the damped wave equation

$$\begin{cases} \epsilon u^\epsilon_{tt} + \chi_\omega u^\epsilon_t - \Delta u^\epsilon + \lambda u^\epsilon = f^\epsilon(t,x) & \text{in } \Omega \times (0,T) \\ u^\epsilon = 0 & \text{on } \Gamma \times (0,T) \\ u^\epsilon(0) = u^\epsilon_0, \quad u^\epsilon_t(0) = v^\epsilon_0 \end{cases} \tag{1.1}$$

where χ_ω denotes the characteristic function of the open set $\omega \subset \Omega$, which is typically, but not necessarily, a neighborhood of the boundary of Ω, Γ. Note that ω defines the set in which the damping term in (1.1) is effective.

Here, we are interested in studying the behavior of solutions of (1.1) as $\epsilon \to 0$, provided the initial data u^ϵ_0, v^ϵ_0 and f^ϵ somehow converge. For this problem the corresponding formal singular perturbation at $\epsilon = 0$ is

$$\begin{cases} \chi_\omega u_t - \Delta u + \lambda u = f(t,x) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0 \end{cases} \tag{1.2}$$

which is formally equivalent to solving

$$\begin{cases} -\Delta u + \lambda u = f(t,x) & \text{in } \Omega \setminus \overline{\omega} \times (0,T) \\ u_t - \Delta u + \lambda u = f(t,x) & \text{in } \omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0 \end{cases} . \tag{1.3}$$

Note that in this formulation boundary condition are missing on $\Gamma_1 = \partial\omega \setminus \Gamma = \partial(\Omega \setminus \overline{\omega}) \setminus \Gamma$, the transmission interface, see Figure 1 below. Since there would be several ways of connecting the solutions of the elliptic and the parabolic equations in (1.3) along that boundary, we will have to find out which one would be the appropriate for the singular perturbation problem that we want to consider in this paper. In this direction we make the following remark. As solutions of (1.1), for smooth enough $f(t,x)$ and initial data will be smooth on Ω, it would be natural then to attach to (1.3) a boundary condition on Γ_1 that ensures maximal smoothness of solutions. This is achieved by imposing the classical transmission conditions on Γ_1, that is, no jump of the u and its normal derivative across Γ_1

$$[u]_{\Gamma_1} = [\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0. \tag{1.4}$$

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These boundary conditions will be shown to be the right ones when passing to the limit in (1.1), as $\epsilon \to 0$.

Note that the singular perturbation problem that we intend to study is in between those one gets when passing to the limit, as $\epsilon \to 0$ and in the homogeneous case $f = 0$, $\lambda = 0$, in the equations

$$\epsilon u_{tt} - \Delta u = 0, \qquad \text{and} \qquad \epsilon u_{tt} + u_t - \Delta u = 0$$

plus boundary conditions on Γ. The limit problems are respectively

$$-\Delta u = 0, \qquad \text{and} \qquad u_t - \Delta u = 0.$$

Therefore the problem we consider makes the transition. Note that an additional difficulty here comes from the missing boundary condition on Γ_1.

Hence, in this paper we study solutions of (1.2) which are smooth across Γ_1 and reveal the parabolic structure of this problem, giving a suitable functional framework for this equation. We also show the smoothing effect of the limit linear semigroup and the decay of solutions. Also, we will show that, when ω is bounded, there exists a natural eigenvalue problem associated to (1.2) that allows one to carry out a Fourier analysis of solutions. All these will be carried out in Section 4 after some technical preparation in Section 3. In order to motivate the techniques and ideas that we use, we have included in Section 2 a formal discussion which sets most of the topics we will discuss rigorously later on. Then, by means of general semigroup techniques, in Section 5 we study the well posedness of (1.1) and prove the existence and regularity of solutions. We will also give some suitable characterization of mild solutions. In Section 6 we will give sufficient conditions on the nonhomogeneous term f and the initial data of (1.1) ensuring that the solution converges, as $\epsilon \to 0$, to a solution of (1.2). For this, assuming Ω is bounded, we first obtain estimates on the solutions which are uniform in ϵ. We then use these bounds combined with compactness arguments to prove that $u^\epsilon(t)$ converges in some sense to $u(t)$. Later on, using different techniques we will show that, under some stronger conditions on the data, the convergence is uniform in time. Finally, in Section 7 we show how the previous analysis can be applied to the case of more general distributed damping terms in (1.1) of the form $\rho(x)\chi_\omega u_t$, where ρ is a given positive bounded function in ω.

Among all results obtained in this paper we would like to emphasize the following ones which are stated here in less generality than in the text. Concerning the parabolic limit problem we will prove, see Proposition 4.4 and Theorem 4.9:

Theorem 1.1
i) Assume first $f = 0$, then for every $v_0 \in L^2(\omega)$ there exists a unique solution of (1.2) that satisfies

$$u \in C((0,\infty), H^2(\Omega)), \quad u_t \in C((0,\infty), L^2(\omega)), \quad u(t) \to v_0 \text{ in } L^2(\omega) \text{ as } t \to 0$$

and satisfies (1.3) and (1.4) on Γ_1, for $t > 0$. In particular

$$u \in C^\infty((\Omega \setminus \Gamma_1) \times (0,\infty)).$$

ii) Assume f is given such that either
a) $f \in W^{1,1}((0,T), L^2(\Omega))$
or
b) $f \in L^2((0,T), L^2(\omega)) = L^2(\omega \times (0,T))$ and $f \in W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$.
Assume also $u_0 \in H^1_0(\Omega)$ satisfies

$$-\Delta u_0 + \lambda u_0 = f(0) \qquad \text{in } \Omega \setminus \overline{\omega}.$$

Then there exists a unique solution of (1.2) that satisfies

$$u \in C([0,T), H^1_0(\Omega)) \cap L^2((0,T), H^2(\Omega)), \quad u_t \in L^2(\omega \times (0,T)), \quad u(0) = u_0$$

and satisfies (1.3) and (1.4) a.e. $t \in (0,T)$.

Concerning passing to the limit in (1.1), as $\epsilon \to 0$, we will prove, see Theorems 6.6, 6.7 and 6.10

Theorem 1.2 *Assume $u_0^\epsilon \in H_0^1(\Omega)$ and $v_0^\epsilon \in L^2(\Omega)$ satisfy*

$$E_\epsilon(u_0^\epsilon, v_0^\epsilon) = \epsilon \|v_0^\epsilon\|_{L^2}^2 + \|\nabla u_0^\epsilon\|_{L^2}^2 + \lambda \|u_0^\epsilon\|_{L^2}^2 \leq M < \infty$$

and $u_0^\epsilon \longrightarrow u_0$ in $H_0^1(\Omega)$ with

$$-\Delta u_0 + \lambda u_0 = f(0) \qquad in\ \Omega \setminus \overline{\omega}$$

and f^ϵ satisfies either one of the following

a)

$$\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0 \quad in\ L^1((0,T), L^2(\Omega)) \qquad or$$

$$\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0 \quad in\ L^1((0,T), L^2(\Omega \setminus \overline{\omega})) \qquad and \qquad f^\epsilon \to f \quad in\ L^2(\omega \times (0,T))$$

b)

$$f^\epsilon \to f \quad in\ L^2(\omega \times (0,T)) \qquad and \qquad f^\epsilon \to f \quad in\ W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$$

Then

$$u^\epsilon \to u \quad in\ L^2((0,T), H_0^1(\Omega)), \quad \sqrt{\epsilon} u_t^\epsilon \to 0 \quad in\ L^2(\Omega \times (0,T)), \quad and \quad u_t^\epsilon \to u_t \quad in\ L^2(\omega \times (0,T))$$

If moreover $u_0^\epsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v_0^\epsilon \in H_0^1(\Omega)$ and

$$\chi_\omega v_0^\epsilon - \Delta u_0^\epsilon + \lambda u_0^\epsilon - f^\epsilon(0) = O(\sqrt{\epsilon}) \qquad in\ L^2(\Omega)$$

and either

a)

$$f^\epsilon = O(\sqrt{\epsilon})\ in\ W^{1,1}((0,T), L^2(\Omega)) \qquad or$$

$$f^\epsilon = O(\sqrt{\epsilon})\ in\ W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega})) \quad and \quad f^\epsilon \to f\ in\ L^2(\omega \times (0,T))\ and\ weakly\ in\ H^1((0,T), L^2(\omega))$$

b)

$$f^\epsilon \to f\ in\ L^2(\omega \times (0,T))\ and\ weakly\ in\ H^1((0,T), L^2(\omega)) \quad and \quad f^\epsilon \to f\ in\ W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))).$$

Then

$$u^\epsilon \to u \quad in\ C([0,T], H_0^1(\Omega)) \cap L^2((0,T), H^2(\Omega)).$$

The techniques and result of this paper are closely related to the ones in [11] where the case of boundary dissipation for a singularly perturbed wave equation was considered. In that case the limit problem was obtained as a parabolic equation on the boundary of Ω. Here the limiting problem is a parabolic equation in ω. We may then say that in both cases the limit problem is a parabolic equation on the support of the damping mechanism.

Note that concerning the asymptotic behavior of solutions, as $t \to \infty$, for $f = 0$, it is known that, to have a uniform decay rate of the energy of the solutions of (1.1), some geometric conditions (the so called "geometric control conditions") on the damping region ω are needed. Namely, every ray of the geometric optics inside Ω must intersect ω in a finite uniform time, see [2]. On the other hand, regardless of the relative geometries of Ω and ω we will show that (1.2) has a parabolic structure and in fact all solutions decay exponentially to zero in several strong norms. This serves as an illustration of the singular character of the limiting process. It would be then interesting to prove that when the geometric control conditions hold there is a uniform exponential decay rate with respect to ϵ. On the other hand, when the geometric control conditions do not hold, it would be also interesting to introduce some splitting of the solutions of the ϵ–systems into the decaying and non–decaying components and to observe how, as $\epsilon \to 0$ the decaying component becomes dominant eventually absorbing the whole solution. This type of analysis has been carried out recently in [9] in the case in which $\omega = \Omega$ and in connection with the null controllability problem with controls on an open subset θ of Ω. It was indeed proved in this paper that: (a) when θ is a neighborhood of the boundary, the controllability property is uniform in ϵ; (b) without any restriction in θ, as $\epsilon \to 0$ the controllable component of the solution grows so as to become the whole solution in the limit. The techniques in [9] might be of some help when addressing the stabilization as well. Due to length considerations this analysis falls out of the scope of this paper and will be pursued somewhere else.

2 A formal discussion on the homogeneous parabolic problem

In this section, in order to motivate our general approach below, we present a preliminary and formal presentation of the ideas that we will employ later on for (1.2). For the sake simplicity, we will consider the homogeneous case, $f = 0$ and $\lambda = 0$. The presence of a nonzero f introduces some additional difficulties that will be solved in later sections.

Assume $\omega \subset \Omega$ is an open set and denote the interface between ω and $\Omega \backslash \overline{\omega}$ by $\Gamma_1 = \partial\omega \backslash \Gamma = \partial(\Omega\backslash\overline{\omega})\backslash\Gamma$. Given u on ω we compute its trace on Γ_1, $\gamma(u)$ and then solve

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \setminus \overline{\omega} \\ v = \gamma(u) & \text{on } \Gamma_1 \end{cases} \tag{2.1}$$

with suitable boundary conditions on Γ and away from Γ_1 such that the problem is well posed. Then we compute $\frac{\partial}{\partial \vec{n}} v$ on Γ_1, where $\vec{n}$ denotes the inward unit normal to $\Omega \setminus \overline{\omega}$ along Γ_1 or, equivalently, the outward unit normal to ω along Γ_1. In this way we have defined the map $\mathcal{L}$

$$H(\omega) \ni u \mapsto \mathcal{L}(u) = \frac{\partial}{\partial \vec{n}} v \in V(\Gamma_1)$$

where $H(\omega)$ and $V(\Gamma_1)$ stand for suitable spaces, to be made precise, of functions defined in ω and on Γ_1 respectively. In fact, the operator $\mathcal{L}$ is well defined if instead of u in ω we are only given its trace on Γ_1. Thus we can view $\mathcal{L}$ as an operator from $H(\Gamma_1)$ into $V(\Gamma_1)$.

Now we analyze the evolution problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{\omega} \times (0, \infty) \\ u_t - \Delta u = 0 & \text{on } \omega \times (0, \infty) \\ u(0) = u_0 \end{cases} \tag{2.2}$$

with suitable boundary conditions on Γ and on Γ_1. Our goal is to make precise in which sense the boundary conditions have to be understood on the interface Γ_1. The highest regularity of the solution is achieved if we impose the two no-jump conditions on u and its normal derivative on Γ_1:

$$[u]_{\Gamma_1} = [\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0 \tag{2.3}$$

where $\vec{n}$ denotes the normal vector to Γ_1 and where by $[\cdot]$ we denote the difference between the values of a function computed from the two different sides of Γ_1.

Observe that, using the operator $\mathcal{L}$ defined above, the no–jump condition on the normal derivative on Γ_1 can be recast as

$$\frac{\partial}{\partial \vec{n}} u = \mathcal{L}(u)$$

where again $\vec{n}$ is the direction of the outward unit normal to ω along Γ_1. Therefore, the evolution problem with the no–jump conditions above, (2.2), (2.3), can be reduced to an evolution problem for functions defined only on ω:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \omega \times (0, \infty) \\ \frac{\partial}{\partial \vec{n}} u = \mathcal{L}(u) & \text{on } \Gamma_1 \times (0, \infty) \\ u(0) = u_0 \end{cases} \tag{2.4}$$

with suitable boundary conditions on Γ and away from Γ_1. Extending u to $\Omega \setminus \overline{\omega}$ as in (2.1) we obtain solutions of (2.2) which are smooth across Γ_1.

Observe that in the one–dimensional case all the computations are explicit. For example consider $\Omega = (0,1)$, $\omega = (0,1/2)$, so $\Gamma_1 = \{1/2\}$, and consider Dirichlet boundary conditions on the boundary of Ω, i.e. $u(0) = u(1) = 0$. Then a simple computation reveals that $\mathcal{L}(u) = \mathcal{L}(u(1/2)) = -2u(1/2)$ and therefore the parabolic problem in ω, (2.4), reads

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,1/2) \times (0,\infty) \\ u_x(1/2,t) + 2u(1/2,t) = 0 & t \in (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(0) = u_0 \end{cases}$$

which is a Dirichlet–Robin problem in $(0,1/2)$.

Similar computations can be carried out in the case of the two dimensional square $\Omega = (0,\pi) \times (0,\pi)$, with obvious extensions to the case of rectangles and higher dimensions. For example consider $\omega = (0,\pi/2) \times (0,\pi)$, so $\Gamma_1 = \{\pi/2\} \times (0,\pi)$ and assume Dirichlet boundary conditions on the boundary of Ω. Therefore we can look for solutions of the parabolic–elliptic problem in Ω as $u(x,y,t) = \sum_{k=1}^{\infty} u_k(x,t) \sin(ky)$ where $z = u_k(x,t)$ solves

$$\begin{cases} z_t - z_{xx} + k^2 z = 0, & 0 < x < \pi/2, \ t > 0 \\ -z_{xx} + k^2 z = 0, & \pi/2 < x < \pi, \ t > 0 \\ z(0,t) = 0 \end{cases}$$

The same arguments as before allow to show that $z = u_k$ may be explicitly computed in $(0,\pi/2)$ obeying a heat equation with mixed boundary conditions

$$\begin{cases} z_t - z_{xx} + k^2 z = 0, & 0 < x < \pi/2, \ t > 0 \\ z_x(\pi/2,t) + \alpha_k z(\pi/2,t) = 0 \\ z(0,t) = 0 \end{cases}$$

where $\alpha_k = k \left(\frac{e^{3k\pi/2} + e^{k\pi/2}}{e^{3k\pi/2} - e^{k\pi/2}} \right) = k \left(\frac{e^{k\pi} + 1}{e^{k\pi} - 1} \right)$.

Also, similar explicit computations may be done using Fourier transform in $\mathbb{R}^N$ when the interface is a hyperplane. In arbitrary geometries and dimensions the evolution problem in ω,(2.4), can be solved by means of Fourier series once the spectrum of the linear operator is known. In this case, and assuming suitable boundary conditions on $\Gamma_0 = \omega \cap \Gamma$, the spectral problem reads

$$\begin{cases} -\Delta u = \mu u & \text{in } \omega \\ \frac{\partial}{\partial \vec{n}} u = \mathcal{L}(u) & \text{on } \Gamma_1 \\ u \text{ satisfies boundary conditions on } \Gamma_0 \end{cases} \tag{2.5}$$

which, by extending u to $\Omega \setminus \overline{\omega}$ as in (2.1), is equivalent to

$$\begin{cases} -\Delta u = \mu u & \text{in } \omega \\ -\Delta u = 0 & \text{in } \Omega \setminus \overline{\omega} \\ u \text{ satisfies boundary conditions on } \Gamma \end{cases} \tag{2.6}$$

and satisfying (2.3).

Finally observe that, if we assume Dirichlet boundary conditions on the boundary of Ω and multiply (2.4) by u and integrate by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\omega u^2 + \int_\omega |\nabla u|^2 - \int_{\Gamma_1} \frac{\partial u}{\partial \vec{n}} u = 0$$

where $\vec{n}$ is the direction of the outward unit normal to ω along Γ_1. But since u has been extended in a harmonic way to $\Omega \setminus \overline{\omega}$, using the definition of $\mathcal{L}$ and manipulating the boundary term above we get

$$\frac{1}{2}\frac{d}{dt}\int_{\omega} u^2 + \int_{\omega} |\nabla u|^2 + \int_{\Omega\setminus\overline{\omega}} |\nabla u|^2 = 0.$$

Therefore the $L^2(\omega)$ norm of the solutions decay exponentially as soon Ω is bounded or if the boundary of ω intersects the boundary of Ω and the Poincaré inequality holds in ω.

This property is in sharp contrast with the behavior of the original hyperbolic problem, (1.2), since, as indicated in the Introduction, in this case it is well know that the exponential decay of energy holds only for suitable subdomains ω of Ω.

3 Notation and preliminary results

We now introduce some notations that will be used throughout the paper. All along the paper and specially in proofs c_i will denote generic positive constants. We will also obtain some technical preliminary results that will be used later on. In particular the operator A_0 constructed below in (3.6) and (3.7) will play a crucial role in the parabolic limit problem in the next section; see Theorem 4.1 below.

Now we make precise the assumptions on $\lambda \geq 0$, Ω and ω. As said above, $\Omega \subset \mathbb{R}^N$ is a regular connected open set with boundary Γ. Note however that we will not always assume that Ω, nor its boundary Γ, are bounded. At the same time $\omega \subset \Omega$ is a smooth domain, which is neither assumed to be bounded, unless explicitly stated. For the sake of simplicity in the exposition we will assume that $\Omega \setminus \overline{\omega}$ is also a smooth open subset of Ω although it will be clear from the proofs that in several of the results this assumption is not really needed. In case Ω and/or ω are unbounded we will assume that they are uniformly regular domains in the sense of [4], so we can use elliptic regularity results for the Dirichlet problem.

For $t \leq \infty$, we denote, $Q_t = \Omega \times (0,t)$ and $q_t = \omega \times (0,t)$, and identify $L^2(Q_t) = L^2((0,t), L^2(\Omega))$ and $L^2(q_t) = L^2((0,t), L^2(\omega))$. When working with a fixed time interval $(0,T)$ we will simply write Q and q instead of Q_T and q_T. Space–time integrals will be denoted by integrals over Q and q.

We will denote $\Gamma_0 = \partial\omega \cap \Gamma$, $\Gamma_1 = \partial\omega \setminus \Gamma = \partial(\Omega \setminus \overline{\omega}) \setminus \Gamma$ and $\Gamma_2 = \partial(\Omega \setminus \overline{\omega}) \cap \Gamma$. In this way we have $\partial\omega = \Gamma_0 \cup \Gamma_1$, $\partial(\Omega \setminus \overline{\omega}) = \Gamma_1 \cup \Gamma_2$ and $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_2$; see Figure 1 below. We will assume that Γ_0, Γ_1 and Γ_2 are union of connected components of the boundaries of Ω, ω and $\Omega \setminus \overline{\omega}$. This nonessential assumption is made in order to avoid regularity problems and to ensure the well posedness of some Dirichlet problems with data on some of these boundaries. Note that when ω is a neighborhood of the boundary, then $\Gamma_0 = \Gamma$ and $\Gamma_2 = \emptyset$ while $\Gamma_0 = \emptyset$ and $\Gamma_2 = \Gamma$, when ω is interior to Ω. Finally, when ω is a neighborhood of a proper connected component of the boundary of Ω then all Γ_0, Γ_1 and Γ_2 are nonempty.

Insert Figure 1 here

Concerning functional spaces, we will use the standard Sobolev spaces $H_0^s(\Omega)$ and $H_{\Gamma_0}^s(\omega)$ for $s \geq 0$, where the subscript Γ_0 means that the traces vanish on that part of the boundary of ω. In case $\Gamma_0 = \emptyset$ we set $H_{\Gamma_0}^s(\omega) = H^s(\omega)$. We will also consider the space $H_0^1(\Omega \setminus \overline{\omega})$ as embedded in $H_0^1(\Omega)$ by identifying an element of the former space with its extension by zero to Ω, as an element of the latter.

Also, we will denote by $H_{\Gamma_0}^{-s}$ the dual space of $H_{\Gamma_0}^s$. Note that this notation introduces some ambiguity when $\Gamma_0 = \emptyset$, since we set $H_{\Gamma_0}^s = H^s$ and then the dual space is denoted H^{-s}, but this symbol is usually reserved to denote the dual space of H_0^s. However, this notation should produce no confusion. When the space H_0^s appears, its dual is denoted by H_0^{-s}. The duality pairing between the spaces above, will be denoted $< \cdot, \cdot >_{-s,s}$ or $< \cdot, \cdot >_{-s,s}^{\omega}$ and $< \cdot, \cdot >_{-s,s}^{\Omega\setminus\omega}$ for functions defined on ω and $\Omega \setminus \overline{\omega}$ respectively. In particular the scalar product in L^2 will be denoted by $< \cdot, \cdot >$. If there is no possible confusion, we will not indicate if the spaces or duality products are referred to functions on Ω, ω or even $\Omega \setminus \overline{\omega}$. When required, we will write $< \cdot, \cdot >_\Omega$, $< \cdot, \cdot >_{\Omega \setminus \overline{\omega}}$ and $< \cdot, \cdot >_\omega$ to differentiate both cases.

Also, for $f \in L^2(\Omega)$ we will denote by f_ω and $f_{\Omega \setminus \overline{\omega}}$ its restriction to ω and $\Omega \setminus \overline{\omega}$ respectively.

Below we shall make use of a special class of elements of $H^{-1}_{\Gamma_0}(\omega)$ that we will denote by $h \stackrel{def}{=} f_\omega + g_{\Gamma_1} \in H^{-1}_{\Gamma_0}(\omega)$, where $f \in L^2(\omega)$ and $g \in H^{-1/2}(\Gamma_1)$, and which are defined by

$$< h, \phi >_{-1,1} = < f, \phi >_\omega + < g, \gamma(\phi) >_{\Gamma_1}$$

for every $\phi \in H^1_{\Gamma_0}(\omega)$, where γ denotes the trace operator.

Finally, we will freely make use of the properties of the normal derivative operator for which the reader is referred to [11], for example. As a general notation when a normal derivative appears on Γ_1 we will always assume that $\vec{n}$ is the outward unit normal to ω along Γ_1.

If $\lambda > 0$ no further assumptions are made on Ω or ω (for example concerning their boundedness) nor on Γ_0 that, as noted above, could be empty. However, if $\lambda = 0$ then we assume that $\Gamma_0 \neq \emptyset$ and moreover, that in $H^1_0(\Omega)$ and $H^1_{\Gamma_0}(\omega)$ the Poincaré inequality holds.

With these notations and since as usual we identify $L^2(\Omega)$ with its dual, we have the embeddings

$$H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}_0(\Omega).$$

Also, under the assumptions above, we define the scalar product in $H^1_0(\Omega)$

$$\int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi \tag{3.1}$$

for every $u, \phi \in H^1_0(\Omega)$, which at the same time allows us to define the canonical isometric isomorphism, L, between $H^1_0(\Omega)$ and its dual, $H^{-1}_0(\Omega)$, such that for every $u, \phi \in H^1_0(\Omega)$

$$< L(u), \phi >_{-1,1} = \int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi. \tag{3.2}$$

Observe that L appears naturally when setting the variational formulation for the Dirichlet problem in Ω

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

which can be read as $L(u) = f$, provided $f \in H^{-1}_0(\Omega)$ and so $u \in H^1_0(\Omega)$. Note that from the regularity assumptions on Ω, if $f \in L^2(\Omega)$ then $u \in H^2(\Omega) \cap H^1_0(\Omega)$.

We also identify $L^2(\omega)$ with its dual and consider the scalar product in $H^1_{\Gamma_0}(\omega)$ and the corresponding isomorphism L_ω between $H^1_{\Gamma_0}(\omega)$ and $H^{-1}_{\Gamma_0}(\omega)$, defined by

$$< L_\omega(u), v >_{-1,1} = \int_\omega \nabla u \nabla v + \lambda \int_\omega uv \tag{3.3}$$

for every $u, v \in H^1_{\Gamma_0}(\omega)$.

We finally make the following definitions, motivated by the scalar product above and the fact that $H^1_0(\Omega \setminus \overline{\omega})$ is a closed subspace of $H^1_0(\Omega)$.

Definition 3.1
i) The set of λ–Harmonic functions on $\Omega \setminus \overline{\omega}$ is the orthogonal set to $H^1_0(\Omega \setminus \overline{\omega})$ in $H^1_0(\Omega)$ with respect to the scalar product (3.1). That is, $u \in Har_\Gamma(\Omega \setminus \overline{\omega}) \stackrel{def}{=} (H^1_0(\Omega \setminus \overline{\omega}))^\perp$ iff $u \in H^1_0(\Omega)$ and

$$\int_{\Omega \setminus \overline{\omega}} \nabla u \nabla \phi + \lambda \int_{\Omega \setminus \overline{\omega}} u\phi = 0 \tag{3.4}$$

for every $\phi \in H_0^1(\Omega \setminus \overline{\omega})$, *i.e.* $L(u)_{|H_0^1(\Omega\setminus\overline{\omega})} = 0$. *In particular* $-\Delta u + \lambda u = 0$ *in the sense of distributions in* $\Omega \setminus \overline{\omega}$ *and* $u = 0$ *on* Γ.
ii) For a given function u *in* $H^1_{\Gamma_0}(\omega)$, *we define the "*λ*–Harmonic lifting" of* u *to* $\Omega \setminus \overline{\omega}$, $v = B(u) \in H^1_{\Gamma_2}(\Omega \setminus \overline{\omega})$, *as the solution of*

$$\begin{cases} -\Delta v + \lambda v = 0 & \text{in } \Omega \setminus \overline{\omega} \\ v = u & \text{on } \Gamma_1 \\ v = 0 & \text{on } \Gamma_2 \end{cases}$$

in the sense that

$$\int_{\Omega\setminus\overline{\omega}} \nabla v \nabla \phi + \lambda \int_{\Omega\setminus\overline{\omega}} v\phi = 0$$

for every $\phi \in H_0^1(\Omega \setminus \overline{\omega})$ *and* v *satisfies the boundary data.*

We also define

$$B_0(u) = \begin{cases} B(u) & \text{in } \Omega \setminus \overline{\omega} \\ u & \text{in } \omega \end{cases} . \tag{3.5}$$

Therefore, $B_0(u) \in Har_\Gamma(\Omega)$ *and* B_0 *defines a linear mapping between* $H^1_{\Gamma_0}(\omega)$ *and* $H_0^1(\Omega)$.
iii) For functions defined on Ω *we define the "restriction" operator to* ω *by* R. *We will also use the following notation for functions defined in* Ω *or* ω

$$\chi_\omega u = \begin{cases} 0 & \text{in } \Omega \setminus \overline{\omega} \\ u & \text{in } \omega \end{cases} .$$

Note that $R(L^2(\Omega)) = L^2(\omega)$ *and* $R(H_0^s(\Omega)) \subset H^s_{\Gamma_0}(\omega)$ *for* $s > 0$ *and that we have in fact an equality since* ω *and* $\Omega \setminus \overline{\omega}$ *are assumed to be smooth enough.*

Now the scalar product above induces the following orthogonal decomposition of $H_0^1(\Omega)$ and $H_0^{-1}(\Omega)$ and the corresponding splitting of the isomorphism L:

Proposition 3.2
i) We have the orthogonal decomposition $H_0^1(\Omega) = Har_\Gamma(\Omega \setminus \overline{\omega}) \oplus H_0^1(\Omega \setminus \overline{\omega})$ *and each* $u \in H_0^1(\Omega)$ *can be split accordingly as* $u = u_1 + u_2$ *where*

$$u_1 = B_0(R(u)) \in Har_\Gamma(\Omega \setminus \omega), \quad u_2 = u - B_0(R(u)) \in H_0^1(\Omega \setminus \overline{\omega})$$

Moreover, $B_0 : H^1_{\Gamma_0}(\omega) \to Har_\Gamma(\Omega \setminus \overline{\omega})$ *is an isomorphism, whose inverse is given by the operator* R.
ii) Acting by restriction, we have the decomposition $H_0^{-1}(\Omega) = H^{-1}_{\Gamma_0}(\omega) \oplus H_0^{-1}(\Omega \setminus \overline{\omega})$ *and therefore every* $h \in H_0^{-1}(\Omega)$ *can be split in a unique way as* $h = h_1 + h_2$ *with* $h_1 \in H^{-1}_{\Gamma_0}(\omega)$ *and* $h_2 \in H_0^{-1}(\Omega \setminus \overline{\omega})$.
iii) The operators

$$L_D = L_{|H_0^1(\Omega\setminus\overline{\omega})} : H_0^1(\Omega \setminus \overline{\omega}) \to H_0^{-1}(\Omega \setminus \overline{\omega})$$

and

$$A_0 = LB_0 : H^1_{\Gamma_0}(\omega) \to H^{-1}_{\Gamma_0}(\omega)$$

are isomorphisms. Therefore, given $h \in H_0^{-1}(\Omega)$, *then* $u \in H_0^1(\Omega)$ *satisfies* $L(u) = h$ *iff* $u = u_1 + u_2 = B_0(R(u)) + u_2$, *as in i), with*

$$R(u) = A_0^{-1}(h_1), \quad u_2 = D_0(h_2)$$

where $D_0 \overset{def}{=} L_D^{-1}$ *and* $h = h_1 + h_2$ *as in ii).*

The isomorphism $A_0 = LB_0 : H^1_{\Gamma_0}(\omega) \to H^{-1}_{\Gamma_0}(\omega)$ *is given by*

$$A_0 = LB_0 = L_\omega - \left(\frac{\partial B}{\partial \vec{n}}\right)_{\Gamma_1} \tag{3.6}$$

in the sense that for every $u, v \in H^1_{\Gamma_0}(\omega)$ one has

$$< A_0(u), v >^{\omega}_{-1,1} = \int_\omega \nabla u \nabla v + \lambda \int_\omega uv - \int_{\Gamma_1} \frac{\partial B(u)}{\partial \vec{n}} B(v) = \int_\Omega \nabla B_0(u) \nabla B_0(v) + \lambda \int_\Omega B_0(u) B_0(v) \quad (3.7)$$

where $\vec{n}$ denotes the outward unit normal to ω along Γ_1 and L_ω has been defined in (3.3).

Proof

i) The direct sum decomposition of $H^1_0(\Omega)$ follows by definition of $Har_\Gamma(\Omega \setminus \overline{\omega})$ and the choice of the scalar product in $H^1_0(\Omega)$. Therefore for every $u \in H^1_0(\Omega)$, $u = u_1 + u_2$ where $u_2 \in H^1_0(\Omega \setminus \overline{\omega})$ and then $R(u) = R(u_1)$, and consequently $u_1 = B_0(R(u))$. Since the projections associated to the orthogonal decomposition are continuous, then $B_0 : H^1_{\Gamma_0}(\omega) \to Har_\Gamma(\Omega \setminus \overline{\omega})$ is an isomorphism, whose inverse is given by the operator R.

ii) The decomposition of $H^{-1}_0(\Omega)$ is obvious.

iii) Since L is an isomorphism between $H^1_0(\Omega)$ and $H^{-1}_0(\Omega)$, using that B_0 is an isomorphism and the direct sum decomposition of $H^1_0(\Omega)$, we get that $A_0 = LB_0$ and L_D are isomorphisms.

Now, given $h \in H^{-1}_0(\Omega)$, $u \in H^1_0(\Omega)$ satisfies $L(u) = h$ iff $L(u_1 + u_2) = h_1 + h_2$, and this is equivalent to $L(u_i) = h_i$, $i = 1, 2$ and therefore $A_0(R(u)) = h_1$ and $L_D(u_2) = h_2$.

Let $v \in H^1_{\Gamma_0}(\omega)$ and $\phi \in H^1_0(\Omega)$. Decomposing $\phi = \phi_1 + \phi_2$ as above, we get

$$< A_0(v), \phi >^{\omega}_{-1,1} = < L(B_0(v)), \phi >^{\Omega}_{-1,1} = < L(B_0(v)), \phi_1 >^{\Omega}_{-1,1} + < L(B_0(v)), \phi_2 >^{\Omega}_{-1,1} .$$

From the definition of L, B_0 and ϕ_2, we get $< L(B_0(v)), \phi_2 >^{\Omega}_{-1,1} = 0$ while

$$< L(B_0(v)), \phi_1 >^{\Omega}_{-1,1} = \int_\omega \nabla v \nabla \phi_1 + \lambda \int_\omega v \phi_1 + \int_{\Omega \setminus \overline{\omega}} \nabla B(v) \nabla B(\phi_1) + \lambda \int_{\Omega \setminus \overline{\omega}} B(v) B(\phi_1)$$

where we have used that $\phi_1 = B_0(\phi_1)$. Therefore, from the definition of B, we get

$$\int_{\Omega \setminus \overline{\omega}} \nabla B(v) \nabla B(\phi_1) + \lambda \int_{\Omega \setminus \overline{\omega}} B(v) B(\phi_1) = -\int_{\Gamma_1} \frac{\partial B(v)}{\partial \vec{n}} B(\phi),$$

since $\phi = \phi_1$ on Γ_1, which proves (3.7).□

Remark 3.3 *Note that the fact that L_D is an isomorphism is equivalent to the well posedness of the Dirichlet problem in $\Omega \setminus \overline{\omega}$*

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \setminus \overline{\omega} \\ u = 0 & \text{on } \partial(\Omega \setminus \overline{\omega}) = \Gamma_1 \cup \Gamma_2 \end{cases}$$

in the sense that $f \in H^{-1}_0(\Omega \setminus \overline{\omega})$, $u \in H^1_0(\Omega \setminus \overline{\omega})$ and for every $\phi \in H^1_0(\Omega \setminus \overline{\omega})$,

$$< L(u), \phi >^{\Omega \setminus \overline{\omega}}_{-1,1} = < f, \phi >^{\Omega \setminus \overline{\omega}}_{-1,1} .$$

Also, from the regularity assumption on $\Omega \setminus \overline{\omega}$ we ge that if $f \in L^2(\Omega \setminus \overline{\omega})$ then $u \in H^2(\Omega \setminus \overline{\omega}) \cap H^1_0(\Omega \setminus \overline{\omega})$

On the other hand, note that in the boundary term in (3.7) one could also write v instead of $B(v)$ since they have the same trace on Γ_1.

The operator A_0 above will be the key in solving, in an appropriate sense, the limiting problem (1.2). To show this we will need the following result which makes precise the decomposition in $H^{-1}_0(\Omega)$ of elements in $L^2(\Omega)$.

Lemma 3.4 *If $h = f \in L^2(\Omega) \subset H_0^{-1}(\Omega)$ then $h = h_1 + h_2$ where*

$$h_1 = f_\omega + (B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1} \in L^2(\omega) + H^{-1/2}(\Gamma_1) \subset H_{\Gamma_0}^{-1}(\omega), \quad h_2 = f_{\Omega\setminus\overline{\omega}} \in L^2(\Omega\setminus\overline{\omega}) \subset H_0^{-1}(\Omega\setminus\overline{\omega})$$

and

$$(B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1} = (\frac{\partial D_0(f)}{\partial \vec{n}})_{\Gamma_1} \in H^{1/2}(\Gamma_1) \subset H_{\Gamma_0}^{-1}(\omega)$$

where $\vec{n}$ denotes the outward unit normal to ω along Γ_1.

Proof If $h = f \in L^2(\Omega)$ and $\phi \in H_0^1(\Omega)$, we decompose $\phi = \phi_1 + \phi_2$ as in Proposition 3.2, $\phi_1 = B_0(R(\phi_1))$, $\phi_2 \in H_0^1(\Omega\setminus\overline{\omega})$ and then

$$< h, \phi >_{-1,1} = < f, \phi_1 >_\omega + < f, B(\phi_1) >_{\Omega\setminus\overline{\omega}} + < f, \phi_2 >_{\Omega\setminus\overline{\omega}} =$$

$$= < f_\omega, \phi_1 >_\omega + < B^* f_{\Omega\setminus\overline{\omega}}, \phi_1 >_{\Gamma_1} + < f, \phi_2 >_{\Omega\setminus\overline{\omega}}$$

and we get the expressions for h_1 and h_2. On the other hand, the expression $B^* f_{\Omega\setminus\overline{\omega}} = (\frac{\partial D_0(f)}{\partial \vec{n}})_{\Gamma_1}$ was obtained in [11], Lemma 1.1.□

Now we are in a position to give a more detailed view of the action of the isomorphism A_0 restricted to a suitable subset of $H_{\Gamma_0}^{-1}(\omega)$.

Proposition 3.5 *Assume $h \stackrel{def}{=} f_\omega + g_{\Gamma_1} \in H_{\Gamma_0}^{-1}(\omega)$ is defined by*

$$< h, \phi >_{-1,1} = < f, \phi >_\omega + < g, \gamma(\phi) >_{\Gamma_1}$$

for every $\phi \in H_{\Gamma_0}^1(\omega)$, where $f \in L^2(\omega)$ and $g \in H^{-1/2}(\Gamma_1)$. Let $v \in H_{\Gamma_0}^1(\omega)$ be the solution of

$$A_0(v) = h = f_\omega + g_{\Gamma_1}. \tag{3.8}$$

Then $v \in Y_0 = \{z \in H_{\Gamma_0}^1(\omega),\ \Delta z \in L^2(\omega)\}$ and it satisfies

$$-\Delta v + \lambda v = f \in L^2(\omega), \quad \text{in } \omega$$

$$\frac{\partial v}{\partial \vec{n}} - \frac{\partial B(v)}{\partial \vec{n}} = g \in H^{-1/2}(\Gamma_1), \quad \text{on } \Gamma_1, \qquad \text{and} \qquad v = 0, \quad \text{on } \Gamma_0.$$

Conversely, for every $v \in Y_0$, $A_0(v) = f_\omega + g_{\Gamma_1}$, where $f_\omega = -\Delta v + \lambda v \in L^2(\omega)$ and $g_{\Gamma_1} = \frac{\partial v}{\partial \vec{n}} - \frac{\partial B(v)}{\partial \vec{n}} \in H^{-1/2}(\Gamma_1)$. That is,

$$A_0 : Y_0 = \{z \in H_{\Gamma_0}^1(\omega),\ \Delta z \in L^2(\omega)\} \longrightarrow L^2(\omega) + H^{-1/2}(\Gamma_1) \subset H_{\Gamma_0}^{-1}(\omega)$$

is an isomorphism, and on this space $A_0 = (-\Delta + \lambda)_\omega + (\frac{\partial}{\partial \vec{n}} - \frac{\partial B(u)}{\partial \vec{n}})_{\Gamma_1}$.

Proof

The result follows from (3.6) and the results in [11] applied to the isomorphism L_ω between $H_{\Gamma_0}^1(\omega)$ and $H_{\Gamma_0}^{-1}(\omega)$.□

Remark 3.6 *Observe that if $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ satisfies*

$$L(u) = f$$

then, from the elliptic regularity for the Dirichlet problem in Ω we get $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

At the same time we have de decomposition $u = B_0(v) + u_2$, with $v = Ru$. From Proposition 3.2 and Lemma 3.4 we have that

$$u_2 = D_0(f) \in H^2(\Omega\setminus\overline{\omega}) \cap H_0^1(\Omega\setminus\overline{\omega}).$$

On the other hand we also have

$$A_0(v) = f_\omega + (\frac{\partial D_0(f)}{\partial \vec{n}})_{\Gamma_1} \in L^2(\omega) + H^{-1/2}(\Gamma_1) \subset H^{-1}_{\Gamma_0}(\omega)$$

and Proposition 3.5 applies for this problem.

In particular observe that, although $u_2 = D_0(f)$ is not smooth across Γ_1, $u = B_0(v) + u_2$ is globally $H^2(\Omega)$ regular.

As a consequence we get the following result

Corollary 3.7 *Assume $f \in L^2(\omega) \subset H^{-1}_{\Gamma_0}(\omega)$ and let $v \in H^1_{\Gamma_0}(\omega)$ be the solution of*

$$A_0(v) = f.$$

Then $u = B_0(v)$ satisfies

$$-\Delta u + \lambda u = f, \quad \text{in } \omega, \qquad \text{and} \qquad -\Delta u + \lambda u = 0, \quad \text{in } \Omega \setminus \overline{\omega}$$

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial B(u)}{\partial \vec{n}}, \quad \text{on } \Gamma_1, \qquad \text{and} \qquad u = 0, \quad \text{on } \Gamma_0 \cup \Gamma_2 = \Gamma$$

and therefore

$$u \in H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega}).$$

In particular u coincides with the solution of

$$\begin{cases} -\Delta u + \lambda u = \chi_\omega f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

given by $L(u) = \chi_\omega f \in L^2(\Omega)$.

Proof The result follows from Proposition 3.5 and the properties of B_0 in Proposition 3.2. The $H^2(\Omega)$ regularity of u follows from $\frac{\partial u}{\partial \vec{n}} = \frac{\partial B(u)}{\partial \vec{n}}$ on Γ_1.□

4 The limit parabolic problem

We are concerned in this section with the well posedness of the parabolic problem (1.2)

$$\begin{cases} \chi_\omega u_t - \Delta u + \lambda u = f(t,x) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0 \end{cases} . \tag{4.1}$$

Our approach is trying to solve an evolution problem in $H^1_{\Gamma_0}(\omega)$ and then use the operator B_0 in (3.5) to extend, for each positive t, the solution in a harmonic way to $\Omega \setminus \overline{\omega}$. The difficulty comes from the fact that a boundary condition is missing on a part of the boundary of ω, namely, on Γ_1. Note that the lack of this boundary condition affects solving both the evolution problem and performing the harmonic extension.

One could think of giving an arbitrary, artificial, time depending, boundary data, $g(t,x)$, on $\Gamma_1 \times (0,T)$, then solve the heat equation on ω with this Dirichlet boundary data on Γ_1 and perform the harmonic extension. In this way one would obtain a solution of (4.1) for each given $g(t,x)$.

Another idea, that we follow here and that we will show to be well suited for the study of the singular limit problem stated in the introduction, is looking for the most regular solution of (4.1). That is, we try to solve the heat equation on ω (lacking the boundary data on Γ_1) and impose the condition that the matching between it and the harmonic extension to $\Omega \setminus \overline{\omega}$ be smooth up to the normal derivative across

Γ_1. In this way we look for solution of (4.3) which are globally $H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega})$. Thus, we look for solutions of (4.1) which are smooth across Γ_1, that is, satisfying the two no–jump conditions

$$[u]_{\Gamma_1} = [\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0 \tag{4.2}$$

where $\vec{n}$ denotes the normal vector to Γ_1, see (2.3).

This problem will be analyzed by means of the operator A_0 introduced in the previous section and making use of semigroup techniques and the variations of constants formula. We will start with the homogeneous case $f = 0$ and then we will make precise all formal considerations in Section 2. Then we will consider the case of nonzero f that introduces some additional difficulties on the smoothness of the solutions across Γ_1.

We first start with the homogeneous case, $f(t, x) = 0$, that is

$$\left\{ \begin{array}{ll} \chi_\omega u_t - \Delta u + \lambda u = 0 & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \Gamma \times (0, \infty) \\ u(0) = u_0 & \end{array} \right. \tag{4.3}$$

with u satisfying (4.2). This problem can be written as

$$\left\{ \begin{array}{ll} -\Delta u + \lambda u = 0 & \text{in } \Omega \setminus \overline{\omega} \times (0, \infty) \\ u_t - \Delta u + \lambda u = 0 & \text{in } \omega \times (0, \infty) \\ u = 0 & \text{on } \Gamma \times (0, \infty) \\ u(0) = u_0 & \end{array} \right.$$

with u smooth across Γ_1. In view of Proposition 3.5 and Corollary 3.7, we notice that if we can solve, for v_0 defined only on ω, the equation

$$v_t + A_0 v = 0, \quad v(0) = v_0 \tag{4.4}$$

in such a way that the solution satisfies $v_t \in L^2(\omega)$ for $t > 0$, then we would get

$$v_t - \Delta v + \lambda v = 0, \quad \text{on } \omega,$$

$$v = 0, \quad \text{on } \Gamma_0 \quad \text{and} \quad \frac{\partial v}{\partial \vec{n}} = \frac{\partial B(v)}{\partial \vec{n}}, \quad \text{on } \Gamma_1.$$

Therefore, $u = B_0(v)$ would be a solution of (4.3), with a smooth matching across Γ_1, i.e. satisfying (4.2), and so

$$u \in H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega}), \quad \text{for } t > 0.$$

Note again that the $H^2(\Omega)$ regularity of u comes from the condition $\frac{\partial v}{\partial \vec{n}} = \frac{\partial B(v)}{\partial \vec{n}}$, on Γ_1, which is equivalent to $[\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0$. In this direction, we have the following result

Theorem 4.1 *The operator A_0 defined by the bilinear form (3.7) in $H^1_{\Gamma_0}(\omega)$, i.e.*

$$a(u, v) = \int_\omega \nabla u \nabla v + \lambda \int_\omega uv - \int_{\Gamma_1} \frac{\partial B(u)}{\partial \vec{n}} B(v) = \int_\Omega \nabla B_0(u) \nabla B_0(v) + \lambda \int_\Omega B_0(u) B_0(v)$$

for every $u, v \in H^1_{\Gamma_0}(\omega)$, induces an unbounded, positive, selfadjoint operator in $H = L^2(\omega)$ that we still denote A_0, which is given by

$$D(A_0) = \{v \in H^2(\omega) \cap H^1_{\Gamma_0}(\omega),\ B_0(v) \in H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega})\} \tag{4.5}$$

$$A_0 v = -\Delta v + \lambda v \text{ on } D(A_0). \tag{4.6}$$

Moreover A_0 has compact resolvent if ω is bounded, *In particular, A_0 is a sectorial operator in H, so $-A_0$ generates an analytic semigroup in H, $\{e^{-A_0 t}\}_t$, and the fractional power spaces $\{X^\alpha\}_\alpha$ are well defined. In particular,*

$$X^1 = D(A_0), \quad X^{1/2} = H^1_{\Gamma_0}(\omega), \quad X^0 = L^2(\omega)$$

and

$$X^m = \{v,\ \Delta^j v \in H^2(\omega) \cap H^1_{\Gamma_0}(\omega),\ B_0(\Delta^j v) \in H^2(\Omega),\ \textit{for } j = 0, \dots, m-1\} \tag{4.7}$$

$$X^{m+1/2} = \{v \in X^m,\ \Delta^m v \in H^1_{\Gamma_0}(\omega)\} \tag{4.8}$$

for $m \in \mathbb{N}$.

Therefore, for every α and $v_0 \in X^\alpha$, there exists a unique solution of

$$v_t + A_0 v = 0, \quad v(0) = v_0$$

in X^α given by $v(t) = e^{-A_0 t} v_0$, that satisfies $v \in C([0,\infty), X^\alpha) \cap C^\omega((0,\infty), X^\beta)$ for every $\beta \geq \alpha$. Since A_0 is strictly positive, $v(t) = e^{-A_0 t} v_0$ decays exponentially to zero in X^β for every β.

Proof From Proposition 3.2 we know $A_0 = LB_0$ is an isomorphism between $V = H^1_{\Gamma_0}(\omega)$ and its dual. If we take the "part of A_0 in H", that is, the operator $\hat{A}$ on H with domain $D(\hat{A}) = \{v \in V,\ A_0 v \in H\}$ and $\hat{A} v = A_0 v$ on $D(\hat{A})$, then $\hat{A}$ is an unbounded, selfadjoint, positive linear operator on H. Since A_0 is an extension of $\hat{A}$ we will keep using A_0 unless some confusion may arise.

Note that when ω is bounded then A_0 has compact resolvent, since in this case the inclusion $H^1_{\Gamma_0}(\omega) \subset L^2(\omega)$ is compact. Also, from Proposition 3.5 and Corollary 3.7 the description of the domain follows.

Since the bilinear form a is coercive we get $\sigma(A_0) \subset \mathbb{R}^+$ and in fact $\inf \sigma(A_0) > \delta > 0$, and the decaying to zero follows, [7]. The description of X^m and $X^{m+1/2}$ follows from induction and the fact that $v \in X^m$ iff $v \in X^{m-1}$ and $A_0 v = -\Delta v + \lambda v \in X^{m-1}$. The rest follows easily.□

Remark 4.2
i) Observe that the intermediate spaces between X^1 and X^0 satisfy, by interpolation, $X^\alpha \subset H^{2\alpha}(\omega)$ for $0 \leq \alpha < 1/2$ and $X^\alpha \subset H^{2\alpha}(\omega) \cap H^1_{\Gamma_0}(\omega)$ for $1/2 \leq \alpha \leq 1$.

ii) Observe that from the Theorem, when ω is bounded then the eigenvalue problem

$$A_0 v = \mu v \tag{4.9}$$

has nontrivial solutions for a discrete sequence of positive semisimple eigenvalues of finite multiplicity, $\{\mu_k\}_{k=1}^\infty \to \infty$ and there exist an orthonormal basis of $L^2(\omega)$ of eigenfunctions $\{v_k\}_{k=1}^\infty$. Therefore $\{u_k = B_0(v_k)\}_{k=1}^\infty$ is an orthogonal basis of $Har_\Gamma(\Omega \setminus \overline{\omega})$ which satisfy

$$\begin{cases} -\Delta u + \lambda u = \mu u & \textit{in } \omega \\ -\Delta u + \lambda u = 0 & \textit{in } \Omega \setminus \overline{\omega} \\ [u]_{\Gamma_1} = [\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0 & \textit{on } \Gamma_1 \\ u = 0 & \textit{on } \Gamma \end{cases} \tag{4.10}$$

and which are in $H^2(\Omega)$. Also, we have the variational min–max characterization of the eigenvalues by means of the Raleigh quotients and in particular

$$\mu_1 = \inf_{v \in H^1_{\Gamma_0}(\omega)} \frac{\int_\Omega |\nabla B_0(v)|^2 + \lambda \int_\Omega |B_0(v)|^2}{\int_\omega |v|^2} > 0.$$

Therefore, solutions of $A_0 v = f$ can be expanded in Fourier series as

$$v = \sum_{k=1}^{\infty} a_k v_k, \quad \textit{with} \quad a_k = \frac{b_k}{\mu_k}$$

where $f = \sum_{k=1}^{\infty} b_k v_k$. *This implies in particular that solutions of* $\left\{ \begin{array}{ll} -\Delta u + \lambda u = \chi_\omega f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{array} \right.$ *can be expanded in Fourier series as*

$$u = \sum_{k=1}^{\infty} a_k u_k = \sum_{k=1}^{\infty} a_k B_0(v_k), \quad \text{with} \quad a_k = \frac{b_k}{\mu_k},$$

see Corollary 3.7.

In view of the above we make the following definition.

Definition 4.3 *The function* u *is a solution of (4.3) iff for* $t > 0$, $u(t) \in H_0^1(\Omega)$, $u(t) = B_0(v(t))$, *where* $v(t) = R(u(t)) \in H^1_{\Gamma_0}(\omega)$ *is a solution of* $v_t + A_0 v = 0$ *and* $v(0) = v_0$ *is given in* ω.

Note that with this definition $u(t) = B_0(v(t))$ automatically satisfies the first no–jump condition in (4.2), namely, that u does not jump across Γ_1. Also, from the remarks before Theorem 4.1 the normal derivative of u does not jump either on Γ_1 if the derivative v_t belongs to $L^2(\omega)$ for $t > 0$. This property will come out from the regularizing effect of the semigroup generated by $-A_0$. Therefore, we can prove the following result on the solutions of (4.3)

Proposition 4.4
i) The function u *is a solution of (4.3) iff for every* $t > 0$, $u(t) \in H_0^1(\Omega)$, $u(t) = B_0(v(t))$, *where* $v(t) = R(u(t)) = e^{-A_0 t} v_0 \in H^1_{\Gamma_0}(\omega)$ *and* v_0 *is given in* ω.
ii) For any α *and* $v_0 \in X^\alpha$, $u(t) = B_0 e^{-A_0 t} v_0$ *is a solution of (4.3) that satisfies (4.2) and*

$$u \in C((0,\infty), H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega})), \quad u_t \in C((0,\infty), L^2(\omega)), \quad Ru(t) \to v_0 \text{ in } X^\alpha \text{ as } t \to 0$$

and

$$\chi_\omega u_t + L(u) = 0, \quad \text{in } H_0^{-1}(\Omega) \quad \text{for } t > 0.$$

In particular, for $t > 0$,

$$-\Delta u + \lambda u = 0, \quad \text{in } \Omega \setminus \overline{\omega}, \qquad u_t - \Delta u + \lambda u = 0, \quad \text{in } \omega$$

$$u = 0, \quad \text{on } \Gamma, \qquad \frac{\partial u}{\partial \vec{n}} = \frac{\partial B(u)}{\partial \vec{n}}, \quad \text{on } \Gamma_1.$$

iii) If $v_0 \in H^1_{\Gamma_0}(\omega)$ *then* $u \in C([0,\infty), H^1_{\Gamma_0}(\Omega))$ *and the one parameter family of linear operators in* $Har_\Gamma(\Omega \setminus \overline{\omega})$ *given by*

$$S_0(t) = B_0 e^{-A_0 t} R$$

defines an analytic semigroup on $Har_\Gamma(\Omega \setminus \overline{\omega})$ *whose infinitesimal generator is given by* $-B_0 A_0 R$. *Moreover, its fractional power spaces are given by* $X_0^\alpha = B_0(X^{\alpha+1/2})$. *In particular,* $X_0^0 = Har_\Gamma(\Omega \setminus \overline{\omega})$ *and* $X_0^{1/2} = H^2(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega})$.
iv) More generally, if $u_0 \in X_0^\alpha$ *then* $u(t) = S_0(t) u_0$ *satisfies (4.3), (4.2) and*

$$u \in C([0,\infty), X_0^\alpha) \cap C^\omega((0,\infty), X_0^\beta)$$

for every $\beta > \alpha$ *and, in particular*

$$u \in C^\infty((\Omega \setminus \Gamma_1) \times (0,\infty)).$$

Finally, solutions of (4.3) satisfy (4.2) and decay exponentially to zero in the norm of X_0^β *for every* β.

Proof The first part follows from Theorem 4.1 and the definition of solution of (4.3). Now, for every α and $v_0 \in X^\alpha$, from the regularizing effect, we know that, in particular, $v(t) = e^{-A_0 t}v_0 \in C((0,\infty), H^1_{\Gamma_0}(\omega))$ and $v_t = (Ru)_t \in C((0,\infty), L^2(\omega))$. Hence from Proposition 3.5 and Corollary 3.7 we conclude the regularity of $u = B_0 v$.

Now, since for $t > 0$ we have $v_t + A_0 v = 0$, we get that $u = B_0(v)$ satisfies $< L(u), \phi >= 0$ for every $\phi \in H^1_0(\Omega \setminus \overline{\omega})$ and, on the other hand for every $\phi \in Har_\Gamma(\Omega \setminus \overline{\omega})$, we have $< L(u), \phi >_{-1,1} =< A_0(Ru), \phi >^\omega_{-1,1} =< -(Ru)_t, \phi >_\omega =< -\chi_\omega u_t, \phi >$. Putting these together we get $\chi_\omega u_t + L(u) = 0$ in $H^{-1}_0(\Omega)$.

The proofs of iii) and iv) are straightforward. Just note that the diagram

$$\begin{array}{ccc} X_0^\alpha & \xrightarrow{S_0} & X_0^\alpha \\ B_0 \uparrow & & \uparrow B_0 \\ X^{\alpha+1/2} & \xrightarrow{e^{-A_0 t}} & X^{\alpha+1/2} \end{array}$$

is commutative and the regularity of the spaces X_0^β, for large β gives the regularity of u away from Γ_1.□

Remark 4.5

i) Observe that by interpolation, for $\alpha \in [0, 1/2]$, we have $X_0^\alpha = B_0(X^{\alpha+1/2}) \subset H^{2\alpha+1}(\Omega) \cap Har_\Gamma(\Omega \setminus \overline{\omega})$.

ii) In case ω is bounded, from the spectral properties of A_0 in Theorem 4.1, we have that solutions of

$$v_t + A_0 v = 0, \quad v(0) = v_0$$

can be expanded in Fourier series as

$$v = \sum_{k=1}^{\infty} a_k e^{-\mu_k t} v_k, \quad \textit{for } t > 0,$$

where $v_0 = \sum_{k=1}^\infty a_k v_k$. This implies in particular that solutions of (4.3) with $u_0 = B_0(v_0)$, or more generally, with $R(u(t)) \to v_0$ as $t \to 0^+$, can be expanded in Fourier series as

$$u = \sum_{k=1}^{\infty} a_k e^{-\mu_k t} u_k = \sum_{k=1}^{\infty} a_k e^{-\mu_k t} B_0(v_k) \quad \textit{for } t > 0$$

and satisfy (4.2). Finally observe that in this case the decay rate of the solutions is given by the first eigenvalue μ_1 of the eigenvalue problem (4.9).

Now for solving the nonhomogeneous problem

$$\begin{cases} \chi_\omega u_t - \Delta u + \lambda u = f(t, x) & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u_0 \end{cases} \tag{4.11}$$

observe that assuming that for each $t > 0$ we have $u(t) \in H^1_0(\Omega)$ and using the decomposition of this space given in Proposition 3.2, one must have

$$u(t) = B_0(R(u(t))) + D_0(f(t)), \quad \text{in } \Omega$$

while the smooth matching condition across Γ_1, (4.2), should now read

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial B(u)}{\partial \vec{n}} + \frac{\partial D_0(f)}{\partial \vec{n}}, \quad \text{on } \Gamma_1 \tag{4.12}$$

where $\vec{n}$ denotes the outward unit normal to ω along Γ_1.

Therefore, in view of the properties of the operator A_0, Proposition 3.5, we are lead to solve an evolution problem of the form

$$\left\{ \begin{array}{ll} u(t) = B_0(v(t)) + D_0(f(t)) & \text{in } \Omega \times (0,T) \\ v_t + A_0 v = f_\omega + (\frac{\partial D_0(f)}{\partial \vec{n}})_{\Gamma_1} & \text{in } \omega \times (0,T) \end{array} \right. \tag{4.13}$$

at least when $f(t) \in L^2(\Omega)$.

Recall that from Lemma 3.4, if $h = f \in L^2(\Omega)$ then its decomposition in $H_0^{-1}(\Omega)$ is given by $h_1 = f_\omega + (B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1}$ and $B^* f_{\Omega\setminus\overline{\omega}} = \frac{\partial D_0(f)}{\partial \vec{n}} \in H^{1/2}(\Gamma_1)$ and $h_2 = f_{\Omega\setminus\overline{\omega}} \in L^2(\Omega \setminus \overline{\omega})$. So, to be more precise, we make the following

Definition 4.6 *Assume $h(t) = h_1(t) + h_2(t) \in H_0^{-1}(\Omega)$ is given a.e. $t \in (0,T)$, with $h_1 \in H_{\Gamma_0}^{-1}(\omega)$ and $h_2 \in H_0^{-1}(\Omega \setminus \overline{\omega})$. Then a solution of (4.11), with h replacing $f(t,x)$ is a function $t \mapsto u(t) \in H_0^1(\Omega)$ such that for $t \in (0,T)$*

$$u(t) = B_0(v(t)) + D_0(h_2(t)) \in H_0^1(\Omega) \tag{4.14}$$

and $v(t) = Ru(t) \in H_{\Gamma_0}^1(\omega)$ satisfies

$$v(t) = e^{-A_0 t} v_0 + \int_0^t e^{-A_0(t-s)} h_1(s)\, ds \tag{4.15}$$

where v_0 is given in ω.

Remark 4.7 *Since h_1 takes values in $H_{\Gamma_0}^{-1}(\omega)$, then for (4.15) to make sense we need the minimal regularity assumptions $v_0 \in H_{\Gamma_0}^{-1}(\omega)$ and $h_1 \in L^1((0,T), H_{\Gamma_0}^{-1}(\omega))$, and in that case (4.15) gives $v \in C([0,T], H_{\Gamma_0}^{-1}(\omega))$. Therefore, to obtain solutions of (4.11) we need to impose conditions on v_0 and h_1 to obtain $v(t) \in H_{\Gamma_0}^1(\omega)$.*

Also, note that with this definition $u(t)$ automatically satisfies the first no–jump condition in (4.2), namely, that u does not jump across Γ_1.

Then, we have

Lemma 4.8 *i) Assume v_0 and $h_1(t)$ are such that for $t \in (0,T)$, $v(t) \in H_{\Gamma_0}^1(\omega)$, $v_t(t) \in H_{\Gamma_0}^{-1}(\omega)$ and*

$$v_t + A_0 v = h_1$$

is satisfied a.e. $t \in (0,T)$ in $H_{\Gamma_0}^{-1}(\omega)$. Then $u(t)$, given by (4.14), satisfies $u(t) \in H_0^1(\Omega)$ and

$$\chi_\omega u_t + L(u) = h, \quad \text{a.e. } t \in (0,T) \tag{4.16}$$

as an equality in $H_0^{-1}(\Omega)$.

Conversely, assume u takes values in $H_0^1(\Omega)$ and satisfies (4.16) with $\chi_\omega u_t \in H_0^{-1}(\Omega)$. Then u is given by (4.14), where $v = Ru$ is given by (4.15) and satisfies $v(t) \in H_{\Gamma_0}^1(\omega)$, $v_t(t) \in H_{\Gamma_0}^{-1}(\omega)$ and $v_t + A_0 v = h_1$ a.e. $t \in (0,T)$ in $H_{\Gamma_0}^{-1}(\omega)$.

ii) Assume u takes values in $H_0^1(\Omega)$, satisfies (4.16), $\chi_\omega u_t \in L^2(\Omega)$ and $h(t) = f(t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$. Then, we have

$$\left\{ \begin{array}{ll} -\Delta u + \lambda u = f(t) & \text{in } L^2(\Omega \setminus \overline{\omega}) \\ u_t - \Delta u + \lambda u = f(t) & \text{in } L^2(\omega) \\ \frac{\partial u}{\partial \vec{n}} = \frac{\partial B(u)}{\partial \vec{n}} + \frac{\partial D_0(f)}{\partial \vec{n}}, & \text{on } \Gamma_1 \\ u = 0, & \text{on } \Gamma \end{array} \right. \tag{4.17}$$

a.e. $t \in (0,T)$. Therefore u satisfies (4.11) and (4.2).

Proof The proof is simple. For the first part, note that from $u(t) = B_0(v(t)) + D_0(h_2(t))$ we get $L(u(t)) = LB_0(v(t)) + h_2 = -v_t + h(t) = -\chi_\omega u_t + h(t)$, since $A_0 = LB_0$.

The second part follows from Proposition 3.5.$\square$

Therefore, to have solutions of (4.11) and (4.2), we have to make sure in (4.14) and (4.15) that $u_t \in L^2(q)$. This property will be obtained from the next two results.

Theorem 4.9 *Assume $f(t) \in L^2(\Omega)$ is given a.e. $t \in (0,T)$ such that either*
i) $f \in W^{1,1}((0,T), L^2(\Omega))$
or
ii) $f \in L^2((0,T), L^2(\omega)) = L^2(q)$ and $f \in W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$.
Assume also $u_0 \in H^1_0(\Omega)$ is given such that u_0 satisfies

$$-\Delta u_0 + \lambda u_0 = f(0) \qquad \text{in } \Omega \setminus \overline{\omega}, \tag{4.18}$$

i.e. $u_0 = B_0(Ru_0) + D_0(f(0))$.
Then u given by (4.14) and (4.15), with $v_0 = R(u_0)$ satisfies

$$u \in C([0,T), H^1_0(\Omega)), \quad u(0) = u_0, \quad u_t \in L^2(q), \quad \chi_\omega u_t + L(u) = f(t)$$

as an equality in $H^{-1}_0(\Omega)$, a.e. $t \in (0,T)$. In particular $u \in L^2((0,T), H^2(\Omega) \cap H^1_0(\Omega))$ and satisfies (4.11) and (4.2) a.e. $t \in (0,T)$.

Proof Recall again that from Lemma 3.4, since $f(t) \in L^2(\Omega)$ a.e. $t \in (0,T)$, then $h_1 = f_\omega + (B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1}$ and $B^* f_{\Omega\setminus\overline{\omega}} = \frac{\partial D_0(f)}{\partial \vec{n}} \in H^{1/2}(\Gamma_1)$ and $h_2 = f_{\Omega\setminus\overline{\omega}} \in L^2(\Omega \setminus \overline{\omega})$.

Now it is clear that in either case i) or ii) we have $f \in C([0,T], L^2(\Omega \setminus \overline{\omega}))$ and therefore, in (4.14) we get $D_0(f) \in C([0,T], H^2(\Omega \setminus \overline{\omega}) \cap H^1_0(\Omega \setminus \overline{\omega})) \subset C([0,T], H^1_0(\Omega))$.

Also, in either case we have $f \in W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$ which now gives $B^* f_{\Omega\setminus\overline{\omega}} = \frac{\partial D_0(f)}{\partial \vec{n}} \in W^{1,1}((0,T), H^{1/2}(\Gamma_1)) \subset W^{1,1}((0,T), H^{-1}_{\Gamma_0}(\omega))$.

Now we split the integral term in (4.15) into two terms, one with f_ω and the other with $(B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1}$. Since we have $B^* f_{\Omega\setminus\overline{\omega}} = \frac{\partial D_0(f)}{\partial \vec{n}} \in W^{1,1}((0,T), H^{1/2}(\Gamma_1)) \subset L^\infty((0,T), X^{-\beta}))$ for all $\beta \in (1/4, 1/2]$, where X^β denotes the fractional power space of A_0, then Proposition 5.5 in [11] gives that the latter term is in $C([0,T], H^1_{\Gamma_0}(\omega))$. We will postpone until the next Proposition proving that this term has a derivative in $L^2((0,T), L^2(\omega)) = L^2(q)$.

For the integral term in (4.15) containing f_ω we have the following. In either case i) or ii), $f \in L^2((0,T), L^2(\omega))$ and again Proposition 5.5 in [11] gives that this term is in $C([0,T], H^1_{\Gamma_0}(\omega)) \cap L^2((0,T), D(A_0))$ and has a derivative in $L^2((0,T), L^2(\omega)) = L^2(q)$.

Finally the term in (4.15) containing $v_0 = Ru_0$ verifies all the above, due to the smoothing effect of the semigroup proved before.

Putting all these together and using Lemma 4.8 we get the result.

Finally since $L(u) = -\chi_\omega u_t + f(t) \in L^2((0,T), L^2(\Omega))$, from the regularity for the Dirichlet problem in Ω, we get $u \in L^2((0,T), H^2(\Omega) \cap H^1_0(\Omega))$.$\square$

In the following proposition we will obtain some energy estimates on the solutions of (4.11) that in particular will complete the proof of Theorem 4.9. This energy estimates will also be needed when studying the convergence of solutions of (1.1) to solutions of (1.2) as $\epsilon \to 0$.

Proposition 4.10 *Assume, as above, that $u_0 \in H^1_0(\Omega)$ satisfying (4.18) and $f(t) \in L^2(\Omega)$ a.e. $t \in (0,T)$, are given.*
i) If $f \in W^{1,1}((0,T), L^2(\Omega))$, then

$$\begin{aligned} \|\nabla u(t)\|^2_{L^2(\Omega)} &+ \lambda \|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \int_\omega u_t^2 = \|\nabla u_0\|^2_{L^2(\Omega)} + \lambda \|u_0\|^2_{L^2(\Omega)} + \\ &+ 2\left(\int_\Omega f(t) u(t) - \int_\Omega f(0) u_0 - \int_0^t \int_\Omega f_t u \right). \end{aligned} \tag{4.19}$$

Therefore, the mapping $(u_0, f) \longmapsto (u, u_t)$ *is Lipschitz from* $H_0^1(\Omega)\times W^{1,1}((0,T),L^2(\Omega))$ *into* $C([0,T],H_0^1(\Omega))\times L^2(q)$.
ii) If $f \in L^2(q)$ *and* $f \in W^{1,1}((0,T), L^2(\Omega\setminus\overline{\omega}))$, *then*

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t\int_\omega u_t^2 = \|\nabla u_0\|^2_{L^2(\Omega)} + \lambda\|u_0\|^2_{L^2(\Omega)} + 2\left(\int_0^t\int_\omega f u_t + \int_{\Omega\setminus\overline{\omega}} f(t)u(t) - \int_{\Omega\setminus\overline{\omega}} f(0)u_0 - \int_0^t\int_{\Omega\setminus\overline{\omega}} f_t u\right). \quad (4.20)$$

Therefore, the mapping $(u_0, f_\omega, f_{\Omega\setminus\overline{\omega}}) \longmapsto (u, u_t)$ *is Lipschitz from* $H_0^1(\Omega)\times L^2(q)\times W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$ *into* $C([0,T],H_0^1(\Omega))\times L^2(q)$.

Proof It is enough to prove the result for very smooth f and initial data such that we have $\chi_\omega u_t + L(u) = f$ and $u_t = B_0(v_t) + D_0(f_t) \in H_0^1(\Omega)$ for $t > 0$. To see that this construction is possible, observe that if $f \in C^1([0,T],L^2(\Omega))$ then the term $D_0(f_t)$ is as regular as needed. On the other hand note that

$$v_t(t) = e^{-A_0 t}v_t(0) + \int_0^t e^{-A_0(t-s)}(f_\omega + (\frac{\partial D_0(f)}{\partial\vec{n}})_{\Gamma_1})_t\, ds$$

with $v_t(0) = -A_0v_0 + f_\omega(0) + (\frac{\partial D_0(f)}{\partial\vec{n}})_{\Gamma_1}(0)$. Therefore, we have $(f_\omega)_t \in L^\infty((0,T),L^2(\omega))$ and $((\frac{\partial D_0(f)}{\partial\vec{n}})_{\Gamma_1})_t \in L^\infty((0,T),X^{-\beta}))$ for all $\beta \in (1/4, 1/2]$, where X^β denotes the fractional power space of A_0, and assuming v_0 is smooth enough such that $v_t(0) \in H^1_{\Gamma_0}(\omega)$, then Proposition 5.5 in [11] concludes that $v_t \in C([0,T],H^1_{\Gamma_0}(\omega))$ which gives the desired regularity for u_t.

Therefore, using u_t as a test function in the equation for u, we get

$$\int_\omega u_t^2 + \frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)}\right) = \int_\Omega f u_t.$$

Then (4.19) and (4.20) follow by integrating in time and integrating by parts in the term $\int_\Omega f u_t$.

In either case we claim that the Lipschitzness of the mapping $(u_0, f) \mapsto (u, u_t)$ follows.

Assumed this for a moment, we can pass simultaneously to the limit in (4.14), (4.15), (4.19) and (4.20) as sequences of smooth functions u_0^n and f^n approach, respectively, u_0 and f as in the statement, and we get the result. To see this, note that in case i) we take $f^n \in C^1([0,T],L^2(\Omega))$ converging to f in $W^{1,1}([0,T],L^2(\Omega))$ and in case ii) we take $f^n \in C^1([0,T],L^2(\Omega))$, such that $f^n(0) = 0$ in ω and converging to f in $L^2(q)$ and in $W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$.

With this, it is enough to define u_0^n as satisfying $u_0^n = B_0(v_0^n) + D_0(f^n(0))$ were v_0^n satisfies $-A_0v_0^n + f_\omega^n(0) + (\frac{\partial D_0(f^n)}{\partial\vec{n}})_{\Gamma_1}(0) = w^n \in H^1_{\Gamma_0}(\omega)$ such that w^n converges in $H^{-1}_{\Gamma_0}(\omega)$ to $-A_0v_0 + f_\omega(0) + (\frac{\partial D_0(f)}{\partial\vec{n}})_{\Gamma_1}(0)$ in case i) or to $-A_0v_0 + (\frac{\partial D_0(f)}{\partial\vec{n}})_{\Gamma_1}(0)$ in case ii). In either case we get that v_0^n converges in $H^1_{\Gamma_0}(\omega)$ to v_0.

Now we prove our claim. In case i), denoting $E_0 = \|\nabla u_0\|^2_{L^2(\Omega)} + \lambda\|u_0\|^2_{L^2(\Omega)} + 2\|f(0)\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)}$ and $K(f) = 2\|f\|_{L^\infty(L^2(\Omega))} + 2\|f_t\|_{L^1(L^2(\Omega))}$ and $z(t) = \sup_{0\le s\le t}\|u(s)\|_{L^2(\Omega)}$, we get, from (4.19),

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + 2\int_{q_t} u_t^2 \le E_0 + K(f)z(t).$$

From here we get, $Cz^2(s) \le E_0 + K(f)z(t)$ for every $0 \le s \le t$ and some constant $C = C(\Omega,\lambda) > 0$. Consequently, using Young's inequality, we get $Cz^2(t) \le 2E_0 + \frac{1}{C}K^2(f)$. Again Young's inequality, gives $\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + 2\int_{q_t} u_t^2 \le E_0 + \frac{1}{2C}K^2(f) + \frac{C}{2}z^2(t)$ and using the bound for $z(t)$ we get

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + 2\int_{q_t} u_t^2 \le 2E_0 + \frac{1}{C}K^2(f)$$

which proves the claim.

For case ii), from (4.20) and using Young's inequality, now we get

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + \int_{q_t} u_t^2 \leq E_0 + K(f)z(t)$$

with $E_0 = \|\nabla u_0\|^2_{L^2(\Omega)} + \lambda\|u_0\|^2_{L^2(\Omega)} + 2\|f\|^2_{L^2(q)} + \|f(0)\|_{L^2(\Omega\setminus\overline{\omega})}\|u_0\|_{L^2(\Omega\setminus\overline{\omega})}$ and $K(f) = 2\|f\|_{L^\infty(L^2(\Omega\setminus\overline{\omega}))} + 2\|f_t\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}$ and $z(t) = \sup_{0\leq s\leq t}\|u(s)\|_{L^2(\Omega\setminus\overline{\omega})}$.

As before, from here we get $Cz^2(s) \leq E_0 + K(f)z(t)$ for every $0 \leq s \leq t$ and some constant $C = C(\Omega,\lambda) > 0$. Proceeding as before, we get

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \lambda\|u(t)\|^2_{L^2(\Omega)} + \int_{q_t} u_t^2 \leq 2E_0 + \frac{1}{C}K^2(f)$$

which proves the claim.□

5 The damped hyperbolic problem

We are now concerned about the solutions of (1.1), i.e.

$$\begin{cases} \epsilon u_{tt} + \chi_\omega u_t - \Delta u + \lambda u = f(t,x) & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \Gamma\times(0,T) \\ u(0) = u_0, \quad u_t(0) = v_0 \end{cases} \tag{5.1}$$

for $0 < \epsilon \leq \epsilon_0$.

For the well posedness of this problem, we will use some general semigroup techniques and prove the existence and regularity of solutions. We will also give some suitable characterization of mild solutions. Later, in Section 6, we will obtain suitable energy estimates on solutions of (5.1).

We consider the operator

$$A_\epsilon = \begin{pmatrix} 0 & -I \\ \frac{1}{\epsilon}(-\Delta+\lambda) & \frac{1}{\epsilon}\chi_\omega \end{pmatrix}$$

where $\epsilon_0 > \epsilon > 0$, acting on the space $E = H^1_0(\Omega)\times L^2(\Omega)$ and domain given by

$$D(A_\epsilon) = (H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega).$$

With these notations (5.1) can be written as

$$U_t + A_\epsilon U = F(t) = \begin{pmatrix} 0 \\ \frac{1}{\epsilon}f(t) \end{pmatrix}, \qquad U(0) = U_0 = (u_0, v_0)^T \tag{5.2}$$

with $U = (u,u_t)^T$. Now by applying general abstract results on C_0 semigroups, see Proposition 5.1 in [11] and [8, 10, 5], we get

Theorem 5.1 *The operator $-A_\epsilon$ generates a C_0 semigroup in E, denoted $S_\epsilon(t)$, and there exists a scalar product in E such that $S_\epsilon(t)$ is a C_0 semigroup of contractions. Therefore, if $U_0 \in E$, then $U(t) = S_\epsilon(t)U_0 = (u(t),v(t))^T$ is a "mild solution" of (5.2), with $f = 0$, and satisfies $U \in C([0,\infty),E)$ or equivalently*

$$u \in C([0,\infty),H^1_0(\Omega)), \quad v \in C([0,\infty),L^2(\Omega)).$$

If moreover, $U_0 \in D(A_\epsilon)$, then $U(t) = S_\epsilon(t)U_0 = (u(t),v(t))^T$ is a "strict solution" of (5.2), with $f = 0$, and satisfies $U \in C([0,\infty),D(A_\epsilon))\cap C^1([0,\infty),E)$ and satisfies (5.2), with $f = 0$, pointwise. Therefore $v(t) = u_t(t)$ and u is a solution of (5.1) such that

$$u \in C([0,\infty),H^2(\Omega)\cap H^1_0(\Omega)), \quad u_t \in C([0,\infty),H^1_0(\Omega)), \quad u_{tt} \in C([0,\infty),L^2(\Omega)).$$

On the other hand for the case $f \neq 0$, we have

Theorem 5.2 *Assume $f : [0,T] \to L^2(\Omega)$. Then*
i) If $f \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0)^T \in E = H_0^1(\Omega) \times L^2(\Omega)$ there exist a unique "mild solution", $U(t) = (u,v)^T$, of (5.1) verifying $U(0) = U_0$, which is given by the variation of constants formula

$$U(t) = S_\epsilon(t)U_0 + \int_0^t S_\epsilon(t-s)F(s)\,ds. \tag{5.3}$$

In this case, $U \in C([0,T], E)$, or equivalently

$$u \in C([0,T], H_0^1(\Omega)), \qquad v \in C([0,T], L^2(\Omega)).$$

Moreover the mapping $(U_0, f) \mapsto U$ is Lipschitz between $E \times L^1((0,T), L^2(\Omega))$ and $C([0,T], E)$.
ii) If $f \in W^{1,1}([0,T], L^2(\Omega))$ or $f \in C([0,T], H_0^1(\Omega))$ and $U_0 \in D(A)$, then the mild solution is a "strict solution", that is, $U \in C([0,T], D(A)) \cap C^1([0,T], E)$ and satisfies (5.2) pointwise. Therefore $v(t) = u_t(t)$ and u is a solution of (5.1) such that

$$u \in C([0,T], H^2(\Omega) \cap H_0^1(\Omega)), \quad u_t \in C([0,T], H_0^1(\Omega)), \quad u_{tt} \in C([0,T], L^2(\Omega)).$$

Moreover, in the first case for f, $U_t = (u_t, u_{tt})^T$, with $\epsilon u_{tt}(0) = -\chi_\omega v_0 + \Delta u_0 - \lambda u_0 + f(0) \in L^2(\Omega)$, is a mild solution of (5.1) in E, with right hand side f_t.

Proof The result for $f \in L^1((0,T), L^2(\Omega))$, $f \in C^1([0,T], L^2(\Omega))$ or $f \in C([0,T], H_0^1(\Omega))$ follows from the references above. The case $f \in C^1([0,T], L^2(\Omega))$ is obtained by density.□

By using the results in [1], we can give a characterization of mild solutions of (5.1). See also Proposition 5.2 in [11]. These result will be used ins Section 6 when proving the convergence of solutions.

Proposition 5.3 *Assume $f \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0) \in E$, and consider $U = (u,v)^T$ be the mild solution of (5.1) given by (5.3), with $F = (0, \frac{1}{\epsilon} f)^T$. Then, U is characterized by: $U \in C([0,T], E)$*

$$v = u_t$$

as a weak derivative in $L^2(\Omega)$, that is, for every $\psi \in L^2(\Omega)$, $\frac{d}{dt} < u, \psi > = < v, \psi >$ and for every $\phi \in H_0^1(\Omega)$, $< \epsilon u_t, \phi >$ is absolutely continuous and

$$\frac{d}{dt}(< \epsilon u_t, \phi >) + < \chi_\omega u_t, \phi > + < \nabla u, \nabla \phi > + \lambda < u, \phi > = < f, \phi > \tag{5.4}$$

a.e. $t \in (0,T)$. In particular

$$(\epsilon u_t)_t + \chi_\omega u_t + L(u) = f \text{ in } H_0^{-1}(\Omega) \text{ a.e. } t \in (0,T) \tag{5.5}$$

where L was defined in (3.2).

Proof From the result in [1], we get that mild solutions of (5.1), given by (5.3) are characterized by $U \in C([0,T], E)$ and for every $W \in D(A^*) \subset E'$, $< U(t), W >_{E,E'}$ is absolutely continuous and

$$\frac{d}{dt} < U, W >_{E,E'} + < U, A^* W >_{E,E'} = < F, W >_{E,E'} .$$

Note that in this case $E' = L^2(\Omega) \times H_0^{-1}(\Omega)$ and we claim that $D(A^*) = H_0^1(\Omega) \times L^2(\Omega) = E \subset E'$ and $A^* = \begin{pmatrix} \frac{1}{\epsilon}\chi_\omega & -I \\ \frac{1}{\epsilon} L & 0 \end{pmatrix}$ on $D(A^*)$. Assumed this for a moment and taking $W = (\phi, \psi) \in D(A^*)$, we get

$$\frac{d}{dt}(< \epsilon u, \psi >_{-1,1} + < \epsilon v, \phi >) + < \nabla u, \nabla \phi > + \lambda < u, \phi > + < v, \chi_\omega \phi - \epsilon \psi > = < f, \phi > .$$

First, taking $\phi = 0$ and $\psi \in L^2(\Omega)$, we get $\frac{d}{dt} < u, \psi >=< v, \psi >$ and hence $v = u_t$ is the weak derivative in $L^2(\Omega)$. Now we have $< v, \chi_\omega \phi - \epsilon\psi >=< \chi_\omega u_t, \phi > -\frac{d}{dt} < \epsilon u, \psi >=< \chi_\omega u_t, \phi > -\frac{d}{dt} < \epsilon u, \psi >_{-1,1}$ and plugging this into the previous expression, we get (5.4) and (5.5).

Therefore, it only remains to prove the claim about A^*. If $U = (u, v) \in D(A)$ and $W = (\phi, \psi) \in E'$, then

$$< AU, W >_{E,E'}= \frac{1}{\epsilon} < -\Delta u + \lambda u, \phi > + \frac{1}{\epsilon} < \chi_\omega v, \phi > - < \psi, v >_{-1,1}$$

and since $< \chi_\omega v, \phi >=< v, \chi_\omega \phi >$, for this to be continuous in U for the topology of E, one needs $\chi_\omega \phi - \psi \in L^2(\Omega)$, that is $\psi \in L^2(\Omega)$, and $\phi \in H^1_0(\Omega)$. In that case

$$< AU, W >_{E,E'}= \frac{1}{\epsilon} < \nabla u, \nabla \phi > + \frac{\lambda}{\epsilon} < u, \phi > + \frac{1}{\epsilon} < v, \chi_\omega \phi > - < \psi, v >_{-1,1}=$$

$$=< \frac{1}{\epsilon} L(\phi), u >_{-1,1} + < \frac{1}{\epsilon}\chi_\omega \phi - \psi, v > .$$

Therefore, $< AU, W >_{E,E'}=< U, \begin{pmatrix} \frac{1}{\epsilon}\chi_\omega \phi - \psi \\ \frac{1}{\epsilon} L(\phi) \end{pmatrix} >_{E,E'}$ and the claim is proved.□

Remark 5.4 *Note that under the assumption of point ii) of Theorem 5.2 one has that u_{tt} is defined in $L^2(\Omega)$. Therefore, in (5.5), one can write $(u_t)_t = u_{tt}$*

6 Convergence of solutions

In this section we show that solutions of

$$\begin{cases} \epsilon u^\epsilon_{tt} + \chi_\omega u^\epsilon_t - \Delta u^\epsilon + \lambda u^\epsilon = f^\epsilon(t) & \text{in } \Omega \times (0, T) \\ u^\epsilon = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u^\epsilon_0, \quad u_t(0) = v^\epsilon_0 \end{cases} \tag{6.1}$$

approach solutions of

$$\begin{cases} \chi_\omega u_t - \Delta u + \lambda u = f(t) & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u_0 \end{cases} \tag{6.2}$$

as ϵ goes to zero, assumed f^ϵ and u^ϵ_0 converge respectively to f and u_0, in some sense, and v^ϵ_0 is not too large. For this, we first obtain estimates on the solutions which are uniform in ϵ. We then use these bounds combined with compactness arguments to prove that $u^\epsilon(t)$ converges in some sense to $u(t)$. Later on, using different techniques we will show that, under some stronger conditions on the data, the convergence is uniform in time.

Therefore, from now on we will assume that Ω is bounded.

6.1 Uniform Energy Estimates

Now we fix our attention on some energy estimates on solutions of the above problem.

We define the energy

$$E_\epsilon(u, v) = \epsilon\|v\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \lambda\|u\|^2_{L^2}$$

and observe that $E^{1/2}_\epsilon$ is an equivalent norm to that of the ambient space $H^1_0(\Omega) \times L^2(\Omega)$, hereafter denoted "energy space". Moreover, when $f = 0$, the semigroup generated by $-A$, $S_\epsilon(t)$, is a contraction semigroup for this norm, so we have

$$E_\epsilon(S_\epsilon(t)(u_0, v_0)) \le E_\epsilon(u_0, v_0).$$

More generally, assume f takes values in $L^2(\Omega)$, then we formally have, by multiplying the equation by u_t, integrating by parts in Ω, using the boundary conditions, and then integrating in time

$$E_\epsilon(u,u_t)+2\int_0^t\int_\omega u_t^2=E_\epsilon(u_0,v_0)+2\int_0^t\int_\Omega fu_t. \tag{6.3}$$

Observe that this equation holds, provided solutions of (5.1) are sufficiently smooth, so one can perform all the computations leading to (6.3). Then, we have the following

Proposition 6.1 *Assume $f\in L^1((0,T),L^2(\Omega))$, $T\le\infty$ and $(u_0,v_0)\in E$. Then the following estimates hold for the mild solutions of (5.1).*
i) Assume first $f=0$. Then

$$u_t\in L^2(\omega\times(0,\infty))$$

and, for any $t>0$, (6.3) holds true, that is,

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2=E_\epsilon(u_0,v_0).$$

Therefore the mapping $(u_0,v_0)\longmapsto((u,u_t),u_t)$ is Lipschitz from E into $C_b([0,\infty),E)\times L^2(\omega\times(0,\infty))$.
ii) Assume $f\ne 0$ and $T\le\infty$. Then

$$u_t\in L^2(\omega\times(0,T))=L^2(q)$$

and, for any $0<t<T$, (6.3) holds true and moreover

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2\le 2E_\epsilon(u_0,v_0)+\frac{4}{\epsilon}\|f\|^2_{L^1(L^2(\Omega))}. \tag{6.4}$$

Therefore, the mapping $(u_0,v_0,f)\longmapsto((u,u_t),u_t)$ is Lipschitz from $E\times L^1((0,T),L^2(\Omega))$ into $C([0,T],E)\times L^2(q)$.
iii) If $f\in L^2(\omega\times(0,T))=L^2(q)$, then for any $0<t<T$

$$E_\epsilon(u(t),u_t(t))+\int_0^t\int_\omega u_t^2\le 2E_\epsilon(u_0,v_0)+2\|f\|^2_{L^2(q)}+\frac{4}{\epsilon}\|f\|^2_{L^1(L^2(\Omega\setminus\overline{\omega}))}. \tag{6.5}$$

Therefore, the mapping $(u_0,v_0,f_\omega,f_{\Omega\setminus\overline{\omega}})\longmapsto((u,u_t),u_t)$ is Lipschitz from $E\times L^2(q)\times L^1((0,T),L^2(\Omega\setminus\overline{\omega}))$ into $C([0,T],E)\times L^2(q)$.
iv) If $f_t\in L^1((0,T),L^2(\Omega))$, then

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2=E_\epsilon(u_0,v_0)+2\left(\int_\Omega f(t)u(t)-\int_\Omega f(0)u_0-\int_0^t\int_\Omega f_tu\right) \tag{6.6}$$

and

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2\le 2E_\epsilon(u_0,v_0)+4\|f(0)\|\|u_0\|+\frac{4}{C(\Omega,\lambda)}\left(\|f\|_{L^\infty(L^2(\Omega))}+\|f_t\|_{L^1(L^2(\Omega))}\right)^2 \tag{6.7}$$

for some constant $C(\Omega,\lambda)>0$.
Therefore, the mapping $(u_0,v_0,f)\longmapsto((u,u_t),u_t)$ is Lipschitz from $E\times W^{1,1}((0,T),L^2(\Omega))$ into $C([0,T],E)\times L^2(q)$.
v) If $f\in L^2(q)$ and $f_t\in L^1((0,T),L^2(\Omega\setminus\overline{\omega}))$, then

$$\begin{aligned}E_\epsilon(u(t),u_t(t)) \quad + \quad 2\int_0^t\int_\omega u_t^2 &= E_\epsilon(u_0,v_0)+2\Bigg(\int_0^t\int_\omega fu_t+\int_{\Omega\setminus\overline{\omega}}f(t)u(t)\\ &\quad -\int_{\Omega\setminus\overline{\omega}}f(0)u_0-\int_0^t\int_{\Omega\setminus\overline{\omega}}f_tu\Bigg)\end{aligned} \tag{6.8}$$

and

$$E_\epsilon(u(t),u_t(t))+\int_0^t\int_\omega u_t^2\le 2E_\epsilon(u_0,v_0)+2\|f\|^2_{L^2(q)}+4\|f(0)\|_{L^2(\Omega\setminus\overline{\omega})}\|u_0\|_{L^2(\Omega\setminus\overline{\omega})}+$$
$$\frac{4}{C(\Omega,\lambda)}\left(\|f\|_{L^\infty(L^2(\Omega\setminus\overline{\omega}))}+\|f_t\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}\right)^2 \tag{6.9}$$

for some constant $C(\Omega,\lambda)>0$.

Therefore, the mapping $(u_0,v_0,f_\omega,f_{\Omega\setminus\overline{\omega}})\longmapsto((u,u_t),u_t)$ *is Lipschitz from* $E\times L^2(q)\times W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$ *into* $C([0,T],E)\times L^2(q)$.
vi) If $f_t\in L^1((0,T),L^2(\omega))$ *then*

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2=E_\epsilon(u_0,v_0)+2\int_0^t\int_{\Omega\setminus\overline{\omega}}fu_t+$$
$$+2\left(\int_\omega f(t)u(t)-\int_\omega f(0)u_0-\int_0^t\int_\omega f_tu\right) \tag{6.10}$$

and

$$E_\epsilon(u(t),u_t(t))+2\int_0^t\int_\omega u_t^2\le 2E_\epsilon(u_0,v_0)+4\|f(0)\|_{L^2(\omega)}\|u_0\|_{L^2(\omega)}+$$
$$\frac{4}{C(\Omega,\lambda)}\left(\|f\|_{L^\infty(L^2(\omega))}+\|f_t\|_{L^1(L^2(\omega))}\right)^2+\frac{4}{\epsilon}\|f\|^2_{L^1(L^2(\Omega\setminus\overline{\omega}))} \tag{6.11}$$

for some constant $C(\Omega,\lambda)>0$.

Therefore, the mapping $(u_0,v_0,f_\omega,f_{\Omega\setminus\overline{\omega}})\longmapsto((u,u_t),u_t)$ *is Lipschitz from* $E\times W^{1,1}((0,T),L^2(\omega))\times L^1((0,T),L^2(\Omega\setminus\overline{\omega}))$ *into* $C([0,T],E)\times L^2(q)$.

Proof Assume first that, as in Theorem 5.2, $f\in C^1([0,T],L^2(\Omega))$ and $U_0\in D(A)$. Then we can multiply the equation by u_t, which is continuous with values in $H^1_0(\Omega)$, and integrating in space and time, we get (6.3). For simplicity we will replace below $\int_0^t\int_\omega$ by $\int_{q_t}$.
i) The case $f=0$ is obvious and one easily obtains from (6.3) that the mapping $(u_0,v_0)\mapsto((u,u_t),u_t)$ is Lipschitz.
ii) For nonzero f, we use in (6.3) the bound $\int_0^t\int_\Omega fu_t\le\|f\|_{L^1(L^2(\Omega))}\sup_{0\le s\le t}\|u_t(s)\|_{L^2(\Omega)}$. Therefore

$$E_\epsilon(u(t),u_t(t))+2\int_{q_t}u_t^2\le E_0+K(f)y(t)$$

with $y(t)=\sup_{0\le s\le t}\|u_t(s)\|_{L^2(\Omega)}$ where we have set $E_0=E_\epsilon(u_0,v_0)$ and $K(f)=2\|f\|_{L^1(L^2(\Omega))}$. From here we get

$$\epsilon y^2(s)\le E_0+K(f)y(t)$$

for every $0\le s\le t$ and consequently, using Young's inequality, we get $\epsilon y^2(t)\le 2E_0+\frac{1}{\epsilon}K^2(f)$.

Again by Young's inequality, we get the bound $E_\epsilon(u,u_t)+2\int_{q_t}u_t^2\le E_0+\frac{1}{2\epsilon}K^2(f)+\frac{\epsilon}{2}y^2(t)$ and using the bound for $y(t)$ we get $E_\epsilon(u,u_t)+2\int_{q_t}u_t^2\le 2E_0+\frac{1}{\epsilon}K^2(f)$ which gives (6.4).
iii) Now we use the bounds

$$\int_{q_t}fu_t\le 1/2\|f\|^2_{L^2(q)}+1/2\|u_t\|^2_{L^2(q_t)},\qquad \int_0^t\int_{\Omega\setminus\overline{\omega}}fu_t\le\|f\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}\sup_{0\le s\le t}\|u_t(s)\|_{L^2(\Omega\setminus\overline{\omega})}$$

and therefore

$$E_\epsilon(u(t),u_t(t))+\int_{q_t}u_t^2\le E_0+K(f)y(t)$$

where now $y(t)=\sup_{0\le s\le t}\|u_t(s)\|_{L^2(\Omega\setminus\overline{\omega})}$, $E_0=E_\epsilon(u_0,v_0)+\|f\|^2_{L^2(q)}$ and $K(f)=2\|f\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}$. From here we now get $\epsilon y^2(s)\le E_0+K(f)y(t)$, for every $0\le s\le t$. Proceeding as before, we get $E_\epsilon(u,u_t)+\int_{q_t}u_t^2\le 2E_0+\frac{1}{\epsilon}K^2(f)$ which gives (6.5).

iv) It is clear that, from (6.3) and partial integration, we get (6.6). Therefore, denoting $E_0 = E_\epsilon(u_0, v_0) + 2\|f(0)\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)}$ and $K(f) = 2\|f\|_{L^\infty(L^2(\Omega))} + 2\|f_t\|_{L^1(L^2(\Omega))}$ and $z(t) = \sup_{0\le s\le t}\|u(s)\|_{L^2(\Omega)}$, we get

$$E_\epsilon(u(t), u_t(t)) + 2\int_{q_t} u_t^2 \le E_0 + K(f)z(t).$$

From here we get, $Cz^2(s) \le E_0 + K(f)z(t)$ for every $0 \le s \le t$ and some constant $C = C(\Omega, \lambda) > 0$. Proceeding as before, we get $E_\epsilon(u, u_t) + 2\int_{q_t} u_t^2 \le 2E_0 + \frac{1}{C}K^2(f)$ which gives (6.7).
v) From (6.3) and integrating by parts in $\Omega \setminus \overline{\omega}$ we get (6.8). By Young's inequality now we get

$$E_\epsilon(u(t), u_t(t)) + \int_{q_t} u_t^2 \le E_0 + K(f)z(t)$$

with $E_0 = E_\epsilon(u_0, v_0) + \|f\|^2_{L^2(q)} + 2\|f(0)\|_{L^2(\Omega\setminus\overline{\omega})}\|u_0\|_{L^2(\Omega\setminus\overline{\omega})}$ and $K(f) = 2\|f\|_{L^\infty(L^2(\Omega\setminus\overline{\omega}))} + 2\|f_t\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}$ and $z(t) = \sup_{0\le s\le t}\|u(s)\|_{L^2(\Omega\setminus\overline{\omega})}$.

As before, from here we get $Cz^2(s) \le E_0 + K(f)z(t)$ for every $0 \le s \le t$ and some constant $C = C(\Omega, \lambda) > 0$. Proceeding as before, we get $E_\epsilon(u, u_t) + \int_{q_t} u_t^2 \le 2E_0 + \frac{1}{C}K^2(f)$ which gives (6.9).
vi) Again from (6.3) and partial integration we get (6.10). Now, we get

$$E_\epsilon(u(t), u_t(t)) + 2\int_{q_t} u_t^2 \le E_0 + K_1(f)z(t) + K_2(f)y(t)$$

with $z(t) = \sup_{0\le s\le t}\|u(s)\|_{L^2(\omega)}$ and $y(t) = \sup_{0\le s\le t}\|u_t(s)\|_{L^2(\Omega\setminus\overline{\omega})}$ and $E_0 = E_\epsilon(u_0, v_0) + 2\|f(0)\|_{L^2(\omega)}\|u_0\|_{L^2(\omega)}$, $K_1(f) = 2\|f\|_{L^\infty(L^2(\omega))} + 2\|f_t\|_{L^1(L^2(\omega))}$ and $K_2(f) = 2\|f\|_{L^1(L^2(\Omega\setminus\overline{\omega}))}$,

From here we get $\epsilon y^2(s) + Cz^2(s) \le E_0 + K_1(f)z(t) + K_2(f)y(t)$ for every $0 \le s \le t$ and some constant $C = C(\Omega, \lambda) > 0$, which gives $\epsilon y^2(t) + Cz^2(t) \le E_0 + K_1(f)z(t) + K_2(f)y(t)$. Now from Young's inequality we get $\frac{\epsilon}{2}y^2(t) + \frac{C}{2}z^2(t) \le E_0 + \frac{1}{2C}K_1^2(f) + \frac{1}{2\epsilon}K_2^2(f)$. Again, using Young's inequality we get

$$E_\epsilon(u(t), u_t(t)) + 2\int_{q_t} u_t^2 \le E_0 + \frac{1}{2C}K_1^2(f) + \frac{1}{2\epsilon}K_2^2(f) + \frac{\epsilon}{2}y^2(t) + \frac{C}{2}z^2(t)$$

and using the bound for y and z we get

$$E_\epsilon(u(t), u_t(t)) + 2\int_{q_t} u_t^2 \le 2E_0 + \frac{1}{C}K_1^2(f) + \frac{1}{\epsilon}K_2^2(f)$$

and then we get (6.11).

In either case above we get that the mapping $(u_0, v_0, f) \mapsto ((u, u_t), u_t)$ is Lipschitz. Using this and the Lipschitzness of the mapping $(U_0, f) \mapsto U$ established in Theorem 5.2, we can pass to the limit, simultaneously in (5.3), (6.3) and (6.4)–(6.11), as sequences $U_0^n \in D(A)$ and $f_n \in C^1([0,T], L^2(\Omega))$ approach, respectively, $U_0 \in E$ and $f \in L^1((0,T), L^2(\Omega))$ (with the corresponding regularity of each case), as $n \to \infty$, and we get the result.□

Remark 6.2 *Observe that the assumptions in point vi) imply the ones in point iii) and those of point iv) imply the ones in point v). Therefore, although all cases above are interesting by themselves, when obtaining uniform estimates on the solutions below, we will just consider the situations of points ii), iii) and v).*

With all the above, we get at once

Corollary 6.3
i) Assume that a family of initial data for (6.1), $(u_0^\epsilon, v_0^\epsilon) \in E$ are given, such that the energies $E_\epsilon(u_0^\epsilon, v_0^\epsilon)$ remain uniformly bounded, for $0 < \epsilon \le \epsilon_0$. Moreover, on the nonhomogeneous terms in (6.1), assume either

a) $\frac{1}{\sqrt{\epsilon}}f^\epsilon$ *remain uniformly bounded in* $L^1((0,T),L^2(\Omega))$ *or in* $L^1((0,T),L^2(\Omega\setminus\overline{\omega}))$ *and in this case* f^ϵ *remain uniformly bounded in* $L^2(q)=L^2((0,T),L^2(\omega))$
or

b) f^ϵ *remain uniformly bounded in* $L^2(q)$ *and in* $W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$.

Then

$$u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon}u_t^\epsilon \in C([0,T],L^2(\Omega)) \quad \text{and} \quad u_t^\epsilon \in L^2(q) \tag{6.12}$$

and remain uniformly bounded in the norm of these spaces.
ii) If the boundedness assumptions above are replaced by convergence to 0, as $\epsilon\to 0$, *then*

$$u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon}u_t^\epsilon \to 0 \quad \text{in} \quad C([0,T],L^2(\Omega)) \quad \text{and} \quad u_t^\epsilon \to 0 \quad \text{in} \quad L^2(q).\square \tag{6.13}$$

Note that under the regularity assumptions on f^ϵ and $(u_0^\epsilon, v_0^\epsilon)$ of Theorem 5.2, $(u_t^\epsilon, u_{tt}^\epsilon)$ is a mild solution of (6.1) with right hand side f_t^ϵ and initial data $(v_0^\epsilon, u_{tt}^\epsilon(0))$ such that $\epsilon u_{tt}^\epsilon(0) = -\chi_\omega v_0^\epsilon + \Delta u_0^\epsilon - \lambda u_0^\epsilon + f^\epsilon(0) \in L^2(\Omega)$. Therefore, Corollary 6.3 allows us to obtain some further energy estimates for this class of more regular solutions that, in particular, imply that the term ϵu_{tt}^ϵ in (6.1) is small.

Corollary 6.4 *Assume* $f^\epsilon \in W^{1,1}((0,T),L^2(\Omega))$ *satisfy either*

a) $\frac{1}{\sqrt{\epsilon}}f^\epsilon$ *remain uniformly bounded in* $W^{1,1}((0,T),L^2(\Omega))$ *or in* $W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$ *and in this case* f^ϵ *remain uniformly bounded in* $H^1((0,T),L^2(\omega))$
or

b) f^ϵ *remain uniformly bounded in* $H^1((0,T),L^2(\omega))$ *and in* $W^{2,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$.

Even more, assume $(u_0^\epsilon, v_0^\epsilon)$ *belong to* $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, *and*

$$\frac{1}{\sqrt{\epsilon}}\|\chi_\omega v_0^\epsilon - \Delta u_0^\epsilon + \lambda u_0^\epsilon - f^\epsilon(0)\|_{L^2(\Omega)} \tag{6.14}$$

remain uniformly bounded. Then

$$u_t^\epsilon, |\nabla u_t^\epsilon|, \sqrt{\epsilon}u_{tt}^\epsilon \in C([0,T],L^2(\Omega)) \quad \text{and} \quad u_{tt}^\epsilon \in L^2(q) \tag{6.15}$$

and remain uniformly bounded in the norm of these spaces. $\square$

Remark 6.5 *Observe that, in particular, if* $\chi_\omega v_0^\epsilon - \Delta u_0^\epsilon + \lambda u_0^\epsilon = f^\epsilon(0)$ *then* v_0^ϵ *can be freely defined on* $\Omega\setminus\overline{\omega}$ *while its restriction to* ω *parameterizes* u_0^ϵ *in* Ω.

6.2 Convergence in $L^p((0,T),H_0^1(\Omega))$, $p<\infty$.

We start by showing the weak * convergence in $L^\infty((0,T),H_0^1(\Omega))$ for solutions of (6.1). For this, assume that for (6.1) and $0<\epsilon\le\epsilon_0$, like in point i) in Corollary 6.3 we have that $E_\epsilon(u_0^\epsilon, v_0^\epsilon)$ remain uniformly bounded, and either

a) $\frac{1}{\sqrt{\epsilon}}f^\epsilon$ remain uniformly bounded in $L^1((0,T),L^2(\Omega))$ or in $L^1((0,T),L^2(\Omega\setminus\overline{\omega}))$ and in this case f^ϵ remain uniformly bounded in $L^2(q)$
or

b) f^ϵ remain uniformly bounded in $L^2(q)$ and in $W^{1,1}((0,T),L^2(\Omega\setminus\overline{\omega}))$.

Then by compactness we can assume, by taking subsequences if necessary, that we have, as $\epsilon\to 0$,

$$u_0^\epsilon \to u_0, \quad \text{weakly in } H_0^1(\Omega) \quad \text{and weakly in} \quad H_{\Gamma_0}^1(\omega)$$

and that, we also have, in case a),

$$f^\epsilon = O(\sqrt{\epsilon}) \text{ in } L^1((0,T),L^2(\Omega)) \quad \text{or}$$

$$f^\epsilon = O(\sqrt{\epsilon}) \text{ in } L^1((0,T),L^2(\Omega\setminus\overline{\omega})), \quad \text{and} \quad f^\epsilon \to f \text{ weakly in } L^2(q)$$

while in case b) we have

$$f^\epsilon \to f \text{ weakly in } L^2(q) \quad \text{and} \quad f^\epsilon \to f \text{ in } L^1((0,T), L^2(\Omega \setminus \overline{\omega}))).$$

Note that in case a) the limit function f vanishes in Ω or in $\Omega \setminus \overline{\omega}$. Then, we have the following result

Theorem 6.6 *With the assumptions above, for a suitable subsequence of $\epsilon \to 0$, u^ϵ converges weak–$*$ in $L^\infty((0,T), H_0^1(\Omega))$ to a function u, which is a solution of*

$$\chi_\omega u_t + L(u) = f, \quad \chi_\omega u(0) = \chi_\omega u_0 \quad \textit{and} \quad u_t \in L^2(q).$$

With no further assumptions on the limiting function f in case a), or assuming further in case b) that $f \in W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$, assume furthermore that u_0 satisfies (4.18), that is

$$-\Delta u_0 + \lambda u_0 = f(0) \qquad \textit{in } \Omega \setminus \overline{\omega},$$

i.e. $u_0 = B_0(Ru_0) + D_0(f(0))$. Then the function u above is unique and all the family u^ϵ converges weak–$*$ in $L^\infty((0,T), H_0^1(\Omega))$ to u as $\epsilon \to 0$.

Proof From Corollary 6.3, $u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon} u_t^\epsilon \in C([0,T], L^2(\Omega))$ and $u_t^\epsilon \in L^2(q)$ remain uniformly bounded in the norm of these spaces. By compactness, we can assume, by taking subsequences if necessary, that there exists $u \in L^\infty((0,T), H_0^1(\Omega))$ such that $u_t \in L^2(q)$ and such that

$$u^\epsilon \to u, \quad \text{weak-* in } L^\infty((0,T), H_0^1(\Omega)) \text{ and } \quad \text{weak-* in } L^\infty((0,T), H_{\Gamma_0}^1(\omega))$$

$$u_t^\epsilon \to u_t, \quad \text{weak in } L^2(q), \qquad L(u^\epsilon) \to L(u), \quad \text{weak-* in } L^\infty((0,T), H_0^{-1}(\Omega)).$$

From (5.5) we have $(\epsilon u_t^\epsilon)_t + \chi_\omega u_t^\epsilon + L(u^\epsilon) = f^\epsilon$ and this is equivalent to: for every $\phi \in H_0^1(\Omega)$, $< \epsilon u_t^\epsilon, \phi >$ is absolutely continuous and

$$\frac{d}{dt}(< \epsilon u_t^\epsilon, \phi >) + < \chi_\omega u_t^\epsilon, \phi > + < L(u^\epsilon), \phi >_{-1,1} = < f^\epsilon, \phi > \tag{6.16}$$

a.e. $t \in (0,T)$. Now, take $\psi \in C^\infty[0,T]$ such that $\psi(T) = \psi'(T) = 0$, then from (6.16) we get

$$\int_0^T \left(\frac{d}{dt} < \epsilon u_t^\epsilon, \phi > + < \chi_\omega u_t^\epsilon, \phi > \right) \psi(s)\, ds + \int_0^T < L(u^\epsilon), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < f^\epsilon, \phi > \psi(s)\, ds. \tag{6.17}$$

Note that from the assumptions on f^ϵ we have $\int_0^T < f^\epsilon, \phi > \psi(s)\, ds \to \int_0^T < f, \phi > \psi(s)\, ds$ while from the assumptions above, we have $\int_0^T < L(u^\epsilon), \phi >_{-1,1} \psi(s)\, ds \to \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds$.

Now we manipulate the term

$$\int_0^T \left(\frac{d}{dt} < \epsilon u_t^\epsilon, \phi > + < \chi_\omega u_t^\epsilon, \phi > \right) \psi(s)\, ds. \tag{6.18}$$

From the absolute continuity, using that $< \chi_\omega u_t^\epsilon, \phi >= \frac{d}{dt} < \chi_\omega u^\epsilon, \phi >$ and integrating by parts, we get that (6.18) equals $< \epsilon u_t^\epsilon + \chi_\omega u^\epsilon, \phi > \psi(s)]_{s=0}^{s=T} - \int_0^T < \epsilon u_t^\epsilon + \chi_\omega u^\epsilon, \phi > \psi'(s)\, ds$ and integrating again by parts the term containing u_t^ϵ, using $\int_0^T < \epsilon u_t^\epsilon, \phi > \psi'(s)\, ds = \int_0^T \frac{d}{dt} < \epsilon u^\epsilon, \phi > \psi'(s)\, ds$ we get that (6.18) equals

$$\begin{aligned} &< \epsilon u_t^\epsilon + \chi_\omega u^\epsilon, \phi > \psi(s)]_{s=0}^{s=T} - < \epsilon u^\epsilon, \phi > \psi'(s)]_{s=0}^{s=T} + \\ &+ \int_0^T < \epsilon u^\epsilon, \phi > \psi''(s)\, ds - \int_0^T < \chi_\omega u^\epsilon, \phi > \psi'(s)\, ds. \end{aligned} \tag{6.19}$$

On the other hand, integrating by parts the first term containing u_t^ϵ in (6.18) we get that (6.18) also equals

$$
\begin{aligned}
&< \epsilon u_t^\epsilon, \phi > \psi(s)]_{s=0}^{s=T} - < \epsilon u^\epsilon, \phi > \psi'(s)]_{s=0}^{s=T} + \\
&+ \int_0^T < \epsilon u^\epsilon, \phi > \psi''(s)\, ds + \int_0^T < (\chi_\omega u^\epsilon)_t, \phi > \psi(s)\, ds.
\end{aligned} \tag{6.20}
$$

Using (6.19) in (6.17) and passing to the limit, we get

$$
\begin{aligned}
&- \int_0^T < \chi_\omega u, \phi > \psi'(s)\, ds - < \chi_\omega u_0, \phi > \psi(0) + \\
&+ \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < f, \phi > \psi(s)\, ds
\end{aligned} \tag{6.21}
$$

and, by integration by parts, that equals

$$
\begin{aligned}
&\int_0^T \frac{d}{dt} < \chi_\omega u, \phi > \psi(s)\, ds + < \chi_\omega u(0) - \chi_\omega u_0, \phi > \psi(0) + \\
&+ \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < f, \phi > \psi(s)\, ds.
\end{aligned} \tag{6.22}
$$

On the other hand, using (6.20) in (6.17) and passing to the limit, we get

$$
\int_0^T \frac{d}{dt} < \chi_\omega u, \phi > \psi(s)\, ds + \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < f, \phi > \psi(s)\, ds \tag{6.23}
$$

i.e. $\chi_\omega u_t + L(u) = f$, since $u_t \in L^2(q)$.

By comparing (6.22) and (6.23) we get $\chi_\omega u(0) = \chi_\omega u_0$ and the first part of the theorem is proved. Note that when passing to the limit we have used the fact that $\epsilon v_0^\epsilon \to 0$ in $L^2(\Omega)$ and $\epsilon u_t^\epsilon \to 0$ in $C([0,T], L^2(\Omega))$. Also note that since $u_t \in L^2(q)$, $\chi_\omega u(0)$ makes sense in $L^2(\Omega)$.

With the further assumptions on f and u_0, we have that the assumptions of Theorem 4.9 are satisfied and therefore, from Lemma 4.8, u is the unique solution of the problem

$$
\chi_\omega u_t + L(u) = f, \quad u(0) = u_0
$$

and from the uniqueness of the limiting problem we get that the full family u^ϵ converges to u.□

We will show now that if f^ϵ and u_0^ϵ converge strongly and v_0^ϵ is not too large, then we have strong convergence in $L^2((0,T), H_0^1(\Omega))$. To be more precise, we will prove the following

Theorem 6.7 *With the assumptions of Theorem 6.6, assume moreover that either one of the following holds.*

i)

$$
\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0 \quad \text{in } L^1((0,T), L^2(\Omega)) \quad \text{or}
$$

$$
\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0 \quad \text{in } L^1((0,T), L^2(\Omega \setminus \overline{\omega})) \qquad \text{and} \qquad f^\epsilon \to f \quad \text{in } L^2(q)
$$

ii)

$$
f^\epsilon \to f \quad \text{in } L^2(q) \qquad \text{and} \qquad f^\epsilon \to f \quad \text{in } W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))
$$

Also assume

$$
u_0^\epsilon \longrightarrow u_0 \quad \text{in } H_0^1(\Omega)
$$

where $u_0 = B_0(Ru_0) + D_0(f(0))$ is the initial data for the limiting problem, $\chi_\omega u_t + L(u) = f$. Then

$$
u^\epsilon \to u \quad \text{in } L^2((0,T), H_0^1(\Omega)), \qquad \sqrt{\epsilon} u_t^\epsilon \to 0 \quad \text{in } L^2(Q), \qquad u_t^\epsilon \to u_t \quad \text{in } L^2(q)
$$

Proof Since weak$-*$ convergence in L^∞ implies weak convergence in L^2, then it suffices to prove convergence of the norms to have strong convergence in the latter space, i,.e. it suffices to prove

$$\|u^\epsilon\|^2_{L^2(H^1_0(\Omega))} = \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1} \to \int_0^T < L(u), u >_{-1,1} = \|u\|^2_{L^2(H^1_0(\Omega))}.$$

Also, from the weak convergence and lower semicontinuity we have

$$\int_0^T < L(u), u >_{-1,1} \leq \liminf_\epsilon \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1} \quad \text{and} \quad \int_{q_t} |u_t|^2 \leq \liminf_\epsilon \int_{q_t} |u^\epsilon_t|^2.$$

Note that the energy estimates for $(u^\epsilon, u^\epsilon_t)$ in Proposition 6.1 were obtained from (6.3), and that all different cases came from a different treatment of the term $\int_0^t < f, u_t >$. Therefore, in order to simplify the notations we shall keep on using the notation $\int_0^t < f, u_t >$ to denote any of the expressions appearing in the equations mentioned above. Thus, integrating on $(0, T)$ in (6.3) we get, respectively

$$\int_0^T \epsilon \|u^\epsilon_t\|^2 + \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1} + 2\int_0^T \int_{q_t} |u^\epsilon_t|^2 = TE_\epsilon(u^\epsilon_0, v^\epsilon_0) + 2\int_0^T \int_0^t < f^\epsilon, u^\epsilon_t > \tag{6.24}$$

and from Proposition 4.10 and integrating in time, we have

$$\int_0^T < L(u), u >_{-1,1} + 2\int_0^T \int_{q_t} |u_t|^2 = T < L(u_0), u_0 >_{-1,1} + 2\int_0^T \int_0^t < f, u_t > . \tag{6.25}$$

Note that the $\liminf_\epsilon$ of the left hand side of (6.24) is greater or equal than the left hand side of (6.25) and from the hypotheses we have $\epsilon v^\epsilon_0 \to 0$ in $L^2(\Omega)$ and $u^\epsilon_0 \to u_0$ in $H^1_0(\Omega)$ which implies that $E_\epsilon(u^\epsilon_0, v^\epsilon_0) \to < L(u_0), u_0 >_{-1,1}$.

On the other hand, from the assumptions on f^ϵ we claim that

$$\int_0^T \int_0^t < f^\epsilon, u^\epsilon_t > \to \int_0^T \int_0^t < f, u_t > .$$

Assumed this for a moment, we get that the right hand side of (6.24) converges to that of (6.25) and therefore the same must happen with the left hand side. Therefore we get $\sqrt{\epsilon} u^\epsilon_t \to 0$ in $L^2(Q)$, $u^\epsilon \to u$ in $L^2((0,T), H^1_0(\Omega))$ and $\int_0^T \int_{q_t} |u^\epsilon_t|^2 \to \int_0^T \int_{q_t} |u_t|^2$. But since for a.e. $t \in (0,T)$ we have $\int_{q_t} |u_t|^2 \leq \liminf_\epsilon \int_{q_t} |u^\epsilon_t|^2$, then we conclude $u^\epsilon_t \to u_t$ in $L^2(q)$.

Therefore to have the theorem proved it only remains to prove the claim.

If $\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0$ in $L^1((0,T), L^2(\Omega))$ then, since $\sqrt{\epsilon} u^\epsilon_t$ remains uniformly bounded in $L^\infty((0,T), L^2(\Omega))$, clearly $\int_0^t < f^\epsilon, u^\epsilon_t >$ goes to zero, for each fixed t. Since $\left|\int_0^t \int_\Omega f^\epsilon u^\epsilon_t\right|$ are uniformly bounded in $(0,T)$, from Lebesgue's dominated convergence theorem we get the result.

If, on the other hand, $\frac{1}{\sqrt{\epsilon}} f^\epsilon \to 0$ in $L^1((0,T), L^2(\Omega \setminus \overline{\omega}))$ and $f^\epsilon \to f$ in $L^2(q)$, then

$$\int_0^t < f^\epsilon, u^\epsilon_t > = \int_0^t \int_{\Omega \setminus \overline{\omega}} f^\epsilon u^\epsilon_t + \int_0^t \int_\omega f^\epsilon u^\epsilon_t.$$

Again, since $\sqrt{\epsilon} u^\epsilon_t$ remains uniformly bounded in $L^\infty((0,T), L^2(\Omega))$, the first term goes to zero for every $t \in (0,T)$. On the other hand, since $f^\epsilon \to f$ in $L^2(q)$ and $u^\epsilon_t \to u_t$ weakly in $L^2(q)$ then the second term converges to $\int_0^t \int_\omega f u_t$ and $\int_0^t < f^\epsilon, u^\epsilon_t >$ converges to $\int_0^t \int_\omega f u_t$ for every $t \in (0,T)$. Since, $\left|\int_0^t \int_{\Omega \setminus \overline{\omega}} f^\epsilon u^\epsilon_t\right|$ and $\left|\int_0^t \int_\omega f^\epsilon u^\epsilon_t\right|$ are uniformly bounded in $(0,T)$, from Lebesgue's dominated convergence theorem we get the result.

Finally, if $f^\epsilon \to f$ in $L^2(q)$ and $f^\epsilon \to f$ in $W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$, then

$$\int_0^t < f^\epsilon, u_t^\epsilon >= \int_{q_t} f^\epsilon u_t^\epsilon + \int_{\Omega\setminus\overline{\omega}} f^\epsilon(t) u^\epsilon(t) - \int_{\Omega\setminus\overline{\omega}} f^\epsilon(0) u_0^\epsilon - \int_0^t \int_{\Omega\setminus\overline{\omega}} f_t^\epsilon u^\epsilon$$

and we conclude as above, since the first term can be handled as above and the strong convergence of f^ϵ and the weak convergence of u^ϵ allow us to pass to the limit in the other terms. Again Lebesgue's theorem concludes.□

Using the convergence in $L^2((0,T), H_0^1(\Omega))$ and the boundedness in $L^\infty((0,T), H_0^1(\Omega))$ and by interpolation, we get

Corollary 6.8 *Under the hypotheses of the Theorem, for every* $2 \le p < \infty$

$$u^\epsilon \to u \quad \text{in } L^p((0,T), H_0^1(\Omega)).$$

6.3 Uniform convergence in time

We will show now that under stronger conditions on f^ϵ and u_0^ϵ, the convergence $u^\epsilon \to u$ is uniform in time. We first outline the main idea behind our approach. Let u^ϵ be the solution of

$$(\epsilon u_t^\epsilon)_t + \chi_\omega u_t^\epsilon + L(u^\epsilon) = f^\epsilon \tag{6.26}$$

with initial data $u^\epsilon(0) = u_0^\epsilon$, $u_t^\epsilon(0) = v_0^\epsilon$ and assume the data is smooth enough so that u_{tt}^ϵ exists in $L^2(\Omega)$. Therefore, the first term in (6.26) can be written as ϵu_{tt}^ϵ. On the other hand, let u the solution of

$$\chi_\omega u_t + L(u) = f \tag{6.27}$$

with initial data $u_0 = B_0(Ru_0) + D_0(f(0))$. Note that both equations are verified in $H_0^{-1}(\Omega)$. Since the solution of (6.26) is in $H_0^1(\Omega)$, for fixed $t \ge 0$, using the splitting in Proposition 3.2, we can write

$$u^\epsilon = B_0(Ru^\epsilon) + u_2^\epsilon$$

while, according to Theorem 4.9, for the solution of (6.27) we have

$$u = B(v) + D_0(f)$$

where v is given by (4.15). Then our goal is to compare u^ϵ and u by comparing their projections $v^\epsilon = Ru^\epsilon$ with v and u_2^ϵ with $D_0(f)$. We start with the following result

Lemma 6.9 *With the notations above, assume u^ϵ is a solution of (6.26) such that for every $t \in [0,T]$, $u_{tt}^\epsilon \in L^2(\Omega)$. Then for every $t \in [0,T]$, u^ϵ can be split as*

$$u^\epsilon = B_0(v^\epsilon) + u_2^\epsilon$$

where

$$u_2^\epsilon = D_0(f^\epsilon - \epsilon u_{tt}^\epsilon) \in H^2(\Omega \setminus \overline{\omega}) \cap H_0^1(\Omega \setminus \overline{\omega})$$

and $v^\epsilon = Ru^\epsilon \in H_{\Gamma_0}^1(\omega)$ satisfies $v^\epsilon(0) = Ru_0^\epsilon$ and

$$v_t^\epsilon + A_0 v^\epsilon = f_\omega^\epsilon - \epsilon(u_{tt}^\epsilon)_\omega + (B^* f_{\Omega\setminus\overline{\omega}}^\epsilon)_{\Gamma_1} - \epsilon(B^*(u_{tt}^\epsilon)_{\Omega\setminus\overline{\omega}})_{\Gamma_1} \quad \text{in } H_{\Gamma_0}^{-1}(\omega)$$

Proof First, taking test functions $\phi \in H_0^1(\Omega \setminus \overline{\omega})$ in (6.26), we get

$$< \epsilon u_{tt}^\epsilon, \phi >_{\Omega\setminus\overline{\omega}} + < L(u^\epsilon), \phi >_{-1,1} = < f^\epsilon, \phi >_{\Omega\setminus\overline{\omega}}$$

Since $< L(u^\epsilon), \phi >_{-1,1} = < L_D(u_2^\epsilon), \phi >_{-1,1}$, we get

$$L_D(u_2^\epsilon) = f^\epsilon - \epsilon u_{tt}^\epsilon \in L^2(\Omega \setminus \overline{\omega}).$$

This and the regularity for the Dirichlet problem in $\Omega \setminus \overline{\omega}$ proves the first part.

Now, taking test functions, $\phi \in Har_\Gamma(\Omega \setminus \overline{\omega})$ in (6.26), so $\phi = B_0(R\phi)$, we get

$$< u_{tt}^\epsilon, \phi > = < u_{tt}^\epsilon, \phi >_\omega + < u_{tt}^\epsilon, B(\phi) >_{\Omega\setminus\overline{\omega}} = < u_{tt}^\epsilon, \phi >_\omega + < B^* u_{tt}^\epsilon, \phi >_{\Gamma_1}$$

and analogously $< f^\epsilon, \phi > = < f^\epsilon, \phi >_\omega + < B^* f^\epsilon, \phi >_{\Gamma_1}$. On the other hand, $< \chi_\omega u_t^\epsilon, \phi > = < u_t^\epsilon, \phi >_\omega = < v_t^\epsilon, \phi >_\omega$ and

$$< L(u^\epsilon), \phi >_{-1,1} = < L B_0(v^\epsilon), \phi >_{-1,1} = < A_0(v^\epsilon), \phi >_{-1,1}^{\omega} .$$

With all these we get the second part.□

Therefore, if ϵu_{tt}^ϵ is small, see (6.15), if f^ϵ converges to f in some sense and if u_0^ϵ converges to u_0, we may expect proving that $z^\epsilon = u^\epsilon - u$ becomes small for all times. Note that for $z^\epsilon = u^\epsilon - u$ we have

$$z^\epsilon = u^\epsilon - u = B_0(v^\epsilon - v) + u_2^\epsilon - D_0(f) = B_0(Rz^\epsilon) + z_2^\epsilon.$$

To go ahead with this idea we will require suitable smoothness and boundedness of f^ϵ and compatibility conditions on the initial data of (6.26) to guarantee that ϵu_{tt}^ϵ is small, see Corollary 6.4 and (6.15). To be more precise, we assume, as in Corollary 6.4, that $f^\epsilon \in W^{1,1}((0,T), L^2(\Omega))$ satisfy either

a) $\frac{1}{\sqrt{\epsilon}} f^\epsilon$ remain uniformly bounded in $W^{1,1}((0,T), L^2(\Omega))$ or in $W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$ and in this case f^ϵ remain uniformly bounded in $H^1((0,T), L^2(\omega))$

or

b) f^ϵ remain uniformly bounded in $H^1((0,T), L^2(\omega))$ and in $W^{2,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$.

Even more, assume $(u_0^\epsilon, v_0^\epsilon)$ belong to $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and

$$\frac{1}{\sqrt{\epsilon}} \|\chi_\omega v_0^\epsilon - \Delta u_0^\epsilon + \lambda u_0^\epsilon - f^\epsilon(0)\|_{L^2(\Omega)}$$

remain uniformly bounded. Observe that under these assumptions, we can assume, by taking subsequences if necessary, that we have as $\epsilon \to 0$, in case a),

$$f^\epsilon = O(\sqrt{\epsilon}) \text{ in } W^{1,1}((0,T), L^2(\Omega)) \quad \text{or}$$

$$f^\epsilon = O(\sqrt{\epsilon}) \text{ in } W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega})) \quad \text{and} \quad f^\epsilon \to f \text{ in } L^2(q) \text{ and weakly in } H^1((0,T), L^2(\omega))$$

while in case b) we have

$$f^\epsilon \to f \text{ in } L^2(q) \text{ and weakly in } H^1((0,T), L^2(\omega)) \quad \text{and} \quad f^\epsilon \to f \text{ in } W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))).$$

In particular, in either case the limiting function f satisfies the assumptions in Theorem 4.9.

We also assume that

$$u_0^\epsilon \to u_0 \text{ in } H_0^1(\Omega)$$

where u_0 satisfies (4.18), that is

$$-\Delta u_0 + \lambda u_0 = f(0) \qquad \text{in } \Omega \setminus \overline{\omega},$$

i.e. $u_0 = B_0(Ru_0) + D_0(f(0))$. Therefore, the initial data for the limiting problem also satisfies the assumptions in Theorem 4.9.

Theorem 6.10 *Under the assumptions above we have*

$$u^\epsilon \to u \quad \text{in } C([0,T], H_0^1(\Omega)) \cap L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega)).$$

Proof First note that since from the assumptions we have $f^\epsilon \to f$ uniformly in $L^2(\Omega \setminus \overline{\omega})$, then $D_0(f^\epsilon) \to D_0(f)$ uniformly in $H^2(\Omega \setminus \overline{\omega}) \cap H^1_0(\Omega \setminus \overline{\omega})$. This implies that

$$z_2^\epsilon = u_2^\epsilon - D_0(f) = D_0(f^\epsilon - f - \epsilon u^\epsilon_{tt}) \to 0$$

uniformly in $H^2(\Omega \setminus \overline{\omega}) \cap H^1_0(\Omega \setminus \overline{\omega})$, since ϵu^ϵ_{tt} is of order $\sqrt{\epsilon}$ in $L^2(\Omega)$, uniformly in $[0,T]$.

On the other hand, note that $(B^* f^\epsilon_{\Omega\setminus\overline{\omega}})_{\Gamma_1} - (B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1}$ and $B^*(\epsilon u^\epsilon_{tt})$ are uniformly small in $H^{1/2}(\Gamma_1)$ and so they are uniformly small in $X^{-\beta}$ for all $\beta \in (1/4, 1/2]$, where X^β denotes the fractional power space of A_0, see Theorem 4.1. Also, from (4.15) and the Lemma above, we have

$$Rz^\epsilon = e^{-A_0 t}(Ru_0^\epsilon - Ru_0) + \int_0^t e^{-A_0(t-s)} \left(f^\epsilon_\omega - \epsilon(u^\epsilon_{tt})_\omega + (B^* f^\epsilon_{\Omega\setminus\overline{\omega}})_{\Gamma_1} - \epsilon(B^*(u^\epsilon_{tt})_{\Omega\setminus\overline{\omega}})_{\Gamma_1} - f_\omega - (B^* f_{\Omega\setminus\overline{\omega}})_{\Gamma_1} \right) ds.$$

Therefore, if $f^\epsilon \to f$ in $L^2(q)$ and $Ru^\epsilon \to Ru_0$ in $H^1_{\Gamma_0}(\omega)$, from point ii) of Proposition 5.5 in [11], we get $Rz^\epsilon \to 0$ uniformly in $H^1_{\Gamma_0}(\omega)$ and then $B_0(Rz^\epsilon)$ is uniformly small in $H^1_0(\Omega)$.

On the other hand, from Theorem 6.7, we have $u^\epsilon_t \to u_t$ in $L^2(q)$ and from the assumption $f^\epsilon \to f$ in $L^2(Q)$ and since

$$L(u^\epsilon) = -\epsilon u^\epsilon_{tt} - \chi_\omega u^\epsilon_t + f^\epsilon, \quad \text{and} \quad L(u) = -\chi_\omega u_t + f.$$

Then we get $u^\epsilon \to u$ in $L^2((0,T), H^2(\Omega) \cap H^1_0(\Omega))$.□

7 Final remarks

The techniques of previous sections can also be used, with slight modifications, to problems of the form

$$\begin{cases} \epsilon u^\epsilon_{tt} + \rho(x)\chi_\omega u^\epsilon_t - \Delta u^\epsilon + \lambda u^\epsilon = f^\epsilon(t,x) & \text{in } \Omega \times (0,T) \\ u^\epsilon = 0 & \text{on } \Gamma \times (0,T) \\ u^\epsilon(0) = u^\epsilon_0, \quad u^\epsilon_t(0) = v^\epsilon_0 \end{cases} \tag{7.1}$$

in which a positive and bounded damping weight function is considered. That is, we assume

$$0 < m \le \rho(x) \le M, \quad \text{for } x \in \omega.$$

For this problem, the corresponding singular limit, as $\epsilon \to 0$, is

$$\begin{cases} \rho(x)\chi_\omega u_t - \Delta u + \lambda u = f(t,x) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0 \end{cases} \tag{7.2}$$

which, as before, must be understood in the sense that

$$\begin{cases} -\Delta u + \lambda u = f(t,x) & \text{in } \Omega \setminus \overline{\omega} \times (0,T) \\ \rho(x) u_t - \Delta u + \lambda u = f(t,x) & \text{in } \omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0 \end{cases}$$

with the no–jump conditions on Γ_1

$$[u]_{\Gamma_1} = [\frac{\partial}{\partial \vec{n}} u]_{\Gamma_1} = 0.$$

In what follows we will just indicate the main changes one has to perform in the functional setting above. We first define the extended weight function

$$\rho_0(x) = \begin{cases} \rho(x) & x \in \omega \\ 1 & x \in \Omega \setminus \omega \end{cases}$$

and then consider the Hilbert space $L^2(\Omega, \rho_0)$ defined as the set $L^2(\Omega)$ endowed with the scalar product

$$< f, g >= \int_\Omega f(x)g(x)\rho_0(x)\, dx.$$

This space, which is identified with its dual, will be playing the role of $L^2(\Omega)$ in the analysis of previous sections. Analogously we define the Hilbert space $L^2(\omega, \rho)$ defined as the set $L^2(\omega)$ endowed with the scalar product

$$< f, g >= \int_\omega f(x)g(x)\rho(x)\, dx$$

which will play the role of $L^2(\omega)$ above. Note that with respect of our former analysis we are only changing the metrics of the pivot spaces $L^2(\Omega)$ and $L^2(\omega)$.

With this the definitions of L and L_ω and the scalar products in $H^1_0(\Omega)$ and $H^1_{\Gamma_0}(\omega)$ in (3.1), (3.2) and (3.3) remain the same, as well as the definition of λ–Harmonic functions on $\Omega \setminus \overline{\omega}$, and the operators B and B_0. Note that the underlying operator in $L^2(\Omega, \rho_0)$, obtained by restriction of L is the unbounded, positive and selfadjoint operator, $(A, D(A))$ given by

$$D(A) = \{u \in H^1_0(\Omega),\ L(u) \in L^2(\Omega, \rho_0)\} \qquad \text{and} \qquad A(u) = L(u), \quad \text{for } u \in D(A).$$

Then it turns out that $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $A(u) = \frac{1}{\rho_0}(-\Delta u + \lambda u)$ for $u \in D(A)$. Therefore, solutions of

$$\begin{cases} \frac{1}{\rho_0}(-\Delta u + \lambda u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

are obtained, in a variational formulation, by solving $L(u) = f$. Thus if $f \in H^{-1}_0(\Omega)$ then $u \in H^1_0(\Omega)$ and, from the regularity assumptions on Ω, if $f \in L^2(\Omega, \rho_0)$ then $u \in H^2(\Omega) \cap H^1_0(\Omega)$. To see this, note that if $u \in H^1_0(\Omega)$ is such that $L(u) = f$ with $f \in L^2(\Omega, \rho_0)$ then we have

$$\int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi =< f, \phi >_{L^2(\Omega,\rho_0)}= \int_\Omega f\phi\rho_0$$

for every $\phi \in H^1_0(\Omega)$. Then taking $\phi \in C^\infty_0(\Omega)$ and integration by parts gives $-\Delta u + \lambda u = f\rho_0$ and the rest follows easily.

With these, in Section 3, Proposition 3.2, Lemma 3.4, Proposition 3.5 and Corollary 3.7 will remain almost the same. The only change in these results is in the last two in which the equation satisfied in ω is $\frac{1}{\rho}(-\Delta u + \lambda u) = f$, in ω, the operator A_0 satisfies

$$A_0 = \left(\frac{1}{\rho}(-\Delta + \lambda)\right)_\omega + (\frac{\partial}{\partial \vec{n}} - \frac{\partial B(u)}{\partial \vec{n}})_{\Gamma_1}$$

and, in Corollary 3.7, $u = B_0(v)$ coincides with the solutions of

$$\begin{cases} \frac{1}{\rho_0}(-\Delta u + \lambda u) = \chi_\omega f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

given by $L(u) = \chi_\omega f$. Also (7.2) can be solved along the same lines as in Section 4 and the equation is solved in the sense that

$$\chi_\omega u_t + L(u) = \frac{1}{\rho_0} f(t), \quad \text{in } H^{-1}_0(\Omega)$$

with f taking values in $L^2(\Omega)$. The rest of that Section, as well as Sections 5 and 6 remain the same with obvious modifications.

In the same way we could consider (7.1) with a more general second order elliptic operator in divergence form replacing the Laplace operator.

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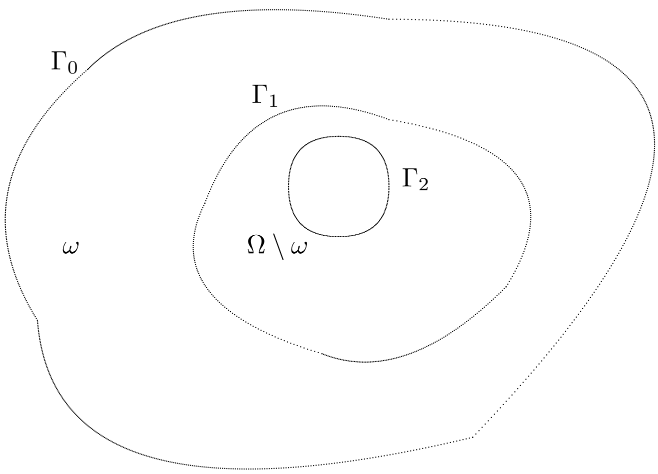

Figure 1: A typical domain