BOUNDARY CONTROLLABILITY OF A HYBRID SYSTEM CONSISTING IN TWO FLEXIBLE BEAMS CONNECTED BY A POINT MASS

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Abstract. We consider a hybrid system consisting on two flexible beams connected by a point mass. The constant of rotational inertia is assumed not to be zero. In a previous paper we have proved that, in the presence of the point mass, the system is well-posed in asymmetric spaces in which solutions have one more degree of regularity to one side of the mass.

We are interested in the problem of controllability when the control acts on the free extreme of one of the beams. We prove that when the control time is large enough the system is exactly controllable in an asymmetric space. This result is sharp. The proofs combine classical techniques from asymptotic analysis and the theory of non-harmonic Fourier series.

Key words. Flexible beams, point mass, asymmetric spaces, Fourier series, controllability.

AMS subject classifications. 35L30, 35P15, 42A55

1. Introduction. In this paper we study the boundary controllability of a linear system modelling the vibrations of two flexible beams connected by a point mass.

We assume that the beams occupy the intervals $(-1,0)$ and $(0,1)$ and that the point mass is located at $x=0$. By means of the scalar function $u=u(x,t)$ defined for $x\in(-1,1)$ and $t>0$ we describe the vertical displacements of the beams and the point mass. The linear equations describing the small vibrations of this system can be written as follows

$$\begin{cases} \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, & \text{for } x\in(-1,0), t>0 \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, & \text{for } x\in(0,1), t>0 \\ [u]\,(0,t) = [\partial u](0,t) = 0, & \text{for } t>0 \\ Mu_{tt}(0,t) + [\partial^3 u](0,t) = 0, & \text{for } t>0 \\ M\gamma_0\partial u_{tt}(0,t) - [\partial^2 u](0,t) = 0 & \text{for } t>0. \end{cases} \tag{1}$$

where ∂ denotes partial derivation with respect to x and the index t derivation with respect to time. $[u](0) = u(0^+) - u(0^-)$ denotes the jump of the function u at the point $x=0$ where the mass is located. Assuming that the beams are hinged at their extremes, system (1) has to be completed with the following boundary conditions:

$$u(\pm 1,t) = \partial^2 u(\pm 1,t) = 0, \quad \text{for } t>0. \tag{2}$$

In (1) the dynamic of the beams is described by the Rayleigh beam equation where $\gamma\geq 0$ represents the constant of rotational inertia. The third equation guarantees that u and ∂u are continuous across $x=0$ while the last two equations describe the vibrations of the point mass at $x=0$. M is the total mass concentrated in $x=0$ and γ_0 the rotational inertia at this point.

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To simplify the exposition we assume $M = 1$ and $\gamma = \gamma_0$ though the analysis is valid for other values of the parameters.

It is worth noting that, in the particular case in which the constant γ of rotational inertia vanishes ($\gamma = 0$) $\partial^2 u$ is continuous across $x = 0$ too. This implies that the effect of the mass point is weaker on the behavior of the system when $\gamma = 0$ than when $\gamma > 0$. Thus the properties of system (1.1)-(1.2) when $\gamma = 0$ are much closer to the case in which the point mass is not present. We refer to [5] and [9] for precise statements and the details of the proofs.

All along this paper we assume that $\gamma > 0$.

System (1)-(2) has to be completed with suitable initial conditions for $u(x,t)$, $u(0,t)$ and $\partial u(0,t)$. The last two quantities will be denoted by y and z respectively, i.e.

$$u(0,t) = y(t); \qquad \partial u(0,t) = z(t). \tag{3}$$

The initial conditions are then:

$$\begin{cases} u(x,0) = u^0(x) & \text{in } (-1,0) \cup (0,1)\,; y(0) = y^0,\ z(0) = z^0 \\ u_t(x,0) = u^1(x) & \text{in } (-1,0) \cup (0,1);\ y_t(0) = y^1,\ z_t(0) = z^1. \end{cases} \tag{4}$$

System (1)-(2) has been studied in [4] where it is proved that, with appropriate regularity and compatibility conditions, on the initial data it admits a unique solution in a suitable class. On the other hand, its energy

$$E(t) = \int_{-1}^{1} \left[\left|\partial^2 u(x,t)\right|^2 + \gamma \left|\partial u_t(x,t)\right|^2 + \left|u_t(x,t)\right|^2 \right] dx + \left|u_t(0,t)\right|^2 + \gamma \left|\partial u_t(0,t)\right|^2 \tag{5}$$

is constant along trajectories.

In this paper we assume that a control function $q = q(t)$ acts on the system through the extreme $x = 1$ on the quantity $\partial^2 u(1,t)$. Then the boundary conditions in (2) have to be replaced by

$$u(\pm 1, t) = \partial^2 u(-1,t) = 0; \quad \partial^2 u(1,t) = q(t) \text{ for } t > 0. \tag{6}$$

The problem of exact controllability can be formulated as follows: *Given* $T > 0$, *find the class* H *of initial conditions for which there exists a control* q, *say, in* $L^2(0,T)$ *such that the solution of* (1), (3) *with boundary conditions* (6) *is at rest at time* $t = T$, i.e. *it satisfies*

$$\begin{cases} u(x,T) = 0 & \textit{for } x \in (-1,0) \cup (0,1), \quad y(T) = 0, \quad z(T) = 0 \\ u_t(x,T) = 0 & \textit{for } x \in (-1,0) \cup (0,1), \quad y_t(T) = 0, \quad z_t(T) = 0. \end{cases} \tag{7}$$

In this formulation of the control problem we have chosen the control to belong to $L^2(0,T)$. This is not, of course, the unique choice but it is the one that comes more naturally when studying the problem of controllability by means of J.-L. Lions' HUM method (see [8]).

It turns that the space H of controllable initial data, can not be found among the family of energy spaces in which system (1)-(2) is well-posed. Indeed, all the energy spaces have in common the fact that solutions in those classes have the same regularity to both sides of the point mass. For instance, the energy E in (5) corresponds to solutions u in H^2 to both sides of $x = 0$ and such that u_t belongs to $H^1(-1,1)$. However the space of controllable data turns out to be asymmetric in the sense that its elemets have one more degree of regularity to the left of $x = 0$.

The same phenomena was observed in [6] in the case of two flexible strings connected by a point mass. In [6] this was proved by using the explicit formula for solutions of the one-dimensional wave equation in terms of its initial data and it was seen that this is a consequence of the fact that solutions gain one derivative when crossing the mass. In [6] it was also observed that the spectral gap of the wave equation vanishes in the presence of a point mass and it was conjectured these two facts (i.e. the asymmetry of the controllable space and the lack of the spectral gap) to be closely related. Later on, in [3], it was proved that these two properties are equivalent (see also [1]).

For the fourth order system that we are considering here it has been observed that, in presence of the point mass, the spectral gap vanishes too (see [4]). Using Fourier developments of solutions it is also proved in [4] that system (1)-(2) is well posed in asymmetric spaces in which the solutions have one more degree of regularity to one side of $x = 0$. This result applies only when $\gamma > 0$ since, as we said above, when $\gamma = 0$ the presence of the mass has a much weaker effect on the behavior of the system. In this case ($\gamma = 0$) system (1)-(2) is not well-posed in asymmetric spaces of this kind.

Thanks to the existence of the asymmetric spaces in which system (1)-(2) is well posed and by means of the theory of non-harmonic Fourier series and more precisely, of some results by D. Ulrich [10], we prove sharp observability results. These results stablish the equivalence between a suitable asymmetric norm of the initial data and the quantity $\int_0^T |\partial u(1,t)|^2 \, dt$ which measures the amount of energy concentrated at $x = 1$ during the time interval $t \in (0,T)$. This result requires the time T to be sufficiently large and, more precisely, $T \geq 4\sqrt{\gamma}$. This is due to the finite speed of propagation underlying in system (1) when $\gamma > 0$, and therefore it is a natural restriction to the observability to hold.

By means of HUM and as a direct consequence of this observability result we prove the exact controllability of the system. We show, roughly, that when $\gamma > 0$ the space of controllable data coincides with the subspace of $H^2(-1,1) \times H^1(-1,1)$ of those elements that restricted to $(-1,0)$ have one more degree of regularity, i.e. belong to $H^3(-1,0) \times H^2(-1,0)$.

It is worth mentionning that, in the absence of mass, the space of controllable data coincides with $H^2(-1,1) \times H^1(-1,1)$ (see [7]). Thus, as in the context of flexible strings with a point mass (see [6]), the presence of the point mass reduces the space of controllable data by one derivative on the opposite side of the mass with respect to the extreme in which the control is located.

The rest of the paper is organized as follows. In section 2 we recall without proofs some basic analytical results of the uncontrolled system (1)-(2) given in [4]. In particular we state the well-posedness of the system in asymmetric spaces using Fourier series. In section 3 we prove suitable observability properties. In section 4 we solve system (1) with non-homogeneous boundary conditions to give sense to the solutions of the controlled problem where we have to consider the boundary condition (6). Finally, in section 6 we obtain the main controllability results.

2. Preliminary results. In this section we set some analytical properties of the solutions of system (1) which will be used along this work. The proofs of the results can be seen in [4].

2.1. Spectral analysis. When decomposing solutions of (1)-(2) in Fourier series one is led to consider solutions in separated variables $u = e^{i\lambda t}\varphi(x)$. In this class of solutions system (1.1)-(1.2) becomes:

$$\left\{\begin{array}{ll} \partial^4\varphi = \lambda^2\varphi - \gamma\lambda^2\partial^2\varphi\,, & \text{for } x\in(-1,0)\\ \partial^4\varphi = \lambda^2\varphi - \gamma\lambda^2\partial^2\varphi\,, & \text{for } x\in(0,1)\\ [\varphi](0) = [\partial\varphi](0) = 0 & \\ [\partial^2\varphi](0) = -\gamma\lambda^2\partial\varphi(0) & \\ [\partial^3\varphi](0) = \lambda^2\varphi(0) & \\ \varphi(\pm1) = \partial^2\varphi(\pm1) = 0. & \end{array}\right. \tag{8}$$

We introduce the operator $(I-\gamma\partial^2)^{-1} : L^2(-1,1) \to H^2\cap H_0^1(-1,1)$ such that $u = (I-\gamma\partial^2)^{-1}F$ if and only if $u\in H^2\cap H_0^1(-1,1)$ and satisfies $u-\gamma\partial^2 u = F$.

If we define the vector valued eigenfunction $\phi = (\varphi, \varphi(0), \partial\varphi(0))$, system (8) can be written as

$$\left\{\begin{array}{l} K\phi = \lambda^2\phi\\ \partial^2\varphi(\pm1) = 0 \end{array}\right. \tag{9}$$

where K is the linear operator given by

$$K = \begin{pmatrix} (I-\gamma\partial^2)^{-1}\partial^4_{(-1,1)/\{0\}} & 0 & 0\\ \partial^3_+ - \partial^3_- & 0 & 0\\ -\frac{1}{\gamma}\left(\partial^2_+ - \partial^2_-\right) & 0 & 0 \end{pmatrix}.$$

Here $\partial^4_{(-1,1)/\{0\}}$ represents the operator which assigns to each function, not necessarily continuous in $x=0$, the fourth order derivative to both sides of $x=0$ and $\partial^k_\pm$ represents the distribution which assigns to a function u the value $\partial^k u(0^\pm)$.

The following result is proved in [4]:

PROPOSITION 2.1. *The eigenvalues* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of system (9) are simple and constitute a sequence of positive real numbers:*

$$0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\to\infty.$$

Moreover, the corresponding eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ *may be normalized to form an orthonormal basis of the space*

$$H = \left\{\phi = (\varphi,\varphi(0),\partial\varphi(0)) \in H^2\cap H_0^1(-1,1)\times\mathbb{R}\times\mathbb{R}\right\}$$

with the norm

$$\|\phi\|_H = \left[\int_{-1}^1 |\partial^2\varphi|^2 dx\right]^{1/2}.$$

REMARK 1. *All the results of the above Proposition but the simplicity of the eigenvalues can be proved using classical theory on compact selfadjoint operators. The simplicity of the eigenvalues requires a detailed analysis of the system under consideration.*

From Proposition 2.1 the space H can also be written as follows

$$H = \left\{ u : u = \sum_{k\in I\!N} a_k \phi_k, \| u \|_H^2 = \sum_{k \in I\!N} | a_k |^2 < \infty \right\}. \tag{10}$$

We can also define the following fractional Hilbert spaces $(H_\alpha, \| \cdot \|_\alpha)_{\alpha \in I\!R}$:

$$H_\alpha = \left\{ u = \sum_{k\in I\!N} a_k \phi_k : \| u \|_\alpha^2 = \sum_{k \in I\!N} | a_k |^2 \lambda_k^{4\alpha} < \infty \right\}. \tag{11}$$

We will denote by $< \cdot, \cdot >_\alpha$ the scalar product in H_α.

Clearly $H_0 = H$ and $\| \cdot \|_H = \| \cdot \|_0$.

Observe that, if $u = \sum_{k\in I\!N} a_k \phi_k$, then $Ku = \sum_{k\in I\!N} \frac{a_k}{\lambda_k^2} \phi_k$. Clearly K is an isomorphism from H_α into $H_{\alpha+1}$. We can also write explicitely K^{-1}:

$$K^{-1} u = \sum_{k\in I\!N} \lambda_k^2 a_k \phi_k$$

which is continuous from $H_{\alpha+1}$ into H_α.

We need to identify the spaces H_α for some values of the parameter $\alpha \in I\!R$. To do that we denote by $H^s((-1,1)\setminus\{0\}) \cap H^2 \cap H_0^1(-1,1)$ the subspace of $H^2 \cap H_0^1(-1,1)$ constituted by the elements such that its restrictions to $(-1,0)$ and $(0,1)$ belong to H^s.

We have the following characterizations of the fractional spaces H_α:

PROPOSITION 2.2. **(a)** *$H_{1/2}$ coincides algebraically and topologically with the subspace of*

$$H^3((-1,1)\setminus\{0\}) \cap H^2 \cap H_0^1(-1,1) \times I\!R \times I\!R$$

constituted by the elements (u, y, z) such that

$$\partial^2 u(\pm 1) = 0, \quad u(0) = y, \quad \partial u(0) = z. \tag{12}$$

(b) *$H_{-1/2}$ coincides with the subspace of $H_0^1(-1,1) \times I\!R \times I\!R$ constituted by the elements (u, y, z) such that $u(0) = y$.*

Moreover

$$\| (u, y, z) \|_{-1/2}^2 = \int_{-1}^{1} \left[\gamma \mid \partial u \mid^2 + \mid u \mid^2 \right] dx + \mid y \mid^2 + \gamma \mid z \mid^2 . \tag{13}$$

(c) *H_{-1} coincides algebraically and topologically with the quotient space of $L^2(-1,1) \times I\!R \times I\!R$ constituted by the classes (u, y, z) characterized in the following way: Two elements (u^1, y^1, z^1) and (u^2, y^2, z^2) belong to the same class if and only if*

$$(u^1 - u^2, y^1 - y^2, z^1 - z^2) = \alpha(m, -1, 0) + \beta(n, 0, \gamma^{-1})$$

where $\alpha, \beta \in I\!R$ and m and n are the functions

$$m(x) = \begin{cases} \frac{\sinh(\frac{1+x}{\sqrt{\gamma}})}{2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})} & \textit{if } x \in [-1, 0] \\ \frac{\sinh(\frac{1-x}{\sqrt{\gamma}})}{2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})} & \textit{if } x \in [0, 1] \end{cases}, \quad n(x) = \begin{cases} \frac{\sinh(\frac{1+x}{\sqrt{\gamma}})}{2\gamma\sinh(\frac{1}{\sqrt{\gamma}})} & \textit{if } x \in [-1, 0] \\ \frac{-\sinh(\frac{1-x}{\sqrt{\gamma}})}{2\gamma\sinh(\frac{1}{\sqrt{\gamma}})} & \textit{if } x \in [0, 1] \end{cases}. \tag{14}$$

(d) *$H_{-3/2}$ coincides with the quotient space of $H^{-1}(-1,1)\times I\!R\times I\!R$ constituted by the classes (u,y,z) characterized in the following way: Two elements (u^1,y^1,z^1) and (u^2,y^2,z^2) belong to the same class if and only if*

$$(u^1-u^2,y^1-y^2,z^1-z^2)=\alpha(m,-1,0)+\beta(n,0,\gamma^{-1})$$

where $\alpha,\beta\in I\!R$, m and n are the functions given in (14).

Let us recall now how solutions of (1)-(2) can be developed in Fourier series.

Consider de energy space $\mathcal{H}=H_0\times H_{-1/2}$ and define $\bar\phi_k=(\phi_k,i\lambda_k\phi_k)$ where $\lambda_{-k}=-\lambda_k$ and $\phi_{-k}=\phi_k$. The set $(\bar\phi_k)_{k\in Z\!\!Z}$ constitutes an orthonormal basis in $\mathcal{H}$. Then, for any initial data $((u^0,y^0,z^0),(u^1,y^1,z^1))\in\mathcal{H}$ we can find a sequence of coefficients (a_k) such that

$$((u^0,y^0,z^0),(u^1,y^1,z^1))=\sum_{k\in Z\!\!Z}a_k\bar\phi_k$$

and the vector valued solution $U=((u,y,z),(u_t,y_t,z_t))$ of (1), (2), (3) and (4) is given by

$$U(t)=\sum_{k\in Z\!\!Z}a_ke^{i\lambda_kt}\bar\phi_k. \tag{15}$$

The conservation of the energy E in (5) is equivalent to the fact that system (1)-(2) generates a group of isometries in $\mathcal{H}$. More precisely

$$E(t)=\parallel U(t)\parallel_{\mathcal{H}}^2=\sum_{k\in Z\!\!Z}\left|a_ke^{i\lambda_kt}\bar\phi_k\right|^2=\sum_{k\in Z\!\!Z}\mid a_k\mid^2=\parallel U(0)\parallel_{\mathcal{H}}^2=E(0). \tag{16}$$

Obviously, one can also obtain developments in Fourier series of the form (15) for solutions of (1)-(2) in other classes $\mathcal{H}_\alpha=H_\alpha\times H_{\alpha-1/2}$.

2.2. Asymptotics of the spectrum. In this section we recall the main results concerning the asymptotic behavior of the eigenvalues and eigenfunctions of (9) that we will need later to prove the observability inequalities.

PROPOSITION 2.3. *We have*

$$\lambda_{2k-1}=\frac{k\pi-\pi/2}{\sqrt{\gamma}}-\frac{c_1(\gamma)}{\sqrt{\gamma}\,(k\pi-\pi/2)}+O(k^{-2}),\ \ as\ k\to\infty \tag{17}$$

$$\lambda_{2k}=\frac{k\pi-\pi/2}{\sqrt{\gamma}}-\frac{c_2(\gamma)}{\sqrt{\gamma}\,(k\pi-\pi/2)}+O(k^{-2}),\ \ as\ k\to\infty \tag{18}$$

where $c_1(\gamma)=\left(2\gamma+\sqrt{\gamma}\tanh\left(\gamma^{-1/2}\right)\right)^{-1}+(2\gamma)^{-1}$ and $c_2(\gamma)=\gamma^{-1/2}\coth\gamma^{-1/2}-2+(2\gamma)^{-1}$.

Moreover,

$$\lambda_{2k}-\lambda_{2k-1}=\frac{C(\gamma)}{(k\pi-\pi/2)\,\gamma^{1/2}}+O(k^{-2}),\ \ as\ k\to\infty \tag{19}$$

where $C(\gamma)=c_1(\gamma)-c_2(\gamma)>0$, for all $\gamma>0$.

REMARK 2. *In the absence of mass the asymptotic behavior of eigenvalues is as follows:*

$$\lambda_k=\frac{k\pi}{2\sqrt{\gamma}}+O(k^{-2}),\ \ as\ k\to\infty.$$

This shows that the first term of the asymptotic expansion of λ_{2k} is affected by the presence of the mass but not the first term of λ_{2k-1}.

In view of (19) the asymptotic gap $\lambda_{2k} - \lambda_{2k-1}$ decays like $1/k$ as $k \to \infty$ in the presence of the mass. Note however that in the absence of the mass the eigenvalues are uniformly separated and the gap is of the order of $\pi/(2\sqrt{\gamma})$ for all k.

Concerning the eigenfunctions we have:

PROPOSITION 2.4. *The eigenfunctions of (9) normalized in $H^2 \cap H_0^1(-1,1)$ are*

$$\varphi_{2k-1}(x) = \frac{\rho_{2k-1}}{(\mu_{2k-1}^+)^2}\left[\sin\left(\mu_{2k-1}^+(1-|x|)\right) - \frac{\mu_{2k-1}^+ \cos\mu_{2k-1}^+}{\mu_{2k-1}^- \cosh\mu_{2k-1}^-}\sinh\left(\mu_{2k-1}^-(1-|x|)\right)\right] \tag{20}$$

$$\varphi_{2k}(x) = -\frac{\rho_{2k}}{(\mu_{2k}^+)^2}\left[\sin\left(\mu_{2k}^+(\frac{x}{|x|} - x)\right) - \frac{\sin\mu_{2k}^+}{\sinh\mu_{2k}^-}\sinh\left(\mu_{2k}^-(\frac{x}{|x|} - x)\right)\right] \tag{21}$$

where $\rho_k = 1 + \mathcal{O}(k^{-1})$ and

$$\mu_k^+ = \lambda_k\sqrt{\gamma}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\gamma^2\lambda_k^2}}}, \quad \mu_k^- = \frac{1}{\sqrt{\gamma}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\gamma^2\lambda_k^2}}}}. \tag{22}$$

Moreover, $\partial\varphi_k(1) \neq 0$.

REMARK 3. *We observe that the eigenfunctions φ_{2k-1} are even while φ_{2k} are odd. This fact is due to the symmetry of the problem with respect to $x = 0$. On the other hand, as the mass only affects the first term in the asymptotic expansion of the eigenvalues corresponding to odd eigenfuntions, these are the only eigenfunctions which are affected in a significant way by the point mass.*

2.3. Asymmetric spaces. In this section we are going to introduce and characterize some asymmetric spaces. It is easy to see that these spaces are stable under the flow generated by system (1)-(2) and they are natural spaces to solve the boundary control problem.

With the notations of section 2.1 we set

$$Y_\alpha = \left\{ U = \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k\bar{\phi}_k \in H \ : \ \| U \|_{Y_\alpha}^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{|\, a_{2k-\sigma_k} \,|^2}{\delta_k^{4\alpha}} + \frac{|\, a_{2k} - a_{2k-\sigma_k} \,|^2}{\delta_k^{4\alpha+2}} \right) < \infty \right\} \tag{23}$$

where $\delta_k = \lambda_{2k} - \lambda_{2k-\sigma_k}$, $\sigma_k = \operatorname{sgn} k$, i.e. $\sigma_k = 1$ if $k > 0$ and $\sigma_k = -1$ if $k < 0$.

Clearly Y_α endowed with the norm $\|\cdot\|_{Y_\alpha}$ is a Hilbert space. On the hand, it is clear that if all the δ_k were uniformly positive and bounded above, then Y_α would coincide algebraically and topologically with one of the energy spaces $\mathcal{H}_\alpha$.

Since, in view of Proposition 2.3, $\delta_k = \mathcal{O}(k^{-1}) = \mathcal{O}(\lambda_k^{-1}) \to 0$ as $k \to \infty$ we deduce that Y_α is a strict subspace of $\mathcal{H}_\alpha$.

We have the following result:

PROPOSITION 2.5. *Let $U^0 = \left((u^0, y^0, z^0), (u^1, y^1, z^1)\right)$ be an element of Y_α. Then, the solution $U(t) = \left((u(t), y(t), z(t)), (u_t(t), y_t(t), z_t(t))\right)$ of (1)-(2) with initial data*

U^0 belongs to Y_α for every $t>0$ and $\alpha \in I\!R$. Furthermore, for any $T>0$ there exists a constant $C(T)>0$ such that

$$\| U(t) \|_{Y_\alpha} \leq C(T) \| U^0 \|_{Y_\alpha}, \forall 0 \leq t \leq T, \forall U^0 \in Y_\alpha. \tag{24}$$

The following theorem provides a precise characterization of the spaces Y_0 and Y_{-1}:

THEOREM 2.6. **(a)** *Y_0 is the subspace of elements $U^0 = \left((u^0, y^0, z^0), (u^1, y^1, z^1)\right)$ of $\mathcal{H}$ such that the restriction of (u^0, u^1) to (0,1) belongs to $H^3(0,1) \times H^2(0,1)$ and, in addition to the compatibility conditions of $\mathcal{H}$ $\left(u^0(0) = y^0, \partial u^0(0) = z^0, u^1(0) = y^1\right)$, the following hold:*

$$\partial u^1(0^+) = z^1, \partial^2 u^0(1) = 0. \tag{25}$$

Furthermore, the norm $\| \cdot \|_{Y_0}$ is equivalent to

$$\left[\| U \|^2_{\mathcal{H}} + \left\| \left(u^0 \mid_{(0,1)}, u^1 \mid_{(0,1)} \right) \right\|^2_{H^3 \times H^2(0,1)} \right]^{1/2}.$$

(b) *Y_{-1} is the subspace of elements $U^0 = \left((u^0, y^0, z^0), (u^1, y^1, z^1)\right)$ of $\mathcal{H}_{-1}$ such that the restriction of (u^0, u^1) to (0,1) belongs to $H^1(0,1) \times L^2(0,1)$ and verify:*

$$u^0(0^+) = y^1, u^0(1) = 0. \tag{26}$$

REMARK 4. *The spaces Y_α are asymmetric in the sense that its elements have one more degree of regularity to the left of $x=0$.*

The characterization of Y_α as asymmetric spaces can be explained as follows: The vectors $p_k = \delta_k^{2\alpha}(\bar{\phi}_{2k-1} + \bar{\phi}_{2k})/2$, $q_k = \delta_k^{2\alpha+1}(\bar{\phi}_{2k-1} - \bar{\phi}_{2k})/2$ constitute a Riesz basis of Y_α. We observe that p_k and q_k are constituted by the functions $(\varphi_{2k-1} + \varphi_{2k})/2$, $(\varphi_{2k-1} - \varphi_{2k})/2$ weighted differently. As it can be seen in Figure 1 for $k=1$, due to the presence of the mass, the profiles of $(\varphi_{2k-1} + \varphi_{2k})/2$ and $(\varphi_{2k-1} - \varphi_{2k})/2$ are essentially one reflection of the other with respect to $x=0$.

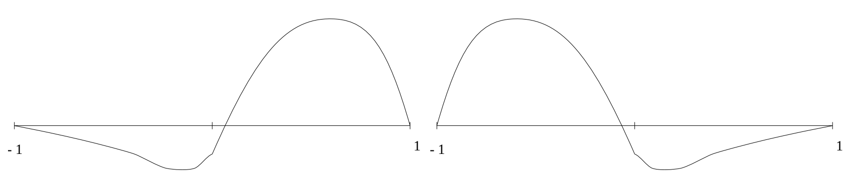

FIG. 1. $(\varphi_1 - \varphi_2)/2$ *and* $(\varphi_1 + \varphi_2)/2$

This explains the asymmetric structure of Y_α.

3. Observability. As we mentioned in the introduction, using HUM [8], the controllability of system (1) with controls of the form (6) can be reduced to the obtention of suitable observability estimates for the system in the absence of control:

$$\left\{\begin{array}{l}\gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \text{ for } x \in (-1,0),\ 0 < t < T \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \text{ for } x \in (0,1),\ 0 < t < T \\ [u]\,(0,t) = [\partial u]\,(0,t) = 0, \text{ for } 0 < t < T \\ u_{tt}(0,t) + \left[\partial^3 u\right](0,t) = 0, \text{ for } 0 < t < T \\ \gamma\partial u_{tt}(0,t) - \left[\partial^2 u\right](0,t) = 0, \text{ for } 0 < t < T \\ u(\pm 1,t) = \partial^2 u(\pm 1,t) = 0, \text{ for } 0 < t < T \\ (u(x,0), u(0,0), \partial u(0,0)) = \left(u^0, y^0, z^0\right), \\ (u_t(x,0), u_t(0,0), \partial u_t(0,0)) = \left(u^1, y^1, z^1\right). \end{array}\right. \tag{27}$$

As in previous sections, we identify the solution u of (1) with the vector valued unknown U. The following holds:

Lemma 3.1. *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T |\partial u(1,t)|^2\, dt \leq C \parallel U^0 \parallel^2_{Y_{-1}},\ \forall U^0 \in Y_{-1}. \tag{28}$$

Moreover, if $T \geq 4\sqrt{\gamma}$ there exists $C(T) > 0$ such that

$$\parallel U^0 \parallel^2_{Y_{-1}} \leq C \int_0^T |\partial u(1,t)|^2\, dt,\ \forall U^0 \in Y_{-1}. \tag{29}$$

Remark 5. *The first estimate (28) of this lemma establishes a hidden regularity result since, in view of Proposition 2.5 and Theorem 2.6, the fact that $U^0 \in Y_{-1}$ implies $u\,\big|_{(0,1)} \in C\left([0,T]; H^1(0,1)\right) \cap C^1\left([0,T]; L^2(0,1)\right)$ but this is not sufficient to guarantee that $\partial u(1,t) \in L^2(0,T)$.*

The second estimate (29) of the lemma guarantees that the norm of the initial data in Y_{-1} can be observed continuously in terms of the $L^2(0,T)$-norm of $\partial u(1,t)$. In view of Proposition 2.5 this implies that the norm of the solution U in $C([0,T], Y_{-1})$ can be observed too. Due to the finite speed of propagation of the system the time required for the uniform observability (29) has to be large enough. The lower bound $4\sqrt{\gamma}$ is sharp.

In order to prove Lemma 3.1 we need the following result on non-harmonic Fourier series due to Ulrich [10].

Theorem 3.2. *Let $(\sigma_n)_{n\in\mathbb{Z}}$ and $(\tau_n)_{n\in\mathbb{Z}}$ be two sequences of distinct complex numbers such that $\sigma_n \neq \tau_n$ for all $n \in \mathbb{Z}$ and*

$$\lim_{|n|\to\infty} |\sigma_n - n| = \lim_{|n|\to\infty} |\tau_n - n| = 0. \tag{30}$$

Then $\left\{e^{i\sigma_n t}\right\}_{n\in\mathbb{Z}}$ forms a Riesz basis of $L^2(0,2\pi)$ and moreover

$$\left\{e^{i\sigma_n t}\right\}_{n\in\mathbb{Z}} \cup \left\{\frac{e^{i\sigma_n t} - e^{i\tau_n t}}{\sigma_n - \tau_n}\right\}_{n\in\mathbb{Z}} \tag{31}$$

forms a Riesz basis of $L^2(0,4\pi)$.

Remark 6. *We refer to [3] and [1] for a generalization of this result.*

As an immediate consequence of this result the following holds:

Corollary 3.3. *Let $\gamma > 0$. Then, if $\left\{\lambda_k^2\right\}_{k\in\mathbb{Z}}$ denote the eigenvalues of system (27),*

$$\left\{e^{i\lambda_{2k} t}\right\}_{k\in\mathbb{Z}} \cup \left\{\frac{e^{i\lambda_{2k} t} - e^{i\lambda_{2k-\sigma_k} t}}{\lambda_{2k} - \lambda_{2k-\sigma_k}}\right\}_{k\in\mathbb{Z}} \tag{32}$$

form a Riesz basis of $L^2\left(0,4\sqrt{\gamma}\right)$.

Proof. We introduce the change of variables $s = t\pi/\sqrt{\gamma}$ that transforms the functions in (32) into:

$$e^{-is/2}\left\{e^{i\left(\lambda_{2k}\sqrt{\gamma}/\pi+1/2\right)s}\right\}_{k\in\mathbb{Z}} \cup e^{-is/2}\left\{\frac{e^{i\left(\lambda_{2k}\sqrt{\gamma}/\pi+1/2\right)s}-e^{i\left(\lambda_{2k-\sigma_k}\sqrt{\gamma}/\pi+1/2\right)s}}{\delta_k}\right\}_{k\in\mathbb{Z}}$$

and the interval $t\in\left(0,4\sqrt{\gamma}\right)$ into $s\in(0,4\pi)$. Obviously the common multiplicative factor $e^{-is/2}$ does not affect the fact that these functions constitute a Riesz basis of $L^2(0,4\pi)$ or not.

By setting

$$\tau_k = \lambda_{2k}\frac{\sqrt{\gamma}}{\pi}+\frac{1}{2},\ \sigma_k = \lambda_{2k-\sigma_k}\frac{\sqrt{\gamma}}{\pi}+\frac{1}{2}$$

we are in the conditions of Theorem 3.2 in view of the asymptotic form of the eigenvalues proved in Propositions 2.3.

Undoing the change of variables we deduce that Corollary 3.3 holds. □

The following characterization of Riesz basis (which can be seen in Young [11]) will also be used:

THEOREM 3.4. *Let H be a separable Hilbert space. The following two properties are equivalent:*

(a) $\{e_n\}_{n\in\mathbb{Z}}$ *forms a Riesz basis of H;*

(b) $\{e_n\}_{n\in\mathbb{Z}}$ *is a complete sequence in H and there exists two positive constants $A,B>0$ such that*

$$A\sum_{i=1}^{n}\mid c_i\mid^2\leq\left\|\sum_{i=1}^{n}c_ie_i\right\|_H^2\leq B\sum_{i=1}^{n}\mid c_i\mid^2$$

for any $n\in\mathbb{N}$ and $c_i, i=1,\cdots,n$.

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1 Recall that the solution u of (27) can be written as

$$(u,u(0),\partial u(0))=\sum_{k\in\mathbb{Z}\setminus\{0\}}a_ke^{i\lambda_kt}\phi_k(x)$$

where $\phi_k=(\varphi_k,\varphi_k(0),\partial\varphi_k(0))$ are the eigenfunctions of the eigenvalue problem (8) with φ_k normalized in $H^2\cap H_0^1(-1,1)$ such that $(-1)^k\partial\varphi_k(1)>0$.

They are of the form

$$\begin{aligned}\varphi_{2k-\sigma_k}(x)=\frac{\rho_{2k-\sigma_k}}{\left(\mu^+_{2k-\sigma_k}\right)^2}&\left[\sin\left(\mu^+_{2k-\sigma_k}(1-x)\right)\right.\\&\left.-\frac{\mu^+_{2k-\sigma_k}\cos\mu^+_{2k-\sigma_k}}{\mu^-_{2k-\sigma_k}\cosh\mu^-_{2k-\sigma_k}}\sinh\left(\mu^-_{2k-\sigma_k}(1-x)\right)\right]\\ \varphi_{2k}(x)=-\frac{\rho_{2k}}{\left(\mu^+_{2k}\right)^2}&\left[\sin\left(\mu^+_{2k}(1-x)\right)-\frac{\sin\left(\mu^+_{2k}\right)}{\sinh\left(\mu^-_{2k}\right)}\sinh\left(\mu^-_{2k}(1-x)\right)\right]\end{aligned}$$

in the interval $(0,1)$.

We introduce the following change in the Fourier coefficients of the solutions: $\widetilde{a}_k = a_k \lambda_k |\partial\varphi_k(1)|$. Then

$$u(x,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widetilde{a}_k}{\lambda_k \mid \partial\varphi_k(1)\mid} e^{i\lambda_k t}\varphi_k(x).$$

Observe that $\{a_k\} \in \ell^2$ if and only if $\{\widetilde{a}_k\} \in \ell^2$. For that it is sufficient to see that there exist positive constants $A, B > 0$ such that $A \leq \mid \lambda_k \mid\mid \partial\varphi_k(1) \mid \leq B$ for all k. To do that we observe that

$$(33)\qquad |\lambda_{2k-\sigma_k}\partial\varphi_{2k-\sigma_k}(1)| = \frac{\lambda_{2k-\sigma_k}\rho_{2k-\sigma_k}}{\left(\mu^+_{2k-\sigma_k}\right)^2}\left(\mu^+_{2k-\sigma_k} - \mu^+_{2k-\sigma_k}\frac{\cos\mu^+_{2k-\sigma_k}}{\cosh\mu^-_{2k-\sigma_k}}\right)$$
$$= \frac{1}{\sqrt{\gamma}} + O(k^{-1}),$$

$$(34)\qquad |\lambda_{2k}\partial\varphi_{2k}(1)| = \frac{\lambda_{2k}\rho_{2k}}{\left(\mu^+_{2k}\right)^2}\left(\mu^+_{2k} - \frac{\sin\left(\mu^+_{2k}\right)\mu^-_{2k}}{\sinh\left(\mu^-_{2k}\right)}\right) = \frac{1}{\sqrt{\gamma}} + O(k^{-1})$$

in view of (22) and the asymptotic results of section 2.2. Thus, $\mid \lambda_k\partial\varphi_k(1) \mid\to 1/\sqrt{\gamma}$ as $\mid k \mid\to \infty$ and, on the other hand, $\mid \lambda_k\partial\varphi_k(1) \mid\neq 0$ for all $k \neq 0$. Therefore, constants $A, B > 0$ exist.

Consequently

$$(35)\qquad A\sum_{k\in\mathbb{Z}\setminus\{0\}} \mid a_k \mid^2 \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \mid \widetilde{a}_k \mid^2 \leq B\sum_{k\in\mathbb{Z}\setminus\{0\}} \mid a_k \mid^2 .$$

On the other hand, the norm in the asymmetric space Y_{-1} can also be written in an equivalent form in terms of the coefficients $\widetilde{a}_k$. Indeed,

$$(36)\qquad A^2\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^4 \left|a_{2k-\sigma_k}\right|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid a_{2k} - a_{2k-\sigma_k} \mid^2\right)$$
$$\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^4 \left|\widetilde{a}_{2k-\sigma_k}\right|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid \widetilde{a}_{2k} - \widetilde{a}_{2k-\sigma_k} \mid^2$$
$$\leq B^2\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^4 \mid a_{2k-\sigma_k} \mid^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid a_{2k} - a_{2k-\sigma_k} \mid^2\right).$$

We start with the second inequality:

$$(37)\qquad \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid \widetilde{a}_{2k} - \widetilde{a}_{2k-\sigma_k} \mid^2$$
$$= \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid a_{2k}\lambda_{2k} \mid \partial\varphi_{2k}(1) \mid - a_{2k-\sigma_k}\lambda_{2k-\sigma_k} \mid \partial\varphi_{2k-\sigma_k}(1) \mid\mid^2$$
$$\leq 4\sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\mid a_{2k} \mid^2 \delta_k^4 \left|\frac{\lambda_{2k} \mid \partial\varphi_{2k}(1) \mid - 1/\sqrt{\gamma}}{\delta_k}\right|^2\right.$$
$$\left. + \mid a_{2k-\sigma_k} \mid^2 \delta_k^4 \left|\frac{\lambda_{2k-\sigma_k} \mid \partial\phi_{2k-\sigma_k}(1) \mid - 1/\sqrt{\gamma}}{\delta_k}\right|^2\right] + \frac{2}{\gamma}\sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k^2 \mid a_{2k} - a_{2k-\sigma_k} \mid^2$$

$$\leq C\left\{\sum_{k\in\mathbb{Z}}\delta_k^4\left|a_{2k-\sigma_k}\right|^2+\sum_{k\in\mathbb{Z}\setminus\{0\}}\delta_k^4\mid a_{2k}\mid^2\right\}+\frac{2}{\gamma}\sum_{k\in\mathbb{Z}\setminus\{0\}}\delta_k^2\mid a_{2k}-a_{2k-\sigma_k}\mid^2$$

since, in view of (33)-(34), $\mid \lambda_{2k} \mid \partial\varphi_{2k}(1) \mid -1/\sqrt{\gamma}\mid$, $\mid \lambda_{2k-\sigma_k} \mid \partial\varphi_{2k-\sigma_k}(1) \mid -1/\sqrt{\gamma}\mid$ and δ_k are of the order of $1/k$.

Clearly, the last term in (37) can be bounded in terms of

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\delta_k^4\mid a_{2k-\sigma_k}\mid^2+\sum_{k\in\mathbb{Z}\setminus\{0\}}\delta_k^2\mid a_{2k}-a_{2k-\sigma_k}\mid^2 .$$

The first inequality in (36) can be proved in a similar way.

Now, taking into account that $\partial u(1,t)=\sum_{k\in\mathbb{Z}\setminus\{0\}}a_k\partial\varphi_k(1)e^{i\lambda_k t}$ we deduce that

$$\int_0^T\left|\partial u(1,t)\right|^2dt=\int_0^T\left|\sum_{k\in\mathbb{Z}\setminus\{0\}}a_k\partial\varphi_k(1)e^{i\lambda_k t}\right|^2dt=\int_0^T\left|\sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^k\frac{\widetilde{a}_k}{\lambda_k}e^{i\lambda_k t}\right|^2dt$$

$$=\int_0^T\left|\sum_{k\in\mathbb{Z}\setminus\{0\}}\left[\left(\frac{\widetilde{a}_{2k}}{\lambda_{2k}}-\frac{\widetilde{a}_{2k-\sigma_k}}{\lambda_{2k-\sigma_k}}\right)e^{i\lambda_{2k}t}+\delta_k\left(\frac{\widetilde{a}_{2k-\sigma_k}}{\lambda_{2k-\sigma_k}}\right)\frac{e^{i\lambda_{2k}t}-e^{i\lambda_{2k-\sigma_k}t}}{\delta_k}\right]\right|^2dt$$

$$\leq C\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(\left|\frac{\widetilde{a}_{2k}}{\lambda_{2k}}-\frac{\widetilde{a}_{2k-\sigma_k}}{\lambda_{2k-\sigma_k}}\right|^2+\delta_k^2\left|\frac{\widetilde{a}_{2k-\sigma_k}}{\lambda_{2k-\sigma_k}}\right|^2\right)$$

$$\leq C\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(\frac{\left|\widetilde{a}_{2k}-\widetilde{a}_{2k-\sigma_k}\right|^2}{\lambda_{2k}^2}+\left|\widetilde{a}_{2k-\sigma_k}\right|^2\left|\frac{1}{\lambda_{2k}}-\frac{1}{\lambda_{2k-\sigma_k}}\right|^2+\frac{\delta_k^2}{\lambda_{2k-\sigma_k}}\left|\widetilde{a}_{2k-\sigma_k}\right|^2\right)$$

$$\leq C\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(\delta_k^2\left|\widetilde{a}_{2k}-\widetilde{a}_{2k-\sigma_k}\right|^2+\delta_k^4\left|\widetilde{a}_{2k-\sigma_k}\right|^2\right)\leq C\parallel U^0\parallel_Y^2$$

in view of Corollary 3.3, inequalities (36) and the fact that $\lambda_{2k}\sim 1/\delta_k$ and $\lambda_{2k-\sigma_k}\sim 1/\delta_k$.

A similar computation shows that (29) holds too.

Inequalities (36) imply the statement of Lemma 3.1 in view of Theorem 3.4. ◻

4. On the solvability of the system with non-homogeneous data. In this section we analyze the existence, uniqueness and regularity of non-homogeneous boundary value problems that appear when addressing the control problem.

In order to state the main results of this section it is convenient to introduce the following asymmetric spaces:

(38) $$V_0^+=\left\{(\varphi,\eta,\xi)\in H_{-3/2}\ :\ \varphi|_{(0,1)}\in L^2(0,1)\right\};$$

(39) $$V_1^+=\left\{(\varphi,\eta,\xi)\in H_{-1}\ :\ \varphi|_{(0,1)}\in H^1(0,1),\ \varphi(0^+)=\eta,\ \varphi(1)=0\right\}.$$

In these notations the superscripts + indicate that the elements of these spaces are more regular to the right of $x=0$, while the subscripts 0 (resp. 1) indicates that the maximal regularity is L^2 (resp. H^1).

Remark 7. *In view of the characterization of Y_{-1} given in Theorem 2.6 we observe that $Y_{-1}=V_1^+\times V_0^+$.*

This section is divided in two parts. In the first one we analyze systems with non-zero right hand side terms. In the second one we address non-homogeneous boundary value problems by transposition.

4.1. Systems with non-zero right hand side. Let us consider the non homogeneous system:

$$
(40)\qquad \begin{cases} \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = f,\ x \in (-1,0), 0<t<T \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = f,\ x \in (0,1), 0<t<T \\ [u]\,(0,t) = [\partial u]\,(0,t) = 0,\ 0<t<T \\ u_{tt}(0,t) + \left[\partial^3 u\right](0,t) = g,\ 0<t<T \\ \gamma \partial u_{tt}(0,t) - \left[\partial^2 u\right](0,t) = h,\ 0<t<T \\ u(\pm 1,t) = \partial^2 u(\pm 1,t) = 0,\ 0<t<T \\ (u(x,0), u(0,0), \partial u(0,0)) = \left(u^0, y^0, z^0\right), \\ (u_t(x,0), u_t(0,0), \partial u_t(0,0)) = (u^1, y^1, z^1). \end{cases}
$$

Observe that the boundary conditions at $x=\pm 1$ vanish.

We have the following result:

THEOREM 4.1. *Assume that* $U^0 = \left((u^0,y^0,z^0),(u^1,y^1,z^1)\right) \in H_{-1/2}\times H_{-1}$ *and*

$$
(41)\qquad ((1-\gamma\partial^2)^{-1} f, g, h) \in L^2\left(0,T; H_{-1}\right).
$$

Then, there exists an unique solution $U \in C\left([0,T]; H_{-1/2}\times H_{-1}\right)$ *of (40).*

Moreover, there exists $C(T)>0$ *such that*

$$
(42)\qquad \int_0^T \left|\partial u(1,t)\right|^2 dt \\ \leq C\left(\|\ ((1-\gamma\partial^2)^{-1} f, g, h)\ \|^2_{L^1(0,T;H_0(-1,1))} + \|\ U^0\ \|^2_{H_{-1/2}\times H_{-1}}\right)
$$

for all U^0 *and* (f,g,h) *as above.*

REMARK 8. *The first result of this theorem is classical and provides the existence of solutions with values in* $H_{-1/2}\times H_{-1}$*, which is a natural symmetric energy space for solving (40).*

Inequality (42) extends the "hidden regularity" result of Lemma 3.1 to the solutions of the non-homogeneous system. However, (42) is not sharp since it requires the same degree of regularity at both sides of $x=0$ *while in Lemma 3.1 this degree of regularity is only required on* $(0,1)$*. The next theorem provides a sharp result.*

THEOREM 4.2. *Assume that* $U^0 \in Y_{-1}$ *and* $((1-\gamma\partial^2)^{-1} f, g, h) \in L^1(0,T;V_0^+)$*. Then, there exists an unique solution* $U \in C\left([0,T]; Y_{-1}\right)$ *of (40).*

Moreover, there exists $C(T)>0$ *such that*

$$
(43)\qquad \int_0^T \left|\partial u(1,t)\right|^2 dt \leq C\left[\|\ ((1-\gamma\partial^2)^{-1} f, g, h)\ \|^2_{L^1(0,T;V_0^+)} + \|\ U^0\ \|^2_{Y_{-1}}\right],
$$

for every U^0 *and* (f,g,h) *as above.*

The proof of both theorems is rather similar. For simplicity we only prove Theorem 4.2.

Proof of Theorem 4.2. Taking into account that the system is linear and in view of Lemma 3.1 it is sufficient to prove it when $U^0 \equiv 0$.

To simplify the notation we identify $(I-\gamma\partial^2)^{-1} f$ and the vector valued function $\left((I-\gamma\partial^2)^{-1} f, g, h\right)$.

We observe that $(I-\gamma\partial^2)^{-1} f \in L^1(0,T;V_0^+)$ and then, in view of Remark 7, we have $\left(0, (I-\gamma\partial^2)^{-1} f\right) \in L^1(0,T;Y_{-1})$

On the other hand, composing system (27) with the operator $(I-\gamma\partial^2)^{-1}$ and identifying the unknown vector $(u, u(0), \partial u(0))$ with u, it can be written as:

$$u_{tt} + Au = (I-\gamma\partial^2)^{-1} f$$

where $A = K^{-1}$ is the underlying elliptic operator.

Since $U^0 \equiv 0$, by the variation of constants formula $u(t) = \int_0^t v(t-s; s)ds$ where $v(\cdot,\cdot; s)$ satisfies

$$\text{(44)} \qquad \begin{cases} v_{tt} + Av = 0 \\ v(0;s) = 0, \quad v_t(0;s) = (I-\gamma\partial^2)^{-1} f(s). \end{cases}$$

In view of Proposition 2.5 it is easy to see that u or, more precisely, its corresponding vector-valued solution U, belongs to $C([0,T]; Y_{-1})$.

On the other hand, in view of Lemma 3.1 we have

$$\int_0^T |\partial v(1,t;s)|^2 dt \le C \left\| \left(0, \left[(I-\gamma\partial^2)^{-1} f\right](s)\right)\right\|^2_{Y_{-1}}$$
$$= C \parallel \left[(I-\gamma\partial^2)^{-1} f\right](s) \parallel^2_{V_0^+}, \quad \forall s \in [0,T].$$

By Minkowski's inequality we deduce that

$$\parallel \partial u(1,t) \parallel_{L^2(0,T)} = \left\| \int_0^t \partial v(1, t-s; s) ds \right\|_{L^2(0,T)} \le C \left\| (I-\gamma\partial^2)^{-1} f \right\|_{L^1(0,T;V_0^+)} . \square$$

The following result is also needed:

Theorem 4.3. *Assume that* $U^0 \equiv 0$ *and* $(f,g,h) = \partial_t(F,G,H)$ *satisfying* $((I-\gamma\partial^2)^{-1}F, G, H) \in L^1\left(0,T; V_1^+\right)$. *Then the solution of (40) verifies* $U \in C([0,T]; Y_{-1})$. *Moreover, there exists* $C(T) > 0$ *such that*

$$\text{(45)} \qquad \int_0^T |\partial u(1,t)|^2 dt \le C \parallel ((I-\gamma\partial^2)^{-1} F, G, H) \parallel^2_{L^1(0,T;V_1^+)}$$

for every (F,G,H) *as above.*

Proof of Theorem 4.3. As in Theorem 4.2 above we identify $(I-\gamma\partial^2)^{-1}F$ (resp. $(I-\gamma\partial^2)^{-1}F_t$) with $((I-\gamma\partial^2)^1 F, G, H)$ (resp. $(I-\gamma\partial^2)^{-1}F_t, G_t, H_t)$) to simplify the notation.

On the other hand, $u = v_t$ where v, which is also identified with the unknown vector $(v, v(0), \partial v(0))$, is the solution of

$$\text{(46)} \qquad \begin{cases} v_{tt} + Av = (I-\gamma\partial^2)^{-1} F \\ v(0) = v_t(0) = 0. \end{cases}$$

Observe that we have taken null initial data for v. This is due to the fact that in this proof we may assume $((1-\gamma\partial^2)^{-1}F, G, H)$ to be of compact support in time. Indeed, if Theorem 4.3 is proved for those $((1-\gamma\partial^2)^{-1}F, G, H)$, it can then be extended by density to all $((1-\gamma\partial^2)^{-1}F, G, H) \in L^1(0,T; V_1^+)$.

With this in mind we see that the appropriate initial conditions for v are as follows:

$$v(0) = u_t(0) = 0; \; v_t(0) = u_{tt}(0) = (I-\gamma\partial^2)^{-1}F(0) - Au(0) = 0.$$

To complete the proof of Theorem 4.3 it is sufficient to prove that the following lemma holds:

LEMMA 4.4. *Assume that* $U^0 = 0$ *and* $((1-\gamma\partial^2)^{-1}f, g, h) \in L^1(0,T;V_1^+)$. *Then, there exists* $C > 0$ *such that the solution of (40) satisfies*

$$\int_0^T |\partial u_t(1,t)|^2 \, dt \leq C \parallel ((1-\gamma\partial^2)^{-1}f, g, h) \parallel^2_{L^1(0,T;V_1^+)} \tag{47}$$

for all (f, g, h) *as above.*

Proof of Lemma 4.4. First of all we observe that in view of the characterization of the asymmetric space Y in Theorem 2.6 it is easy to see that $((I-\gamma\partial^2)^{-1}f, 0) \in L^1(0,T;Y_{-1})$. Note that we identify $(I-\gamma\partial^2)^{-1}f$ with the vector $((I-\gamma\partial^2)^{-1}f, g, h)$ to simplify the notation.

As in Theorem 4.2 above $u = \int_0^t v(x, t-s; s)ds$ where v solves (44). Then $\omega = v_t$ verifies

$$\begin{cases} \omega_{tt} + A\omega = 0 \\ \omega(0; s) = (I-\gamma\partial^2)^{-1}f(s), \quad \omega_t(0; s) = 0. \end{cases} \tag{48}$$

In view of Lemma 3.2 we have

$$\int_0^T |\partial\omega(1,t)|^2 \, dt \leq C \parallel ((I-\gamma\partial^2)^{-1}f, 0) \parallel^2_{Y_{-1}} = C \parallel (I-\gamma\partial^2)^{-1}f \parallel^2_{V_1^+} . \tag{49}$$

On the other hand

$$\partial u_t = \int_0^t \partial v_t(t-s; s)ds = \int_0^t \partial\omega(t-s; s)ds$$

and therefore

$$\int_0^T |\partial u_t(1,t)|^2 \, dt = \left\| \int_0^t \partial\omega(1, t-s; s)ds \right\|^2_{L^2_t(0,T)} . \tag{50}$$

Now, by Minkowski's inequality and (49) we deduce that

$$\left\| \int_0^t \partial\omega(1, t-s; s)ds \right\|_{L^2_t(0,T)} \leq C \parallel (I-\gamma\partial^2)^{-1}f \parallel_{L^1(0,T;V_1^+)} . \tag{51}$$

Combining (50)-(51) we deduce that (47) holds.

This concludes the proof of Lemma 4.4 and Theorem 4.3. □

4.2. Non-homogeneous boundary conditions. Let us consider now the system

$$\begin{cases} \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \ x \in (-1,0), \ 0 < t < T \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \ x \in (0,1), \ 0 < t < T \\ [u](0,t) = [\partial u](0,t) = 0, \ 0 < t < T \\ u_{tt}(0,t) + \left[\partial^3 u\right](0,t) = 0, \ 0 < t < T \\ \partial u_{tt}(0,t) - \left[\partial^2 u\right](0,t) = 0, \ 0 < t < T \\ u(\pm 1, t) = \partial^2 u(-1,t) = 0, \ \partial^2 u(1,t) = q(t), \ 0 < t < T \\ (u(x,0), u(0,0), \partial u(0,0)) = \left(u^0, y^0, z^0\right), \\ (u_t(x,0), u_t(0,0), \partial u_t(0,0)) = \left(u^1, y^1, z^1\right). \end{cases} \tag{52}$$

Observe that the boundary conditions in (52) are non-homogeneous since the boundary condition $\partial^2 u(1,t)$ takes the value q.

We assume that

$$q \in L^2(0,T) \tag{53}$$

and

$$U^0 \in Y^- = \{U^0 \in H_0 \times H_{-1/2} \ : \ u^0|_{(-1,0)} \in H^3(-1,0),\ \partial^2 u^0(-1) = 0,\\ u^1|_{(-1,0)} \in H^2(-1,0),\ \partial u^1(0^-) = z^1\}. \tag{54}$$

Observe that Y^- is the reflection of the space Y_0 (characterized in Theorem 2.6) with respect to $x = 0$. In other words, $U^0 \in Y^-$ if and only if $V^0(x) = U^0(-x) \in Y_0$.

In view of Proposition 2.5 and taking into account that system (52) is symmetric with respect to $x = 0$ when $q = 0$ the following holds:

Lemma 4.5. *When $q \equiv 0$ and $U^0 \in Y^-$ system (6.16) admits an unique solution $U \in C([0,T];Y^-)$. Moreover, there exists $C(T) > 0$ such that*

$$\| U \|_{L^\infty(0,T;Y^-)} \leq C(T) \| U^0 \|_{Y^-},\ \forall U^0 \in Y^-. \tag{55}$$

As a consequence of this Lemma and since system (52) is linear it is sufficient to analyze solutions of (52) when $U^0 \equiv 0$ and q is as in (53). Therefore, in the sequel we assume that $U^0 \equiv 0$.

Solutions of (52) can be understood in the sense of transposition. To make precise this notion we consider the adjoint system:

$$\begin{cases} \gamma\partial^2\varphi_{tt} - \varphi_{tt} - \partial^4\varphi = f,\ x \in (-1,0),\ 0 < t < T \\ \gamma\partial^2\varphi_{tt} - \varphi_{tt} - \partial^4\varphi = f,\ x \in (0,1),\ 0 < t < T \\ [\varphi](0,t) = [\partial\varphi](0,t) = 0,\ 0 < t < T \\ \varphi_{tt}(0,t) + \left[\partial^3\varphi\right](0,t) = g(t),\ 0 < t < T \\ \gamma\partial\varphi_{tt}(0,t) - \left[\partial^2\varphi\right](0,t) = h(t),\ 0 < t < T \\ \varphi(\pm 1,t) = \partial^2\varphi(\pm 1,t) = 0,\ 0 < t < T \\ (\varphi(x,T), \varphi(0,T), \partial\varphi(0,T)) = (\varphi_t(x,T), \varphi_t(0,T), \partial\varphi_t(0,T)) \equiv 0. \end{cases} \tag{56}$$

Given $((I-\gamma\partial^2)^{-1}f, g, h) \in L^1(0,T;V_0^+)$ and taking into account that system (56) is time-reversible, in view of Theorem 4.2 we deduce that (56) admits an unique solution $\Phi \in C([0,T];Y_{-1})$ (by Φ we denote the vector-valued unknown associated to φ).

On the other hand, $\partial\varphi_t(1,t) \in L^2(0,T)$.

Multiplying in (52) by φ and integrating by parts we get, at least formally, the following identity:

$$\int_0^T < (I-\gamma\partial^2)u, (I-\gamma\partial^2)^{-1}f >_{H_0^1,H^{-1}} dt + \int_0^T y(t)g(t)dt + \int_0^T z(t)h(t)dt \\ = \int_0^T \partial\varphi(1,t)q(t)dt, \tag{57}$$

where $y(t) = u(0,t)$, $z(t) = \partial u(0,t)$ and $<,>_{H_0^1,H^{-1}}$ represents the duality product between $H_0^1(-1,1)$ and H^{-1}.

Note that we have used the selfadjointness of $(I-\gamma\partial^2)^{-1}$ in the identity

$$\int_{-1}^{1} uf = \int_{-1}^{1} (I-\gamma\partial^2)u(I-\gamma\partial^2)^{-1}f = < (I-\gamma\partial^2)u, (I-\gamma\partial^2)^{-1}f >_{H_0^1, H^{-1}} .$$

We adopt (57) as definition of solution of (52) when $U^0 \equiv 0$.

Definition. *We say that $U \in C(0,T;Y^-)$ is a solution of (52) with $U^0 \equiv 0$ in the sense of transposition if (57) holds for any $((I-\gamma\partial^2)^{-1}f, g, h) \in L^1(0,T;V_0^+)$.*

REMARK 9. *As we will see below, solutions in the sense of transposition are more regular on the left hand side of $x = 0$. They satisfy*

$$u|_{(-1,0)} \in C\left([0,T]; H^3(-1,0)\right) \cap C^1\left([0,T]; H^2(-1,0)\right) \tag{58}$$

and the compatibility conditions

$$\partial u_t(0^-, t) = z_t(t), \quad \partial^2 u(-1) = 0. \tag{59}$$

Observe that the initial condition $U^0 \equiv 0$ is implicit in (57).

THEOREM 4.6. *When $U^0 \equiv 0$ and q is as in (53), system (52) admits an unique solution in the sense of transposition.*

Moreover, there exists $C(T) > 0$ such that

$$\| U \|_{C([0,T];Y^-)} \leq C(T) \| q \|_{L^2(0,T)}, \tag{60}$$

for every q as above.

Proof of Theorem 4.6. In view of Theorem 4.2 the right hand side of (57) defines a linear continuous operator in $L^1(0,T;V_0^+)$. Therefore, by duality, we deduce that there exists an unique $(u,y,z) \in L^\infty\left(0,T;(V_0^+)'\right)$ solution of (57) where $\left(V_0^+\right)'$ denotes the dual of V_0^+. Moreover, there exists $C > 0$ such that

$$\| ((I-\gamma\partial^2)u, y, z) \|_{L^\infty\left(0,T;(V_0^+)'\right)} \leq C \| q \|_{L^2(0,T)} . \tag{61}$$

Furthermore, as a consequence of Theorem 4.2 we deduce that $(u_t, y_t, z_t) \in L^\infty\left(0,T;\left(V_1^+\right)'\right)$ and the estimate

$$\left\| \left((I-\gamma\partial^2)u_t, y_t, z_t\right) \right\|_{L^\infty\left(0,T;\left(V_1^+\right)'\right)} \leq C \| q \|_{L^2(0,T)} \tag{62}$$

holds for every q as in (53).

Observe now that the duals of V_0^+, $\left(V_0^+\right)'$, and V_1^+, coincide respectively with the spaces

$$\begin{aligned} V_1^- = \big\{ &(v,y,z) \in L^2(-1,1) \times I\!R \times I\!R \ : v|_{(-1,0)} \in H^1(-1,0),\ v(-1) = 0, \\ &[(I-\gamma\partial^2)^{-1}v](0) = y,\ [\partial(I-\gamma\partial^2)^{-1}v](0) = z \big\} \\ V_0^- = \big\{ &(v,y,z) \in H^{-1}(-1,1) \times I\!R \times I\!R \ : \ v|_{(-1,0)} \in L^2(-1,0), \\ &[(1-\gamma\partial^2)^{-1}v](0) = y,\ [\partial(1-\gamma\partial^2)^{-1}v](0^-) = z \big\} . \end{aligned}$$

On the other hand, $V_1^- \times V_0^-$ is the image of the space Y^- by the operator $\mathcal{L}$ defined as

$$\mathcal{L}((u^0,y^0,z^0),(u^1,y^1,z^1)) = (((I-\gamma\partial^2)u^0, y^0, z^0), ((I-\gamma\partial^2)u^1, y^1, z^1)). \tag{63}$$

We observe that $\mathcal{L}$ is in fact an isomorphism from Y^- to $V_1^- \times V_0^-$.

With the above considerations and (61)-(62) we deduce

$$\| U \|_{L^\infty(0,T;Y^-)} \leq C(T) \| q \|_{L^2(0,T)} .$$

Now by density it follows that (60) hold. To see this it is sufficient to observe that when q is smooth enough and of compact support, solutions of (52) belong to $C\left([0,T]; H_{1/2} \times H_0\right)$. □

5. Controllability. In this section we prove the main controllability result for the system

$$\left\{ \begin{array}{l} \gamma \partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \; x \in (-1,0), \; 0 < t < T \\ \gamma \partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \; x \in (0,1), \; 0 < t < T \\ \left[u\right](0,t) = \left[\partial u\right](0,t) = 0, \; 0 < t < T \\ u_{tt}(0,t) + \left[\partial^3 u\right](0,t) = 0, \; 0 < t < T \\ \gamma \partial u_{tt}(0,t) - \left[\partial^3 u\right](0,t) = 0, \; 0 < t < T \\ u(\pm 1,t) = \partial^2 u(-1,t) = 0, \; \partial^2 u(1,t) = q(t), \; 0 < t < T \\ (u(x,0), u(0,0), \partial u(0,0)) = \left(u^0, y^0, z^0\right), \\ (u_t(x,0), u_t(0,0), \partial u_t(0,0)) = \left(u^1, y^1, z^1\right). \end{array} \right. \tag{64}$$

The following holds:

Theorem 5.1. *Assume that $T \geq 4\sqrt{\gamma}$. Then for every $\left(\left(u^0,y^0,z^0\right), \left(u^1,y^1,z^1\right)\right) \in Y^-$ there exists a control $q \in L^2(0,T)$ such that the solution of (64) in the sense of transposition satisfies*

$$((u(x,T), u(0,T), \partial u(0,T)), (u_t(x,T), u_t(0,T), \partial u_t(0,T))) \equiv 0. \tag{65}$$

Moreover, there exists $C > 0$ such that

$$\| q \|_{L^2(0,T)} \leq C \| U^0 \|_{Y^-}, \; \forall U^0 \in Y^-. \tag{66}$$

Remark 10. *Theorem 5.1 states the exact controllability of (64) in the space Y^- with controls in $L^2(0,T)$ provided $T \geq 4\sqrt{\gamma}$.*

The functional frame we have chosen for the control problem ($U^0 \in Y^-$ and $q \in L^2(0,T)$) is not unique. A similar result holds for U^0 in Y^-_{-1} where

$$\begin{aligned} Y^-_{-1} = \big\{ & U^0 \in H_{-1} \times H_{-3/2} \; : \; u^0|_{(-1,0)} \in H^1(-1,0), \\ & u^1|_{(-1,0)} \in L^2(-1,0), u^0(0^-) = y^0, \; u^0(-1) = 0 \big\} \end{aligned}$$

with controls $q \in H^{-2}(0,T)$. In this case exact controllability holds at time $T > 4\sqrt{\gamma}$ because it is convenient to take controls q of compact support in order to avoid further singularities in the solutions at $t = 0$ and $t = T$.

Proof of Theorem 5.1. In view of the observability results of Lemma 3.1 it is a direct application of HUM (see [8]).

Given $T \geq 4\sqrt{\gamma}$, for any $\Phi^0 = \left((\varphi^0, \psi^0, \xi^0), (\varphi^1, \psi^0, \xi^1)\right) \in Y_{-1}$ we solve the adjoint system

$$\text{(67)}\qquad \begin{cases} \gamma\partial^2\varphi_{tt} - \varphi_{tt} - \partial^4\varphi = 0, \ -1 < x < 0, \ 0 < t < T \\ \gamma\partial^2\varphi_{tt} - \varphi_{tt} - \partial^4\varphi = 0, \ 0 < x < 1, \ 0 < t < T \\ [\varphi]\,(0,t) = [\partial\varphi]\,(0,t) = 0, \ 0 < t < T \\ \varphi_{tt}(0,t) + \left[\partial^3\varphi\right](0,t) = 0, \ 0 < t < T \\ \gamma\partial\varphi_{tt}(0,T) - \left[\partial^2\varphi\right](0,t) = 0, \ 0 < t < T \\ \varphi(\pm 1,t) = \partial^2\varphi(\pm 1,t) = 0, \ 0 < t < T \\ \Phi(0) \equiv \left((\varphi(x,0),\varphi(0,0),\partial\varphi(0,0)), (\varphi_t(x,0),\varphi_t(0,0),\partial\varphi_t(0,0))\right) \\ \quad = \left((\varphi^0,\psi^0,\xi^0),(\varphi^1,\psi^0,\xi^1)\right) = \Phi^0. \end{cases}$$

In view of Proposition 2.5, system (67) admits an unique solution $\Phi \in C([0,T;Y_{-1})$. Moreover, thanks to Lemma 3.1, $\partial\varphi(1,t) \in L^2(0,T)$.

We then solve

$$\text{(68)}\qquad \begin{cases} \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \ -1 < x < 0, \ 0 < t < T \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, \ 0 < x < 1, \ 0 < t < T \\ [u]\,(0,t) = [\partial u]\,(0,t) = 0, \ 0 < t < T \\ u_{tt}(0,t) + \left[\partial^3 u\right](0,t) = 0, \ 0 < t < T \\ \gamma\partial u_{tt}(0,T) - \left[\partial^2 u\right](0,t) = 0, \ 0 < t < T \\ u(\pm 1,t) = 0, \partial^2 u(-1,t) = 0, \partial^2 u(1,t) = \partial\varphi(1,t) \\ (u(x,T),u(0,T),\partial u(0,T)) \equiv (u_t(x,T),u_t(0,T),\partial u_t(0,T)) \equiv 0. \end{cases}$$

In view of Theorem 4.6 and the time-reversibility of system (68) we deduce that it has an unique solution defined by transposition.

We define the linear map

$$\Lambda\Phi^0 = \left((u(x,0),u(0,0),\partial u(0,0)), (u_t(x,0),u_t(0,0),\partial u_t(0,0))\right).$$

Multiplying in (68) by φ and integrating by parts (this a formal computation that may be done rigorous by the definition of solution in the sense of transposition) it follows that

$$\text{(69)}\qquad \langle L\Lambda\Phi^0, \Phi^0\rangle = \int_0^T \left|\partial\varphi(1,t)\right|^2 dt, \ \forall \Phi^0 \in Y_{-1}$$

where

$$L\Phi^0 = \left(\left((I-\gamma\partial^2)\varphi^1(x),\psi^1,\xi^1\right), -\left((I-\gamma\partial^2)\varphi^0(x),\psi^0,\xi^0\right)\right).$$

In view of identity (69) it follows that $L\Lambda$ is an isomorphism from Y_{-1} into its dual Y'_{-1} and therefore Λ is an isomorphism from Y_{-1} into $L^{-1}Y'_{-1}$.

Now we observe that, as it was pointed out in Remark 7, $Y_{-1} = V_1^+ \times V_0^+$ and then $Y'_{-1} = V_0^- \times V_1^-$ where V_0^- and V_1^- are the spaces introduced in the proof of Theorem 4.6.

It is easy to see that $L^{-1}(V_0^- \times V_1^-) \equiv Y^-$ algebraically and topologically.

This implies that $\Lambda : Y_{-1} \to Y^-$ is an isomorphism. Therefore, for any $U^0 \in Y^-$ there exists an unique $\Phi^0 \in Y_{-1}$ such that $\Lambda\Phi^0 = U^0$. This means that the solution U of (68) with control $q = \partial\varphi(1,t)$ where φ is the solution of (67) with initial data $\Phi^0 = \Lambda^{-1}U^0$ is such that $U(0) \equiv U^0$. Therefore q is the control we were looking for.

We also have by construction and in view of identity (69) and the observability inequalities of Lemma 3.1 that

$$\int_0^T |\partial\varphi(1,t)|^2\,dt = \left|\langle L\Lambda\Phi^0, \Phi^0\rangle\right| = \left|\langle LU^0, \Phi^0\rangle\right|$$

$$\leq C \parallel \Phi^0 \parallel_{Y_{-1}} \parallel U^0 \parallel_{Y^-} \leq C \left(\int_0^T |\partial\varphi(1,t)|^2\,dt\right)^{1/2} \parallel U^0 \parallel_{Y^-} .$$

Therefore

$$\int_0^T |\partial\varphi(1,t)|^2\,dt \leq C \parallel U^0 \parallel_{Y^-}^2$$

and consequently (66) holds. □

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