High frequency asymptotic analysis of a string with rapidly oscillating density

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Abstract: We consider the eigenvalue problem associated to the vibrations of a string with rapidly oscillating periodic density. In a previous paper we stated asymptotic formulas for the eigenvalues and eigenfunctions when the size of the microstructure ϵ is shorter than the wavelength of the eigenfunctions $1/\sqrt{\lambda^\epsilon}$. On the other hand, it has been observed that when the size of the microstructure is of the order of the wavelength of the eigenfunctions ($\epsilon \sim 1/\sqrt{\lambda^\epsilon}$) singular phenomena may occur.

In this paper we study the behavior of the eigenvalues and eigenfunctions when $1/\sqrt{\lambda^\epsilon}$ is larger than the critical size ϵ. We use the WKB approximation which allows us to find an explicit formula for eigenvalues and eigenfunctions with respect to ϵ. Our analysis provides all order correction formulas for the limit eigenvalues and eigenfunctions above the critical size.

Each term of the asymptotic expansion requires one more derivative of the density. Thus, a full description requires the density to be C^∞ smooth.

1 Introduction

Consider the following eigenvalue problem:

$$\begin{cases} u''(x) + \lambda\, \rho\left(\frac{x}{\epsilon}\right) u(x) = 0, & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\rho(x)$ is a periodic function with $0 < \rho_m \leq \rho(x) \leq \rho_M < \infty$ and ϵ is a small parameter which measures the size of the microstructure. To fix ideas and without loss of generality we take ρ of period 1.

Let us denote by $\{\lambda_k^\epsilon\}_{k\in I\!N}$ the set of eigenvalues of (1.1) ordered in an increasing way, i.e.

$$0 < \lambda_1^\epsilon < \lambda_2^\epsilon < \cdots < \lambda_k^\epsilon < \cdots \to \infty.$$

The associated eigenfunctions $\{\varphi_k^\epsilon\}_{k\in I\!N}$ can be chosen to constitute an orthogonal basis of $H_0^1(0,1)$. However, it will be convenient for us to normalize them so that $(\varphi_k^\epsilon)'(0) = 1$. We are interested in the behavior of eigenvalues and eigenfunctions for small values of the parameter ϵ.

The limit of system (1.1) as $\epsilon \to 0$ is given by

$$\begin{cases} u''(x) + \lambda\, \bar{\rho} u(x) = 0, & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where we have changed the oscillating coefficient $\rho(x/\epsilon)$ by its average $\bar{\rho} = \int_0^1 \rho(x)\, dx$.

The eigenpairs (λ_k, φ_k) of (1.2) can be computed explicitly:

$$\lambda_k = \frac{k^2\pi^2}{\bar{\rho}}, \qquad k \in I\!N \tag{1.3}$$

$$\varphi_k(x) = \frac{\sin(k\pi x)}{k\pi}, \qquad k \in I\!N. \tag{1.4}$$

It is well known that the eigenpairs $(\lambda_k^\epsilon, \varphi_k^\epsilon)$ of system (1.1) converge to those of system (1.2) as $\epsilon \to 0$. In fact, in a previous paper [7] we have proved an asymptotic expression for the eigenvalues and eigenfunctions in which the first order terms were the eigenvalues and eigenfunctions of the limit problem (1.2). However, this expression is only valid for the eigenpairs $(\lambda_k^\epsilon, \varphi_k^\epsilon)$ with $k \leq c\epsilon^{-1}$ with c small enough.

When the size of the microstructure is of the same order as the wavelength of the vibrations ($k \sim \epsilon^{-1}$ or $\lambda_k^\epsilon \sim \epsilon^{-2}$) the eigenfunctions φ_k^ϵ can exhibit a singular behavior and concentrate most of its energy near one of the extremes of the interval $[0,1]$ (see [2] and [7]). This effect has also been observed in a different eigenvalue problem, similar to (1.1), but with oscillating coefficients in the principal part (see [1]).

Eigenfunctions (1.4) do not exhibit any concentration of energy in the boundary. This allowed us to prove in [7] that, below the critical size $k \leq c\epsilon^{-1}$ none of the eigenfunctions of (1.1) concentrate on the boundary.

In this paper we complement the analysis of the low frequencies in [7] with a study of the high ones which correspond to $k >> \epsilon^{-1}$. Assuming that the coefficient $\rho \in W^{N+1,\infty}(I\!R)$ (the space of bounded measurable functions with bounded derivatives up to the order $N+1$) with $N \geq 1$, we give an approximate formula for the eigenvalues λ_k^ϵ and eigenfunctions φ_k^ϵ valid for $k \geq C_N \epsilon^{-1-1/N}$ with C_N large enough. In the particular case $N = 1$, we obtain the first order term in the expansion of λ_k^ϵ, actually $\lambda_k^\epsilon \sim (k\pi/\rho_\epsilon^*)^2$ where $\rho_\epsilon^* = \int_0^1 \sqrt{\rho(x/\epsilon)}dx$ valid for $k \geq C_1\epsilon^{-2}$. Note however that the range of validity of this approximation $k \geq C_1\epsilon^{-2}$ is far from covering the whole range of high frequencies $k >> \epsilon^{-1}$. In this case ($k \geq C_1\epsilon^{-2}$) the associated eigenfunctions φ_k^ϵ are close to

$$\varphi_k^\epsilon \sim \frac{\rho_\epsilon^*}{k\pi\,[\rho(x/\epsilon)\rho(0)]^{1/4}} \sin\left(\frac{k\pi}{\rho_\epsilon^*}\int_0^x \sqrt{\rho(s/\epsilon)}ds\right)$$

in $W^{1,\infty}(0,1)$ which of course do not exhibit any concentration of energy. For lower high frequencies, i.e. for $\epsilon^{-1} << k < C_1\epsilon^{-2}$, it is necessary to introduce some correctors which depend on ρ and its derivatives. We prove in particular that, if $\rho \in C^\infty(I\!R)$, eigenfunctions corresponding to eigenvalues λ_k^ϵ with $k \geq \epsilon^{-\alpha}$, for any $\alpha > 1$ do not exhibit any localization of energy. This result is sharp.

Observe that the behavior of the eigenvalues for low and high frequencies is different. Looking at the first term in the expansion in each case we have

$$\lambda_k^\epsilon \sim \frac{(k\pi)^2}{\bar{\rho}} \quad \text{for the low frequencies while } \lambda_k^\epsilon \sim \left(\frac{k\pi}{\rho_\epsilon^*}\right)^2 \quad \text{for the high ones.}$$

Note that $\rho_\epsilon^* = \int_0^1 \sqrt{\rho(x/\epsilon)}dx$ approaches $\int_0^1 \sqrt{\rho(x)}dx = \overline{\sqrt{\rho}}$ as $\epsilon \to 0$. On the other hand, $\int_0^1 \sqrt{\rho(x)}dx \leq \sqrt{\bar{\rho}}$ and the equality holds if and only if ρ is a constant function. Thus, the effective limit equation for the high frequencies has density $(\overline{\sqrt{\rho}})^2$ while for the low ones, the corresponding density is $\bar{\rho}$.

This work was motivated by the problem of the uniform boundary controllability of the one-dimensional wave equation with rapidly oscillating coefficients. The results in [2] show that the controllability property is not uniform due to the existence of eigenfunctions that behave in a singular way for $k \sim \epsilon^{-1}$. The results of the present work combined with the theory of non-harmonic Fourier series allow us to prove sharp uniform controllability results. We refer to [6] for some preliminary results and to [5] for a detailed discussion of this problem in the low frequency case.

All along this work we implicitly use the fact that λ_k^ϵ and k^2 are of the same order. In fact,

$$\frac{k^2\pi^2}{\rho_M} \leq \lambda_k^\epsilon \leq \frac{k^2\pi^2}{\rho_m}, \quad \text{for all } k \in I\!\!N. \tag{1.5}$$

This can be checked easily by means of some rough estimates in the Rayleigh formula. Indeed, observe that if $u \in H_0^1(0,1)$ we have

$$\frac{\int_0^1 |u'(x)|^2 dx}{\rho_M \int_0^1 |u(x)|^2 dx} \leq \frac{\int_0^1 |u'(x)|^2 dx}{\int_0^1 \rho(\frac{x}{\epsilon})|u(x)|^2 dx} \leq \frac{\int_0^1 |u'(x)|^2 dx}{\rho_m \int_0^1 |u(x)|^2 dx}. \tag{1.6}$$

Taking into account that

$$\lambda_k^\epsilon = \max_{\substack{E_k \subset H_0^1(0,1) \\ \dim E_k = k}} \min_{u \in E_k^\perp} \frac{\int_0^1 |u'(x)|^2 dx}{\int_0^1 \rho(\frac{x}{\epsilon})|u(x)|^2 dx}$$

and applying (1.6) we obtain (1.5).

The rest of the paper is organized as follows: in Section 2 we state the main approximation result for the eigenvalues and eigenfunctions. In Sections 3 and 4 we discuss the consequences for the eigenvalues and eigenfunctions respectively. In Section 5 we show how our analysis may be applied to the problem with the oscillating coefficient in the principal part of the operator: $(a(x/\epsilon)u')' + \lambda u = 0$. In Section 6 we prove our main result stated in Section 2. Finally, two technical results used in the proofs are proved in the Appendix.

2 Statement of the main result

The main result in this paper is the following:

Theorem 2.1 *Let ρ be a 1-periodic function with $0 < \rho_m \leq \rho(x) \leq \rho_M < \infty$ and such that $\rho \in W^{N+1,\infty}(I\!\!R)$ for some $N \geq 1$. Given $\delta > 0$ there exists a constant $C = C(\delta) > 0$ such that if $k \geq C\epsilon^{-1-1/N}$ with $0 < \epsilon < 1$ then*

$$\left| \sqrt{\lambda_k^\epsilon} - \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + i \sum_{n=1}^{[N/2]} \frac{\left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^{2n} \int_0^1 S_t^{2n}(s/\epsilon)ds}{k^{2n-1}\epsilon^{2n}\pi^{2n-1}} \right| \leq \delta, \tag{2.1}$$

$$\left\| \varphi_k^\epsilon - A_k^\epsilon \exp\left(\sum_{n=0}^{[N/2-1]} \frac{S^{2n+1}(x/\epsilon)}{\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon\right)^{2n}} \right) \sin\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon \sum_{n=0}^{[N/2]} \frac{S^{2n}(x/\epsilon)}{\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon\right)^{2n}} \right) \right\|_{W^{1,\infty}} \leq \delta, \tag{2.2}$$

where $[\cdot]$ denotes the integer part and

$$\sqrt{\widetilde{\lambda}_k^\epsilon} = \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + i \sum_{n=1}^{[N/2]} \frac{\left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^{2n} \int_0^1 S_t^{2n}(s/\epsilon)ds}{k^{2n-1}\epsilon^{2n}\pi^{2n-1}}. \tag{2.3}$$

Here A_k^ϵ is a normalization constant and the coefficients $S^n(t)$ can be computed explicitly by the following recurrence formula

$$\begin{cases} S^0(t) = i\int_0^t \sqrt{\rho(s)}ds, \quad S_t^n(t) = -\frac{S_{tt}^{n-1}(t)+\sum_{\substack{i+j=n\\ i,j\neq n}} S_t^i(t)S_t^j(t)}{2S_t^0(t)}, & \forall n\geq 1,\\ S^n(0)=0, & \forall n \geq 0. \end{cases}$$

Moreover,

(a) *$S_t^n(t)$ (the derivative of S^n) are 1-periodic functions which only depend on ρ,*
(b) *$S^{2n+1}(t)$ are 1-periodic functions while $S^{2n}(t)$ grow linearly with respect to t,*
(c) *$S^{2n+1}(t)$ are real functions while $S^{2n}(t)$ are purely imaginary.*

Remark 2.2 *The main difference between this result and the one we found for the low frequencies $k \leq c\epsilon^{-1}$ (see [7]) is that here the approximation is not valid just above the critical size ($k\sim\epsilon^{-1}$), but only above $k\sim\epsilon^{-1-1/N}$. Note that N depends on the regularity of ρ so that ρ has to be assumed to be C^∞ if we want to cover the whole region $k\geq\epsilon^{-\alpha}$ with any $\alpha>1$.*

There exist a number of asymptotic methods to study the behavior of the solutions of (1.1) as $\epsilon\to 0$. For example, when using the so called multiple scales method we assume that $u_\epsilon(x)$ depends on the slow and fast variables x and $X = x/\epsilon$ as follows:

$$u_\epsilon(x) = u_0(x) + \epsilon^2 u_1(x, x/\epsilon) + \dots$$

where the functions $u_j(x,X)$ are $1-$periodic with respect to X. The functions u_0 and u_1 must satisfy

$$\begin{cases} \frac{\partial^2 u_1}{\partial X^2} + \frac{d^2u_0}{dx^2} + \lambda\rho(X)u_0(x) = 0,\\ U_1(x,X) \quad 1-\text{periodic on } X. \end{cases}$$

Therefore,

$$u_0(x) \sim \exp\left(\pm i\sqrt{\lambda\bar\rho}x\right),$$

and we obtain (1.3) after the boundary conditions in (1.1). This method provides good approximations when $\epsilon^2\lambda$ is sufficiently small, i.e. for the low frequencies.

When $\epsilon^2\lambda$ is large, which is the case in Theorem 2.1, the multiple scales method is no longer valid. Instead, we transform the equation in (1.1) by means of the change of variables $x/\epsilon = t$ into

$$-w_\epsilon''(t) = \epsilon^2\lambda\,\rho(t)w_\epsilon(t) \tag{2.4}$$

where $w_\epsilon(t) = u_\epsilon(t\epsilon)$. Then we use the WKB method to obtain a good approximation of the solutions of (2.4) for large $\epsilon^2\lambda$

$$w_\epsilon(t) \sim \exp\left(\pm i\sqrt{\lambda}\epsilon\int_0^t \sqrt{\rho(s)}ds\right),$$

and therefore

$$u_\epsilon(x) \sim \exp\left(\pm i\sqrt{\lambda}\int_0^x \sqrt{\rho(s/\epsilon)}ds\right).$$

This is the approach we follow in the proof of Theorem 2.1. As we will see, the main difficulty is to prove the uniform approximation of the asymptotic formulas. We leave the details to section 6.

3 Analysis of eigenvalues

As we said in the introduction, our work is motivated by the uniform observability property of the wave equation with oscillating density. This property can be obtained from two spectral properties: the uniform gap between two consecutive eigenvalues and a uniform observability property for the eigenfunctions (see [5] and [6]). In this section and the following one we prove that these two properties can be obtained from Theorem 2.1. We also discuss the first order terms in the asymptotic expansion of the eigenvalues and eigenfunctions obtained in Theorem 2.1.

3.1 First order approximation

Looking at the first order term in the formula (2.1) we have

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\int_0^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds} + \mathcal{O}((\epsilon^2 k)^{-1}). \tag{3.1}$$

This means that the first order term is valid for $k \geq B\epsilon^{-2}$ with B large enough provided $\rho \in W^{2,\infty}(\mathbb{R})$. The constant $\rho_\epsilon^* = \int_0^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds$ in the denominator converges to $\rho^* = \int_0^1 \sqrt{\rho(x)} ds$ when $\epsilon \to 0$. In fact we have the following estimate:

Lemma 3.1 *If ρ is a continuous 1-periodic function,*

$$\left| \int_0^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds - \int_0^1 \sqrt{\rho(s)}\, ds \right| \leq 2\|\sqrt{\rho}\|_\infty \epsilon.$$

Proof of Lemma 3.1 Define $n = [\frac{1}{\epsilon}]$, where $[\cdot]$ represents the integer part. Then

$$\int_0^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds - \int_0^1 \sqrt{\rho(s)}\, ds = \sum_{i=0}^{n-1} \int_{i\epsilon}^{(i+1)\epsilon} \sqrt{\rho(\frac{s}{\epsilon})}\, ds - \int_0^1 \sqrt{\rho(s)}\, ds + \int_{n\epsilon}^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds.$$

We observe that

$$\left| \sum_{i=0}^{n-1} \int_{i\epsilon}^{(i+1)\epsilon} \sqrt{\rho(\frac{s}{\epsilon})}\, ds - \int_0^1 \sqrt{\rho(s)}\, ds \right| = \left| \epsilon \sum_{i=0}^{n-1} \int_i^{(i+1)} \sqrt{\rho(y)}\, dy - \int_0^1 \sqrt{\rho(s)}\, ds \right| =$$

$$= \left| (\epsilon n - 1) \int_0^1 \sqrt{\rho(s)}\, ds \right| \leq \epsilon \|\sqrt{\rho}\|_\infty$$

because ρ is 1-periodic. Then,

$$\left| \int_0^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds - \int_0^1 \sqrt{\rho(s)}\, ds \right| \leq \epsilon \|\sqrt{\rho}\|_\infty + \left| \int_{n\epsilon}^1 \sqrt{\rho(\frac{s}{\epsilon})}\, ds \right| \leq 2\epsilon \|\sqrt{\rho}\|_\infty.$$

□

However it is not possible to replace ρ_ϵ^* by ρ^* in Theorem 2.1 or in (3.1) because

$$\left| \frac{k\pi}{\rho_\epsilon^*} - \frac{k\pi}{\rho^*} \right| \leq \left| k\pi \frac{\rho^* - \rho_\epsilon^*}{\rho^* \rho_\epsilon^*} \right| = \mathcal{O}(k\epsilon)$$

which is not small when $k \sim \epsilon^{-2}$, which is the region of validity of formula (3.1).

In Figure 1 we show the behavior of the high frequencies $\sqrt{\lambda_k^\epsilon}$ for $k = 100, ..., 105$ with respect to the parameter ϵ (that ranges between 1/10 and 2/10) when $\rho(t) = (2 + \sin 2\pi t)^2$. This behavior is only valid when $k \geq \epsilon^{-2}$, i.e. for $\epsilon \geq 1/10$.

FIGURE 1. $\sqrt{\lambda_{100}^\epsilon}, \cdots, \sqrt{\lambda_{105}^\epsilon}$ as functions of $\epsilon \in (0.1, 0.2)$ when $\rho(t) = (2 + \sin 2\pi t)^2$.

It is interesting to compare (3.1) with the first term in the approximation of the low frequencies $k \leq C\epsilon^{-1}$ (see [7]). Indeed, for the low frequencies we proved that $\sqrt{\lambda_k^\epsilon} \sim k\pi/\sqrt{\overline{\rho}}$ which is the limit of $k\pi/\sqrt{\int_0^1 \rho(s/\epsilon)ds}$. Note that

$$\sqrt{\int_0^1 \rho(s)ds} > \int_0^1 \sqrt{\rho(s)}ds$$

and that equality only holds when ρ is identically constant. Roughly speaking it can be said that the low frequencies approach the solutions of the wave equation

$$\overline{\rho} u_{tt} - u_{xx} = 0$$

while the high frequencies obey to

$$\left(\int_0^1 \sqrt{\rho(s)}ds \right)^2 u_{tt} - u_{xx} = 0.$$

3.2 Higher order approximation

Formula (2.1) provides higher order approximations for the eigenvalues. In this section we analyze these approximations in order to simplify (2.1).

Note that all the terms in the approximation but the first order one contain the factors

$$\int_0^1 \sqrt{\rho(s/\epsilon)}ds \quad \text{and} \quad \int_0^1 S_t^{2n}(s/\epsilon)ds. \tag{3.2}$$

From Lemma 3.1 we have

$$\left| \int_0^1 \sqrt{\rho(s/\epsilon)}ds - \int_0^1 \sqrt{\rho(s)}ds \right| \leq \|\sqrt{\rho}\|_\infty \epsilon. \tag{3.3}$$

On the other hand, since S_t^{2n} is $1-$periodic we also have:

$$\left| \int_0^1 S_t^{2n}(s/\epsilon)ds - \int_0^1 S_t^{2n}(s)ds \right| \leq \|S_t^{2n}\|_\infty \epsilon. \tag{3.4}$$

So, taking (3.3) and (3.4) into account we can simplify formula (2.1) into the following one: For any $\delta > 0$, there exists a constant $C > 0$ (which depends on ρ, N and δ) such that if $k \geq C\epsilon^{-1-1/N}$ with $0 < \epsilon < 1$ then

$$\left| \sqrt{\lambda_k^\epsilon} - \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + i \sum_{n=1}^{[N/2]} \frac{\left(\int_0^1 \sqrt{\rho(s)}ds\right)^{2n} \int_0^1 S_t^{2n}(s)ds}{k^{2n-1}\epsilon^{2n}\pi^{2n-1}} \right| \leq \delta. \tag{3.5}$$

Observe that the oscillation behavior of the eigenvalues λ_k^ϵ with respect to ϵ is relevant only in the first order approximation. Then, higher correctors do not exhibit this oscillation behavior. For instance, looking to the second order approximation we have

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \frac{\int_0^1 \sqrt{\rho(s)}ds}{32\pi k\epsilon^2} \int_0^1 \frac{5(\rho'(s))^2 - 4\rho''(s)\rho(s)}{\rho^{5/2}(s)} ds + \mathcal{O}(\epsilon^{-4}k^{-3}), \tag{3.6}$$

which is valid when $\rho \in C^3$.

3.3 Estimates on the gap between two consecutive eigenvalues

Theorem 2.1 allows us to obtain the following property on the separability of eigenvalues:

Proposition 3.1 *Let ρ be a 1-periodic function such that $\rho \in W^{N+1,\infty}(R)$ for some $N \geq 1$. Given $\delta > 0$, there exists a constant $C = C(\delta) > 0$ such that if $k \geq C\epsilon^{-1-1/N}$ we have*

$$\sqrt{\lambda_{k+1}^\epsilon} - \sqrt{\lambda_k^\epsilon} \geq \frac{\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} - \delta.$$

Proof of Proposition 3.1 From formula (2.1) we deduce that given $\delta/3 > 0$ there exists $C > 0$ such that if $k \geq C\epsilon^{-1-1/N}$ we have

$$\begin{aligned}
\sqrt{\lambda_{k+1}^\epsilon} - \sqrt{\lambda_k^\epsilon} \geq & \left| \frac{(k+1)\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} - \sum_{n=1}^{N/2} \frac{\pi \int_0^1 S_t^{2n}(s/\epsilon)ds}{(k+1)^{2n-1}\epsilon^{2n} i \left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^2} \right. \\
& \left. - \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \sum_{n=1}^{N/2} \frac{\pi \int_0^1 S_t^{2n}(s/\epsilon)ds}{k^{2n-1}\epsilon^{2n} i \left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^2} \right| - 2\frac{\delta}{3} \\
= & \left| \frac{\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \sum_{n=1}^{N/2} \frac{\pi \int_0^1 S_t^{2n}(s/\epsilon)ds}{\epsilon^{2n} i \left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^2} \left(\frac{1}{k^{2n-1}} - \frac{1}{(k+1)^{2n-1}}\right) \right| - 2\frac{\delta}{3} \\
= & \frac{\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \sum_{n=1}^{N/2} \mathcal{O}(\epsilon^{-2n}k^{-2n}) - 2\frac{\delta}{3} = \frac{\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} - \delta,
\end{aligned}$$

since we can choose C so that $\sum_{n=1}^{N/2} \mathcal{O}(\epsilon^{-2n}k^{-2n}) \leq \delta/3$, for $k \geq C\epsilon^{-1-1/N}$. □

As we mentioned above, the gap between consecutive eigenvalues is essential when analyzing the observability or controllability of waves from the boundary (see [5] and [6]).

4 Analysis of eigenfunctions

4.1 Uniform observability of eigenfunctions

The following result is another key ingredient when analyzing the uniform observability or controllability of the high frequency solutions of the wave equation associated to (1.1) (see for instance [5] and [6]).

Proposition 4.1 *Let ρ be a 1-periodic function such that $\rho \in W^{N+1,\infty}(R)$ for some $N \geq 1$. There exist $C, c > 0$ such that the following estimates hold for the eigenfunctions φ_k^ϵ with $k \geq c\epsilon^{-1-1/N}$:*

$$\frac{1}{C}\left(|(\varphi_k^\epsilon)'(0)|^2 + |(\varphi_k^\epsilon)'(1)|^2\right) \leq \int_0^1 |(\varphi_k^\epsilon)'(x)|^2 dx \leq C\left(|(\varphi_k^\epsilon)'(0)|^2 + |(\varphi_k^\epsilon)'(1)|^2\right).$$

Observe that, as a consequence of Proposition 4.1, when $\rho \in W^{N+1,\infty}(0,1)$ the eigenfunctions corresponding to eigenvalues λ_k^ϵ with $k \geq c\epsilon^{-1-1/N}$ (c large enough) can not exhibit any concentration of energy on the extremes of the interval.

According to the asymptotic formula for the eigenvalues given in Theorem 1, the proof of Proposition 4.1 is similar to the one of Proposition 3.5 in [7] and we omit it.

4.2 Approximation formulas for the eigenfunctions

Theorem 2.1 provides also approximation formulas for the eigenfunctions. For example, when we consider the case $N = 3$ in Theorem 1, we obtain

$$\varphi_k^\epsilon(x) \sim \left(\frac{\rho(1/\epsilon)}{\rho(x/\epsilon)}\right)^{1/4} \sin\left(\sqrt{\tilde{\lambda}_k^\epsilon}\int_0^x \sqrt{\rho(s/\epsilon)}ds + \epsilon\tilde{\lambda}_k^\epsilon \int_0^x \frac{4\rho''(s/\epsilon)\rho(s/\epsilon) - 5(\rho'(s/\epsilon))^2}{32\rho^2(s/\epsilon)\sqrt{\rho(s/\epsilon)}}ds\right)$$

where

$$\sqrt{\tilde{\lambda}_k^\epsilon} = \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \frac{\left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^2}{32k\epsilon^2\pi}\int_0^1 \frac{4\rho''(s/\epsilon)\rho(s/\epsilon) - 5(\rho'(s/\epsilon))^2}{\rho^2(s/\epsilon)\sqrt{\rho(s/\epsilon)}}ds,$$

which is valid for $k \geq C\epsilon^{-4/3}$ with C sufficiently large provided $\rho \in W^{4,\infty}(\mathbb{R})$.

5 The case where the oscillating coefficient is in the principal part

In this section we study the eigenvalue problem

$$\begin{cases} (a(x/\epsilon)u')' + \lambda u = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where $a(x) \in L^\infty(R)$ is a periodic function with $0 < a_m \leq a(x) \leq a_M < \infty$ and ϵ is small. We assume, without loss of generality, that the period of a is 1.

We observe that the oscillating coefficient is now in the principal part of the operator.

Let us denote by $\{\lambda_k^\epsilon\}_{k\in N}$ the eigenvalues of (5.1) ordered in an increasing way.

The eigenvalues and eigenfunctions of (5.1) converge, as $\epsilon \to 0$, to those of the limit system

$$\begin{cases} \widehat{a}u'' + \lambda u = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.2}$$

where $\widehat{a} = 1/\int_0^1 \frac{dr}{a(r)}$. This is a classical result in homogenization theory (see for instance [3]). The eigenpairs of (5.2) can be also computed explicitly:

$$\begin{aligned} \lambda_k &= \widehat{a}k^2\pi^2, \quad k \in N, \\ \varphi_k(x) &= \sin(k\pi x), \quad k \in N. \end{aligned} \tag{5.3}$$

We refer to [7] for a complete description of the convergence of the spectrum and its correctors.

We show that the analysis of the previous section can be also applied to this system to obtain a full description of the convergence of the high frequencies. The idea is to reduce system (5.1) to one in the form (1.1) for which we can apply Theorem 2.1.

Consider the following change of variables

$$\begin{aligned} y(x) &= \frac{\int_0^{x/\epsilon} \frac{dr}{a(r)}}{\int_0^{\epsilon^{-1}} \frac{dr}{a(r)}}, \quad \delta(\epsilon) = \frac{\int_0^1 \frac{dr}{a(r)}}{\int_0^{\epsilon^{-1}} \frac{dr}{a(r)}}, \\ b(s) &= a(t(s)), \quad v(y) = u(x(y)), \\ \mu &= \lambda \left(\epsilon \int_0^{\epsilon^{-1}} \frac{dr}{a(r)} \right)^2, \quad s(t) = \frac{\int_0^t \frac{dr}{a(r)}}{\int_0^1 \frac{dr}{a(r)}}, \end{aligned} \tag{5.4}$$

where $x(y)$ represents the inverse function of $y(x)$, i.e. $x(y) = x$ if and only if $y(x) = y$, and $t(s)$ the inverse function of $s(t)$. Obviously, this change of variables depends on ϵ but, for the sake of simplicity, we do not make this fact explicit in the notation.

This change of variables transforms system (5.1) into

$$\begin{cases} v_{yy}(y) + \mu b(y/\delta)v(y) = 0, \quad y \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{5.5}$$

Let us see that the function $b(s)$ is 1-periodic. Note that $t(s) + 1 = t(s+1)$. Indeed,

$$s(t) + 1 = s(t) + \frac{\int_t^{t+1} \frac{dr}{a(r)}}{\int_0^1 \frac{dr}{a(r)}} = \frac{\int_0^{t+1} \frac{dr}{a(r)}}{\int_0^1 \frac{dr}{a(r)}} = s(t+1),$$

Taking this fact and the periodicity of a into account we deduce that

$$b(s+1) = a(t(s+1)) = a(t(s)+1) = a(t(s)) = b(s), \qquad \forall s \geq 0.$$

Therefore, b is $1-$periodic and system (5.5) is equivalent to system (1.1).

By formula (3.5) if $b(s) \in W^{N+1,\infty}(0,1)$, i.e. $a(t) \in W^{N+1,\infty}(0,1)$, then the eigenvalues μ_k^δ of (5.5) with $k \geq C\delta^{-1-1/N}$ satisfy:

$$\sqrt{\mu_k^\delta} \sim \frac{k\pi}{\int_0^1 \sqrt{b(s/\delta)}ds} + \sum_{n=1}^{[N/2]} \frac{\left(\int_0^1 \sqrt{b(s)}ds\right)^{2n} \int_0^1 iS_t^{2n}(s)ds}{k^{2n-1}\delta^{2n}\pi^{2n-1}}. \tag{5.6}$$

Recall that the functions $S_t^{2n}(x)$ can be computed from the coefficient b. On the other hand

$$\begin{aligned} \int_0^1 \sqrt{b(s/\delta)}ds &= \delta \int_0^{\delta^{-1}} \sqrt{b(s)}ds = \delta \int_0^{\epsilon^{-1}} \sqrt{a(t)}s'(t)dt \\ &= \frac{\delta}{\int_0^1 \frac{dr}{a(r)}} \int_0^{\epsilon^{-1}} \frac{1}{\sqrt{a(t)} \int_0^1 \frac{dr}{a(r)}} dx = \widehat{a}\frac{\delta}{\epsilon} \int_0^1 \frac{dt}{\sqrt{a(t/\epsilon)}} = \frac{\int_0^1 \frac{dt}{\sqrt{a(t/\epsilon)}}}{\int_0^1 \frac{dt}{a(t/\epsilon)}}. \end{aligned} \tag{5.7}$$

Consider the first order approximation which we obtain truncating the series in the first term:

$$\sqrt{\mu_k^\delta} = \frac{k\pi}{\int_0^1 \sqrt{b(s/\delta)}ds} + \mathcal{O}(\delta^2 k)^{-1}. \tag{5.8}$$

Coming back to the original variables λ and ϵ we have

$$\sqrt{\lambda_k^\epsilon}\epsilon \int_0^{\epsilon^{-1}} \frac{dr}{a(r)} = k\pi \frac{\int_0^1 \frac{dt}{a(t/\epsilon)}}{\int_0^1 \frac{dt}{\sqrt{a(t/\epsilon)}}} + \mathcal{O}(\delta^2 k)^{-1}. \tag{5.9}$$

Due to the definition of $\delta(\epsilon)$ we have that

$$\epsilon a_m/\hat{a} \leq \delta(\epsilon) \leq \epsilon a_M/\hat{a} \tag{5.10}$$

and therefore,

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\int_0^1 \frac{dt}{\sqrt{a(t/\epsilon)}}} + \mathcal{O}(\epsilon^2 k)^{-1}. \tag{5.11}$$

We observe that the square root of the eigenvalues $\sqrt{\lambda_k^\epsilon}$ show an oscillatory behavior as a function of ϵ in the first term which is completely similar to the case where the oscillation occurs in the density ρ.

6 Proof of Theorem 2.1

The proof of Theorem 1 consists in 5 steps: In the first step we use a shooting method to transform the eigenvalue problem in an initial value one (IVP). In the second step we obtain a formal asymptotic expansion of the solutions of the IVP using the classical WKB method (see [4] for a detailed description of this method). In the third step we prove that the first N terms in the asymptotic expansion constitute a uniform approximation of the solution of the IVP. This approximation is improved as N grows. We use a classical asymptotic method whose general description can be found in [10]. Roughly, it consists in estimating the difference between the solution and its approximation comparing the equations satisfied by each of them. In step 4 we deduce the formula for the eigenvalues. The shooting method gives us a characterization of the eigenvalues in terms of the solution of the IVP. This characterization and a perturbation argument involving the Theorem of Rouché provides us the desired result. Finally, in step 5 we deduce the formula for the eigenfunctions.

STEP 1: *Shooting method.* We start solving the following Cauchy problem:

$$\begin{cases} -y_\epsilon''(x) = \lambda\, \rho^\epsilon(x) y_\epsilon(x) \\ y_\epsilon(0) = 0, \qquad y_\epsilon'(0) = 1 \end{cases} \tag{6.1}$$

so that the eigenvalues and eigenfunctions of (1.1) are characterized as the pairs (λ, y_ϵ) satisfying (6.1) and $y_\epsilon(1) = 0$.

As the eigenvalue problem (6.1) is self-adjoint and positive, all the eigenvalues are real and positive. So, we assume $\lambda \in \mathbb{R}$.

STEP 2: *Formal asymptotic expansion.* Here we look for a formal expansion of solutions of (6.1), when $\lambda >> 1$, based in the WKB method.

Consider the change of variables $x/\epsilon = t$ which transforms (6.1) into:

$$\begin{cases} -w_\epsilon''(t) = \epsilon^2\lambda\, \rho(t) w_\epsilon(t) \\ w_\epsilon(0) = 0, \qquad w_\epsilon'(0) = \epsilon \end{cases} \tag{6.2}$$

where $w_\epsilon(t) = y_\epsilon(t\epsilon)$. As we are supposing $\lambda\epsilon^2$ to be large, we introduce $\lambda\epsilon^2 = \frac{1}{\delta^2}$ where δ is a small parameter, i.e. $\delta << 1$.

The basic idea in the WKB method is to assume that the solution can be written in the form:

$$w_\epsilon(t) = \mathrm{Im}\left(\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S^n(t)\right)\right). \tag{6.3}$$

Then the coefficients $S^n(t)$ must verify:

$$\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S^n_{tt} + \left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S^n_t\right)^2 + \frac{1}{\delta^2}\rho(t) = 0.$$

Equating the terms with the same powers in δ we obtain the following system:

$$\begin{cases} (S^0_t)^2 + \rho(t) = 0 \\ 2S^0_t S^1_t + S^0_{tt} = 0 \\ (S^1_t)^2 + 2S^0_t S^2_t + S^1_{tt} = 0 \\ \cdots \\ S^n_{tt} + \sum_{i+j=n+1} S^i_t S^j_t = 0 \\ \cdots \end{cases} \tag{6.4}$$

To integrate the system we assume that $S^i(0) = 0$. From the first equation in (6.4) we have $S^0_t = \pm i\sqrt{\rho}$ and then

$$S^0(t) = \pm i \int_0^t \sqrt{\rho(s)} ds. \tag{6.5}$$

We take the + sign in (6.5). From the second equation in (6.4) we obtain

$$S^1_t = -S^0_{tt}/(2S^0_t) \tag{6.6}$$

and then

$$S^1(t) = -\frac{1}{4}\log\left(\frac{\rho(t)}{\rho(0)}\right). \tag{6.7}$$

From the third equation in (6.4) we obtain

$$S^2_t = \frac{-(S^1_t)^2 - S^1_{tt}}{2S^0_t} = \frac{-\frac{(\rho')^2}{16\rho^2} + \frac{\rho''}{4\rho} - \frac{(\rho')^2}{4\rho^2}}{-2i\sqrt{\rho}} = \frac{-5(\rho')^2 + 4\rho''\rho}{-i32\rho^2\sqrt{\rho}},$$

which is periodic of period 1. Then,

$$S^2(t) = \frac{i}{32}\int_0^t \left(\frac{5(\rho')^2 - 4\rho''\rho}{\rho^2\sqrt{\rho}}\right).$$

In general

$$S^n_t = -\frac{S^{n-1}_{tt} + \sum_{\substack{i+j=n \\ i,j\neq n}} S^i_t S^j_t}{2S^0_t}$$

which is 1-periodic in t.

We easily check the following properties for S_t^n:

$$\begin{array}{l} -S_t^n \text{ is 1-periodic for all } n \text{ and does not depend on } \delta, \\ -S_t^n \text{ is real if } n \text{ is odd and purely imaginary if } n \text{ is even,} \\ -S_t^n = f_n(\rho^{-1/2}, \rho^{1/2}, \rho', \ldots, \rho^{n)}) \text{ for some polynomial } f_n. \end{array} \tag{6.8}$$

Here and in the sequel we denote by $\rho^{n)}$ the derivative of order n of ρ.

We also have the following property which is the key in our analysis:

Lemma 6.1 *The coefficients $S^{2n+1}(t)$ are 1-periodic functions for all $n \geq 0$.*

The proof of this technical lemma is given in the Appendix A at the end of the paper.

In view of the initial conditions $w_\epsilon(0) = 0$, $w_\epsilon'(0) = \epsilon$, the formal solution of (6.2) takes the form

$$w_\epsilon(t) = A_\epsilon e^{\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^{2n+1} S^{2n+1}(t)\right)} \sin\left(\frac{1}{i\delta}\sum_{n=0}^{\infty} \delta^{2n} S^{2n}(t)\right) \tag{6.9}$$

where $A_\epsilon = i\delta\epsilon\left[\sum_{n=0}^{\infty}\delta^{2n} S_t^{2n}(0)\right]^{-1}$. Observe that as $S^n(0) = 0$, we have $w_\epsilon(0) = 0$ while we have chosen A_ϵ so that the second boundary condition in (6.2) holds, i.e. $w_\epsilon'(0) = \epsilon$.

Coming back to the original variable $x = \epsilon t$ we obtain the following asymptotic formula for the solution $y_\epsilon(x)$ of (6.1):

$$y_\epsilon(x) = A_\epsilon e^{\left(\sqrt{\lambda}\epsilon \sum_{n=0}^{\infty}(\sqrt{\lambda}\epsilon)^{-2n-1} S^{2n-1}(x/\epsilon)\right)} \sin\left(\frac{\sqrt{\lambda}\epsilon}{i}\sum_{n=0}^{\infty}(\sqrt{\lambda}\epsilon)^{-2n} S^{2n}(x/\epsilon)\right). \tag{6.10}$$

The developments above are purely formal. In order to get a rigorous justification we would need to get uniform bounds on all the coefficients S^n. We do not pursue this approach but rather analyze the proximity of the solution of (6.2) towards the expression we obtain when truncating the above series at the level $n = N$.

STEP 3: *Uniform approximation of the first N terms in the asymptotic formula.* Here we show that the first N terms of the asymptotic expansion (6.10) constitute a uniform approximation of the solution of (6.1). As we will see later, we need estimates not only for λ real but also for λ complex. Therefore, from now on we assume that λ is complex. We use an asymptotic method to estimate the difference between two functions using the differential equations they satisfy. So, our first aim is to find the differential equation satisfied by the first N terms in the asymptotic expansion (6.10) extended to complex values of λ:

$$y_{\epsilon,N}(x) = A_{\epsilon,N} e^{\left(\sqrt{\lambda}\epsilon \sum_{n=0}^{N/2-1}(\sqrt{\lambda}\epsilon)^{-2n-1} S^{2n+1}(x/\epsilon)\right)} \sin\left(\frac{\sqrt{\lambda}\epsilon}{i}\sum_{n=0}^{N/2}(\sqrt{\lambda}\epsilon)^{-2n} S^{2n}(x/\epsilon)\right), \tag{6.11}$$

with

$$A_{\epsilon,N} = \frac{i}{\sqrt{\lambda}}\left[\sum_{n=0}^{N/2}\frac{S_t^{2n}(0)}{(\sqrt{\lambda}\epsilon)^{2n}}\right]^{-1}. \tag{6.12}$$

To simplify the notation we introduce

$$R_1(x/\epsilon) = \sqrt{\lambda}\epsilon \sum_{n=0}^{N/2-1}(\sqrt{\lambda}\epsilon)^{-2n-1} S^{2n+1}(x/\epsilon), \tag{6.13}$$

$$R_2(x/\epsilon) = \frac{\sqrt{\lambda}\epsilon}{i} \sum_{n=0}^{N/2} (\sqrt{\lambda}\epsilon)^{-2n} S^{2n}(x/\epsilon). \tag{6.14}$$

Note that R_1 and R_2 depend also on N, ρ, ϵ and λ but we do not make this explicit in the notation.

A straightforward computation shows that $y_{\epsilon,N}$ satisfies the following equation:

$$y''_{\epsilon,N}(x) + \lambda\rho(x/\epsilon)y_{\epsilon,N}(x) + \frac{L^1_N(x/\epsilon)}{\epsilon^2} y_{\epsilon,N}(x) + \frac{L^2_N(x/\epsilon)}{\epsilon^2} z_{\epsilon,N}(x) = 0, \tag{6.15}$$

where

$$z_{\epsilon,N}(x) = e^{R_1(x/\epsilon)} \cos R_2(x/\epsilon), \tag{6.16}$$

$$L^1_N(x/\epsilon) = -\sum_{k=0}^{N/2-2} (\sqrt{\lambda}\epsilon)^{-N-2k} \left[\sum_{\substack{i+j=N+2k+2 \\ i,j\le N}} S^i_t(x/\epsilon) S^j_t(x/\epsilon) \right], \tag{6.17}$$

$$\begin{aligned} L^2_N(x/\epsilon) = & -\frac{(\sqrt{\lambda}\epsilon)^{-N+1}}{i} \left[S^N_{tt}(x/\epsilon) + \sum_{\substack{i+j=N+1 \\ i,j\le N}} S^i_t(x/\epsilon) S^j_t(x/\epsilon) \right] \\ & -\frac{1}{i} \sum_{k=0}^{N/2-2} (\sqrt{\lambda}\epsilon)^{-N-2k-1} \left[\sum_{\substack{i+j=N+2k+3 \\ i,j\le N}} S^i_t(x/\epsilon) S^j_t(x/\epsilon) \right]. \end{aligned} \tag{6.18}$$

Observe that the functions $L^1_N(t)$ and $L^2_N(t)$ depend also on ϵ and ρ but we do not make explicit this dependence to simplify the notation.

On the other hand, note that $y_{\epsilon,N}(t)$ and $z_{\epsilon,N}(t)$ are related by the identity

$$y'_{\epsilon,N}(t) = \frac{1}{\epsilon} \left[R'_1(x/\epsilon) y_{\epsilon,N}(x) + R'_2(x/\epsilon) z_{\epsilon,N}(x) \right]. \tag{6.19}$$

Eliminating $z_{\epsilon,N}$ in (6.15) and (6.19) we obtain that $y_{\epsilon,N}$ is solution of

$$y''(x) + \lambda\rho(x/\epsilon)v(x) + p_\epsilon(x)y'(x) + q_\epsilon(x)y(x) = 0 \tag{6.20}$$

where

$$p_\epsilon(x) = \frac{L^2_N(x/\epsilon)}{\epsilon R'_2(x/\epsilon)}, \tag{6.21}$$

$$q_\epsilon(x) = \frac{L^1_N(x/\epsilon)}{\epsilon^2} - \frac{L^2_N(x/\epsilon) R'_1(x/\epsilon)}{\epsilon^2 R'_2(x/\epsilon)}. \tag{6.22}$$

The coefficients p_ϵ and q_ϵ depend on N and ρ but we do not make explicit this dependence in the notation.

In a similar way, we deduce that $z_{\epsilon,N}$ is also solution of (6.20). Then $\{y_{\epsilon,N}(t), z_{\epsilon,N}(t)\}$ constitutes a linear independent set of solutions of (6.20).

Note that, due to the normalization constant $A_{\epsilon,N}$, $y_{\epsilon,N}$ satisfies also the initial conditions

$$y_{\epsilon,N}(0) = 0, \qquad y'_{\epsilon,N}(0) = 1, \tag{6.23}$$

while $z_{\epsilon,N}(t)$ satisfies

$$z_{\epsilon,N}(0) = 1, \qquad z'_{\epsilon,N}(0) = 0.$$

Until now we have introduced the approximation $y_{\epsilon,N}$ obtained with the first N terms in the formal asymptotic expansion (6.10). We have also deduced the differential equation (6.20) satisfied by $y_{\epsilon,N}$. Let us see now that $y_{\epsilon,N}$ is a good approximation of the solutions of (6.1).

We rewrite system (6.1) in the form:

$$\begin{cases} y_\epsilon''(x) + \lambda\rho^\epsilon(x)y_\epsilon(x) + p_\epsilon(x)y_\epsilon'(x) + q_\epsilon(x)y_\epsilon(x) = p_\epsilon(x)y_\epsilon'(x) + q_\epsilon(x)y_\epsilon(x) \\ y_\epsilon(0) = 0, \qquad y_\epsilon'(0) = 1. \end{cases} \tag{6.24}$$

Taking into account that $y_{\epsilon,N}$ is a solution when the second term in (6.24) vanishes, we assume that the solution y_ϵ of (6.24) has the form

$$y_\epsilon(x) = C_0(x) + \sum_{j\geq 1} C_j(x) \tag{6.25}$$

where $C_0(x) = y_{\epsilon,N}(x)$. Substituting this expression in (6.24) and equating terms in a suitable way we obtain the system:

$$\begin{cases} C_0'' + \lambda\rho^\epsilon C_0 + p_\epsilon C_0' + q_\epsilon C_0 = 0, \qquad x \in (0,1), \\ C_0(0) = 0, \quad C_0'(0) = 1, \end{cases}$$

$$\begin{cases} C_j'' + \lambda\rho^\epsilon C_j + p_\epsilon C_j' + q_\epsilon C_j = p_\epsilon(x)C_{j-1}' + q_\epsilon C_{j-1}, \qquad x \in (0,1), \\ C_j(0) = 0, \quad C_j'(0) = 0, \end{cases} \qquad \forall j \geq 1.$$

Taking into account that both $y_{\epsilon,N}(x)$ and $z_{\epsilon,N}(x)$ are linearly independent solutions of the homogeneous equation (6.20) and by means of the variation of constants method we easily obtain the following result:

Lemma 6.2 *Given $\lambda \in \mathbb{C}$, $\rho \in W^{N+1,\infty}(0,1)$ and $f \in C([0,1])$, the problem*

$$\begin{cases} C''(x) + \lambda\rho^\epsilon(x)C(x) + p_\epsilon(x)C'(x) + q_\epsilon(x)C(x) = f(x), \qquad x \in (0,1), \\ C(0) = 0, \quad C'(0) = -\gamma, \end{cases} \tag{6.26}$$

admits a unique solution given by:

$$C(x) = \gamma y_{\epsilon,N}(x) + \int_0^x \frac{y_{\epsilon,N}(x)z_{\epsilon,N}(s) - z_{\epsilon,N}(x)y_{\epsilon,N}(s)}{y_{\epsilon,N}(s)z_{\epsilon,N}'(s) - y_{\epsilon,N}'(s)z_{\epsilon,N}(s)} f(s)\, ds. \tag{6.27}$$

Corollary 6.3 The solution y_ϵ of (6.24) solves the integral equation

$$y_\epsilon(x) = y_{\epsilon,N}(x) + \int_0^x \frac{y_{\epsilon,N}(x)z_{\epsilon,N}(s) - z_{\epsilon,N}(x)y_{\epsilon,N}(s)}{y_{\epsilon,N}(s)z_{\epsilon,N}'(s) - y_{\epsilon,N}'(s)z_{\epsilon,N}(s)} (p_\epsilon(s)y_\epsilon'(s) + q_\epsilon(s)y_\epsilon(s))ds. \tag{6.28}$$

The coefficients C_j in (6.25) can be explicitly computed using Lemma 6.2. So,

$$\begin{aligned} C_0(x) &= y_{\epsilon,N}(x), \\ C_j(x) &= \int_0^x \frac{y_{\epsilon,N}(x)z_{\epsilon,N}(s) - z_{\epsilon,N}(x)y_{\epsilon,N}(s)}{y_{\epsilon,N}(s)z_{\epsilon,N}'(s) - y_{\epsilon,N}'(s)z_{\epsilon,N}(s)} (p_\epsilon(s)C_{j-1}'(s) + q_\epsilon(s)C_{j-1}(s))ds. \end{aligned} \tag{6.29}$$

We prove that the series (6.25) converges uniformly in bounded subsets of $(x, \lambda, p_\epsilon, q_\epsilon) \in [0,1] \times \mathbb{C} \times L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ and then defines the unique solution of (6.1).

Lemma 6.4 *Suppose that $\rho \in W^{N+1,\infty}(0,1)$. Then there exist $B_N, D > 0$, such that if*

$|\sqrt{\lambda}| \geq B_N \epsilon^{-1}$ the following estimates hold for the coefficients C_j:

$$|C_0(x)| \leq \frac{D}{|\sqrt{\lambda}|} e^{|\ Im\ [R_2(x/\epsilon)]|}, \qquad |C_0'(x)| \leq De^{|\ Im\ [R_2(x/\epsilon)]|}, \tag{6.30}$$

$$|C_j(x)| \leq De^{|\ Im\ [R_2(x/\epsilon)]|} \frac{\left(De^{|Im[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)\right)^j}{|\sqrt{\lambda}|j!}, \tag{6.31}$$

$$|C_j'(x)| \leq De^{|\ Im\ [R_2(x/\epsilon)]|} \frac{\left(De^{|Im[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)\right)^j}{j!}, \tag{6.32}$$

for all $j \geq 1$ and $x \in [0, 1]$.

When $N = 1$, we have $R_2(t) = \sqrt{\lambda}\epsilon \int_0^t \sqrt{\rho(s)}ds$ and these estimates can be slightly improved:

$$|C_j(x)| \leq De^{|\ Im\ [\sqrt{\lambda}]| \int_0^x \sqrt{\rho(s/\epsilon)}ds} \frac{\left(D\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)\right)^j}{|\sqrt{\lambda}|j!},$$

$$|C_j'(x)| \leq De^{|\ Im\ [\sqrt{\lambda}]| \int_0^x \sqrt{\rho(s/\epsilon)}ds} \frac{\left(D\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)\right)^j}{j!},$$

for all $j \geq 1$ and $x \in [0, 1]$.

We leave the proof of this lemma to the Appendix B.

Remark 6.5 *The term $|\ Im\ [R_2(x/\epsilon)]|$ can be estimated in terms of $Im[\sqrt{\lambda}]$, i.e. it is bounded if $Im[\sqrt{\lambda}]$ is bounded. Indeed*

$$|\ Im\ [R_2(x/\epsilon)]| \leq |\ Im\ [\epsilon\sqrt{\lambda}]||S^0(x/\epsilon)| + \sum_{n=1}^{N/2} |\ Im\ [(\epsilon\sqrt{\lambda})^{-2n+1}]||S^{2n}(x/\epsilon)| \tag{6.33}$$

The first term in (6.33) is easily estimated as follows

$$|\ Im\ [\epsilon\sqrt{\lambda}]||S^0(x/\epsilon)| = |\ Im\ [\sqrt{\lambda}]|\epsilon \int_0^{x/\epsilon} \sqrt{\rho(s)}ds \leq \sqrt{\rho_M}|\ Im\ [\sqrt{\lambda}]|, \tag{6.34}$$

while the second term in (6.33) can be bounded by

$$\begin{aligned} \sum_{n=1}^{N/2} |S^{2n}(x/\epsilon)| \frac{|\ Im\ [(\sqrt{\lambda}\epsilon)^{2n-1}]|}{|\sqrt{\lambda}\epsilon|^{4n-2}} &\leq \sum_{n=1}^{N/2} |S^{2n}(x/\epsilon)||\ Im\ [(\sqrt{\lambda}\epsilon)]| \frac{|\sqrt{\lambda}\epsilon|^{2n-2}}{|\sqrt{\lambda}\epsilon|^{4n-2}} \\ &= |\ Im\ [\sqrt{\lambda}]| \sum_{n=1}^{N/2} \frac{\epsilon|S^{2n}(x/\epsilon)|}{|\sqrt{\lambda}\epsilon|^{2n}} \leq D_1|\ Im\ [\sqrt{\lambda}]|, \end{aligned} \tag{6.35}$$

since it is a finite sum of bounded functions ($\epsilon|S^{2n}(x/\epsilon)|$) multiplied by powers of $|\sqrt{\lambda}\epsilon|^{-1}$ which are bounded in the range of ϵ and λ that we are considering.

Proposition 6.1 *Assume that $\rho \in W^{N+1,\infty}(0, 1)$. Given ϵ and λ fixed, the series $\sum_{j\geq 0} C_j(x)$ converges uniformly in $x \in [0, 1]$ to the unique solution y_ϵ of system (6.1).*

Remark 6.6 *Recall that the coefficients $C_j(x)$ depend on ρ, N, λ and ϵ as long as they* *are computed from p_ϵ and q_ϵ by means of formulas (6.29). However, we do not make this fact explicit in the notation.*

Proof of Proposition 6.1 From Lemma 6.4 we have

$$\left|\sum_{j\geq 0} C_j(x)\right| \leq De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\sum_{j\geq 0}\frac{M^j}{j!|\sqrt{\lambda}|} = \frac{De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}}{|\sqrt{\lambda}|}\exp(M), \tag{6.36}$$

where $M = De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)$. Note that (6.36) is bounded for fixed ϵ and λ. In a similar way,

$$\left|\sum_{j\geq 0} C_j'(x)\right| \leq De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\sum_{j\geq 0}\frac{M^j}{j!} = \exp(M) \tag{6.37}$$

which is also bounded for fixed ϵ and λ.

To check that $\sum_{j\geq 0} C_j(x)$ is actually the only solution of (6.1) we observe that we have constructed C_i precisely to guarantee that the series satisfies the integral equation (6.28). □

By Proposition 6.1 the solution y_ϵ of system (6.1) can be written in the form

$$y_\epsilon(x) = C_0(x) + \sum_{j\geq 1} C_j(x), \tag{6.38}$$

C_0 being $y_{\epsilon,N}$.

We now study the range of ϵ and λ in which the second term $\sum_{j\geq 1} C_j(x)$ is small in $W^{1,\infty}(0,1)$. Assume that the imaginary part of λ is bounded so that the term $|\ \mathrm{Im}\ [R_2(x/\epsilon)]|$ is bounded (see Remark 6.5). From estimate (6.37) there exists a constant $D > 0$ such that if $M = De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)$ then

$$\begin{aligned}\left|\sum_{j\geq 1} C_j(x)\right| + \left|\sum_{j\geq 1} C_j'(x)\right| &\leq \left(1 + \frac{1}{|\sqrt{\lambda}|}\right) De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\left[\exp(M) - 1\right] \\ &= \left(1 + \frac{1}{|\sqrt{\lambda}|}\right) De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\mathcal{O}(M).\end{aligned} \tag{6.39}$$

To estimate M we observe that p_ϵ and q_ϵ can be estimated from their definition in terms of L_N^1 and L_N^2 ((6.21) and (6.22)), the definition of L_N^1 and L_N^2 ((6.17) and (6.18)) and the estimates for R_1' and R_2' (see (B 5)). Indeed,

$$\begin{aligned} p_\epsilon(x) &= \frac{L_N^2(x/\epsilon)}{\epsilon R_2'(x/\epsilon)} = \frac{\mathcal{O}((|\sqrt{\lambda}|\epsilon)^{-N+1})}{\epsilon\left[(\epsilon\sqrt{\lambda})\sqrt{\rho(x/\epsilon)} + \mathcal{O}(|\sqrt{\lambda}|\epsilon)^{-2}\right]} = \frac{\mathcal{O}((|\sqrt{\lambda}|\epsilon)^{-N})}{\epsilon}, \\ q_\epsilon(x) &= \frac{L_N^1(x/\epsilon)}{\epsilon^2} - \frac{L_N^2(x/\epsilon)R_1'(x/\epsilon)}{\epsilon^2 R_2'(x/\epsilon)} \\ &= \frac{\mathcal{O}(|\sqrt{\lambda}\epsilon|^{-N})}{\epsilon^2} + \frac{\mathcal{O}(|\sqrt{\lambda}\epsilon|^{-N+1})}{\epsilon^2\left[(\epsilon\sqrt{\lambda})\sqrt{\rho(x/\epsilon)} + \mathcal{O}(|\sqrt{\lambda}\epsilon|^{-2})\right]} = \frac{\mathcal{O}(|\sqrt{\lambda}\epsilon|^{-N})}{\epsilon^2}. \end{aligned} \tag{6.40}$$

Then,

$$M = De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right) = e^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\mathcal{O}(|\lambda|^{-N/2}\epsilon^{N-1}).$$

Substituting in (6.39) we obtain

$$\left\|\sum_{j\geq 1} C_j(x)\right\|_{W^{1,\infty}(0,1)} = \left(1+\frac{1}{|\sqrt{\lambda}|}\right)e^{2|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\mathcal{O}\left(|\lambda|^{-N/2}\epsilon^{N-1}\right),$$

which is small when $|\mathrm{Im}\ (\sqrt{\lambda})|$ is bounded and $(|\sqrt{\lambda}|\epsilon)^{-N}\epsilon^{-1}$ is bounded above by a sufficiently small positive constant, i.e. $|\sqrt{\lambda}| \geq B_N\epsilon^{-1-1/N}$ with B_N large enough.

STEP 4: *Asymptotic formula for the eigenvalues.* The eigenvalues λ_k^ϵ of (1.1) are the roots of the map $\lambda \to y_\epsilon(1,\lambda)$ where $y_\epsilon(x,\lambda)$ is the solution of (6.1). Note that we have changed the notation for y_ϵ (which now is a function of two variables x and λ) to make explicit the dependence of y_ϵ on λ.

Observe also that

$$y_\epsilon(1,\lambda) = C_0(1,\lambda) + \sum_{j\geq 1} C_j(1,\lambda), \tag{6.41}$$

where the second term on the right is small when $|\mathrm{Im}\ \sqrt{\lambda}|$ is bounded and $|\sqrt{\lambda}| \geq B\epsilon^{-1-1/N}$ for a sufficiently large B. In (6.41) we also make explicit in the notation the dependence on λ.

We use the Theorem of Rouché to show that the zeros of (6.41) are close to the zeros of $\lambda \to C_0(1,\lambda) = y_{\epsilon,N}(1,\lambda)$ when $|\sqrt{\lambda}| \geq B\epsilon^{-1-1/N}$ for a large enough constant B. Note that the zeros of $\lambda \to C_0(1,\lambda)$, $(\widetilde{\lambda}_k^\epsilon)_{k\in\mathbb{N}}$ satisfy

$$\frac{\sqrt{\widetilde{\lambda}_k^\epsilon}\,\epsilon}{i}S^0(1/\epsilon) + \frac{\sqrt{\widetilde{\lambda}_k^\epsilon}\,\epsilon}{i}\sum_{n=1}^{N/2}\frac{S^{2n}(1/\epsilon)}{\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\,\epsilon\right)^{2n}} = k\pi, \qquad k\in\mathbb{N}. \tag{6.42}$$

To simplify the notation we introduce the function

$$h_\epsilon(z) = \frac{z\epsilon}{i}S^0(1/\epsilon) + \frac{z\epsilon}{i}\sum_{n=1}^{N/2}\frac{S^{2n}(1/\epsilon)}{(z\epsilon)^{2n}}, \tag{6.43}$$

so that (6.42) takes the form

$$h_\epsilon(\sqrt{\widetilde{\lambda}_k^\epsilon}) = k\pi, \qquad k\in\mathbb{N}. \tag{6.44}$$

Note that, when $|z| \geq B_1\epsilon^{-1-1/N}$ (B_1 large enough), $h_\epsilon(z)$ is an injective (and therefore invertible) function which transforms the square roots of the zeros of $C_0(1,\lambda)$ into $(k\pi)_{k\in\mathbb{N}}$.

We set

$$\Gamma_j = \{\lambda\in\mathbb{C}\colon |h_\epsilon(\sqrt{\lambda}) - j\pi| = r_j\}$$

where r_j is such that the sets enclosed by Γ_j do not intersect and

$$|y_\epsilon(1,\lambda) - C_0(1,\lambda)| < |C_0(1,\lambda)|, \quad \lambda\in\Gamma_j. \tag{6.45}$$

From formula (6.41) and estimates (6.39) and (6.40) we obtain

$$|y_\epsilon(1,\lambda) - C_0(1,\lambda)| = \left|\sum_{j\geq 1} C_j(1,\lambda)\right| \leq \frac{D\exp(2|\text{ Im }[R_2(1/\epsilon)]|)}{|\sqrt{\lambda}|}|\lambda|^{-N/2}\epsilon^{-N-1}, \quad (6.46)$$

for a constant $D > 0$.

Then, to obtain the inequality (6.45) it suffices to show that

$$|C_0(1,\lambda)| > \frac{D\exp(2|\text{ Im }[R_2(1/\epsilon)]|)}{|\sqrt{\lambda}|}|\lambda|^{-N/2}\epsilon^{-N-1}, \qquad \lambda \in \Gamma_j. \quad (6.47)$$

In view of (B 2) and (B 3) in Appendix B, there exist constants α and $B_2 > B_1$ such that

$$|C_0(1,\lambda)| = |A_\epsilon|\exp|R_1(1/\epsilon)|\,|\sin(h_\epsilon(\sqrt{\lambda}))| \geq \frac{\alpha}{|\sqrt{\lambda}|}|\sin(h_\epsilon(\sqrt{\lambda}))|, \text{ when } |\sqrt{\lambda}| \geq B_2\epsilon^{-1} \quad (6.48)$$

This last inequality is a consequence of (B 3) and the uniform boundedness in ϵ of $|R_1(1/\epsilon)|$ (Recall that $|R_1(1/\epsilon)|$ is a linear combination of $S^{2n+1}(x/\epsilon)$ which are uniformly bounded in ϵ due to the periodicity $S^{2n+1}(t)$ stated in Lemma 6.1).

On the other hand, if $\lambda \in \Gamma_j$ then $h_\epsilon(\sqrt{\lambda}) = \pi j + r_j e^{i\theta}$, $0 \leq \theta < 2\pi$ and we have

$$|C_0(1,\lambda)| \geq \frac{\alpha}{|\sqrt{\lambda}|}|\sin(r_j e^{i\theta})| > \frac{\alpha r_j}{\pi|\sqrt{\lambda}|}, \quad \text{when } \lambda \in \Gamma_j, |\sqrt{\lambda}| > B_2\epsilon^{-1} \text{ and } r_j < \pi/2. \quad (6.49)$$

This last inequality comes from the fact that $|\sin(r_j e^{i\theta})| > |\sin(r_j/2)| \geq r_j/\pi$ when $r_j \leq \pi/2$ and $\theta \in [0, 2\pi)$. Therefore, to guarantee (6.47) we must consider r_j so that

$$\frac{\pi}{2} \geq r_j > \frac{\pi D\exp(2|\text{ Im }[R_2(1/\epsilon)]|)}{2\alpha}|\lambda|^{-N/2}\epsilon^{-N-1}, \quad (6.50)$$

for $\lambda \in \Gamma_j$ and $|\sqrt{\lambda}| > B_2\epsilon^{-1-1/N}$.

Note that when $\lambda \in \Gamma_j$ we have

$$\begin{aligned} j\pi + r_j e^{i\theta} = h_\epsilon(\sqrt{\lambda}) &= \sqrt{\lambda}\frac{\epsilon S^0(1/\epsilon)}{i} + \sqrt{\lambda}\sum_{n=1}^{N/2}\frac{\epsilon S^{2n}(1/\epsilon)}{i\left(\sqrt{\lambda}\epsilon\right)^{2n}} \\ &= \sqrt{\lambda}\frac{\epsilon S^0(1/\epsilon)}{i} + \sqrt{\lambda}\mathcal{O}(|\sqrt{\lambda}|\epsilon)^{-2} = \sqrt{\lambda}\int_0^1 \sqrt{\rho(s/\epsilon)}ds(1 + \mathcal{O}(\sqrt{\lambda}\epsilon)^{-2}), \quad \theta \in [0, 2\pi). \end{aligned}$$

From this identity we deduce in particular that

$$|\text{Im}(\sqrt{\lambda})| = \left|\frac{r_j \sin\theta}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds(1 + \mathcal{O}(j\epsilon)^{-2})}\right| \leq D_1,$$

with D_1 independent of j, for all $\lambda \in \Gamma_j$ and then $|\text{ Im }[R_2(1/\epsilon)]|$ in (6.46) is uniformly bounded by a constant D_1 independent of λ (see Remark 6.5). On the other hand, we also deduce that

$$\begin{aligned} |\sqrt{\lambda}| = |h_\epsilon^{-1}(\pi j + r_j e^{i\theta})| &= \frac{|j\pi + r_j e^{i\theta}|}{\epsilon|S^0(1/\epsilon)|}(1 + \mathcal{O}((j\epsilon)^{-2})) \\ &< \frac{j\pi + \pi/2}{\sqrt{\rho_m}}(1 + \mathcal{O}((j\epsilon)^{-2})), \quad \text{when } \lambda \in \Gamma_j. \end{aligned} \quad (6.51)$$

Then, there exists $B_3 > B_2$ such that for $j \geq B_3\epsilon^{-1-1/N}$

$$\frac{\pi}{2} \geq \frac{\pi D \exp(2|\text{ Im }[R_2(1/\epsilon)]|)}{2\alpha}|\lambda|^{-N/2}\epsilon^{-N-1}, \quad \forall \lambda \in \Gamma_j,$$

and we can chose r_j so that (6.50) holds.

By the Theorem of Rouché the number of roots of $y_{\epsilon,1}(1,\lambda)$ and $C_0(1,\lambda)$ inside Γ_j ($j \geq B_3\epsilon^{-1-1/N}$) coincides. The circles Γ_j ($j \geq B_3\epsilon^{-1-1/N}$) contain the j-root of $C_0(1,\lambda)$ and therefore each one contain one and only one root of $y_\epsilon(1,\lambda)$, say $\lambda^\epsilon_{m_j+j}$ ($m_j \in Z$).

Note that a priori we can not guarantee that $m_j = 0$ because we have not proved that the root of $y_\epsilon(1,\lambda)$ inside Γ_j is exactly the $j-$root of $y_\epsilon(1,\lambda)$ yet.

We chose $r_j = \pi/2$ so that the circles Γ_j cover the real line

$$R_J = \Big\{\lambda \in \mathbb{R} \text{ such that } h_\epsilon(\sqrt{\lambda}) \geq J\pi - \pi/2\Big\}$$

where J is the minimum integer such that $J \geq B_3\epsilon^{-1-1/N}$. As the roots of $y_\epsilon(1,\lambda)$ are real there exists $m \in \mathbb{Z}$ such that $\lambda^\epsilon_{m+j} \in R_J$ for all $j \geq J$ and they must be inside any of the circles Γ_j. Therefore two consecutive roots of $y_\epsilon(1,\lambda)$ (λ^ϵ_{m+j} and λ^ϵ_{m+j+1}) must be inside two consecutive circles (Γ_j and Γ_{j+1}) and we have proved the following: There exists $m \in \mathbb{Z}$ (independent of j) such that λ^ϵ_{m+j} is inside Γ_j for all $j \geq J$.

Observe that if we chose $r_j = \mathcal{O}(j^{-N}\epsilon^{-N-1})$ the estimate (6.45) still holds and by the Theorem of Rouché λ^ϵ_{m+j} remains inside Γ_j for all $j \geq J$. Then, the distance between $h_\epsilon(\sqrt{\lambda^\epsilon_{m+j}})$ (for $j > J$) and $j\pi$ is at most $r_j = \mathcal{O}(j^{-N}\epsilon^{-N-1})$, i.e.

$$|h_\epsilon(\sqrt{\lambda^\epsilon_{m+j}}) - j\pi| \leq Dj^{-N}\epsilon^{-N-1}, \quad j \geq J. \tag{6.52}$$

We now investigate $h_\epsilon(\sqrt{\lambda})$ defined in (6.43) in order to obtain a more precise estimate for $\sqrt{\lambda^\epsilon_{m+j}}$.

From (6.52) and taking into account that $\frac{\epsilon}{i}S^0(1) = \int_0^1 \sqrt{\rho(s/\epsilon)}ds$ we have

$$\left|\sqrt{\lambda^\epsilon_{m+k}} - \frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds}\right| \leq \frac{D(k)^{-N}\epsilon^{-N-1}}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds} + \sum_{n=1}^{N/2} \frac{|S^{2n}(1)|}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds\,\left(\sqrt{\lambda^\epsilon_k}\epsilon\right)^{2n-1}}$$
$$= \mathcal{O}((\lambda^\epsilon_{m+k})^{-1/2}\epsilon^{-2}) = \mathcal{O}(k^{-1}\epsilon^{-2}). \tag{6.53}$$

From this first order approximation we deduce that

$$\left(\sqrt{\lambda^\epsilon_{m+k}}\epsilon\right)^{2n} = \left(\frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds}\epsilon + \mathcal{O}(k^{-1}\epsilon^{-2})\right)^{2n}$$
$$= \left(\frac{k\pi}{\int_0^1 \sqrt{\rho(s/\epsilon)}ds}\epsilon\right)^{2n}(1 + \mathcal{O}(k^{-2}\epsilon^{-3})), \quad \forall n \geq 1, \tag{6.54}$$

where we have written down only the higher order terms. Dividing equation (6.52) by $\frac{1}{i}\epsilon S^0(1) = \int_0^1 \sqrt{\rho(s/\epsilon)}ds$ and taking into account (6.54) we deduce

$$\left|\sqrt{\lambda^\epsilon_{m+k}} - \frac{k\pi}{\int_0^1 \rho(s/\epsilon)ds} - \sum_{n=1}^{N/2} \frac{S^{2n}(1)}{i\,(k\epsilon)^{2n-1}\,\pi^{2n-1}\left(\int_0^1 \sqrt{\rho(s/\epsilon)}ds\right)^{-2n}(1 + \mathcal{O}(k^{-2}\epsilon^{-3})}\right|$$
$$= \mathcal{O}(k^{-N}\epsilon^{-1-N}). \tag{6.55}$$

Now we observe that

$$S^{2n}(1)=\int_0^{1/\epsilon} S_t^{2n}(t)dt=\frac{1}{\epsilon}\int_0^1 S_t^{2n}(s/\epsilon)ds$$

which is of the order ϵ^{-1} because of the periodicity of S_t^{2n} and the fact that S_t^{2n} does not depend on ϵ. Substituting in (6.55) we obtain

$$\left|\sqrt{\lambda_{m+k}^{\epsilon}}-\frac{k\pi}{\int_0^1\sqrt{\rho(s/\epsilon)}ds}+\sum_{n=1}^{N/2}\frac{\left(\int_0^1\sqrt{\rho(s/\epsilon)}ds\right)^{2n} i\int_0^1 S_t^{2n}(s/\epsilon)ds}{k^{2n-1}\epsilon^{2n}\pi^{2n-1}}\right|=\mathcal{O}(k^{-N}\epsilon^{-1-N}),$$

which is valid for $\sqrt{\lambda_{m+k}^{\epsilon}}\geq B\epsilon^{-1-1/N}$ with B large enough, i.e. taking $k\geq B\epsilon^{-1-1/N}$. We deduce then that

$$\sqrt{\lambda_{m+k}^{\epsilon}}=\frac{k\pi}{\int_0^1\sqrt{\rho(s/\epsilon)}ds}-\sum_{n=1}^{N/2}\frac{\left(\int_0^1\sqrt{\rho(s/\epsilon)}ds\right)^{2n} i\int_0^1 S_t^{2n}(s/\epsilon)ds}{k^{2n-1}\epsilon^{2n}\pi^{2n-1}}+\mathcal{O}(k^{-N}\epsilon^{-1-N}).$$

To finish the proof of (2.1) it remains to see that $m=0$.

Consider $\epsilon>0$ and $K\in I\!N$ such that

$$\left|\sqrt{\lambda_{m+k}^{\epsilon}}-\frac{k\pi}{\int_0^1\sqrt{\rho(s/\epsilon)}ds}\right|<\frac{\pi}{4},\text{ for all } k\geq K. \tag{6.56}$$

Note that due to (6.53) we can consider $K\geq B\epsilon^{-2}$ with B large enough. Let us introduce the curve

$$\Gamma^K=\left\{\lambda\in \mathsf{C}:\left|\sqrt{\lambda}\right|=\frac{K\pi}{\int_0^1\sqrt{\rho(s/\epsilon)}ds}+\frac{\pi}{4}\right\}$$

We prove that the number of roots of $y_\epsilon(1,\lambda)$ and $y_{\epsilon,1}(1,\lambda)$ inside Γ^K is the same when $K\geq B\epsilon^{-2}$ with B large enough. Observe that this is enough to prove that $m=0$ because the number of zeros of $y_\epsilon(1,\lambda)$ inside Γ^K is $m+K$ while

$$y_{\epsilon,1}(1,\lambda)=\frac{\sin\left(\sqrt{\lambda}\int_0^1\sqrt{\rho(s/\epsilon)}ds\right)}{\sqrt{\lambda\rho(0)}}, \tag{6.57}$$

which is an analytic function in λ, has exactly K zeros inside Γ^K.

To prove that the number of roots of $y_\epsilon(1,\lambda)$ and $y_{\epsilon,1}(1,\lambda)$ inside Γ^K coincides we use the Theorem of Rouché. First of all, observe that formula (6.41) gives us

$$|y_\epsilon(1,\lambda)-y_{\epsilon,1}(1,\lambda)|=|y_\epsilon(1,\lambda)-C_0(1,\lambda)|=\left|\sum_{j=1}^{\infty}C_j(1,\lambda)\right|. \tag{6.58}$$

Here we can estimate the coefficients C_j using the results of Lemma 6.4 with $N=1$. Then,

$$\left|\sum_{j=1}^{\infty}C_j(1,\lambda)\right|\leq\frac{D}{\sqrt{\lambda}}e^{|\text{ Im }[\sqrt{\lambda}]|\int_0^1\sqrt{\rho(s/\epsilon)}ds}\sum_{j=1}^{\infty}\frac{M_1^j}{j!}=\frac{D}{\sqrt{\lambda}}e^{|\text{ Im }[\sqrt{\lambda}]|\int_0^1\sqrt{\rho(s/\epsilon)}ds}\mathcal{O}(M_1), \tag{6.59}$$

where $M_1=D\left(\|p_\epsilon\|_\infty+\frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|}\right)=\mathcal{O}(|\lambda|^{-1/2}\epsilon^{-2})$ (see estimates (6.40)).

On the other hand,

$$|y_{\epsilon,1}(1,\lambda)| > \frac{e^{|\text{ Im } [\sqrt{\lambda}]| \int_0^1 \sqrt{\rho(s/\epsilon)}ds}}{4\sqrt{\lambda\rho(0)}}. \tag{6.60}$$

This last inequality comes from the following simple lemma which is proved in [10] (Lemma 1, Ch.2).

Lemma 6.7 *If* $|z - n\pi| \geq \pi/4$ *for all integers* n *then* $\sin z > e^{|\ Im\ z|}/4$.

From formulas (6.59), (6.60) and the definition of Γ^K we deduce that there exists $K \geq B\epsilon^{-2}$ (with B large enough) such that

$$|y_\epsilon(1,\lambda) - y_{\epsilon,1}(1,\lambda)| < |y_{\epsilon,1}(1,\lambda)|, \quad \text{for all } \lambda \in \Gamma^K.$$

Then, by the Theorem of Rouché, the number of roots of $y_\epsilon(1,\lambda)$ inside Γ^K is the same as the number of roots of $y_{\epsilon,1}(1,\lambda)$ (which is K) and therefore $m = 0$.

STEP 5: *Asymptotic formula for the eigenfunctions.* Now we prove the formula (2.2) for the eigenfunctions. Recall that we are assuming that the eigenfunctions (1.1) are normalized so that $(\varphi_k^\epsilon)'(0) = 1$. Then, the eigenfunctions φ_k^ϵ are the solutions of (6.24) with $\lambda = \lambda_k^\epsilon$ and can be developed in the form (6.25):

$$\varphi_k^\epsilon(x) = C_0(x,\lambda_k^\epsilon) + \sum_{j\geq 1} C_j(x,\lambda_k^\epsilon).$$

Define $\widetilde{\lambda}_k^\epsilon$, as in (2.3), which is the approximate eigenvalue given by formula (2.1). We have to estimate

$$\left\| \varphi_k^\epsilon - A_k^\epsilon \exp\left(\sum_{n=0}^{N/2-1} \frac{S^{2n+1}(x/\epsilon)}{\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon\right)^{2n}} \right) \sin\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon \sum_{n=0}^{N/2} \frac{S^{2n}(x/\epsilon)}{\left(\sqrt{\widetilde{\lambda}_k^\epsilon}\epsilon\right)^{2n}} \right) \right\|_{W^{1,\infty}(0,1)}$$

$$= \left\| C_0(x,\lambda_k^\epsilon) + \sum_{j\geq 1} C_j(x,\lambda_k^\epsilon, p_\epsilon) - C_0(x,\widetilde{\lambda}_k^\epsilon) \right\|_{W^{1,\infty}(0,1)}$$

$$\leq \left\| C_0(x,\lambda_k^\epsilon) - C_0(x,\widetilde{\lambda}_k^\epsilon) \right\|_{W^{1,\infty}(0,1)} + \left\| \sum_{j\geq 1} C_j(x,\lambda_k^\epsilon) \right\|_{W^{1,\infty}(0,1)}. \tag{6.61}$$

To finish the proof of (2.2) we have to see that the two terms in (6.61) converge to zero as $\epsilon \to 0$. We start with the second one. By Lemma 6.4 we have:

$$\| \sum_{j\geq 1} C_j'(x,\lambda_k^\epsilon) \|_\infty \leq \sum_{j\geq 1} \| C_j'(x,\lambda_k^\epsilon) \|_\infty \leq \sum_{j\geq 1} 2^j D^j \frac{\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{\sqrt{\lambda}} \right)^j}{j!} =$$
$$= \left(1 - \exp\left(2D\, \|p_\epsilon\|_\infty + 2D \frac{\|q_\epsilon\|_\infty}{\sqrt{\lambda}} \right) \right)$$

which can be done small, uniformly in $x \in [0,1]$, taking $\lambda_k^\epsilon \geq B\epsilon^{-2-2/N}$ with B large enough as we showed in the proof of Proposition 6.1. We deduce that the quantity

$\| \sum_{j\geq 1} C_j(x,\lambda_k^\epsilon)\|_{W^{1,\infty}(0,1)}$ can be done as small as we want if $k \geq B\epsilon^{-1-1/N}$ with B large enough.

Concerning the first term in (6.61), observe that

$$C_0(x,\lambda) = A_k^\epsilon \exp\left(\sum_{n=0}^{N/2-1} \frac{S^{2n+1}(x/\epsilon)}{\left(\sqrt{\lambda}\epsilon\right)^{2n}} \right) \sin\left(\sqrt{\lambda}\epsilon \sum_{n=0}^{N/2} \frac{S^{2n}(x/\epsilon)}{\left(\sqrt{\lambda}\epsilon\right)^{2n}} \right).$$

We deduce that both $C_0(x,\lambda)$ and $\frac{\partial C_0}{\partial x}(x,\lambda)$ are C^1 functions of $\sqrt{\lambda}$ near the points $\sqrt{\lambda_k^\epsilon}$ so that the result is a consequence of (2.1) and the mean value theorem.

7 Conclusions

We have deduced an asymptotic expansion for the high frequency eigenvalues and eigenfunctions of the one-dimensional problem associated to the vibration of a string with rapidly oscillating periodic density. Our expansions provide good approximations when the frequency of the eigenfunctions ($\sqrt{\lambda^\epsilon}$) is larger than the frequency of the density $1/\epsilon$. This result complements a previous one where an asymptotic expansion of the low frequencies is given, i.e. those frequencies $\sqrt{\lambda^\epsilon}$ lower than $1/\epsilon$ (see [7]). The case where $\sqrt{\lambda^\epsilon} \sim 1/\epsilon$ is critical (see [2] and [7]) and there are not asymptotic formulas which describe the behavior of the spectra.

To illustrate the behavior of the high frequencies a numerical example is given.

As an application we prove the following two spectral properties for the high frequencies:

- There is an asymptotic spectral gap between two consecutive eigenvalues.
- A uniform boundary observability property for the eigenfunctions.

These two properties constitute the two key points to establish the boundary controllability of the high frequencies of the string (see [5]).

Our results can be also applied to the eigenvalue problem

$$\begin{cases} (a(x/\epsilon)u')' + \lambda u = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{7.1}$$

where $a(x)$ is a bounded periodic function.

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Appendix A

In this section we prove a technical result (Lemma 6.1) that we have used in Section 6. Let us recall it:

Lemma 6.1 The coefficients $S^{2n+1}(t)$ with $n \geq 0$ are 1-periodic functions.

Proof of Lemma 6.1 We are going to prove the following formula by induction:

$$S_t^{2n+1} = \sum_{k=1}^{n} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} \frac{(-1)^k}{2k} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_k}}{\left(S_t^0\right)^k} \right]_t, \qquad n \geq 1. \tag{A 1}$$

Observe that formula (A 1) is enough to prove the lemma because S_t^{2n} are all 1-periodic functions.

Consider $n = 1$. Using (6.6) we easily obtain

$$S_t^3 = -\left[\frac{S_{tt}^2}{2S_t^0} + \frac{S_t^2 S_t^1}{S_t^0} \right] = -\left[\frac{S_{tt}^2}{2S_t^0} - \frac{S_t^2 S_{tt}^0}{2\left(S_t^0\right)^2} \right] = -\left[\frac{S_t^2}{2S_t^0} \right]_t.$$

Now, assume that formula (A 1) holds for all $j < n$. We are going to see that it also holds for n:

$$\begin{aligned}
S_t^{2n+1} &= -\frac{S_{tt}^{2n}}{2S_t^0} - \frac{\sum_{i=1}^{n} S_t^{2i} S_t^{2(n-i)+1}}{S_t^0} = -\left[\frac{S_{tt}^{2n}}{2S_t^0} + \frac{S_t^{2n} S_t^1}{S_t^0} \right] - \frac{\sum_{i=1}^{n-1} S_t^{2i} S_t^{2(n-i)+1}}{S_t^0} \\
&= -\left[\frac{S_t^{2n}}{2S_t^0} - \frac{S_t^2 S_{tt}^0}{2\left(S_t^0\right)^2} \right]_t - \sum_{i=1}^{n-1} \frac{S_t^{2i}}{S_t^0} \sum_{k=1}^{n-i} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n-i}} \frac{(-1)^k}{2k} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_k}}{\left(S_t^0\right)^k} \right]_t \\
&= -\left[\frac{S_t^{2n}}{2S_t^0} \right]_t - \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n-i}} \frac{(-1)^k}{2k} \frac{S_t^{2i}}{S_t^0} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_k}}{\left(S_t^0\right)^k} \right]_t \\
&= -\left[\frac{S_t^{2n}}{2S_t^0} \right]_t + \sum_{k=1}^{n-1} \sum_{\substack{i_1,\ldots,i_{k+1} \geq 1 \\ i_1+\ldots+i_k+i_{k+1}=n}} \frac{(-1)^{k+1}}{2(k+1)} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_{k+1}}}{\left(S_t^0\right)^{k+1}} \right]_t
\end{aligned} \tag{A 2}$$

Here the last identity is the most delicate one. Indeed we have to check that

$$\sum_{i=1}^{n-k} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n-i}} \frac{1}{k} \frac{S_t^{2i}}{S_t^0} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_k}}{\left(S_t^0\right)^k} \right]_t = \sum_{\substack{i_1,\ldots,i_{k+1} \geq 1 \\ i_1+\ldots+i_k+i_{k+1}=n}} \frac{1}{k+1} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_{k+1}}}{\left(S_t^0\right)^{k+1}} \right]_t \tag{A 3}$$

for all $1 \leq k \leq n-1$.

The following formula holds:

$$\begin{aligned}
k \left[\frac{S_t^{2i_1} \cdots S_t^{2i_{k+1}}}{\left(S_t^0\right)^{k+1}} \right]_t &= \frac{S_t^{2i_1}}{S_t^0} \left[\frac{S_t^{2i_2} \cdots S_t^{2i_{k+1}}}{\left(S_t^0\right)^k} \right]_t + \frac{S_t^{2i_2}}{S_t^0} \left[\frac{S_t^{2i_1} S_t^{2i_3} \cdots S_t^{2i_{k+1}}}{\left(S_t^0\right)^k} \right]_t + \cdots \\
&\quad + \frac{S_t^{2i_{k+1}}}{S_t^0} \left[\frac{S_t^{2i_1} \cdots S_t^{2i_k}}{\left(S_t^0\right)^k} \right]_t .
\end{aligned} \tag{A 4}$$

Indeed, if we define $f^j = S_t^{2i_j}/S_t^0$ formula (A 4) is just

$$k\left[f^1 \cdots f^{k+1}\right]_t = f^1\left[f^2 \cdots f^{k+1}\right]_t + f^2\left[f^1 f^3 \cdots f^{k+1}\right]_t + f^{k+1}\left[f^1 \cdots f^k\right]_t$$

that may be easily checked.

Then, we obtain

$$\sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} k\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k+1}}}{(S_t^0)^{k+1}}\right]_t = \sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} \frac{S_t^{2i_1}}{S_t^0}\left[\frac{S_t^{2i_2}\cdots S_t^{2i_{k+1}}}{(S_t^0)^{k}}\right]_t$$

$$+\sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} \frac{S_t^{2i_2}}{S_t^0}\left[\frac{S_t^{2i_1}S_t^{2i_3}\cdots S_t^{2i_{k+1}}}{(S_t^0)^{k}}\right]_t$$

$$+\cdots+\sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} \frac{S_t^{2i_{k+1}}}{S_t^0}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k}}}{(S_t^0)^{k}}\right]_t$$

$$=(k+1)\sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} \frac{S_t^{2i_{k+1}}}{S_t^0}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k}}}{(S_t^0)^{k}}\right]_t$$

$$=(k+1)\sum_{i=1}^{n-k}\sum_{\substack{i_1,\dots,i_{k}\geq 1\\ i_1+\dots+i_k=n-i}} \frac{S_t^{2i}}{S_t^0}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k}}}{(S_t^0)^{k}}\right]_t$$

as we wanted to prove. Now, from (A 2) we deduce

$$S_t^{2n+1} = -\left[\frac{S_t^{2n}}{2S_t^0}\right]_t + \sum_{k=1}^{n-1}\sum_{\substack{i_1,\dots,i_{k+1}\geq 1\\ i_1+\dots+i_k+i_{k+1}=n}} \frac{(-1)^{k+1}}{2(k+1)}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k+1}}}{(S_t^0)^{k+1}}\right]_t$$

$$= -\left[\frac{S_t^{2n}}{2S_t^0}\right]_t + \sum_{k=2}^{n}\sum_{\substack{i_1,\dots,i_{k}\geq 1\\ i_1+\dots+i_k+i_k=n}} \frac{(-1)^{k}}{2k}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k}}}{(S_t^0)^{k}}\right]_t$$

$$= \sum_{k=1}^{n}\sum_{\substack{i_1,\dots,i_{k}\geq 1\\ i_1+\dots+i_k+i_k=n}} \frac{(-1)^{k}}{2k}\left[\frac{S_t^{2i_1}\cdots S_t^{2i_{k}}}{(S_t^0)^{k}}\right]_t$$

and the proof of formula (A 1) is finished. □

Appendix B

This appendix is devoted to prove Lemma 6.4.

Proof All along this proof D will denote a constant which depends on ρ and N but which can change from one line to the other. Observe that

$$\begin{aligned}|C_0(x)| = |y_{\epsilon,N}(x)| &\leq |A_\epsilon|\exp\left(|R_1(x/\epsilon)|\right)|\sin\left(R_2(x/\epsilon)\right)| \\ &\leq |A_\epsilon|\exp\left(|R_1(x/\epsilon)| + |\,\mathrm{Im}\,[R_2(x/\epsilon)]|\right).\end{aligned} \tag{B 1}$$

We now estimate these two terms as follows:

$$\begin{aligned}|R_1(x/\epsilon)| &= \left|\sum_{n=0}^{N/2-1}\frac{S^{2n+1}(x/\epsilon)}{(\sqrt{\lambda}\epsilon)^{2n}}\right| = \left|S^1(x/\epsilon) + \sum_{n=1}^{N/2-1}\frac{S^{2n+1}(x/\epsilon)}{(\sqrt{\lambda}\epsilon)^{2n}}\right| \\ &= \frac{1}{4}\left|\log\frac{\rho(x/\epsilon)}{\rho(0)}\right| + \mathcal{O}((|\sqrt{\lambda}|\epsilon)^{-2}).\end{aligned} \tag{B 2}$$

This holds because we are dealing with a finite sum of periodic (and therefore bounded) functions $|S^{2n+1}(x/\epsilon)|$ multiplied by powers of $(|\sqrt{\lambda}|\epsilon)^{-1}$ which are bounded in the range of ϵ and λ that we are considering.

We now estimate the first term in (B 1). From formula (6.12) we obtain

$$\begin{aligned} |A_{\epsilon,N}| &= \frac{1}{|\sqrt{\lambda}|}\left|\sum_{n=0}^{N/2}\frac{S_t^{2n}(0)}{(\sqrt{\lambda}\epsilon)^{2n}}\right|^{-1} \leq \frac{1}{|\sqrt{\lambda}|}\left|\sqrt{\rho(0)}+\sum_{n=1}^{N/2}\frac{S_t^{2n}(0)}{i(\sqrt{\lambda}\epsilon)^{2n}}\right|^{-1} \\ &= \frac{1}{|\sqrt{\lambda}|\sqrt{\rho(0)}}\left(1+\mathcal{O}(|\sqrt{\lambda}|\epsilon)^{-1})\right), \end{aligned} \tag{B 3}$$

which can be bounded uniformly in ϵ taking $(|\sqrt{\lambda}|\epsilon)^{-1} \leq b_N = B_N^{-1}$ with b_N small enough (i.e. B_N large enough). From (B 1) and (B 3) we easily obtain the first inequality in (6.30). Note that the constant B_N depends on N because we do not have any uniform estimate (independent of n) for $S_t^n(0)$.

Finally, we deduce that

$$|C_0(x)| \leq \frac{D}{|\sqrt{\lambda}|}e^{|\text{ Im }[R_2(x/\epsilon)]|}.$$

Now we are going to estimate $C_0'(x)$. Observe that due to the definition of $C_0(x)$ we have:

$$\begin{aligned} |C_0'(x)| = \Big| A_{\epsilon,N}\frac{1}{\epsilon}R_1'(x/\epsilon)] \exp\left(R_1(x/\epsilon)\right)\sin\left(R_2(x/\epsilon)\right) \\ +A_{\epsilon,N}\frac{1}{\epsilon}R_2'(x/\epsilon)\exp\left(R_1(x/\epsilon)\right)\cos\left(R_2(x/\epsilon)\right)\Big| \\ \leq 2\,|A_{\epsilon,N}|\exp\left(|R_1(x/\epsilon)|+|\text{ Im }[R_2(x/\epsilon)]|\right)\frac{|R_1'(x/\epsilon)|+|R_2'(x/\epsilon)|}{\epsilon}. \end{aligned} \tag{B 4}$$

The first two factors on the right hand side of (B 4) have been estimated in (B 3) and (B 2) respectively. Concerning the third factor in (B 4) we have

$$\frac{|R_1'(x/\epsilon)|+|R_2'(x/\epsilon)|}{\epsilon} \leq \frac{1}{\epsilon}\sum_{n=0}^{N/2}\frac{|S_t^n(x/\epsilon)|}{\left(|\sqrt{\lambda}|\epsilon\right)^{n-1}} \leq |\sqrt{\lambda}|\sum_{n=0}^{N/2-1}\frac{|S_t^n(x/\epsilon)|}{\left|\sqrt{\lambda}\epsilon\right|^{n}} \leq D|\sqrt{\lambda}|, \tag{B 5}$$

since it is a finite sum of periodic functions S_t^n multiplied by powers of $|\sqrt{\lambda}\epsilon|^{-1}$ which is bounded in the region of ϵ and λ we are considering. Obviously the constant D in (B 5) depends on N.

From (B 2)-(B 5) we easily deduce the uniform estimate for $C_0'(x)$ in (6.30).

Consider now the coefficients C_j with $j \geq 1$. We start analyzing the kernel arising in (6.29):

$$K(x,s) = \frac{y_{\epsilon,N}(x)z_{\epsilon,N}(s)-z_{\epsilon,N}(x)y_{\epsilon,N}(s)}{y_{\epsilon,N}(s)z_{\epsilon,N}'(s)-y_{\epsilon,N}'(s)z_{\epsilon,N}(s)} = \frac{e^{2R_1(x/\epsilon)}\sin\left(R_2(x/\epsilon)-R_2(s/\epsilon)\right)}{\frac{R_2'(x/\epsilon)}{\epsilon}\exp\left(2R_1(x/\epsilon)\right)}. \tag{B 6}$$

The first factor in the numerator can be estimated using (B 2) while for the denominator we have two terms: the first one is

$$\begin{aligned} \frac{R_2'(x/\epsilon)}{\epsilon} &= \sqrt{\lambda}\frac{S_t^0(x/\epsilon)}{i}+\frac{\sqrt{\lambda}}{\epsilon\sqrt{\lambda}}\sum_{n=1}^{N/2}\frac{S_t^n(x/\epsilon)}{i(\sqrt{\lambda}\epsilon)^{n-1}} \\ &= \sqrt{\lambda}\sqrt{\rho(x/\epsilon)}(1+\mathcal{O}(|\sqrt{\lambda}\epsilon|^{-1})), \end{aligned} \tag{B 7}$$

and the second one is uniformly bounded below in view of (B 2).

Then, we deduce that

$$|K(x,s)| \leq D\,|\sin\left(R_2(x/\epsilon) - R_2(s/\epsilon)\right)| \leq \frac{De^{|\mathrm{Im}[R_2(\frac{x-s}{\epsilon})]|}}{\sqrt{\lambda}} \tag{B 8}$$

in the region of ϵ and λ that we are considering. The last inequality comes from the following identity

$$\begin{aligned} R_2(x/\epsilon) - R_2(s/\epsilon) &= \sqrt{\lambda}\epsilon \sum_{n=0}^{N/2} (\sqrt{\lambda}\epsilon)^{-2n} \left(\int_0^{x/\epsilon} S_t^{2n}(r)dr - \int_0^{s/\epsilon} S_t^{2n}(r)dr \right) \\ &= \sqrt{\lambda}\epsilon \sum_{n=0}^{N/2} (\sqrt{\lambda}\epsilon)^{-2n} \int_0^{(x-s)/\epsilon} S_t^{2n}(r)dr \\ &= \sqrt{\lambda}\epsilon \sum_{n=0}^{N/2} (\sqrt{\lambda}\epsilon)^{-2n} S^{2n}(\frac{x-s}{\epsilon})dr = R_2((x-s)/\epsilon), \end{aligned} \tag{B 9}$$

and the fact that $|\sin(z)| \leq e^{|\mathrm{Im}[z]|}$ for all $z \in \mathbb{C}$.

In a similar way, we deduce that

$$\left| \frac{\partial}{\partial x} K(x,s) \right| \leq De^{|\mathrm{Im}[R_2(\frac{x-s}{\epsilon})]|}. \tag{B 10}$$

Now observe that:

$$\mathrm{Im}[R_2(t)] \leq |\mathrm{Im}[R_2(0)]| + |\mathrm{Im}[R_2(1/\epsilon)]| = |\mathrm{Im}[R_2(1/\epsilon)]|\,, \quad \forall t \in [0, \epsilon^{-1}], \tag{B 11}$$

which is due to the fact that $\mathrm{Im}[R_2(t)]$ is a monotone function, because the leading term in the derivative $\mathrm{Im}\ [R_2'(t)]$ is $\mathrm{Im}\ [\sqrt{\lambda}]\sqrt{\rho(t)} \neq 0$ when $\mathrm{Im}\ [\sqrt{\lambda}] \neq 0$, while $\mathrm{Im}\ [R_2(t)] = 0$ when $\mathrm{Im}\ [\sqrt{\lambda}] = 0$.

Combining (B 11) with estimates (B 8) and (B 10) we have the following:

$$|K(x,s)| \leq \frac{De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}}{\sqrt{\lambda}}, \quad \left| \frac{\partial}{\partial x} K(x,s) \right| \leq De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}.$$

To simplify the notation, we introduce $M = De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|}\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|} \right)$. We have then

$$\begin{aligned} |C_j(x)| &= \left| \int_0^x K(x,s)(p_\epsilon(s)C_{j-1}'(s) + q_\epsilon(s)C_{j-1}(s))ds \right| \\ &\leq \frac{M}{|\sqrt{\lambda}|} \int_0^x \left(|C_{j-1}'(s)| + |\sqrt{\lambda}| \cdot |C_{j-1}(s)| \right) ds, \end{aligned} \tag{B 12}$$

$$\begin{aligned} |C_j'(x)| &= \left| \int_0^x \frac{\partial}{\partial x} K(x,s)(p_\epsilon(s)C_{j-1}'(s) + q_\epsilon(s)C_{j-1}(s))ds \right| \\ &\leq M \int_0^x \left(|C_{j-1}'(s)| + |\sqrt{\lambda}| \cdot |C_{j-1}(s)| \right) ds. \end{aligned} \tag{B 13}$$

Finally, we have

$$\begin{aligned} |\sqrt{\lambda}||C_j(x)| &\leq M \int_0^1 \left(|C_{j-1}'(s)| + |\sqrt{\lambda}||C_{j-1}(s)| \right) ds \\ &\leq 2M^2 \int_0^1 \int_0^{s_1} \left(|C_{j-2}'(s_2)| + |\sqrt{\lambda}||C_{j-2}(s_2)| \right) ds_2 ds_1 \end{aligned}$$

$$\leq 2^{j-1}M^j \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{j-1}} \left(|C_0'(s_j)| + |\sqrt{\lambda}| \cdot |C_0(s_j)| \right) ds_j \cdots ds_1$$

$$= 2^j De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|} M^j \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{j-1}} ds_j \cdots ds_1 \leq \frac{2^j De^{|\mathrm{Im}[R_2(\frac{1}{\epsilon})]|} M^j}{j!} \qquad \text{(B 14)}$$

Estimate (B 14) concludes the proof for C_j. The estimates for C_j' can be derived in a similar way from (B 10) and (B 13).

Now we consider the case $N = 1$ where we can improve the estimates for $|K|$ and $\left|\frac{\partial K}{\partial x}\right|$. Note that we have proved the following:

$$|K(x,s)| \leq \frac{D}{\sqrt{\lambda}} e^{|\mathrm{Im}[R_2(\frac{x-s}{\epsilon})]|} = \frac{D}{\sqrt{\lambda}} e^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x-s} \sqrt{\rho(r/\epsilon)}dr},$$

$$\left| \frac{\partial}{\partial x} K(x,s) \right| \leq De^{|\mathrm{Im}[R_2(\frac{x-s}{\epsilon})]|} = De^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x-s} \sqrt{\rho(r/\epsilon)}dr}.$$

Then, if we denote $M_1 = D\left(\|p_\epsilon\|_\infty + \frac{\|q_\epsilon\|_\infty}{|\sqrt{\lambda}|} \right)$ we have

$$|\sqrt{\lambda}||C_j(x)| \leq M_1 \int_0^1 e^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x-s} \sqrt{\rho(r/\epsilon)}dr} \left(|C_{j-1}'(s)| + |\sqrt{\lambda}| \cdot |C_{j-1}(s)| \right) ds$$

$$\leq 2M_1^2 \int_0^1 \int_0^{s_1} e^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x-s_2} \sqrt{\rho(r/\epsilon)}dr} \left(|C_{j-2}'(s_2)| + |\sqrt{\lambda}||C_{j-2}(s_2)| \right) ds_2 ds_1$$

$$\leq 2^{j-1}M^j \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{j-1}} e^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x-s_j} \sqrt{\rho(r/\epsilon)}dr} \left(|C_0'(s_j)| + |\sqrt{\lambda}||C_0(s_j)| \right) ds_j \cdots ds_1$$

$$\leq \frac{2^j DM^j}{j!} e^{|\mathrm{Im}[\sqrt{\lambda}]| \int_0^{x} \sqrt{\rho(r/\epsilon)}dr}.$$

The estimates for C_j' can be derived in a similar way. □

References

[1] Allaire A. & Conca C. (1995) Analyse asymptotique spectrale de l'équation des ondes. Homogénéisation par ondes de Bloch. *C.R. Acad. Paris* **321**, Série I, 293–298 and 557–562.

[2] Avellaneda, M., Bardos C. & Rauch, J. (1992) Contrôlabilité exacte, homogénéisation et localisation d'ondes dans un milieu non-homogène. *Asymptotic analysis* **5**, 481–494.

[3] Bensoussan, A., Lions, J.-L. & Papanicolau, G. (1978) *Asymptotic analysis for periodic structures.* Amsterdam. North Holland.

[4] Bender, C. M. & Orszag, S. A. (1978) *Advanced Mathematical Methods for Scientists and Engineers,* MacGraw-Hill.

[5] Castro, C., (1999) Boundary controllability of the one-dimensional wave equation with rapidly oscillating density, *Asymptotic Analysis*, **20**, 317-350.

[6] Castro, C. & Zuazua, E. (1997) Contrôlabilité de l'équation des ondes à densité rapidement oscillante à une dimension d'espace *C.R. Acad. Sci. Paris*, **324**, Série I, 1237–1242.

[7] Castro, C. & Zuazua, E. (2000) Low frequency asymptotic analysis of a string with rapidly oscillating density, *SIAM J. Appl. Math.*, **60**, 1205-1233.

[8] Eastham, M. S. P. (1973) *The spectral theory of periodic differential equations.* Edinburgh. Scottish Academic Press.

[9] Oleinik, O. A., Shamaev, A. S. & Yosifian, B. A. (1992) *Mathematical Problems in Elasticity and Homogenization*, North Holland.

[10] Poschel, J. & Trubowitz, E. (1987) *Inverse spectral theory.* Academic Press.

[11] Santosa, F. & Vogelius, M. (1993) First-order corrections to the homogenized eigenvalues of a periodic composite medium, *SIAM J. Appl. Math.* **53**, 1636–1668.

[12] Young, R. M. (1980) *An introduction to Nonharmonic Fourier series*. Academic Press.