# PARABOLIC SINGULAR LIMIT OF A WAVE EQUATION WITH LOCALIZED BOUNDARY DAMPING

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Contents



## 1. Introduction

1.1. **Formulation of the problem.** We will consider the family of wave equations with boundary damping

$$(P_{\epsilon,\lambda,\Gamma_0}) \begin{cases} \epsilon u_{tt} - \Delta u + \lambda u = f & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{cases}$$

where $0 < \epsilon \leq \epsilon_0$, $\Omega \subset \mathbb{R}^N$ is a regular open connected set, $\lambda \geq 0$ and $\Gamma = \Gamma_0 \cup \Gamma_1$ is a partition of the boundary of $\Omega$. We will also consider the case where $\Gamma_0$ is empty (see below for more precise assumptions on $\lambda$, $\Omega$ and $\Gamma_0$, $\Gamma_1$).

For this problem the corresponding formal singular perturbation at $\epsilon = 0$ is

$$(P_{0,\lambda,\Gamma_0}) \begin{cases} -\Delta u + \lambda u = f & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{cases}$$

We are here concerned with the well possedness of both problems for the non–homogeneous case, i.e. $f = f(t,x)$, $g = g(t,x)$, and with the convergence, as $\epsilon$ approaches 0, of the solutions of $(P_{\epsilon,\lambda,\Gamma_0})$ to solutions of $(P_{0,\lambda,\Gamma_0})$.

In past years, equations of the type $(P_{\epsilon,\lambda,\Gamma_0})$ have attracted a lot of attention because of the decay properties as $t \to \infty$ when $f = g = 0$. When $\Omega$ is bounded and $\Gamma_1$ is a "large enough" subset of $\Gamma$ it is well known that the energy of solutions of $(P_{\epsilon,\lambda,\Gamma_0})$ converges exponentially to zero as $t \to \infty$ (see for example [2, 14, 15, 16] and references therein). It is important to notice that the class of subsets $\Gamma_1$ for which the exponential decay holds does not depend on the value of $\epsilon$. See also [18]. However, as we will see below, the limiting problem $(P_{0,\lambda,\Gamma_0})$ has a parabolic structure and therefore solutions decay whatever subset $\Gamma_1$ of $\Gamma$ we choose.

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These problems and the corresponding singular perturbations have been considered by several authors. The limiting problem at $\epsilon = 0$ and $f = 0$ was considered by Friedman and Shinbrot in [7] in relation with the equations of surface water waves. See also [20]. The problem of the approximation as $\epsilon$ goes to zero, was considered by Hale and Raugel in [11], in the case of homogeneous Neumann boundary conditions and damping distributed on $\Omega$, i.e. for the problem

$$\left\{ \begin{array}{ll} \epsilon u_{tt} + u_t - \Delta u + \lambda u = f & \text{on } \Omega \times (0,T) \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma \times (0,T) \end{array} \right. \tag{1.1}$$

and $f$ a suitable nonlinearity. Here the authors studied the upper semicontinuity of attractors as $\epsilon \to 0$. The special case of dimension one was also considered in [22], where the authors proved stronger results by means of inertial manifolds techniques.

There are three main issues we want to discuss in this paper. First of all, the well possedness of problem $(P_{\epsilon,\lambda,\Gamma_0})$. As we will see in Section 2, the case $g = 0$ can be treated in a very simple and natural way. However, the case $g \neq 0$ presents a much more subtle structure. At first sight this case is not immediately seen as a perturbation of the case $g = 0$. Another point of discussion is the concept of solution itself, and in what sense the boundary conditions are verified. It turns out that an adjoint equation in a dual space is the right tool to solve the above questions.

On the other hand, we want to discuss also the well possedness of the "limiting equation", $(P_{0,\lambda,\Gamma_0})$. At first glance it has not a typical form of an evolution equation, since an elliptic equation is involved. In Section 3 we will show that, however, this equation has a parabolic structure in suitable spaces, and we will show its regularizing effect. Finally, in Section 4 we shall prove that $(P_{0,\lambda,\Gamma_0})$ is in fact, not only a formal "limiting equation", but that the solutions of $(P_{\epsilon,\lambda,\Gamma_0})$ actually converge to those of $(P_{0,\lambda,\Gamma_0})$.

1.2. **Notation and elliptic results.** We now introduce some notations that will be used throughout the paper. All along the paper and specially in proofs $c_i$ will denote generic positive constants.

Now we make precise the assumptions on $\lambda \geq 0$, $\Omega$ and $\Gamma_0$, $\Gamma_1$. As said above, $\Omega \subset I\!\!R^N$ is a regular connected open set. It is assumed that $\Omega$ and $\Gamma$ are such that the trace operator and the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$ are well defined, see [10]. Note however that we will not always assume that $\Omega$, nor its boundary $\Gamma$, are bounded. When needed it will be explicitly stated. In some cases we will assume that $\Gamma_1$ is bounded. Also, note that the usual assumption that the $\Gamma_i$ form a disjoint partition of the boundary i.e. $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, which is not made here, allows one to use general elliptic regularity results that are not available otherwise, [10]. Some remarks will be done on this when necessary.

We denote, for $t \leq \infty$, $Q_t = \Omega \times (0,t)$ and $\Sigma_{1,t} = \Gamma_1 \times (0,t)$, and identify $L^2(Q_t) = L^2(0,t,L^2(\Omega))$ and $L^2(\Sigma_{1,t}) = L^2(0,t,L^2_{\Gamma_0}(\Gamma))$. When working with a fixed time interval $(0,T)$ we will simply write $Q$ and $\Sigma_1$. Space–time integrals will be denoted by integrals over $Q$ and $\Sigma_1$.

Concerning functional spaces, we will use the standard Sobolev spaces $H^s_{\Gamma_0}(\Omega)$ and $H^s_{\Gamma_0}(\Gamma)$ for $s \geq 0$, which are closed subspaces of $H^s(\Omega)$ and $H^s(\Gamma)$, respectively, and the subscript $\Gamma_0$ means that, respectively, traces or functions in $\Gamma$, vanish on that part of the boundary of $\Omega$. In case $\Gamma_0 = \emptyset$ we set $H^s_{\Gamma_0} = H^s$. Also, we will denote by $H^{-s}_{\Gamma_0}$ the dual space of $H^s_{\Gamma_0}$, either on $\Omega$ or $\Gamma$. Note that this notation introduces some ambiguity when $\Gamma_0 = \emptyset$, since we set $H^s_{\Gamma_0} = H^s$ and then the dual space is denoted $H^{-s}$, but this simbol is usually reserved to denote the dual space of $H^s_0$. However, this notation should produce no confusion. When the space $H^s_0$ appears, its dual is denoted by $H^{-s}_0$. The duality pairing between the spaces above, will be denoted $< \cdot, \cdot >_{-s,s}$. In particular the scalar product in $L^2$ will be denoted by $< \cdot, \cdot >$. If there is no possible confusion, we will not indicate if the spaces or duality products are referred to functions on $\Omega$ or $\Gamma$. When required, we will write $< \cdot, \cdot >_\Omega$ and $< \cdot, \cdot >_\Gamma$ to differentiate both cases.

Concerning the partition of the boundary we have the following

**Definition 1.1.** *For functions defined on $\Gamma$, we define the "truncation" operator on $\Gamma_0$*

$$R(u) = \left\{ \begin{array}{ll} u & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0 \end{array} \right.$$

*Note that $R(L^2(\Gamma)) = L^2_{\Gamma_0}(\Gamma)$ and $H^s_{\Gamma_0}(\Gamma) = R(H^s_{\Gamma_0}(\Gamma)) \subset R(H^s(\Gamma))$.*

*We say the partition $\Gamma_i$ is "s–regular" iff $R(H^s(\Gamma)) = H^s_{\Gamma_0}(\Gamma)$ and that holds for any $s$ if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. In order to shorten the notations, when the regularity is needed we will just use "regular" to mean "s–regular" for suitable $s$, that will not be explicitly indicated.*

We will denote by $\gamma$ the trace operator defined on $H^s_{\Gamma_0}(\Omega)$ and $H^s(\Omega)$, with values in $H^{s-1/2}_{\Gamma_0}(\Gamma)$ and $H^{s-1/2}(\Gamma)$ respectively, with $s > 1/2$. Moreover, for a given function $f \in H^1_{\Gamma_0}(\Omega)$, we will identify its trace, $\gamma(f) \in H^{1/2}_{\Gamma_0}(\Gamma)$, with the linear form $\gamma(f) \in H^{-1/2}_{\Gamma_0}(\Gamma) \subset H^{-1}_{\Gamma_0}(\Omega)$, such that for every $\phi \in H^1_{\Gamma_0}(\Omega)$

$$< \gamma(f), \phi >_{-1,1} \stackrel{def}{=} < f, \phi >_\Gamma \stackrel{def}{=} \int_\Gamma \gamma(f)\gamma(\phi)$$

that is, we use the embedding $H^{1/2}_{\Gamma_0}(\Gamma) \subset L^2_{\Gamma_0}(\Gamma) \subset H^{-1/2}_{\Gamma_0}(\Gamma) \subset H^{-1}_{\Gamma_0}(\Omega)$.

We will also consider the normal derivative operator defined as follows: if $u \in Y_0 \stackrel{def}{=} \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\}$ then $R\frac{\partial u}{\partial \vec{n}} \in H^{-1/2}_{\Gamma_0}(\Gamma)$ and it is defined as

$$< R\frac{\partial u}{\partial \vec{n}}, \gamma(v) >_{-1/2,1/2} = \int_\Omega \Delta u v + \int_\Omega \nabla u \nabla v = \int_\Omega Div(v\nabla u) \tag{1.2}$$

for every $v \in H^1_{\Gamma_0}(\Omega)$. Note that $R$ stands for the restriction, since $\frac{\partial u}{\partial \vec{n}}$ is also defined as an element in $H^{-1/2}(\Gamma) \stackrel{R}{\hookrightarrow} H^{-1/2}_{\Gamma_0}(\Gamma)$ as soon as $z \in H^1(\Omega)$ and $\Delta z \in L^2(\Omega)$.

If $\lambda > 0$ no other assumptions are made on $\Omega$ (for example concerning its boundedness) nor on $\Gamma_0$ that, as said above, could be empty. However, if $\lambda = 0$ then we assume that $\Gamma_0 \neq \emptyset$ and moreover, that in $H^1_{\Gamma_0}(\Omega)$ the Poincaré inequality holds, namely, there exists a constant $c$ such that

$$\int_\Omega |\nabla u|^2 \geq c \int_\Omega u^2 \tag{1.3}$$

for every $u \in H^1_{\Gamma_0}(\Omega)$, so the square root of $\int_\Omega |\nabla u|^2$ defines a norm in $H^1_{\Gamma_0}(\Omega)$ equivalent to the usual one. For example, for any bounded connected set that condition is verified as soon as $\Gamma_0 \neq \emptyset$. The same holds if $\Omega$ is bounded in one direction and $\Gamma_0$ is sufficiently large. Note that the assumptions on $\lambda$ and $\Gamma_0$ imply that $\int_\Omega \nabla u \nabla v + \lambda \int_\Omega uv$ is a scalar product in $H^1_{\Gamma_0}(\Omega)$ equivalent to the usual one.

Under these conditions, we introduce the canonical isometric isomorphism between $H^1_{\Gamma_0}(\Omega)$ and its dual, $H^{-1}_{\Gamma_0}(\Omega)$, given by $L \approx -\Delta + \lambda$, such that for every $f, \phi \in H^1_{\Gamma_0}(\Omega)$

$$< L(f), \phi >_{-1,1} = \int_\Omega \nabla f \nabla \phi + \lambda \int_\Omega f\phi \tag{1.4}$$

Note that we can then write (1.2) as

$$< R\frac{\partial u}{\partial \vec{n}}, \gamma(v) >_{-1/2,1/2} = < L(u), v >_{-1,1} - < -\Delta u + \lambda u, v > \tag{1.5}$$

We will show below that with this operator the Neuman problem

$$\begin{cases} -\Delta u + \lambda u = f & \text{on } \Omega \\ R\frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_0 \end{cases} \tag{1.6}$$

can be solved in a variational way in $H^{-1}_{\Gamma_0}(\Omega)$. We start with the following

**Definition 1.2.**
*i) The set of $\lambda$–Harmonic functions in $H^1_{\Gamma_0}(\Omega)$ is the orthogonal set to $H^1_0(\Omega)$ in $H^1_{\Gamma_0}(\Omega)$ with respect to the scalar product $\int_\Omega \nabla u \nabla v + \lambda \int_\Omega uv$. That is, $u \in Har_{1/2,\Gamma_0}(\Omega) \stackrel{def}{=} (H^1_0(\Omega))^\perp$ iff*

$$\int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi = 0 \tag{1.7}$$

*for every $\phi \in H^1_0(\Omega)$, i.e. $L(u)_{|H^1_0} = 0$. In particular $-\Delta u + \lambda u = 0$ in the sense of distributions in $\Omega$.*
*ii) For a given function $u$ on $\Gamma$, such that it vanishes on $\Gamma_0$, we define $v = B(u)$, the "$\lambda$–Harmonic lifting" of $u$, defined in $\Omega$ as the solution of the elliptic boundary value problem*

$$\begin{cases} -\Delta v + \lambda v = 0 & \text{on } \Omega \\ v = u & \text{on } \Gamma_1 \\ v = 0 & \text{on } \Gamma_0 \end{cases} \tag{1.8}$$

*in the sense that $v \in Har_{1/2,\Gamma_0}(\Omega)$ and $\gamma(v) = u$.*

With these notations we have the following

**Proposition 1.1.**
*i) We have the decomposition* $H^1_{\Gamma_0}(\Omega) = Har_{1/2,\Gamma_0}(\Omega) \oplus H^1_0(\Omega)$ *and each* $u \in H^1_{\Gamma_0}(\Omega)$ *can be split as* $u = u_1 + u_2$ *where*

$$u_1 = B(\gamma(u)) \in Har_{1/2,\Gamma_0}(\Omega), \quad u_2 = u - B(\gamma(u)) \in H^1_0(\Omega)$$

*Moreover,* $B : H^{1/2}_{\Gamma_0}(\Gamma) \to Har_{1/2,\Gamma_0}(\Omega)$ *is an isomorphism, whose inverse is given by the trace operator* $\gamma$.
*ii) Acting by restriction, we have the decomposition* $H^{-1}_{\Gamma_0}(\Omega) = H^{-1/2}_{\Gamma_0}(\Gamma) \oplus H^{-1}_0(\Omega)$ *and therefore every* $h \in H^{-1}_{\Gamma_0}(\Omega)$ *can be split as* $h = h_1 + h_2$ *with* $h_1 \in H^{-1/2}_{\Gamma_0}(\Gamma)$ *and* $h_2 \in H^{-1}_0(\Omega)$. *In particular,*

$$L_D = L_{|H^1_0} : H^1_0(\Omega) \to H^{-1}_0(\Omega)$$

*and*

$$A_0 = LB : H^{1/2}_{\Gamma_0}(\Gamma) \to H^{-1/2}_{\Gamma_0}(\Gamma)$$

*are isomorphisms and moreover* $A_0 = LB = R(\frac{\partial B}{\partial \vec{n}})$. *That implies that given* $h \in H^{-1}_{\Gamma_0}(\Omega)$, $u \in H^1_{\Gamma_0}(\Omega)$ *verifies* $L(u) = h$ *iff*

$$\gamma(u) = A_0^{-1}(h_1)$$
$$u_2 = D_0(h_2)$$

*where* $D_0 \stackrel{def}{=} L_D^{-1}$.

**Proof**
i) The direct sum decomposition of $H^1_{\Gamma_0}(\Omega)$ follows by definition of $Har_{1/2,\Gamma_0}(\Omega)$ and the choice of the scalar product in $H^1_{\Gamma_0}(\Omega)$. Therefore for every $u \in H^1_{\Gamma_0}(\Omega)$, $u = u_1 + u_2$ where $u_2 \in H^1_0(\Omega)$ and then $\gamma(u) = \gamma(u_1)$, and consequently $u_1 = B(\gamma(u))$. Since the projections are continuous, then $B : H^{1/2}_{\Gamma_0}(\Gamma) \to Har_{1/2,\Gamma_0}(\Omega)$ is an isomorphism, whose inverse is given by the trace operator $\gamma$.
ii) Since $L$ is an isomorphism between $H^1_{\Gamma_0}(\Omega)$ and $H^{-1}_{\Gamma_0}(\Omega)$, using that $B$ is an isomorphism and the direct sum decomposition of $H^1_{\Gamma_0}(\Omega)$, we get that $A_0 = LB$ and $L_D$ are isomorphisms and the decomposition of $H^{-1}_{\Gamma_0}(\Omega)$ follows.

On the other hand, by (1.2) and (1.5), we get for every $u \in H^{1/2}_{\Gamma_0}(\Gamma)$ and $v \in H^1_{\Gamma_0}(\Omega)$,

$$< R(\frac{\partial B(u)}{\partial \vec{n}}), \gamma(v) >_{-1/2,1/2} = < L(B(u)), v >_{-1,1} - < -\Delta B(u) + \lambda B(u), v > = < L(B(u)), v >_{-1,1}$$

Now, given $h \in H^{-1}_{\Gamma_0}(\Omega)$, $u \in H^1_{\Gamma_0}(\Omega)$ verifies $L(u) = h$ iff $L(u_1 + u_2) = h_1 + h_2$, and this is equivalent to $L(u_i) = h_i$, $i = 1, 2$ and therefore $A_0(\gamma(u)) = h_1$ and $L_D(u_2) = h_2$.□

**Remark 1.1.** *Note that the fact that* $L_D$ *and* $A_0$ *are isomorphisms imply the well possedness of the problems*

$$\begin{cases} -\Delta u + \lambda u = f & on\ \Omega \\ u = 0 & on\ \Gamma \end{cases}$$

*in the sense that* $f \in H^{-1}_0(\Omega)$, $u \in H^1_0(\Omega)$ *and for every* $\phi \in H^1_0(\Omega)$, $< L(u), \phi >_{-1,1} = < f, \phi >_{-1,1}$, *and*

$$\begin{cases} -\Delta u + \lambda u = 0 & on\ \Omega \\ R\frac{\partial u}{\partial \vec{n}} = g & on\ \Gamma_1 \\ u = 0 & on\ \Gamma_0 \end{cases}$$

*in the sense that* $g \in H^{-1/2}_{\Gamma_0}(\Gamma)$, $u \in H^1_{\Gamma_0}(\Omega)$ *and for every* $\phi \in H^1_{\Gamma_0}(\Omega)$, $< L(u), \phi >_{-1,1} = < g, \phi >_\Gamma$.

Now and throughout the paper we will find a special class of elements $h \in H^{-1}_{\Gamma_0}(\Omega)$ defined as

$$< h, \phi >_{-1,1} = < f, \phi >_\Omega + < g, \gamma(\phi) >_\Gamma$$

for every $\phi \in H^1_{\Gamma_0}(\Omega)$, where $f \in L^2(\Omega)$ and $g \in H^{-1/2}_{\Gamma_0}(\Gamma)$. So, for short, $h \stackrel{def}{=} f_\Omega + g_\Gamma$. For this special type of elements in $H^{-1}_{\Gamma_0}(\Omega)$ we have

**Lemma 1.1.** *Assume* $h \stackrel{def}{=} f_\Omega + g_\Gamma$ *is given in* $H^{-1}_{\Gamma_0}(\Omega)$*, with* $f \in L^2(\Omega)$ *and* $g \in H^{-1/2}_{\Gamma_0}(\Gamma)$*, then the decomposition of* $h$ *is given by*

$$h_1 = B^* f_\Omega + g_\Gamma \in H^{-1/2}_{\Gamma_0}(\Gamma), \quad h_2 = f_\Omega \in H^{-1}_0(\Omega)$$

*and*

$$B^* f = -R(\frac{\partial D_0(f)}{\partial \vec{n}}) \in R(H^{1/2}(\Gamma)) \subset L^2_{\Gamma_0}(\Gamma) \subset H^{-1/2}_{\Gamma_0}(\Gamma)$$

**Proof** For every $\phi \in H^1_{\Gamma_0}(\Omega)$, we use the splitting $\phi = \phi_1 + \phi_2$, $\phi_1 = B(\gamma(\phi))$, and then

$$< h, \phi >_{-1,1} = < h, \phi_1 >_{-1,1} + < h, \phi_2 >_{-1,1} =$$

$$= < f, \phi_1 > + < g, \gamma(\phi) >_\Gamma + < f, \phi_2 > = < B^* f + g, \gamma(\phi) > + < f, \phi_2 >$$

and we get the expressions for $h_1$ and $h_2$. On the other hand we have $< B^* f, \gamma(\phi) > = < f, B(\gamma(\phi)) > = < f, \phi_1 >$ and from (1.5) and the definition of $D_0$, we also have for every $f \in L^2(\Omega)$ and $\phi \in H^1_{\Gamma_0}(\Omega)$,

$$< R(\frac{\partial D_0(f)}{\partial \vec{n}}), \gamma(\phi) >_{-1/2,1/2} = < LD_0(f), \phi >_{-1,1} - < -\Delta D_0(f) + \lambda D_0(f), \phi > =$$

$$= < f, \phi_2 > - < f, \phi > = - < f, \phi_1 >$$

so, adding up $< B^* f + R(\frac{\partial D_0(f)}{\partial \vec{n}}), \gamma(\phi) > = 0$.□

**Remark 1.2.**
*i) Note that* $B$ *is a well defined linear and bounded operator from* $H^s_{\Gamma_0}(\Gamma)$ *to* $H^{s+1/2}_{\Gamma_0}(\Omega)$ *for* $s > 0$*. Moreover,* $B$ *is an isomorphism onto its image, denoted* $B(H^s_{\Gamma_0}(\Gamma)) \stackrel{def}{=} Har_{s,\Gamma_0}(\Omega)$*,* [19].
*ii) Note that if* $\Gamma_i$ *form a regular partition of the boundary then* $B^*(f) \in H^{1/2}_{\Gamma_0}(\Gamma)$*.*
*iii) By the following duality (or transposition) argument, one can extend* $B$ *to act between* $L^2_{\Gamma_0}(\Gamma)$ *and* $L^2(\Omega)$*: by definition of* $B$ *we have* $B : H^{1/2}_{\Gamma_0}(\Gamma) \to H^1_{\Gamma_0}(\Omega)$ *and therefore by transposition* $B^* : H^{-1}_{\Gamma_0}(\Omega) \to H^{-1/2}_{\Gamma_0}(\Gamma)$*. But Lemma 1.1 states in particular that, by restriction,* $B^* : L^2(\Omega) \to R(H^{1/2}(\Gamma)) \subset L^2_{\Gamma_0}(\Gamma) \subset H^{-1/2}_{\Gamma_0}(\Gamma)$ *and therefore, again by transposition,* $\hat{B} = B^{**} : L^2_{\Gamma_0}(\Gamma) \to L^2(\Omega)$*.*
*iii) In fact if the partition is 1/2–regular then* $R(H^{1/2}(\Gamma)) = H^{1/2}_{\Gamma_0}(\Gamma)$ *and the previous argument gives* $\hat{B} = B^{**} : H^{-1/2}_{\Gamma_0}(\Gamma) \to L^2(\Omega)$*. In the general case, if we take on* $R(H^{1/2}(\Gamma))$ *the topology induced by* $H^{1/2}(\Gamma_1)$*, and denote this Hilbert space by* $Z$*, which is isomorphic to* $H^{1/2}(\Gamma_1)$*, then the previous argument gives* $\hat{B} = B^{**} : Z' \to L^2(\Omega)$*.*

*It is easy to check than in any case the operator* $\hat{B}$ *is an extension of* $B$*. Therefore, when needed, we will simply write* $B$ *instead of* $\hat{B}$ *to denote this extension. Note that* $B = \hat{B} = -(R(\frac{\partial D_0}{\partial \vec{n}}))^*$*.*

Concerning (1.6), we have the following result

**Theorem 1.1.** *Assume* $h \stackrel{def}{=} f_\Omega + g_\Gamma$ *is given in* $H^{-1}_{\Gamma_0}(\Omega)$*, with* $f \in L^2(\Omega)$ *and* $g \in H^{-1/2}_{\Gamma_0}(\Gamma)$*, and let* $u \in H^1_{\Gamma_0}(\Omega)$ *be the solution of*

$$L(u) = h = f_\Omega + g_\Gamma \tag{1.9}$$

*Then* $u \in Y_0 = \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\}$ *and* $u = B(\gamma(u)) + u_2$ *where*

$$\gamma(u) = A_0^{-1}(B^* f_\Omega + g_\Gamma) \in H^{1/2}_{\Gamma_0}(\Gamma) \tag{1.10}$$

$$u_2 = D_0(f_\Omega) \in H^2(\Omega) \cap H^1_0(\Omega) \tag{1.11}$$

*Moreover,* $-\Delta u + \lambda u = f \in L^2(\Omega)$ *and* $R(\frac{\partial u}{\partial \vec{n}}) = g \in H^{-1/2}_{\Gamma_0}(\Gamma)$*.*

*Conversely, for every* $u \in Y_0$*,* $L(u) = f_\Omega + g_\Gamma$*, where* $f_\Omega = -\Delta u + \lambda u \in L^2(\Omega)$ *and* $g_\Gamma = R(\frac{\partial u}{\partial \vec{n}}) \in H^{-1/2}_{\Gamma_0}(\Gamma)$*. That is,*

$$L : Y_0 = \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\} \longrightarrow L^2(\Omega) + H^{-1/2}_{\Gamma_0}(\Gamma) \subset H^{-1}_{\Gamma_0}(\Omega)$$

*is an isomorphism, and on this space* $L = (-\Delta + \lambda)_\Omega + R(\frac{\partial}{\partial \vec{n}})_\Gamma$*.*

**Proof** From Proposition 1.1 and Lemma 1.1, it only remains to prove that $R(\frac{\partial u}{\partial \vec{n}}) = g \in H^{-1/2}_{\Gamma_0}(\Gamma)$. But since $u = B(\gamma(u)) + u_2$ and $u_2 \in H^2(\Omega) \cap H^1_0(\Omega)$ then $u_2$ has normal derivative in $H^{1/2}(\Gamma)$, therefore $R(\frac{\partial u}{\partial \vec{n}}) = R(\frac{\partial B(\gamma(u))}{\partial \vec{n}}) + R(\frac{\partial u_2}{\partial \vec{n}}) = h_1 + R(\frac{\partial D_0(f)}{\partial \vec{n}}) = g$. The converse statement comes out easily from (1.5).□

**Remark 1.3.** *Note that for an arbitrary $h \in H^{-1}_{\Gamma_0}(\Omega)$ the solution of $L(u) = h$ provides a "generalized weak solution" of*

$$\begin{cases} -\Delta u + \lambda u = h_2 \in H^{-1}_0(\Omega) \\ R(\frac{\partial u}{\partial \vec{n}}) = h_1 + R(\frac{\partial D_0(h_2)}{\partial \vec{n}}) \in H^{-1/2}_{\Gamma_0}(\Gamma) \\ u = 0 \ \text{on } \Gamma_0 \end{cases}$$

*but $R(\frac{\partial D_0(h_2)}{\partial \vec{n}})$ and consequently $R(\frac{\partial u}{\partial \vec{n}})$ are not defined unless $h_2 \in L^2(\Omega)$.*

In particular for $g = 0$ and $f \in L^2(\Omega)$ the operator $L$ induces the positive, selfadjoint operator, $-\Delta_N + \lambda$, in $Y = L^2(\Omega)$ with domain

$$D(-\Delta_N + \lambda) = \{u \in H^1_{\Gamma_0}(\Omega),\ \Delta u \in L^2(\Omega),\ R(\frac{\partial u}{\partial \vec{n}}) = 0\} = L^{-1}(L^2(\Omega))$$

which moreover has compact resolvent if $\Omega$ is bounded. In particular, it is a sectorial operator in $Y$ and the fractional power spaces $\{Y^\alpha\}_{\alpha \in I\!R}$ are well defined, see [12], and we have

$$Y^{1/2} = H^1_{\Gamma_0}(\Omega), \quad Y^0 = L^2(\Omega), \quad Y^{-1/2} = H^{-1}_{\Gamma_0}(\Omega) \tag{1.12}$$

When $g \neq 0$ we will consider the spaces associated to the solvability of equation (1.6), e.g. $L^{-1}(H^s(\Omega) + H^r_{\Gamma_0}(\Gamma))$. In particular, we have

$$L^{-1}(H^{-1/2}_{\Gamma_0}(\Gamma)) = Har_{1/2,\Gamma_0}(\Omega), \quad L^{-1}(H^{-1}_0(\Omega)) = H^1_0(\Omega), \quad L^{-1}(H^{-1}_{\Gamma_0}(\Omega)) = H^1_{\Gamma_0}(\Omega)$$

$$L^{-1}(L^2(\Omega) + H^{-1/2}_{\Gamma_0}(\Gamma)) = \{u \in H^1_{\Gamma_0}(\Omega),\ \Delta u \in L^2(\Omega)\}$$

$$L^{-1}(L^2(\Omega)) = \{u \in H^1_{\Gamma_0}(\Omega),\ \Delta u \in L^2(\Omega),\ R(\frac{\partial u}{\partial \vec{n}}) = 0\}$$

Note, moreover, that if the $\Gamma_i$ form a regular partition of the boundary then by elliptic regularity theory we have

$$L^{-1}(L^2(\Omega)) = \{u \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega),\ R\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_1\}$$

$$L^{-1}(H^s(\Omega) + H^{s+1/2}_{\Gamma_0}(\Gamma)) = H^{s+2}_{\Gamma_0}(\Omega), \quad s \geq 0$$

We now show the well possedness of the Dirichlet problem

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ \quad u = g & \text{in } \Gamma_1 \\ \quad u = 0 & \text{in } \Gamma_0 \end{cases} \tag{1.13}$$

Note that for $g = 0$, the operator $L_D : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ induces the positive, selfadjoint operator, $-\Delta_D + \lambda$, in $Y_D = L^2(\Omega)$ with domain

$$D(-\Delta_D + \lambda) = \{u \in H^1_0(\Omega),\ \Delta u \in L^2(\Omega)\} = H^2(\Omega) \cap H^1_0(\Omega) = L^{-1}_D(L^2(\Omega))$$

which moreover has compact resolvent if $\Omega$ is bounded. In particular, it is a sectorial operator in $Y_D$ and the fractional power spaces $\{Y^\alpha_D\}_{\alpha \in I\!R}$ are well defined, see [12], and we have

$$Y^{1/2}_D = H^1_0(\Omega), \quad Y^0_D = L^2(\Omega), \quad Y^{-1/2}_D = H^{-1}_0(\Omega) \tag{1.14}$$

**Definition 1.3.** *Assume $f \in H^{-1}_0(\Omega)$ and $g \in H^{1/2}_{\Gamma_0}(\Omega)$ are given. Then $u \in H^1_{\Gamma_0}(\Omega)$ is a solution of (1.13) iff $\gamma(u) = g$ and*

$$\int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi = < f, \phi >_{-1,1} \tag{1.15}$$

*for every $\phi \in H^1_0(\Omega)$, i.e. $L(u)_{|H^1_0} = f \in H^{-1}_0(\Omega)$.*

Then we have

**Proposition 1.2.** *The equation (1.13) has a unique solution, in the sense above, which is given by*

$$u = B(g) + D_0(f) \tag{1.16}$$

*In particular if* $f \in L^2(\Omega)$ *then* $R(\frac{\partial u}{\partial \vec{n}}) = R(\frac{\partial B(g)}{\partial \vec{n}}) + R(\frac{\partial D_0(f)}{\partial \vec{n}}) \in H^{-1/2}_{\Gamma_0}(\Gamma)$.

**Proof** The proof is almost straightforward. We have $u = B(g)+u_2$ and for every $\phi \in H^1_0(\Omega)$, $< L(u), \phi >=< L_D(u_2), \phi >=< f, \phi >$ and therefore $u_2 = D_0(f)$. The rest is obvious.□

By using the previous results, we can also solve the Robin problem i.e.

$$\left\{ \begin{array}{ll} -\Delta u + \lambda u = f & \text{on } \Omega \\ a(x)u + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_0 \end{array} \right. \tag{1.17}$$

where $a(x)$ is a given $L^\infty$ function on $\Gamma_1$. Now define the bilinear form

$$a_R(u, \phi) = \int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u\phi + \int_{\Gamma_1} a(x)u\phi$$

which is symmetric and continuous on $H^1_{\Gamma_0}(\Omega)$. Moreover, assume $a(x)$ is such that $a_R(\cdot,\cdot)$ is coercive on $H^1_{\Gamma_0}(\Omega)$. For this, note that under the assumptions above it would be sufficient for $a(x)$ to have a small negative part. In that case $a_R(\cdot,\cdot)$ induces an isomorphism, $L_R$, between $H^1_{\Gamma_0}(\Omega)$ and $H^{-1}_{\Gamma_0}(\Omega)$ that can be written as

$$L_R = L + a(x)\gamma$$

Therefore we have the following result.

**Theorem 1.2.** *Assume* $h \stackrel{def}{=} f_\Omega + g_\Gamma$ *is given in* $H^{-1}_{\Gamma_0}(\Omega)$, *with* $f \in L^2(\Omega)$ *and* $g \in H^{-1/2}_{\Gamma_0}(\Gamma)$, *and let* $u \in H^1_{\Gamma_0}(\Omega)$ *be the solution of*

$$L_R(u) = L(u) + a(x)\gamma(u) = h = f_\Omega + g_\Gamma \tag{1.18}$$

*Then* $u \in Y_0 = \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\}$ *and* $u = B(\gamma(u)) + u_2$ *where*

$$\gamma(u) = A_0^{-1}(B^* f_\Omega + (g - a(x)\gamma(u))_\Gamma) \in H^{1/2}_{\Gamma_0}(\Gamma) \tag{1.19}$$

$$u_2 = D_0(f_\Omega) \in H^2(\Omega) \cap H^1_0(\Omega) \tag{1.20}$$

*Moreover,* $-\Delta u + \lambda u = f \in L^2(\Omega)$ *and* $a(x)\gamma(u) + R(\frac{\partial u}{\partial \vec{n}}) = g \in H^{-1/2}_{\Gamma_0}(\Gamma)$.

**Proof** The proof is now obvious, since we read the equation as $L(u) = f_\Omega + (g - a(x)\gamma(u))_\Gamma$ and apply Theorem 1.1.□

## 2. The damped hyperbolic problem.

In this section we provide existence and regularity results for the damped hyperbolic problem. For related results the reader is referred to [2, 5, 14, 16, 21, 24, 25, 26] and references therein.

### 2.1. Homogeneous boundary conditions.

We are now concerned with the solutions of

$$\left\{ \begin{array}{ll} \epsilon u_{tt} - \Delta u + \lambda u = 0 & \text{on } \Omega \times (0,\infty) \\ u_t + \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty) \end{array} \right. \tag{2.1}$$

for $0 < \epsilon \le \epsilon_0$.

Consider the operator $A = A_{\epsilon,\lambda,\Gamma_0} = \begin{pmatrix} 0 & -I \\ \frac{1}{\epsilon}(-\Delta + \lambda) & 0 \end{pmatrix}$, where $\epsilon_0 \ge \epsilon > 0$, acting on the space $E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and domain given by

$$D(A) = \{(u, v) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega), \Delta u \in L^2(\Omega),\ v + R\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_1\}$$

Recall that $\Gamma_0$ may be empty and that when the $\Gamma_i$ form a regular partition of the boundary then $u \in H^2_{\Gamma_0}(\Omega)$. Also, note that the boundary conditions are satisfied in the $H^{-1/2}_{\Gamma_0}(\Gamma)$ sense if $(u,v) \in D(A)$. With these notations (2.1) can be written as

$$U_t + AU = 0 \tag{2.2}$$

with $U = (u, u_t)^T$, and then we have

**Theorem 2.1.** *With the above definitions, the operator $-A$ generates a $C_0$ semigroup in $E$, denoted $S_\epsilon(t)$, and there exists a scalar product in $E$ such that $S_\epsilon(t)$ is a $C_0$ semigroup of contractions.*

*If moreover, $U_0 \in D(A)$, then $U(t) = S_\epsilon(t)U_0$ remains in $D(A)$, is differentiable in $E$, and verifies (2.2) pointwise. Note that the PDE in $\Omega$ is verified in the $L^2(\Omega)$ sense while the boundary conditions are verified in the $H^{-1/2}_{\Gamma_0}(\Gamma)$ sense.*

**Proof** First, $A$ is dissipative on $E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, where in $H^1_{\Gamma_0}(\Omega)$ we take the scalar product $\frac{1}{\epsilon}\int_\Omega \nabla u \nabla v + \frac{\lambda}{\epsilon}\int_\Omega uv$, since for this choice $< A(u,v),(u,v) >_{H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)} = \frac{1}{\epsilon}\int_{\Gamma_1} v^2$.

Clearly, $D(A)$ is dense in $E$, since with the notations in (1.12), $E = Y^{1/2} \times Y^0$ and $D(A)$ contains $Y^1 \times H^1_0(\Omega)$, i.e. $\{u \in H^1_{\Gamma_0}(\Omega), \Delta u \in L^2(\Omega),\ R\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_1\} \times H^1_0(\Omega)$.

It remains to show that $R(I + A) = E$ since, in that case, the Lummer–Philips theorem gives us the generation of a $C_0$ semigroup of contractions, see [13, 23]. So, for given $(f,g) \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ we need to solve $(I + A)(u,v)^T = (f,g)^T$, i.e. $(u,v) \in D(A)$ and

$$\begin{cases} u - v = f \\ \epsilon v - \Delta u + \lambda u = \epsilon g \end{cases}$$

therefore, $v = u - f$ and $u$ satisfies

$$\begin{cases} -\Delta u + (\lambda + \epsilon)u = \epsilon(f + g) & \text{on } \Omega \\ u + \frac{\partial u}{\partial \vec{n}} = f & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_0 \end{cases}$$

From the results in Subsection 1.2 on the Robin problem, there exists a unique solution and it is clear that $(u,v) \in D(A)$ and $(I + A)(u,v)^T = (f,g)^T$.□

By linear perturbation results, [23], we can prove

**Corollary 2.1.** *For any $\beta, \lambda \in \mathbb{R}$, the equation*

$$\begin{cases} \epsilon u_{tt} + \beta u_t - \Delta u + \lambda u = 0 & \text{on } \Omega \times (0,\infty) \\ u_t + \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty) \end{cases} \tag{2.3}$$

*defines a $C_0$ semigroup on $E$.*□

**Proof** We fix $\lambda_0 > 0$ such that all the above applies to the operator $A_{\epsilon,\lambda_0,\Gamma_0}$ and write the matrix operator in (2.3) as $A = A_{\epsilon,\lambda_0,\Gamma_0} + P$, with $P = -\frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ -(\lambda - \lambda_0) & \beta \end{pmatrix}$. Then $P$ is a bounded perturbation in $E$, and therefore so $A$ generates a $C_0$ semigroup, [13, 23].□

Also, by applying general abstract results on $C_0$ semigroups, see the Proposition 5.3 in the Appendix and [13, 23], we get

**Theorem 2.2.** *Assume $f : [0,T] \to L^2(\Omega)$, and consider*

$$\begin{cases} \epsilon u_{tt} - \Delta u + \lambda u = f(t) & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{cases} \tag{2.4}$$

*then*

*i) If $f \in L^1(0,T,L^2(\Omega))$ and $U_0 \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ there exist a unique "mild solution" of (2.4) verifying $U(0) = U_0$, which is given by the variation of constants formula*

$$U(t) = S_\epsilon(t)U_0 + \int_0^t S_\epsilon(t-s)F(s)\,ds \tag{2.5}$$

*where* $U(t) = (u, u_t)^T$, $F(t) = (0, f(t))^T$. *In that case,* $U$ *is continuous in* $E$.
*ii) If* $f \in C^1([0,T], L^2(\Omega))$ *or* $f \in C([0,T], H^1_{\Gamma_0}(\Omega))$ *and* $U_0 \in D(A)$, *then the unique solution is a "strict solution", that is, it is differentiable in* $E$, *remains in* $D(A)$ *and verifies (2.4) pointwise. Note that the PDE in* $\Omega$ *is verified in the* $L^2(\Omega)$ *sense while the boundary conditions are verified in the* $H^{-1/2}_{\Gamma_0}(\Gamma)$ *sense.*□

2.2. **Formal Energy Estimates.** Now we fix our attention on some energy estimates on solutions of

$$\begin{cases} \epsilon u_{tt} - \Delta u + \lambda u = f(t) & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g(t) & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{cases} \tag{2.6}$$

From the results of Subsection 1.2, it is natural to require the minimal regularity assumptions $t \mapsto f(t) \in L^2(\Omega)$ and $t \mapsto g(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$, i.e. $t \mapsto h(t) = f_\Omega(t) + g_\Gamma(t) \in L^2(\Omega) + H^{-1/2}_{\Gamma_0}(\Gamma) \subset H^{-1}_{\Gamma_0}(\Omega)$, although in some cases these conditions may be strengthened.

Most of these estimates will be formal, since we will assume existence and regularity of solutions, except for the case $g = 0$, for which we have already set up a good existence and regularity result. However, those formal estimates, for $g \neq 0$, will be the key in defining the concept of solution of (2.6), and in fact they will play an essential role in the proof of the existence, uniqueness and regularity result for (2.6).

We define the energy

$$E_\epsilon(u, v) = \epsilon \|v\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \lambda \|u\|^2_{L^2}$$

and observe that, in view of (1.3), $E_\epsilon^{1/2}$ is an equivalent norm to that of the ambient space $E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, hereafter denoted "energy space". Moreover, when $f = g = 0$, the semigroup generated by $-A$, $S_\epsilon(t)$, is a contraction semigroup for this norm in the energy space, see Theorem 2.1, so we have

$$E_\epsilon(S_\epsilon(t)(u_0, v_0)) \le E_\epsilon(u_0, v_0).$$

More generally, assume $f$ takes values in $L^2(\Omega)$ and $g$ takes values in $L^2_{\Gamma_0}(\Gamma)$ then we formally have, by multiplying the equation by $u_t$, integrating by parts in $\Omega$, using the boundary conditions, and then integrating in time

$$E_\epsilon(u, u_t) + 2\int_0^t \int_{\Gamma_1} u_t^2 = E_\epsilon(u_0, v_0) + 2\int_0^t \int_{\Gamma_1} g u_t + 2\int_0^t \int_\Omega f u_t \tag{2.7}$$

Observe that this equation holds, provided solutions of (2.6) are sufficiently smooth, so one can perform all the computations leading to (2.7). Also note that

$$\int_0^t \int_{\Gamma_1} g u_t + \int_0^t \int_\Omega f u_t = \int_0^t < h, u_t >_{-1,1}$$

and we have a generalized version of (2.7)

$$E_\epsilon(u, u_t) + 2\int_0^t \int_{\Gamma_1} u_t^2 = E_\epsilon(u_0, v_0) + 2\int_0^t < h, u_t >_{-1,1} \tag{2.8}$$

In the case $g = 0$, and using the results of the previous subsection, one can make (2.7) rigorous for solutions of (2.4), by first working with smooth initial data and $f$ and passing to the limit. However, we will obtain below the rigorous proof of (2.7) under the sharp conditions mentioned before for $f$ and $g$, so we will not give further details here.

Then, we have the following

**Proposition 2.1.** *Assume* $f \in L^1(0, T, L^2(\Omega))$ *and* $g \in L^2(\Sigma_1)$. *Then for any* $(u_0, v_0)$ *in the energy space, the following estimates hold for the corresponding solution of (2.6)*
*i) Assume first* $f = g = 0$ *and* $T \le \infty$, *then*

$$u_t \in L^2(\Sigma_1)$$

*and for any* $0 < t < T$

$$E_\epsilon(u, u_t) + 2\|u_t\|^2_{L^2(\Sigma_{1,t})} \le E_\epsilon(u_0, v_0) \tag{2.9}$$

*Therefore* $(u_0, v_0) \longmapsto ((u, u_t), u_t)$ *is Lipschitz between* $E$ *and* $C([0,T], E) \times L^2(\Sigma_1)$.
*ii) Now, assume* $f = 0$ *and* $T \le \infty$. *Then*

$$u_t \in L^2(\Sigma_1)$$

*and for any* $0 < t < T$

$$E_\epsilon(u,u_t) + \|u_t\|^2_{L^2(\Sigma_{1,t})} \le E_\epsilon(u_0,v_0) + \|g\|^2_{L^2(\Sigma_1)} \tag{2.10}$$

*Therefore* $(u_0,v_0,g) \longmapsto ((u,u_t),u_t)$ *is Lipschitz between* $E \times L^2(\Sigma_1)$ *into* $C([0,T],E) \times L^2(\Sigma_1)$.
*iii) If* $f, g \neq 0$ *and* $T \le \infty$, *then*

$$u_t \in L^2(\Sigma_1)$$

*and for any* $0 < t < T$

$$E_\epsilon(u,u_t) + \|u_t\|^2_{L^2(\Sigma_{1,t})} \le 2E_\epsilon(u_0,v_0) + 2\|g\|^2_{L^2(\Sigma_1)} + \frac{4}{\epsilon}\|f\|^2_{L^1(L^2(\Omega))} \tag{2.11}$$

*Therefore,* $(u_0,v_0,g,f) \longmapsto ((u,u_t),u_t)$ *is Lipschitz between* $E \times L^2(\Sigma_1) \times L^1(0,T,L^2(\Omega))$ *into* $C([0,T],E) \times L^2(\Sigma_1)$.

**Proof** Assuming regularity of solutions, multiplying the equation by $u_t$ and integrating in space and time, we get (2.7), i.e. $E_\epsilon(u,u_t) + 2\int_{\Sigma_{1,t}} u_t^2 = E_\epsilon(u_0,v_0) + 2\int_{\Sigma_{1,t}} gu_t + 2\int_{Q_t} fu_t$. The case $f = g = 0$ is obvious. If $f = 0$, by Young inequality, we get (2.10).

For nonzero $f$, we use the bounds

$$\int_{\Sigma_{1,t}} gu_t \le 1/2\|g\|^2_{L^2(\Sigma_1)} + 1/2\|u_t\|^2_{L^2(\Sigma_{1,t})}, \quad \int_{Q_t} fu_t \le \|f\|_{L^1(L^2(\Omega))} \sup_{0\le s\le t} \|u_t\|_{L^2(\Omega)}$$

and therefore $y(t) = \sup_{0\le s\le t}\|u_t\|_{L^2}$ verifies, $\epsilon y^2(s) \le E_0 + K(f)y(t)$, for every $0 \le s \le t$, where we have set $E_0 = E_\epsilon(u_0,v_0) + \|g\|^2_{L^2(\Sigma_1)}$ and $K(f) = 2\|f\|_{L^1(L^2)}$. Consequently $\epsilon y^2(t) \le E_0 + K(f)y(t)$ and using Young's inequality we get $\epsilon y^2(t) \le 2E_0 + \frac{1}{\epsilon}K^2(f)$. Now we get the bound $E_\epsilon(u,u_t) + \int_{\Sigma_{1,t}} u_t^2 \le E_0 + K(f)y(t)$ and again by Young's inequality and the bound for $y(t)$ we get (2.11).

In either case, the lipschitzness of the mapping $(u_0,v_0,g,f) \mapsto ((u,u_t),u_t)$ follows.□

Note that for $f = g = 0$, (2.9) shows that the energy of the initial data is lost through the boundary of $\Omega$, where the dissipation takes place. Also, (2.10) and (2.11) show that conditions $f \in L^1(0,T,L^2(\Omega))$ and $g \in L^2(\Sigma_1)$ and initial data in the energy space are enough to obtain the inequalities. That suggests to prove an existence result under these conditions. Finally, note there are some difficulties in making the inequalities rigorous. Namely, if one proves that the solution of (2.6) is in the energy space, then, in principle, $u_t$ is in $L^2(\Omega)$, so it can not be restricted to the boundary. At the same time, it is not clear in what sense the solution in the energy space of (2.6), or even (2.4), verify the boundary conditions. Therefore, partial integration needs justification. We will answer satisfactorily both questions in the next section.

Finally, note that if $\lambda = 0$ and $\Gamma_0 = \emptyset$ or $\Gamma_0 \neq \emptyset$ but the Poincaré inequality (1.3) doesn't hold true, then $E_\epsilon$ is not coercive in $E$. Moreover, in the case $\lambda = 0$ and $\Gamma_0 = \emptyset$, the quantity

$$\epsilon \int_\Omega u_t(x,t)\,dx + \int_\Gamma u(\sigma,t)\,d\sigma$$

is invariant in time when $f = g = 0$. This can be easily obtained by integrating the equation on $\Omega$.

2.3. **Non homogeneous boundary conditions.** We are now concerned with existence of solutions of

$$\begin{cases} \epsilon u_{tt} - \Delta u + \lambda u = f(t) & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g(t) & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{cases} \tag{2.12}$$

with $t \mapsto h(t) = f_\Omega(t) + g_\Gamma(t) \in L^2(\Omega) + L^2_{\Gamma_0}(\Gamma) \subset H^{-1}_{\Gamma_0}(\Omega)$ or even more general classes of functions $h$.

Note that problem (2.12), as written, can not be immediately treated as a perturbation of the case $g = 0$ studied above, since $g$ is affecting the boundary conditions. So, one of our main points in this section is *defining* the concept of solution of (2.12). On the other hand, for the case $g = 0$, it is well known that (2.5) is a quite natural way of defining weak or "mild solutions" of (2.4).

Observe that there are several properties one would like to have from a nice concept of solution. For example, from the formal energy estimates of the previous section we would like to have that for initial data in the energy space and $f \in L^1(0,T,L^2(\Omega))$, $g \in L^2(\Sigma_1)$, there exists a unique solution, which is moreover continuous with values in the energy space, and verifies the energy estimates. We would also want to have that smoothness of initial data, $f$ and $g$, implies smoothness of the solution. On the other hand, we would like to have a representation of the solution in the form of a variation of constants formula, so that one might try to solve nonlinear equations via fixed points arguments. We will show below that all these apriori requirements can be met by using standard results on semigroups, working in an appropriate space.

Let $E = H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$ and $E' = L^2(\Omega)\times H^{-1}_{\Gamma_0}(\Omega)$ its dual space. Then according to [4], by transposition, $-A^*$ generates in $E'$ the $C_0$ semigroup $S^*_\epsilon(t)$, i.e. the transposed semigroup of $S_\epsilon(t)$, see Proposition 5.3 in the Appendix.

Then, we can state our main results in this section, which are the following

**Theorem 2.3.** *Let $f \in L^1(0,T,L^2(\Omega))$, $g \in L^2(\Sigma_1)$ and $U_0 = (u_0, v_0) \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. Let $U^*(t) = (u,w)^T$ be defined by the variation of constants formula of the dual semigroup*

$$U^*(t) = S^*_\epsilon(t)U^*_0 + \int_0^t S^*_\epsilon(t-s)H(s)\,ds \tag{2.13}$$

*where $U^*_0 = (u_0, w_0)$, $w_0 = v_0+\frac{1}{\epsilon}\gamma(u_0) \in H^{-1}_{\Gamma_0}(\Omega)$, $H(t) = (0, \frac{1}{\epsilon}h(t))^T$ and $h \overset{def}{=} f_\Omega+g_\Gamma \in L^1_{loc}(0,T,H^{-1}_{\Gamma_0}(\Omega))$.*

*Then $U^* \in C([0,T],E')$, $w = u_t + \frac{1}{\epsilon}\gamma(u)$, and $U(t) = (u,u_t)$ verifies*

*i) $U(t) \in C([0,T],E)$, $\gamma(u) \in C([0,T],H^{1/2}_{\Gamma_0}(\Gamma)) \cap H^1(0,T,L^2_{\Gamma_0}(\Gamma))$ and $u_{tt} \in L^1(0,T,H^{-1}_{\Gamma_0}(\Omega))$.*

*ii) The energy equality (2.7), i.e.*

$$E_\epsilon(u,u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)^2_t = E_\epsilon(u_0,v_0) + 2\int_{\Sigma_{1,t}} g\gamma(u)_t + 2\int_{Q_t} fu_t$$

*holds true.*

*iii) $u(t)$ verifies the following equation*

$$(\epsilon u_t + \gamma(u))_t + L(u) = h = f_\Omega + g_\Gamma \tag{2.14}$$

*a.e. $[0,T]$, as an equality in $H^{-1}_{\Gamma_0}(\Omega)$.*

Observe the remarkable feature of (2.13), that despite $H(s)$ is not in $D(A^*)$ (unless $g = 0$), nor is regular in time, $U^*$ is in $D(A^*)$, see Lemma 2.1. This is due to the particular form of $H$ and a subtle smoothing effect of the semigroup. Also note that from the energy estimates obtained in the previous section, $\gamma(u)_t \in L^2(\Sigma_1)$ and therefore

$$(\epsilon u_t + \gamma(u))_t = \epsilon u_{tt} + \gamma(u)_t$$

where the derivative is to be understood as weak derivative, i.e. as

$$\frac{d}{dt} < \epsilon u_t + \gamma(u), \phi >= \epsilon\frac{d}{dt} < u_t, \phi > + < \gamma(u)_t, \phi >$$

for every $\phi \in H^1_{\Gamma_0}(\Omega)$.

We will also find other classes of functions $h$ with values in $H^{-1}_{\Gamma_0}(\Omega)$ for which a result analogous to Theorem 2.3 can be obtained.

Once this theorem is proved we can make the following definition

**Definition 2.1.** *For given $f \in L^1(0,T,L^2(\Omega))$, $g \in L^2(\Sigma_1)$ and $U_0 = (u_0,v_0) \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, the function $U(t)$ obtained from the function $U^*(t)$, given by the theorem above is, by definition, the solution of (2.12).*

Note that in our setting the solution of (2.12) is unique since it is explicitly given in the theorem above.

Concerning further regularity results, we have

**Theorem 2.4.** *Under the assumptions of the theorem above, if moreover $(u_0, v_0)$ belongs to $H^1_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)$, $f \in W^{1,1}(0,T,L^2(\Omega))$ and $g \in H^1(0,T,L^2_{\Gamma_0}(\Gamma))$, are such that*

$$\gamma(v_0) + L(u_0) - g(0) \in L^2(\Omega) \tag{2.15}$$

*then $U = (u,u_t)^T$ is differentiable in the energy space and $U_t = (u_t,u_{tt})^T$ is a solution of (2.12) in $E$, with right hand sides $f_t$ and $g_t$ on $\Omega$ and $\Gamma$, respectively.*

*In particular, $U_t = (u_t,u_{tt}) \in C([0,T],E)$, $\gamma(u_t) \in C([0,T],H^{1/2}_{\Gamma_0}(\Gamma)) \cap H^1(0,T,L^2_{\Gamma_0}(\Gamma))$ and moreover*

$$E_\epsilon(u_t,u_{tt}) + 2\int_{\Sigma_{1,t}} \gamma(u_t)^2_t = E_\epsilon(v_0,u_{tt}(0)) + 2\int_{\Sigma_{1,t}} g_t\gamma(u_t)_t + 2\int_{Q_t} f_tu_{tt} \tag{2.16}$$

*Note that* $\epsilon u_{tt}(0) = -\gamma(v_0) - L(u_0) + f(0) + g(0) \in L^2(\Omega)$. *Moreover,* $u$ *verifies,* $u \in C([0,T], Y_0)$, *where* $Y_0 = \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\}$, *and*

$$R(\frac{\partial u}{\partial \vec{n}}) = g - \gamma(u)_t \in L^2_{\Gamma_0}(\Gamma) \subset H^{-1/2}_{\Gamma_0}(\Gamma)$$

$$\epsilon u_{tt} + \gamma(u_t) + L(u) = h(t), \quad in\ H^{-1}_{\Gamma_0}(\Omega)$$

$$(\epsilon u_{tt} + \gamma(u_t))_t + L(u_t) = h_t(t), \quad in\ H^{-1}_{\Gamma_0}(\Omega)$$

**Remark 2.1.**
*i) Note that by using the properties of the operator* $L$, *defined in (1.4), (2.15) is a weak formulation, in* $H^{-1}_{\Gamma_0}(\Omega)$, *of the condition*

$$\left\{ \begin{array}{lll} -\Delta u_0 + \lambda u_0 \in L^2(\Omega) & & \\ v_0 + \frac{\partial u_0}{\partial \vec{n}} = g(0) & on & \Gamma_1 \\ u_0, v_0 = 0 & on & \Gamma_0 \end{array} \right. \tag{2.17}$$

*and since under the hypotheses of Theorem 2.4* $u_{tt} \in L^2(\Omega)$, *then* $u$ *satisfies*

$$L(u) = (f(t) - u_{tt})_\Omega + (g(t) - \gamma(u)_t)_\Gamma$$

*i.e.*

$$\left\{ \begin{array}{ll} \epsilon u_{tt} - \Delta u + \lambda u = f & on\ \Omega \\ \gamma(u)_t + \frac{\partial u}{\partial \vec{n}} = g & on\ \Gamma_1 \\ u = 0 & on\ \Gamma_0 \end{array} \right.$$

*On the other hand, under the hypotheses of Theorem 2.3 the term* $u_{tt}$ *is only known to belong to* $H^{-1}_{\Gamma_0}(\Omega)$ *and therefore we can not use the previous argument.*
*ii) Under the hypotheses of Theorem 2.4, when* $\Gamma_i$ *form a regular partition of the boundary, then by elliptic regularity theory, if* $g(t) \in H^{1/2}_{\Gamma_0}(\Gamma)$, *for some fixed* $t$, *then* $u(t) \in H^2_{\Gamma_0}(\Omega)$ *and* $u$ *inherits the time regularity of* $g$.

Before going to the proof of the main theorems we introduce some notation. In what follows we will denote by $U = (u, v)$ a generic element of $E$, while $U^* = (u, w)$ will denote a generic element in $E'$. When going from one space to the other we will often use the following linear injective "change of variables":

$$U \mapsto U^* \tag{2.18}$$

where for given $U = (u, v) \in E$, $U^* = (u, w)$, is defined by $w = v + \frac{1}{\epsilon}\gamma(u) \in H^{-1}_{\Gamma_0}(\Omega)$. Note however that the mapping is not onto. Then we have

**Lemma 2.1.**
*i) With the above notations, we have*

$$D(A^*) = \{(f, h) \in H^1_{\Gamma_0}(\Omega) \times H^{-1}_{\Gamma_0}(\Omega),\ \frac{1}{\epsilon}\gamma(f) - h \in L^2(\Omega)\}$$

*and for* $(f, h) \in D(A^*)$

$$A^*(f, h) = (\frac{1}{\epsilon}\gamma(f) - h, \frac{1}{\epsilon}L(f))$$

*i.e. in matrix form,* $A^* = \begin{pmatrix} \frac{1}{\epsilon}\gamma & -I \\ & \\ \frac{1}{\epsilon}L & 0 \end{pmatrix}$.
*ii)* $U = (u, v) \in E$ *if and only if* $U^* \in D(A^*)$ *and moreover*

$$\|A^*U^*\|^2_{E'} = \|v\|^2 + \frac{1}{\epsilon}\|L(u)\|^2_{-1} = \|v\|^2 + \frac{1}{\epsilon}\|u\|^2_1 = \|U\|^2_E = \frac{1}{\epsilon}E_\epsilon(u, v)$$

*iii)* $U = (u, v) \in D(A)$ *if and only if* $U^* \in D(A^{*2})$.

**Proof**

i) If $U=(u,v)\in D(A)$ and $F=(f,h)\in E'$, then

$$<AU,F>_{E,E'}=\frac{1}{\epsilon}<-\Delta u+\lambda u,f>-<h,v>_{-1,1}$$

and for this to be continuous in $U$ for the topology of $E$, one needs $f\in H^1_{\Gamma_0}(\Omega)$. In that case

$$<AU,F>_{E,E'}=\frac{1}{\epsilon}<\nabla u,\nabla f>+\frac{1}{\lambda}<u,f>+\frac{1}{\epsilon}<v,f>_{\Gamma}-<h,v>_{-1,1}=$$

$$=<\frac{1}{\epsilon}L(f),u>_{-1,1}+<\frac{1}{\epsilon}\gamma(f)-h,v>_{-1,1}$$

Now, this is continuous in $U$ for the $E$ topology iff $f\in H^1_{\Gamma_0}(\Omega)$ and $\frac{1}{\epsilon}\gamma(f)-g\in L^2(\Omega)$, and in that case $<AU,F>_{E,E'}=<U,\begin{pmatrix}\frac{1}{\epsilon}\gamma(f)-h\\ \frac{1}{\epsilon}L(f)\end{pmatrix}>_{E,E'}$ and i) is proved.

ii) The second statement is a simple computation. Just note that the norm in $E$ used above is that for which $-A$ generates a contraction semigroup and we have used that $L$ is an isometric isomorphism between $H^1_{\Gamma_0}(\Omega)$ and its dual $H^{-1}_{\Gamma_0}(\Omega)$.

iii) Just note that $U^*=(u,w)\in D(A^{*2})$ iff $A^*U^*=(-v,\frac{1}{\epsilon}L(u))\in D(A^*)$ and that holds iff $(-v,\frac{1}{\epsilon}\gamma(v)+\frac{1}{\epsilon}L(u))\in E$. Hence, $v\in H^1_{\Gamma_0}(\Omega)$ and $\gamma(v)+L(u)=f\in L^2(\Omega)$. From the properties of the operator $L$, we get $U\in D(A)$.□

We now recall that a function $u$ from an interval of $\mathbb{R}$ into a Banach space $X$ is weakly differentiable iff for every $x'\in X'$, $<x',u(t)>$ is differentiable, where we denote by $<\cdot,\cdot>$ the duality between $X$ and $X'$. Now, by using Proposition 5.2 in the Appendix, we have

**Proposition 2.2.**

*i) Assume $f\in L^1(0,T,L^2(\Omega))$ and $U_0=(u_0,v_0)\in E$, and consider $U=(u,v)^T$ be the mild solution of (2.4) given by (2.5), with $F=(0,\frac{1}{\epsilon}f)^T$. Then, $U$ is characterized by: $U\in C([0,T],E)$*

$$v=u_t$$

*(weak derivative in $L^2(\Omega)$), and for every $\phi\in H^1_{\Gamma_0}(\Omega)$, $<\epsilon u_t,\phi>+<\gamma(u),\phi>_{-1,1}$ is absolutely continuous and*

$$\frac{d}{dt}(<\epsilon u_t,\phi>+<\gamma(u),\phi>_{-1,1})+<\nabla u,\nabla\phi>+\lambda<u,\phi>=<f,\phi> \tag{2.19}$$

*a.e. $t\in(0,T)$; in particular*

$$(\epsilon u_t+\gamma(u))_t+L(u)=f\ \text{in}\ H^{-1}_{\Gamma_0}(\Omega)\ \text{a.e.}\ t\in(0,T)$$

*ii) Assume $h\in L^1(0,T,H^{-1}_{\Gamma_0}(\Omega))$ and $U^*_0=(u_0,w_0)\in E'$, and consider $U^*=(u,w)^T$ be the function given by (2.13) with $H=(0,\frac{1}{\epsilon}h)^T$. Then, $U^*$ is characterized by the following condition: $U^*\in C([0,T],E')$ and for every $(\phi,\psi)\in D(A)$, $<\epsilon u,\psi>+<\epsilon w,\phi>_{-1,1}$ is absolutely continuous and*

$$\frac{d}{dt}(<\epsilon u,\psi>+<\epsilon w,\phi>_{-1,1})-<\epsilon w,\psi>_{-1,1}+<u,-\Delta\phi+\lambda\phi>=<h,\phi>_{-1,1} \tag{2.20}$$

*a.e. $t\in(0,T)$. Moreover, if we assume $u(t)\in H^1_{\Gamma_0}(\Omega)$ a.e. $t\in(0,T)$ and it is weakly differentiable in $H^{-1}_{\Gamma_0}(\Omega)$, then (2.20) is equivalent to*

$$w=u_t+\frac{1}{\epsilon}\gamma(u)$$

$$\frac{d}{dt}(<\epsilon u_t+\gamma(u),\phi>_{-1,1})+<\nabla u,\nabla\phi>+\lambda<u,\phi>=<h,\phi>_{-1,1}$$

*a.e. $t\in(0,T)$, and for every $\phi\in H^1_{\Gamma_0}(\Omega)$, i.e.*

$$(\epsilon u_t+\gamma(u))_t+L(u)=h\ \text{in}\ H^{-1}_{\Gamma_0}(\Omega)\ \text{a.e.}\ t\in(0,T)$$

*iii) Assume $h\in L^1(0,T,H^{-1}_{\Gamma_0}(\Omega))$ and $U^*_0=(u_0,w_0)\in E'$ are such that the dual equation*

$$U^*_t+A^*U^*=(0,\frac{1}{\epsilon}h)^T \tag{2.21}$$

*is verified pointwise. Then, $u$ is differentiable in $L^2(\Omega)$, $u(t) \in H^1_{\Gamma_0}(\Omega)$ for every $t$, $w(t)$ given by $w = u_t + \frac{1}{\epsilon}\gamma(u)$ is continuous and differentiable in $H^{-1}_{\Gamma_0}(\Omega)$ and*

$$(\epsilon u_t + \gamma(u))_t + L(u) = h, \ \ in\ H^{-1}_{\Gamma_0}(\Omega)$$

**Proof**

i) Just by using Proposition 5.2 we get that for every $W = (\phi, \psi) \in D(A^*)$

$$\frac{d}{dt} < U, W > + < U, A^* W >=< F, W >$$

and this gives

$$\frac{d}{dt}\left(< \epsilon u, \psi >_{-1,1} + < \epsilon v, \phi >\right) + < \nabla u, \nabla \phi > + \lambda < u, \phi > + < v, \gamma(\phi) - \epsilon \psi >=< f, \phi >$$

First, taking $\phi = 0$ and $\psi \in L^2(\Omega)$, we get $\frac{d}{dt} < u, \psi >=< v, \psi >$ and hence $v = u_t$. Now we have $< v, \gamma(\phi) - \epsilon\psi >= \frac{d}{dt} < u, \gamma(\phi) - \epsilon\psi >= \frac{d}{dt} < u, \gamma(\phi) - \epsilon\psi >_{-1,1}$ and plugging this into the previous expression, we get the result.

ii) Again Proposition 5.2 gives (2.20). If in addition, $u \in H^1_{\Gamma_0}(\Omega)$ and it is weakly differentiable in $H^{-1}_{\Gamma_0}(\Omega)$, then $\frac{d}{dt} < u, \psi >= \frac{d}{dt} < u, \psi >_{-1,1}=< u_t, \psi >_{-1,1}$. Taking this into (2.20), we get

$$< \epsilon u_t - \epsilon w + \gamma(u), \psi >_{-1,1} + \frac{d}{dt} < \epsilon w, \phi >_{-1,1} + < \nabla u, \nabla \phi > + \lambda < u, \phi >=< h, \phi >_{-1,1}$$

for every $(\phi, \psi) \in D(A)$. Now taking $\psi = 0$ and $\phi \in Y^1 = \{z \in H^1_{\Gamma_0}(\Omega), \Delta z \in L^2(\Omega),\ R\frac{\partial z}{\partial \vec{n}} = 0 \text{ on } \Gamma_1\}$, which is a dense set in $H^1_{\Gamma_0}(\Omega)$, we get $\frac{d}{dt} < \epsilon w, \phi >_{-1,1} + < \nabla u, \nabla \phi > + \lambda < u, \phi >=< h, \phi >_{-1,1}$ and by density, this holds true for any $\phi \in H^1_{\Gamma_0}(\Omega)$.

Therefore, we get $< \epsilon u_t - \epsilon w + \gamma(u), \psi >_{-1,1}= 0$ for any $\psi \in H^1_{\Gamma_0}(\Omega)$, and the result is proved.

iii) The proof is straightforward.□

**Remark 2.2.** *Note that when obtaining the equation $(\epsilon u_t + \gamma(u))_t + L(u) = h$ in $H^{-1}_{\Gamma_0}(\Omega)$, the two terms above can not be, a priori, differentiated separately. However depending on $h$ and the initial data one can, for example through energy estimates, prove the differentiability of both terms.*

We also have

**Corollary 2.2.**

*i) If $U$ is a mild solution of (2.4), then $U^*$ is a mild solution of (2.21), that is, if $U$ verifies (2.5) then $U^*$ verifies (2.13). In particular, $S(t)$ and $S^*(t)$ restricted to $D(A^*)$ coincide, modulo (2.18).*

*ii) Conversely if $h = f \in L^1(0, T, L^2(\Omega)) \subset L^1(0, T, H^{-1}_{\Gamma_0}(\Omega))$, and $U^*$ is a mild solution of (2.21) such that $U^* \in C([0, T], D(A^*))$ and $u$ is weakly differentiable in $L^2(\Omega)$, then $U$ is a mild solution of (2.4).*

**Proof**

i) Let $U$ be a mild solution of (2.4) and $(\phi, \psi) \in D(A)$. Then, from (2.19), we get

$$\frac{d}{dt} < \epsilon w, \phi >_{-1,1} + < u, -\Delta \phi + \lambda \phi > - < \gamma(u), \psi >_{-1,1}=< f, \phi >$$

so, using $v = u_t$ and its relation to $w$, we get (2.20).

ii) The proof is straightforward, since we can use point ii) of the previous Proposition to arrive to (2.19).□

**Corollary 2.3.**

*i) Assume $U^*_0 = (u_0, w_0) \in D(A^*)$ and either $h \in C([0, T], L^2(\Omega))$ or $h \in C^1([0, T], H^{-1}_{\Gamma_0}(\Omega))$. Then, $U^*$, given by (2.13) verifies*

$$U^* \in C([0, T], D(A^*)) \cap C^1([0, T], E')$$

*and verifies (2.21). Therefore*

$$\begin{array}{ll} u \in C([0, T], H^1_{\Gamma_0}(\Omega)) & w \in C([0, T], H^{-1}_{\Gamma_0}(\Omega)) \\ u_t \in C([0, T], L^2(\Omega)) & w_t \in C([0, T], H^{-1}_{\Gamma_0}(\Omega)) \end{array}$$

*and*

$$(\epsilon u_t + \gamma(u))_t + L(u) = h \ in\ H^{-1}_{\Gamma_0}(\Omega)$$

*ii) Moreover, assume $h \in C^2([0,T], H^{-1}_{\Gamma_0}(\Omega))$ and $U_0^* = (u_0, w_0)$ is such that $U_0 = (u_0, v_0)$, with $w_0 = v_0 + \frac{1}{\epsilon}\gamma(u_0)$, belongs to $H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ and*

$$\gamma(v_0) + L(u_0) - h(0) \in L^2(\Omega) \tag{2.22}$$

*then we have*

$$\begin{array}{ll} u \in C([0,T], H^1_{\Gamma_0}(\Omega)) & w \in C([0,T], H^{-1}_{\Gamma_0}(\Omega)) \\ u_t \in C([0,T], H^1_{\Gamma_0}(\Omega)) & w_t \in C([0,T], H^{-1}_{\Gamma_0}(\Omega)) \\ u_{tt} \in C([0,T], L^2(\Omega)) & w_{tt} \in C([0,T], H^{-1}_{\Gamma_0}(\Omega)) \end{array}$$

*Moreover,*

$$\epsilon u_{tt} + \gamma(u_t) + L(u) = h \quad \text{and} \quad (\epsilon u_{tt} + \gamma(u_t))_t + L(u_t) = h_t \ \text{in } H^{-1}_{\Gamma_0}(\Omega)$$

*iii) If $h$ has the special form $h(t) = f_\Omega(t) + g_\Gamma(t) \in H^{-1}_{\Gamma_0}(\Omega)$, with $f(t) \in L^2(\Omega)$ and $g(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$, for each $t \in (0,T)$, and are continuous in time, then under the assumptions in ii), we have $u \in C([0,T], Y_0)$, where $Y_0 = \{z \in H^1_{\Gamma_0}(\Omega),\ \Delta z \in L^2(\Omega)\}$, and*

$$u_{tt} - \Delta u + \lambda u = f(t) \in L^2(\Omega), \quad \gamma(u)_t + R(\frac{\partial u}{\partial \vec{n}}) = g(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$$

**Proof** The proofs of i) and ii) are an immediate consequence of ii) and iv) in Proposition 5.1. Just note that the compatibility condition on the initial data, (2.22), is equivalent to

$$-A^* U_0^* + \begin{pmatrix} 0 \\ \frac{1}{\epsilon} h(0) \end{pmatrix} \in D(A^*)$$

Concerning iii), just note that since $\epsilon u_{tt} + \gamma(u_t) + L(u) = f_\Omega + g_\Gamma$ holds, we read this equation as $L(u) = (f - \epsilon u_{tt})_\Omega + (g - \gamma(u_t))_\Gamma$ and we get the result from the properties of the operator $L$, since $(f - \epsilon u_{tt}) \in L^2(\Omega)$ and $\gamma(u_t) \in H^{1/2}_{\Gamma_0}(\Gamma)$ and are continuous in time.□

Now we are in a position to prove the main results of this subsection.

**Proof of Theorem 2.3**

Take $U_0^n = (u_0^n, v_0^n) \in D(A) \subset E$, $f^n \in C^2_c(0,T, L^2(\Omega))$ and $g^n \in C^2_c(0,T, L^2_{\Gamma_0}(\Gamma))$ be such that $U_0^n \to U_0 = (u_0, v_0)$ in $E$, $f^n \to f$ in $L^1(0,T, L^2(\Omega))$ and $g^n \to g$ in $L^2(\Sigma_1)$. Denote $h^n = f^n + g^n \to h = f + g$ in $L^1(0,T, H^{-1}_{\Gamma_0}(\Omega))$. Since $U_0^n \in D(A)$ and $h^n(0) = 0$, then condition (2.22) holds, namely $\gamma(v_0^n) + L(u_0^n) - h^n(0) \in L^2(\Omega)$, and therefore we have the regularity results in Corollary 2.3 for $u^n$, $u_t^n$, $u_{tt}^n$ and

$$\epsilon u_{tt}^n + \gamma(u_t^n) + L(u^n) = h^n$$

Therefore, we can take $u_t^n \in H^1_{\Gamma_0}(\Omega)$ as a test function and perform partial integration so we get the energy equality (2.8) $E_\epsilon(u^n, u_t^n) + 2\int_{\Sigma_{1,t}} \gamma(u^n)_t^2 = E_\epsilon(u_0^n, v_0^n) + 2\int_0^t < h^n, u_t^n >_{-1,1}$. Hence, the energy estimates of Proposition 2.1 hold on the sequence $(u^n, u_t^n)$.

On the other hand, working on $E'$, we have $U^{*,n}(t) = S^*_\epsilon(t) U_0^{*,n} + \int_0^t S^*_\epsilon(t-s) H^n(s)\, ds$ but since $U_0^{*,n} \in D(A^{*2}) \to U_0^* \in D(A^*)$ and since $H^n$ converges to $H$ in $L^1(0,T,E')$, using ii) in Proposition 5.1, we can pass to the limit in this expression, to get (2.13). But at the same time, using Proposition 2.1 we get that in the limit, $U = (u, u_t)$ lies in the energy space and has a regular trace in $\Sigma_1$. In particular (2.7) holds true.

Finally, note that we can assume, by taking subsequences, if necessary, that $h^n \to h$ a.e. in $[0,T]$, and the same for $\gamma(u_t^n)$. Therefore, we can pass to the limit in $(\epsilon u_t^n + \gamma(u^n))_t + L(u^n) = h^n$ and then $(\epsilon u_t + \gamma(u))_t + L(u) = h = f_\Omega + g_\Gamma$. In particular, since $\gamma(u)_t \in L^2(\Sigma_1)$ then $\epsilon u_{tt} = -\gamma(u)_t - L(u) + h \in L^1(0,T, H^{-1}_{\Gamma_0}(\Omega))$ and the theorem is proved.□

Note that the key point in proving Theorem 2.3 is constructing regular solutions of (2.13) such that $u_t \in H^1_{\Gamma_0}(\Omega)$ and the energy estimate (2.8)

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = E_\epsilon(u_0, v_0) + 2\int_0^t < h, u_t >_{-1,1}$$

holds true. From here the Lipschitz dependence of the mapping

$$(u_0, v_0, h) \longmapsto ((u, u_t), u_t) \in C([0,T], E) \times L^2(\Sigma_1)$$

is obtained and then one passes to the limit simultaneously in (2.13) and in the energy estimates.

To construct these regular solutions it is enough, in view of Corollary 2.4, to consider initial data $(u_0, v_0) \in D(A) \subset E$ and functions $h \in C^2_c(0, T, H^{-1}_{\Gamma_0}(\Omega))$, since then the compatibility condition (2.22) would hold true. Therefore we have, still for $h = f_\Omega + g_\Gamma$

**Corollary 2.4.** *Assume $f \in L^1(0, T, L^2(\Omega))$, $g \in W^{1,1}(0, T, H^{-1/2}_{\Gamma_0}(\Gamma))$ and $(u_0, v_0) \in E$. Then Theorem 2.3 holds true, but instead of (2.7) we have*

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = E_\epsilon(u_0, v_0) + 2\int_Q f u_t +$$
$$+2\left( < g(t), \gamma u(t) >_{-1/2,1/2} - < g(0), \gamma u_0 >_{-1/2,1/2} - \int_0^t < g_t, \gamma(u) >_{-1/2,1/2} \right) \tag{2.23}$$

$$\epsilon \sup_{0\le s\le t} \|u_t\|^2 + \sup_{0\le s\le t} \|u\|_1^2 \le 4E_\epsilon(u_0, v_0) + 8\|g(0)\|_{-1/2}\|u_0\|_1 + 16\|g\|^2_{W^{1,1}} + \frac{16}{\epsilon}\|f\|^2_{L^1(L^2)} \tag{2.24}$$

*and*

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 \le 3E_\epsilon(u_0, v_0) + 6\|g(0)\|_{-1/2}\|u_0\|_1 + 10\|g\|^2_{W^{1,1}} + \frac{10}{\epsilon}\|f\|^2_{L^1(L^2)} \tag{2.25}$$

**Proof** It is clear that from (2.7) and partial integration we get (2.23). Now, denoting $E_0 = E_\epsilon(u_0, v_0) + 2\|g(0)\|_{-1/2}\|u_0\|_1$, $K(g) = 2\|g\|_{W^{1,1}}$ and $K(f) = 2\|f\|_{L^1(L^2)}$, we get

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 \le E_0 + K(g) \sup_{0\le s\le t} \|u\|_1 + K(f) \sup_{0\le s\le t} \|u_t\|$$

and then $\epsilon \sup_{0\le s\le t} \|u_t\|^2 + \sup_{0\le s\le t} \|u\|_1^2 \le 2\left(E_0 + K(g)\sup_{0\le s\le t}\|u\|_1 + K(f)\sup_{0\le s\le t}\|u_t\|\right)$ from here, by Young's inequality, we get (2.24) and plugging this on the previous inequality, we get (2.25).□

For more general $t \mapsto h(t) \in H^{-1}_{\Gamma_0}(\Omega)$, we have

**Corollary 2.5.** *Assume $h \in L^1(0, T, H^{-1}_{\Gamma_0}(\Omega))$ with $h = h_1 + h_2$, $h_1(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$, $h_2(t) \in H^{-1}_0(\Omega)$ and $(u_0, v_0) \in E$.*
*i) If moreover $h \in W^{1,1}(0, T, H^{-1}_{\Gamma_0}(\Omega))$, then Theorem 2.3 holds true but instead of (2.7) we have*

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = E_\epsilon(u_0, v_0) + 2\left( < h(t), u(t) >_{-1,1} - < h(0), u_0 >_{-1,1} - \int_0^t < h_t, u >_{-1,1} \right) \tag{2.26}$$

*and*

$$E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 \le 2E_\epsilon(u_0, v_0) + 2\|h(0)\|_{-1}\|u_0\|_1 + 4\|h\|^2_{W^{1,1}} \tag{2.27}$$

*ii) Assume now $h_1 \in L^2(\Sigma_1)$ and $h_2 \in W^{1,1}(0, T, H^{-1}_0(\Omega))$, then Theorem 2.3 holds true, but instead of (2.7) we have*

$$E_\epsilon(u, u_t) \quad + \quad 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = E_\epsilon(u_0, v_0) + 2\left(\int_{\Sigma_{1,t}} h_1\gamma(u)_t + < (I - (B\gamma)^*)h_2(t), u(t) >_{-1,1}\right.$$
$$\left. - < (I - (B\gamma)^*)h_2(0), u_0 >_{-1,1} - \int_0^t < (I - (B\gamma)^*)h_{2t}, u >_{-1,1}\right) \tag{2.28}$$

*and*

$$E_\epsilon(u, u_t) + \int_{\Sigma_{1,t}} \gamma(u)_t^2 \le 2E_\epsilon(u_0, v_0) + 2\|h_1\|^2_{L^2(\Sigma_1)} + 2\|h_2(0)\|_{-1}\|u_0\|_1 + 4\|h_2\|^2_{W^{1,1}} \tag{2.29}$$

*In particular that holds if $h = f_\Omega + g_\Gamma$, where $f \in W^{1,1}(0, T, L^2(\Omega))$ and $g \in L^2(\Sigma_1)$.*

**Proof**
i) Now is clear that from (2.8) and partial integration, we get (2.26). Denoting $E_0 = E_\epsilon(u_0, v_0) + \|h(0)\|_{-1}\|u_0\|_1$ and $K(h) = 2\|h\|_{W^{1,1}}$ we get $E_\epsilon(u, u_t) + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 \leq E_0 + K(h)\sup_{0\leq s\leq t}\|u\|_1$ and from here we get $\sup_{0\leq s\leq t}\|u\|_1^2 \leq E_0 + K(h)\sup_{0\leq s\leq t}\|u\|_1$. Using Young's inequality, and plugging this into the previous inequality, we get (2.27).
ii) From (2.8) and writing $< h, u_t >_{-1,1}$ as $< h_1, \gamma(u)_t >_\Gamma + < h_2, (I - B\gamma)(u_t) >_\Omega$ and integrating by parts this last term in time, we get (2.28). By Young's inequality and denoting $E_0 = E_\epsilon(u_0, v_0) + \|h_1\|^2_{L^2(\Sigma_1)} + \|h_2(0)\|_{-1}\|u_0\|_1$ and $K(h_2) = 2\|h_2\|_{W^{1,1}}$, we get $E_\epsilon(u, u_t) + \int_{\Sigma_{1,t}} \gamma(u)_t^2 \leq E_0 + K(h_2)\sup_{0\leq s\leq t}\|u\|_1$. Hence, $\sup_{0\leq s\leq t}\|u\|_1^2 \leq E_0 + K(h_2)\sup_{0\leq s\leq t}\|u\|_1$ and using again Young's inequality and plugging this into the previous inequality, we get (2.29). In particular when $f \in W^{1,1}(0, T, L^2(\Omega))$ and $g \in L^2(\Sigma_1)$, then by Lemma 1.1, we have $h_1 = B^* f_\Omega + g_\Gamma \in L^2(\Sigma_1)$ and $h_2 = f_\Omega \in W^{1,1}(0, T, L^2(\Omega)) \subset W^{1,1}(0, T, H_0^{-1}(\Omega))$ and the above applies.□

**Proof of Theorem 2.4**
From Proposition 5.1, we know that if $U_0^* \in D(A^*)$ and $H \in C^1([0, T], E')$, then the function

$$U^*(t) = S_\epsilon^*(t)U_0^* + \int_0^t S_\epsilon^*(t - s)H(s)\,ds$$

is differentiable and $U_t^*(t) = S_\epsilon^*(t)(-A^*U_0^* + H(0)) + \int_0^t S_\epsilon^*(t - s)H_t(s)\,ds$.
But now, provided $-A^*U_0^* + H(0)$ corresponds to an element $W_0 \in E$, through (2.18), and since $H_t = \frac{1}{\epsilon}(0, f_t + g_t)^T$ with $f_t \in L^1(0, T, L^2(\Omega))$ and $g_t \in L^2(\Sigma_1)$, we can apply Theorem 2.3. Note that the compatibility for the initial data is equivalent to (2.15) and that $U^* = (u, w)^T$, with $w = u_t + \frac{1}{\epsilon}\gamma(u)$ and then $U_t^* = (u_t, w_t)^T$ with $w_t = u_{tt} + \frac{1}{\epsilon}\gamma(u_t)$. All these give the regularity for $U$, $U_t$, $\gamma(u_t)$ and (2.16). With this regularity we also get the equations $\epsilon u_{tt} + \gamma(u_t) + L(u) = h(t)$ and $(\epsilon u_{tt} + \gamma(u_t))_t + L(u_t) = h_t(t)$ in $H^{-1}_{\Gamma_0}(\Omega)$. The extra regularity comes, as in Corollary 2.3, from reading the equation as

$$L(u) = (f - \epsilon u_{tt})_\Omega + (g - \gamma(u_t))_\Gamma$$

and using the properties of the operator $L$, since $(f - \epsilon u_{tt}) \in L^2(\Omega)$ and $\gamma(u_t) \in H^{1/2}_{\Gamma_0}(\Gamma)$ and are continuous in time.□

Note that the key in the proof above is using twice Theorem 2.3, first on

$$U^*(t) = S_\epsilon^*(t)U_0^* + \int_0^t S_\epsilon^*(t - s)H(s)\,ds$$

and then on

$$U_t^*(t) = S_\epsilon^*(t)(-A^*U_0^* + H(0)) + \int_0^t S_\epsilon^*(t - s)H_t(s)\,ds$$

Therefore with the help of Corollaries 2.4 and 2.5 we can obtain a similar result as follows.

**Corollary 2.6.** *Assume $h \in W^{1,1}(0, T, H^{-1}_{\Gamma_0}(\Omega))$ is such that $h$ and $h_t$ satisfies either conditions in Theorem 2.3 or Corollaries 2.4 and 2.5. Moreover assume $(u_0, v_0)$ belongs to $H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ and verifies*

$$\gamma(v_0) + L(u_0) - h(0) \in L^2(\Omega)$$

*Then Theorem 2.4 holds true, but (2.16) must be replaced by the corresponding energy equality for $(u_t, u_{tt})$. Note that now $E_\epsilon(u_t, u_{tt})$ is estimated in terms of $E_\epsilon(u_t(0), u_{tt}(0))$ and some norm of $h$. Also $u_t(0) = v_0$ and $\epsilon u_{tt}(0) = h(0) - \gamma(v_0) - L(u_0)$.*

## 3. Evolution problems on Γ

We are concerned in this section with the understanding of the parabolic nature of problems

$$\begin{cases} -\Delta u + \lambda u = f & \text{on } \Omega \times (0, T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1 \times (0, T) \\ u = 0 & \text{on } \Gamma_0 \times (0, T) \end{cases} \tag{3.1}$$

where $f = f(t,x)$ and $g = g(t,x)$. We start by studying the homogeneous case, i.e. $f, g = 0$

$$\begin{cases} -\Delta u + \lambda u = 0 & \text{on } \Omega \times (0,\infty) \\ u_t + \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty) \end{cases} \tag{3.2}$$

and first we make precise the concept of solution of (3.2).

With the aid of the operators $R$ and $B$, as defined in (1.8), we can reduce the study of problem (3.2) to that of an evolution problem on $\Gamma$. In fact, if we can solve for $u_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$

$$\begin{cases} u_t + \frac{\partial B(u)}{\partial \vec{n}} = 0 \\ u(0) = u_0 \text{ on } \Gamma_1 \\ u = 0 \text{ on } \Gamma_0 \end{cases} \tag{3.3}$$

then $z = B(u)$ will be a solution of (3.2) with initial value $z_0 = B(u_0)$. Furthermore, we can still reduce the problem to solve

$$\begin{cases} u_t + R(\frac{\partial B(u)}{\partial \vec{n}}) = 0 \\ u(0) = u_0 \text{ on } \Gamma \end{cases} \tag{3.4}$$

working on the space $H = L^2_{\Gamma_0}(\Gamma)$. Note that the linear operator $A_0(u) = R(\frac{\partial B(u)}{\partial \vec{n}})$ is a nonlocal operator, for functions defined on $\Gamma$. Concerning (3.4) our main result is the following

**Theorem 3.1.** *The operator $A_0(u) = R(\frac{\partial B(u)}{\partial \vec{n}})$ induces an unbounded, positive, selfadjoint operator in $H = L^2_{\Gamma_0}(\Gamma)$ with domain $\gamma L^{-1}(L^2_{\Gamma_0}(\Gamma))$. Moreover, when $\Gamma_1$ is bounded, $A_0$ has compact resolvent.*

*In particular, it is a sectorial operator in $H$, so $-A_0$ generates an analytic semigroup on $H$, $\{e^{-A_0 t}\}_t$, and the fractional power spaces $\{X^\alpha\}_\alpha$ are well defined. In particular*

$$X^1 = \gamma L^{-1}(L^2_{\Gamma_0}(\Gamma)), \quad X^{1/2} = H^{1/2}_{\Gamma_0}(\Gamma), \quad X^0 = L^2_{\Gamma_0}(\Gamma), \quad X^{-1/2} = H^{-1/2}_{\Gamma_0}(\Gamma)$$

*and moreover $X^{\alpha+1} = \gamma L^{-1}(X^\alpha)$. When $\Gamma_i$ form a regular partition of the boundary, then*

$$X^\alpha = H^\alpha_{\Gamma_0}(\Gamma)$$

*Therefore, for every $\alpha$ and $u_0 \in X^\alpha$, there exists a unique solution of (3.4), $u(t) = e^{-A_0 t} u_0$ that verifies*

$$u_t + A_0 u = 0$$

*in $X^\alpha$ and $u \in C([0,\infty), X^\alpha) \cap C^\omega(0,\infty, X^\beta)$ for every $\beta \geq \alpha$. Since $A_0$ is strictly positive, all solutions of (3.4) decay exponentially to zero.*

**Proof** From Proposition 1.1 we know $A_0 = LB$ is an isomorphism between $H^{1/2}_{\Gamma_0}(\Gamma)$ and its dual. If we take the "part of $A_0$ in $H$", that is, the operator $\hat{A}$ on $H$ with domain $D(\hat{A}) = \{v \in V,\ A_0 v \in H\}$ and $\hat{A}v = A_0 v$ on $D(\hat{A})$, then $\hat{A}$ is an unbounded, selfadjoint, positive linear operator on $H$, which moreover has compact resolvent when $\Gamma_1$ is bounded since in this case the inclusion $H^{1/2}_{\Gamma_0}(\Gamma) \subset L^2_{\Gamma_0}(\Gamma)$ is compact. Note that we have the variational description of $A_0$

$$< A_0(u), v >_{-1/2,1/2} = a(u,v) = \int_\Omega \nabla B(u) \nabla B(v) + \lambda \int_\Omega B(u) B(v)$$

and then $\hat{A}$ is positive since the bilinear form $a$ is coercive. Therefore, $\sigma(A_0) \subset \mathbb{R}^+$ and in fact $\inf \sigma(A_0) > \delta > 0$, and the decaying to zero follows, [12]. The rest is obvious.

When $\Gamma_i$ form a regular partition of $\Gamma$, again the regularity results for the mixed Dirichlet–Neumann problem imply $X^n = H^n_{\Gamma_0}(\Gamma)$ and $X^{n+1/2} = H^{n+1/2}_{\Gamma_0}(\Gamma)$, $n \in \mathbb{N}$, and by interpolation we get $X^\alpha = H^\alpha_{\Gamma_0}(\Gamma)$, $\alpha \geq 0$. To see this, note that for $j = 1, 3/2$, $X^j = \{u \in X^{1/2},\ A_0 u \in X^{j-1}\}$ and we already kow that for $k = j-1 = 0, 1/2$, $X^k = H^k_{\Gamma_0}(\Gamma)$. Therefore, if $u \in X^j$ then $v = B(u)$ satisfies $R(\frac{\partial v}{\partial \vec{n}}) \in X^{j-1}$ and

$$\begin{cases} -\Delta v + \lambda v = 0 & \text{on } \Omega \\ \frac{\partial v}{\partial \vec{n}} \in H^{j-1}_{\Gamma_0}(\Gamma) & \text{on } \Gamma_1 \\ v = 0 & \text{on } \Gamma_0 \end{cases}$$

and from elliptic regularity, we get $v \in H^{j+1/2}_{\Gamma_0}(\Omega)$ and consequently $u = \gamma(v) \in H^j_{\Gamma_0}(\Gamma)$.

Conversely, for the other inclusion, if $u \in H^j_{\Gamma_0}(\Gamma)$ then $v = B(u) \in H^{j+1/2}_{\Gamma_0}(\Omega)$ and then $R(\frac{\partial v}{\partial \vec{n}}) = A_0 u \in H^{j-1}_{\Gamma_0}(\Gamma)$, i.e. $u \in X^j$.∎

Again, general results for sectorial operators, [12], allows us to solve (3.4) in the $X^\alpha$ spaces. More precisely

**Corollary 3.1.** *For the problem*

$$\begin{cases} u_t + R(\frac{\partial B(u)}{\partial \vec{n}}) = g \\ u(0) = u_0 \text{ on } \Gamma \end{cases} \tag{3.5}$$

*Propositions 5.1 and 5.5 of the Appendix apply.*

**Corollary 3.2.** *For the problem*

$$u_{tt} + \beta u_t + R(\frac{\partial B(u)}{\partial \vec{n}}) = g \tag{3.6}$$

*Propositions 5.1 and 5.3 of the Appendix apply. In particular, for $g = 0$, the equation defines a semigroup on $H^{1/2}_{\Gamma_0}(\Gamma) \times L^2_{\Gamma_0}(\Gamma)$.*

Once we have solved the evolution problem (3.4) in $\Gamma$, we just need to lift the solution to $\Omega$ to get a solution of (3.2). For this, note that for $\alpha \geq 0$ we have $X^\alpha \subset L^2_{\Gamma_0}(\Gamma)$ and then, from Remark 1.2, we can apply $B$ to lift the solution to the interior of $\Omega$. Also, from the same remark, note that if the partition is regular then for $\alpha \geq -1/2$, $X^\alpha \subset H^{-1/2}_{\Gamma_0}(\Gamma)$ and therefore $B(X^\alpha)$ is well defined.

Then we have

**Definition 3.1.** *The function $u$ is a solution of (3.2) iff for $t > 0$, $u(t) \in H^1_{\Gamma_0}(\Omega)$, $u(t) = B(\gamma(u(t)))$, where $v(t) = \gamma(u(t)) \in H^{1/2}_{\Gamma_0}(\Gamma)$ is a solution of $v_t + R(\frac{\partial B(v)}{\partial \vec{n}}) = 0$*

**Proposition 3.1.** *The function $u$ is a solution of (3.2) iff for every $t > 0$, $u(t) \in H^1_{\Gamma_0}(\Omega)$, $u(t) = B(v(t))$, where $v(t) = e^{-A_0 t} v_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$.*
*i) Therefore, for any $\alpha$ and $v_0 \in X^\alpha$, $u(t) = Be^{-A_0 t} v_0$ is a solution of (3.2) and verifies $u \in C(0, \infty, H^1_{\Gamma_0}(\Omega))$, $\gamma(u(t)) \to v_0$ in $X^\alpha$ as $t \to 0$ and*

$$\gamma(u)_t + L(u) = 0$$

*for $t > 0$.*
*ii) In particular, if $v_0 \in L^2_{\Gamma_0}(\Gamma)$ then $u \in C([0, \infty), L^2(\Omega))$. The same is true if the partition is regular and if $v_0 \in H^{-1/2}_{\Gamma_0}(\Gamma)$. In any case, if $v_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$ then $u \in C([0, \infty), H^1_{\Gamma_0}(\Omega))$ and the one parameter family of linear operators in $Har_{1/2,\Gamma_0}(\Omega)$ given by*

$$S_0(t) = Be^{-A_0 t}\gamma$$

*defines an analytic semigroup on $Har_{1/2,\Gamma_0}(\Omega)$ whose infinitesimal generator is given by $-BA_0\gamma$, with domain $L^{-1}(H^{1/2}_{\Gamma_0}(\Gamma))$. Moreover, its fractional power spaces are given by $X^\alpha_0 = B(X^{\alpha+1/2})$.*
*iii) More generally, if $u_0 \in X^\alpha_0$ then $u(t) = S_0(t)u_0$ verifies*

$$u \in C([0, \infty), X^\alpha_0) \cap C(0, \infty, X^\beta_0)$$

*for every $\beta \geq \alpha$. Finally, solutions of (3.2) decay exponentially to zero.*

**Proof** The first part follows from Theorem 3.1 and the definition of solution of (3.2). Now, for every $\alpha$ and $v_0 \in X^\alpha$, from the regularizing effect, we know that $v(t) = e^{-A_0 t} v_0 \in C(0, \infty, H^{1/2}_{\Gamma_0}(\Gamma))$ and hence $u \in C(0, \infty, H^1_{\Gamma_0}(\Omega))$.

Now, note that for $t > 0$ it holds $v_t + A_0 v = 0$ and therefore $u = B(v)$ verifies $< L(u), \phi >= 0$ for every $\phi \in H^1_0(\Omega)$, and for every $\phi \in Har_{1/2,\Gamma_0}(\Omega)$, $< L(u), \phi >=< R(\frac{\partial B(v)}{\partial \vec{n}}), \phi >_\Gamma =< -\gamma(u)_t, \phi >_\Gamma$. Putting these together we get $\gamma(u)_t + L(u) = 0$.

The proofs of ii) and iii) are straightforward. Just note that the diagram

$$\begin{array}{ccc} X^\alpha_0 & \xrightarrow{S_0} & X^\alpha_0 \\ B\uparrow & & \uparrow B \\ X^{\alpha+1/2} & \xrightarrow{e^{-A_0 t}} & X^{\alpha+1/2} \end{array}$$

is commutative.∎

Note that , solving the nonhomogeneous case, i.e.

$$\left\{\begin{array}{ll} -\Delta u + \lambda u = f(t) & \text{on } \Omega \times (0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g(t) & \text{on } \Gamma_1 \times (0,T) \\ u = 0 & \text{on } \Gamma_0 \times (0,T) \end{array}\right. \tag{3.7}$$

is formally equivalent to solving

$$\left\{\begin{array}{l} u(t) = D(\gamma(u(t)), f(t)) = B(\gamma(u(t))) + D_0(f(t)) \text{ on } \Omega \\ u_t + R(\frac{\partial B(u)}{\partial \vec{n}}) = g - R(\frac{\partial D_0(f)}{\partial \vec{n}}) \text{ on } \Gamma \\ u(0) = u_0 \text{ on } \Gamma \end{array}\right. \tag{3.8}$$

at least when $f(t) \in L^2(\Omega)$, and the equation on $\Gamma$ is a particular form of (3.5). Recall that from Lemma 1.1, if $h = f_\Omega + g_\Gamma$ then $h_1 = B^* f_\Omega + g_\Gamma = g - R(\frac{\partial D_0(f)}{\partial \vec{n}}) \in H^{-1/2}_{\Gamma_0}(\Gamma)$ and $h_2 = f_\Omega \in H^{-1}_0(\Omega)$. So, to be more precise, we make the following

**Definition 3.2.** *Assume $h(t) = h_1(t) + h_2(t) \in H^{-1}_{\Gamma_0}(\Omega)$ is given a.e. $t \in (0,T)$, with $h_1 \in H^{-1/2}_{\Gamma_0}(\Gamma)$ and $h_2 \in H^{-1}_0(\Omega)$. Then a solution of (3.7) is a function $t \mapsto u(t) \in H^1_{\Gamma_0}(\Omega)$ such that for $t \in (0,T)$*

$$u(t) = B(v(t)) + D_0(h_2(t)) \in H^1_{\Gamma_0}(\Omega) \tag{3.9}$$

*and $v(t) = \gamma(u(t)) \in H^{1/2}_{\Gamma_0}(\Gamma)$ verifies*

$$v(t) = e^{-A_0 t} v_0 + \int_0^t e^{-A_0(t-s)} h_1(s)\, ds \tag{3.10}$$

**Remark 3.1.**
*i) Since $h_1$ takes values in $H^{-1/2}_{\Gamma_0}(\Gamma)$, then for (3.10) to make sense we need the minimal regularity assumptions $v_0 \in H^{-1/2}_{\Gamma_0}(\Gamma)$ and $h_1 \in L^1(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$, and in that case (3.10) gives $v \in C([0,T], H^{-1/2}_{\Gamma_0}(\Gamma))$. Therefore, to obtain solutions of (3.7) we need to impose conditions on $v_0$ and $h_1$ to obtain $v(t) \in H^{1/2}_{\Gamma_0}(\Gamma)$.*

*Recall however that $B(v)$ makes sense whenever $v \in L^2_{\Gamma_0}(\Gamma)$ or even $H^{-1/2}_{\Gamma_0}(\Gamma)$ if the partition is regular. In such a case, (3.9) provides a "generalized" solution of (3.7).*
*ii) Assume $v_0$ and $h_1$ are such that for $t \in (0,T)$, $v(t) \in H^{1/2}_{\Gamma_0}(\Gamma)$, $v_t(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$ and*

$$v_t + A_0 v = h_1$$

*is satisfied a.e. $t \in (0,T)$ in $H^{-1/2}_{\Gamma_0}(\Gamma)$, then $u(t)$ given by (3.9) verifies $u(t) \in H^1_{\Gamma_0}(\Omega)$ and*

$$\gamma(u)_t + L(u) = h, \quad a.e.\ t \in (0,T) \tag{3.11}$$

*as an equality in $H^{-1}_{\Gamma_0}(\Omega)$. To see this, note that from $u(t) = B(v(t)) + D_0(h_2(t))$ we get $L(u(t)) = LB(v(t)) + h_2 = -v_t + h(t) = -\gamma(u)_t + h(t)$.*

*Conversely, if $u$ takes values in $H^1_{\Gamma_0}(\Omega)$ and satisfies (3.11), then $u$ is given by (3.9), where $v$ is given by (3.10) and satisfies $v_t + A_0 v = h_1$ a.e. $t \in (0,T)$ in $H^{-1/2}_{\Gamma_0}(\Gamma)$.*
*iii) Finally, assume $u$ takes values in $H^1_{\Gamma_0}(\Omega)$, satisfies (3.11) and $h(t) = f_\Omega(t) + g_\Gamma(t)$, with $f_\Omega(t) \in L^2(\Omega)$ and $g_\Gamma(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$. Then it verifies a.e. $t \in (0,T)$*

$$\left\{\begin{array}{ll} -\Delta u + \lambda u = f(t) & \text{in } L^2(\Omega) \\ \gamma(u)_t + R\frac{\partial u}{\partial \vec{n}} = g(t) & \text{in } H^{-1/2}_{\Gamma_0}(\Gamma) \end{array}\right. \tag{3.12}$$

*iv) With the same ideas, and by using (3.6), we can give a suitable framework for solving*

$$\left\{\begin{array}{ll} -\Delta u + \lambda u = f(t) & \text{in } L^2(\Omega) \\ \gamma(u)_{tt} + \beta\gamma(u)_t + R\frac{\partial u}{\partial \vec{n}} = g(t) & \text{in } H^{-1/2}_{\Gamma_0}(\Gamma) \end{array}\right.$$

*an equation appearing in relation with surface water waves,* [7, 8]. *The solution would be given as in (3.9) and instead of (3.10) we would use the variation of constants formula for the semigroup induced by equation (3.6).*

Then, we have the following result

**Theorem 3.2.**
*i) Assume* $h_1 \in L^1(0,T,H^s_{\Gamma_0}(\Gamma))$ *and* $v_0 \in H^s_{\Gamma_0}(\Gamma)$, *for* $s = -1/2, 0, 1/2$, *then*

$$v \in C([0,T], H^s_{\Gamma_0}(\Gamma))$$

*and the mapping* $(v_0, h) \mapsto v$ *is Lipschitz on these spaces.*
*ii) Assume* $h_1 : (0,T) \to H^{-1/2}_{\Gamma_0}(\Gamma)$ *is locally Lipschitz and integrable and* $v_0 \in H^{-1/2}_{\Gamma_0}(\Gamma)$, *then*

$$v \in C([0,T), H^{-1/2}_{\Gamma_0}(\Gamma)) \cap C(0,T,H^{1/2}_{\Gamma_0}(\Gamma)) \quad v_t \in C(0,T,L^2_{\Gamma_0}(\Gamma))$$

*Moreover, for* $t \in (0,T)$, $u(t) = B(v(t)) + D_0(h_2(t)) \in H^1_{\Gamma_0}(\Omega)$ *and verifies (3.11) for every* $t \in (0,T)$.
*iii) If* $h_1 \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$ *and* $v_0 \in L^2_{\Gamma_0}(\Gamma)$ *then*

$$v \in C([0,T], L^2_{\Gamma_0}(\Gamma)) \cap L^2(0,T,H^{1/2}_{\Gamma_0}(\Gamma)) \quad v_t \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$$

*and the mapping* $(v_0, h) \mapsto (v, v_t)$ *is Lipschitz on these spaces. Moreover,* $u(t) \in H^1_{\Gamma_0}(\Omega)$ *verifies (3.11) a.e.* $t \in (0,T)$.
*iv) Assume* $h_1 : (0,T) \to L^2_{\Gamma_0}(\Gamma)$ *is locally Lipschitz and integrable and* $v_0 \in L^2_{\Gamma_0}(\Gamma)$, *then*

$$v \in C([0,T), L^2_{\Gamma_0}(\Gamma)) \cap C(0,T,\gamma L^{-1}(L^2_{\Gamma_0}(\Gamma))) \quad v_t \in C(0,T,H^{1/2}_{\Gamma_0}(\Gamma))$$

*v) If* $h_1 \in L^2(0,T,L^2_{\Gamma_0}(\Gamma)) = L^2(\Sigma_1)$ *and* $v_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$ *then*

$$v \in C([0,T], H^{1/2}_{\Gamma_0}(\Gamma)) \cap L^2(0,T,\gamma L^{-1}(L^2_{\Gamma_0}(\Gamma))) \quad v_t \in L^2(0,T,L^2_{\Gamma_0}(\Gamma)) = L^2(\Sigma_1)$$

*and the mapping* $(v_0, h) \mapsto (v, v_t)$ *is Lipschitz on these spaces. Moreover,* $u(t) \in H^1_{\Gamma_0}(\Omega)$ *verifies (3.11) a.e.* $t \in (0,T)$.

**Proof** By applying Proposition 5.5 to (3.10) we get all five cases at once. Note that in all cases $u(t) \in H^1_{\Gamma_0}(\Omega)$ and the equation $v_t + A_0 v = h_1$ is verified a.e. $t \in (0,T)$. Therefore Remark 3.1 concludes the result.□

**Corollary 3.3.** *Assume* $h(t) = h_1(t) + h_2(t) \in H^{-1}_{\Gamma_0}(\Omega)$ *is given a.e.* $t \in (0,T)$, *with* $h_1 \in L^2(\Sigma_1)$ *and* $h_2 \in L^p(0,T,H^{-1}_0(\Omega))$ *for some* $1 \le p \le \infty$. *Assume also* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *is given.*

*Then* $u$ *given by (3.9) and (3.10), with* $v_0 = \gamma(u_0)$ *verifies*

$$u \in L^p(0,T,H^1_{\Gamma_0}(\Omega)), \quad \gamma(u)_t \in L^2(\Sigma_1)$$

$$\gamma(u)_t + L(u) = h \tag{3.13}$$

*as an equality in* $H^{-1}_{\Gamma_0}(\Omega)$, *a.e.* $t \in (0,T)$. *Moreover, if* $h_2 \in C([0,T), H^{-1}_0(\Omega))$ *and* $u_0$ *verifies*

$$-\Delta u_0 + \lambda u_0 = h_2(0) \tag{3.14}$$

*in* $\Omega$, *i.e.* $u_0 = B(\gamma(u_0)) + D_0(h_2(0))$, *then*

$$u \in C([0,T), H^1_{\Gamma_0}(\Omega)), \quad u(0) = u_0$$

*In particular, the above applies if* $h(t) = f_\Omega(t) + g_\Gamma(t)$, *with* $f(t) \in L^2(\Omega)$ *and* $g(t) \in H^{-1/2}_{\Gamma_0}(\Gamma)$ *a.e.* $t \in (0,T)$, $g - R(\frac{\partial D_0(f)}{\partial \vec{n}}) \in L^2(\Sigma_1)$ *and* $f \in L^p(0,T,H^{-1}_0(\Omega))$ *or* $f \in C([0,T), H^{-1}_0(\Omega))$.

**Proof** The proof is straightforward, since we can apply point v) of the Theorem. The rest comes from the properties of $h_2$ and (3.9). When $h(t) = f_\Omega(t) + g_\Gamma(t)$ the result comes from Lemma 1.1.□

Concerning energy estimates we have the following.

**Proposition 3.2.** *Assume as above, that* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $h(t) = h_1(t) + h_2(t)$ *a.e.* $t \in (0,T)$ *are given.*
*i) If* $h \in W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$ *and* $u_0$ *verifies (3.14) then*

$$\begin{aligned}\|\nabla u\|^2_{L^2} \quad + \quad & \lambda \|u\|^2_{L^2} + 2\int_{\Sigma_{1,t}} \gamma(u)^2_t = \|\nabla u_0\|^2_{L^2} + \lambda\|u_0\|^2_{L^2} + \\ & +2\left(< h(t), u(t) >_{-1,1} - < h(0), u_0 >_{-1,1} - \int_0^t < h_t, u >_{-1,1}\right)\end{aligned} \tag{3.15}$$

*ii) If $h_1 \in L^2(\Sigma_1)$ and $h_2 \in W^{1,1}(0,T,H_0^{-1}(\Omega))$ and $u_0$ verifies (3.14), then*

$$\|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = \|\nabla u_0\|_{L^2}^2 + \lambda\|u_0\|_{L^2}^2 + 2\left(\int_{\Sigma_1} h_1\gamma(u)_t + \right.$$
$$\left. < (I-(B\gamma)^*)h_2(t), u(t) >_{-1,1} - < (I-(B\gamma)^*)h_2(0), u(0) >_{-1,1} - \int_0^t < (I-(B\gamma)^*)h_{2t}, u >_{-1,1}\right) \quad (3.16)$$

*In particular that holds if $h = f_\Omega + g_\Gamma$, where $f \in W^{1,1}(0,T,L^2(\Omega))$ and $g \in L^2(\Sigma_1)$.*

**Proof** It is enough to prove the result for regular $h$ and initial data satisfying (3.14). For such solutions $\gamma(u)_t + L(u) = h$ holds true and $u_t$ is in $H^1_{\Gamma_0}(\Omega)$. Therefore, using $u_t$ as a test function in the equation, above we get $\int_{\Gamma_1}\gamma(u)_t^2 + \frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2\right) =< h, u_t >_{-1,1}$ and the rest follows by integrating in time and using integration by parts in the term $\int_0^t < h, u_t >_{-1,1}$ as in Corollary 2.5. Just note that both cases for $h$ imply $h_2 \in C([0,T), H_0^{-1}(\Omega))$ and then (3.14) makes sense and we can use Corollary 3.3.□

## 4. Convergence of Solutions

In this section we show how solutions of

$$\begin{cases} \epsilon u^\epsilon_{tt} - \Delta u^\epsilon + \lambda u^\epsilon = f^\epsilon(t) & \text{on } \Omega\times(0,T) \\ u^\epsilon_t + \frac{\partial u^\epsilon}{\partial \vec{n}} = g^\epsilon(t) & \text{on } \Gamma_1\times(0,T) \\ u^\epsilon = 0 & \text{on } \Gamma_0\times(0,T) \end{cases} \quad (4.1)$$

and initial data $(u_0^\epsilon, v_0^\epsilon)$, approach solutions of

$$\begin{cases} -\Delta u + \lambda u = f & \text{on } \Omega\times(0,T) \\ u_t + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma_1\times(0,T) \\ u = 0 & \text{on } \Gamma_0\times(0,T) \end{cases} \quad (4.2)$$

with initial data $u_0$ as $\epsilon$ goes to zero, assumed $h^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$ and $u_0^\epsilon$ converge respectively to $h = f_\Omega + g_\Gamma$ and $u_0$ in some sense. We will also consider the problem for more general $h^\epsilon$.

We first start by recalling the results obtained in previous sections that allow us to obtain estimates on the solutions which are uniform in $\epsilon$. We then use these bounds combined with compactness arguments and other techniques to prove that $u^\epsilon(t)$ converges in some sense to $u(t)$.

4.1. **Uniform energy estimates.** Now we fix our attention on some energy estimates on solutions of (4.1). As an immediate consequence of inequalities (2.11), (2.25), (2.27) and (2.29), we have

**Proposition 4.1.**
*i) Assume $(u_0^\epsilon, v_0^\epsilon) \in E$ are given such that $E_\epsilon(u_0^\epsilon, v_0^\epsilon) = O(1)$, for $0 < \epsilon \le \epsilon_0$. Moreover, assume $h^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$, with $\frac{1}{\sqrt{\epsilon}} f^\epsilon = O(1)$, in $L^1(0,T,L^2(\Omega))$ and either $g^\epsilon = O(1)$ in $L^2(\Sigma_1)$ or in $W^{1,1}(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$. For more general $h^\epsilon$, assume either $h^\epsilon = O(1)$ in $W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$ or $h^\epsilon = h_1^\epsilon + h_2^\epsilon$ with $h_1^\epsilon = O(1)$ in $L^2(\Sigma_1)$ and $h_2^\epsilon = O(1)$ in $W^{1,1}(0,T,H_0^{-1}(\Omega))$.*

*Then*

$$u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon} u^\epsilon_t \in C_b([0,T], L^2(\Omega)) \quad \text{and} \quad \gamma(u)^\epsilon_t \in L^2(\Sigma_1) \quad (4.3)$$

*with bounds independent of $\epsilon$.*
*ii) If in the conditions above, $O(1)$ is replaced everywhere by $o(1)$, then*

$$u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon} u^\epsilon_t = o(1) \quad (4.4)$$

*in $L^2(\Omega)$, uniformly in $[0,T]$ and $\gamma(u)^\epsilon_t = o(1)$ in $L^2(\Sigma_1)$.*□

Now we will obtain further energy estimates for more regular solutions ensuring that the term $\epsilon u^\epsilon_{tt}$ is small. For this we will use the results in Theorem 2.4 and Corollary 2.6 and the corresponding inequalities, similar to (2.16).

**Proposition 4.2.** *Assume $h^\epsilon \in W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$ are as in Theorems 2.4 or Corollary 2.6, and such that $h^\epsilon$ and $h^\epsilon_t$ satisfy either conditions of the type $O(1)$, in Proposition 4.1 above.*

*Even more, assume $(u^\epsilon_0, v^\epsilon_0)$ belong to $H^1_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)$, and*

$$\frac{1}{\sqrt{\epsilon}}\|\gamma(v^\epsilon_0)+L(u^\epsilon_0)-h^\epsilon(0)\|_{L^2(\Omega)} = O(1) \tag{4.5}$$

*Then*

$$u^\epsilon_t, |\nabla u^\epsilon_t|, \sqrt{\epsilon}u^\epsilon_{tt} \in C_b([0,T],L^2(\Omega)) \quad and \quad \gamma(u)^\epsilon_{tt}\in L^2(\Sigma_1) \tag{4.6}$$

*with bounds independent of $\epsilon$.*

**Remark 4.1.** *Observe that the compatibility conditions for the data, (4.5), are a weak formulation of*

$$\begin{cases} -\Delta u^\epsilon_0 + \lambda u^\epsilon_0 - h^\epsilon_2(0) = O(\epsilon^{1/2}) & in\ L^2(\Omega) \\ v^\epsilon_0 + \frac{\partial u^\epsilon_0}{\partial \vec{n}} = h^\epsilon_1(0) & on\ \Gamma_1 \\ u^\epsilon_0, v^\epsilon_0 = 0 & on\ \Gamma_0 \end{cases}$$

*and with these we have*

$$\epsilon u^\epsilon_{tt} = O(\sqrt{\epsilon}) \tag{4.7}$$

*in $L^2(\Omega)$, uniformly in $[0,T]$. Observe also that if $\gamma(v^\epsilon_0)+L(u^\epsilon_0)-h^\epsilon(0)=0$, then $v^\epsilon_0$ can be freely defined on $\Omega$ while its trace parametrizes $u^\epsilon_0$ in $\Omega$.*

### 4.2. **Convergence in** $L^p(0,T,H^1_{\Gamma_0}(\Omega))$.

4.2.1. *Weak * convergence for $p=\infty$.* We have the following result

**Theorem 4.1.** *Assume $h^\epsilon$ and $(u^\epsilon_0, v^\epsilon_0)\in E$ are given, such that (4.3) holds true. Then, by taking subsequences if necessary, $u^\epsilon_0$ converges weakly in $H^1_{\Gamma_0}(\Omega)$ to $u_0$, and there exists a function $h\in L^1(0,T,H^{-1}_{\Gamma_0}(\Omega))$ such that $u^\epsilon$ converges $w-*$ in $L^\infty(0,T,H^1_{\Gamma_0}(\Omega))$ to a function $u$, which is the unique solution of*

$$\gamma(u)_t + L(u) = h, \quad \gamma(u(0)) = \gamma(u_0)$$

**Proof** Since (4.3) holds true, $u^\epsilon, |\nabla u^\epsilon|, \sqrt{\epsilon}u^\epsilon_t \in C_b([0,T],L^2(\Omega))$ and $\gamma(u)^\epsilon_t\in L^2(\Sigma_1)$ with bounds independent of $\epsilon$, by compactness, we can assume, by taking subsequences if necessary, that there exists $u_0\in H^1_{\Gamma_0}(\Omega)$ and $u\in L^\infty(0,T,H^1_{\Gamma_0}(\Omega))$ such that $\gamma(u)_t\in L^2(\Sigma_1)$ and such that

$$u^\epsilon_0 \to u_0, \quad w \text{ in } H^1_{\Gamma_0}(\Omega) \quad \text{and} \quad \gamma(u_0)^\epsilon \to \gamma(u_0), \quad w \text{ in } H^{1/2}_{\Gamma_0}(\Gamma)$$
$$u^\epsilon \to u, \quad w-* \text{ in } L^\infty(0,T,H^1_{\Gamma_0}(\Omega))$$
$$\gamma(u^\epsilon)\to\gamma(u), \quad w-* \text{ in } L^\infty(0,T,H^{1/2}_{\Gamma_0}(\Gamma))$$
$$\gamma(u^\epsilon)_t \to \gamma(u)_t, \quad w \text{ in } L^2(\Sigma_1)$$
$$L(u^\epsilon)\to L(u), \quad w-* \text{ in } L^\infty(0,T,H^{-1}_{\Gamma_0}(\Omega))$$

From (2.14) we have $(\epsilon u^\epsilon_t + \gamma(u^\epsilon))_t + L(u^\epsilon) = h^\epsilon$ and this is equivalent to: for every $\phi\in H^1_{\Gamma_0}(\Omega)$, $<\epsilon u^\epsilon_t,\phi> + <\gamma(u^\epsilon),\phi>_{-1,1}$ is absolutely continuous and

$$\frac{d}{dt}\left(<\epsilon u^\epsilon_t,\phi> + <\gamma(u^\epsilon),\phi>_{-1,1}\right) + <L(u^\epsilon),\phi>_{-1,1} = <h^\epsilon,\phi>_{-1,1} \tag{4.8}$$

a.e. $t\in(0,T)$. Now, take $\psi\in C^\infty[0,T]$ such that $\psi(T)=\psi'(T)=0$, then from (4.8) we get

$$\int_0^T \frac{d}{dt}\left(<\epsilon u^\epsilon_t + \gamma(u^\epsilon),\phi>_{-1,1}\right)\psi(s)\,ds + \int_0^T <L(u^\epsilon),\phi>_{-1,1}\psi(s)\,ds = \int_0^T <h^\epsilon,\phi>_{-1,1}\psi(s)\,ds \tag{4.9}$$

From the assumptions on $h^\epsilon$ note that either $h^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$, with $\frac{1}{\sqrt{\epsilon}}f^\epsilon = O(1)$, in $L^1(0,T,L^2(\Omega))$ and either $g^\epsilon = O(1)$ in $L^2(\Sigma_1)$ or in $W^{1,1}(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$ or $h^\epsilon = O(1)$ in $W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$ or $h^\epsilon = h^\epsilon_1 + h^\epsilon_2$ with $h^\epsilon_1 = O(1)$ in $L^2(\Sigma_1)$ and $h^\epsilon_2 = O(1)$ in $W^{1,1}(0,T,H^{-1}_0(\Omega))$. Therefore in either case, by compactness, there exists $h\in L^1(0,T,H^{-1}_{\Gamma_0}(\Omega))$ such that

$$\int_0^T <h^\epsilon,\phi>_{-1,1}\psi(s)\,ds \to \int_0^T <h,\phi>_{-1,1}\psi(s)\,ds$$

where $h$ is, respectively, of the form $h = g_\Gamma$, with $g$ in $L^2(\Sigma_1)$ or in $C([0,T], H^{-1/2}_{\Gamma_0}(\Gamma))$, or $h \in C([0,T], H^{-1}_{\Gamma_0}(\Omega))$ or $h = h_1 + h_2$ with $h_1 \in L^2(\Sigma_1)$ and $h_2 \in C([0,T], H^{-1}_0(\Omega))$.

Now we manipulate the term

$$\int_0^T \frac{d}{dt}\,(< \epsilon u^\epsilon_t + \gamma(u^\epsilon), \phi >_{-1,1})\, \psi(s)\, ds \tag{4.10}$$

From the absolute continuity, and integrating by parts, we get that (4.10) equals

$$< \epsilon u^\epsilon_t + \gamma(u^\epsilon), \phi >_{-1,1} \psi(s)]^{s=T}_{s=0} - \int_0^T < \epsilon u^\epsilon_t + \gamma(u^\epsilon), \phi >_{-1,1} \psi'(s)\, ds$$

and integrating again by parts the term containing $u^\epsilon_t$, using

$$\int_0^T < \epsilon u^\epsilon_t, \phi >_{-1,1} \psi'(s)\, ds = \int_0^T \frac{d}{dt} < \epsilon u^\epsilon, \phi >_{-1,1} \psi'(s)\, ds$$

we get that (4.10) equals

$$\begin{aligned} &< \epsilon u^\epsilon_t + \gamma(u^\epsilon), \phi >_{-1,1} \psi(s)]^{s=T}_{s=0} - < \epsilon u^\epsilon, \phi >_{-1,1} \psi'(s)]^{s=T}_{s=0} + \\ &+ \int_0^T < \epsilon u^\epsilon, \phi >_{-1,1} \psi''(s)\, ds - \int_0^T < \gamma(u^\epsilon), \phi >_{-1,1} \psi'(s)\, ds \end{aligned} \tag{4.11}$$

On the other hand, since $\gamma(u^\epsilon)_t \in L^2(\Sigma_1)$, we can rewrite (4.10) as

$$\frac{d}{dt}\,(< \epsilon u^\epsilon_t, \phi > + < \gamma(u^\epsilon), \phi >_{-1,1}) = \frac{d}{dt} < \epsilon u^\epsilon_t, \phi > + < \gamma(u^\epsilon)_t, \phi >_{-1,1}$$

and integrating again by parts the term containing $u^\epsilon_t$ we get that (4.10) equals

$$\begin{aligned} &< \epsilon u^\epsilon_t, \phi >_{-1,1} \psi(s)]^{s=T}_{s=0} - < \epsilon u^\epsilon, \phi >_{-1,1} \psi'(s)]^{s=T}_{s=0} + \\ &+ \int_0^T < \epsilon u^\epsilon, \phi >_{-1,1} \psi''(s)\, ds + \int_0^T < \gamma(u^\epsilon)_t, \phi >_{-1,1} \psi(s)\, ds \end{aligned} \tag{4.12}$$

Using (4.11) in (4.9) and passing to the limit, we get

$$\begin{aligned} &- \int_0^T < \gamma(u), \phi >_{-1,1} \psi'(s)\, ds - < \gamma(u_0), \phi >_{-1,1} \psi(0) + \\ &+ \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < h, \phi >_{-1,1} \psi(s)\, ds \end{aligned} \tag{4.13}$$

and, by integration by parts, that equals

$$\begin{aligned} &\int_0^T \frac{d}{dt} < \gamma(u), \phi >_{-1,1} \psi(s)\, ds + < \gamma(u(0)) - \gamma(u_0), \phi >_{-1,1} \psi(0) + \\ &+ \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < h, \phi >_{-1,1} \psi(s)\, ds \end{aligned} \tag{4.14}$$

On the other hand, using (4.12) in (4.9) and passing to the limit, we get

$$\int_0^T \frac{d}{dt} < \gamma(u), \phi >_{-1,1} \psi(s)\, ds + \int_0^T < L(u), \phi >_{-1,1} \psi(s)\, ds = \int_0^T < h, \phi >_{-1,1} \psi(s)\, ds \tag{4.15}$$

i.e. $\gamma(u)_t + L(u) = h$.

By comparing (4.14) and (4.15) we get $\gamma(u(0)) = \gamma(u_0)$ and the Theorem in proved.

Note that when passing to the limit we have used the convergence of $u^\epsilon$ to $u$ and the fact that $\epsilon v^\epsilon_0 \to 0$ in $L^2(\Omega)$. Also, from the regularity of the limiting function $h$, note that the initial data for $u$, i.e. $u(0) = B(\gamma(u_0)) + D_0(h_2(0))$ is well defined in $H^1_{\Gamma_0}(\Omega)$. □

4.2.2. *Strong convergence for $p < \infty$.* We will show now that if $h^\epsilon$ converges strongly then we have strong convergence in $L^2(0,T,H^1_{\Gamma_0}(\Omega))$. To be more precise, we will prove the following

**Theorem 4.2.** *With the assumptions above, assume either*
*i)* $h^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$, *with*

$$f^\epsilon = o(\sqrt{\epsilon}) \quad \text{in } L^1(0,T,L^2(\Omega))$$

*and either*

$$g^\epsilon \to g \quad \text{in } L^2(\Sigma_1) \text{ or in } W^{1,1}(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$$

*ii)*

$$h^\epsilon \to h \quad \text{in } W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$$

*iii)* $h^\epsilon = h^\epsilon_1 + h^\epsilon_2$ *with*

$$h^\epsilon_1 \to h_1 \quad \text{in } L^2(\Sigma_1)$$

*and*

$$h^\epsilon_2 \to h_2 \quad \text{in } W^{1,1}(0,T,H^{-1}_0(\Omega))$$

*Also assume*

$$\|u^\epsilon_0\|_1 \longrightarrow \|u(0)\|_1$$

*where* $u(0) = B(\gamma(u_0)) + D_0(h_2(0))$ *is the initial data for the limiting problem,* $\gamma(u)_t + L(u) = h$. *Then*

$$u^\epsilon \to u \quad \text{in } L^2(0,T,H^1_{\Gamma_0}(\Omega))$$

$$\sqrt{\epsilon}u^\epsilon_t \to 0 \quad \text{in } L^2(Q)$$

$$\gamma(u^\epsilon)_t \to \gamma(u)_t \quad \text{in } L^2(\Sigma_1)$$

**Proof** Since $w-*$ convergence in $L^\infty$ implies $w$ convergence in $L^2$, then it suffices to prove convergence of the norms to have strong convergence in the latter space, i,.e.

$$\|u^\epsilon\|^2_{L^2(H^1_{\Gamma_0}(\Omega))} = \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1} \to \int_0^T < L(u), u >_{-1,1} = \|u\|^2_{L^2(H^1_{\Gamma_0}(\Omega))}$$

Also, from the weak convergence and lower semicontinuity we have

$$\int_0^T < L(u), u >_{-1,1} \le \liminf_\epsilon \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1}$$

and $\int_{\Sigma_{1,t}} |\gamma(u)_t|^2 \le \liminf_\epsilon \int_{\Sigma_{1,t}} |\gamma(u^\epsilon)_t|^2$. Finally note that the energy estimates for $(u^\epsilon, u^\epsilon_t)$ and $u$ were obtained from (2.8), i.e. $E_\epsilon(u,u_t) + 2\int_{\Sigma_{1,t}} u_t^2 = E_\epsilon(u_0,v_0) + 2\int_0^t < h, u_t >_{-1,1}$ and $\|\nabla u\|^2_{L^2} + \lambda\|u\|^2_{L^2} + 2\int_{\Sigma_{1,t}} \gamma(u)_t^2 = \|\nabla u_0\|^2_{L^2} + \lambda\|u_0\|^2_{L^2} + 2\int_0^t < h, u_t >_{-1,1}$ which only hold for smooth solutions. Hence, for handling more general functions $h$ we had to manipulate the term $\int_0^t < h, u_t >_{-1,1}$ in a suitable way. That approach gave us the equalities (2.7), (2.23), (2.26) and (2.28) for the hyperbolic problem and (3.15) and (3.16) for the parabolic one. Also note that in any case for $h^\epsilon$, the limiting $h$ is such that Proposition 3.2 applies.

In order to simplify the notations we shall keep on using the notation $\int_0^t < h, u_t >_{-1,1}$ to denote any of the expressions appearing in the equations mentioned above.

Therefore, integrating on $(0,T)$ we get, respectively

$$\int_0^T \epsilon\|u^\epsilon_t\|^2 + \int_0^T < L(u^\epsilon), u^\epsilon >_{-1,1} + 2\int_0^T \int_{\Sigma_{1,t}} |\gamma(u^\epsilon)_t|^2 = TE_\epsilon(u^\epsilon_0, v^\epsilon_0) + 2\int_0^T \int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} \tag{4.16}$$

and

$$\int_0^T < L(u), u >_{-1,1} + 2\int_0^T \int_{\Sigma_{1,t}} |\gamma(u)_t|^2 = T < L(u(0)), u(0) >_{-1,1} + 2\int_0^T \int_0^t < h, u_t >_{-1,1} \tag{4.17}$$

Note that the $\liminf_\epsilon$ of the left hand side of (4.16) is greater or equal than the left hand side of (4.17). From the hypotheses we have $\epsilon v^\epsilon_0 \to 0$ in $L^2(\Omega)$ and $\|u^\epsilon_0\|_1 \to \|u(0)\|_1$ and from the assumptions on $h^\epsilon$ we claim that

$$\int_0^T \int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} \to \int_0^T \int_0^t < h, u_t >_{-1,1}$$

Assumed this for a moment, we get that the right hand side of (4.16) converges to that of (4.17) and therefore the same must happen with the left hand side. Therefore we get $\sqrt{\epsilon}u^\epsilon_t \to 0$ in $L^2(Q)$, $u^\epsilon \to u$ in $L^2(0,T,H^1_{\Gamma_0}(\Omega))$ and $\int_0^T \int_{\Sigma_{1,t}} |\gamma(u^\epsilon)_t|^2 \to \int_0^T \int_{\Sigma_{1,t}} |\gamma(u)_t|^2$. But since for a.e. $t \in (0,T)$ we have $\int_{\Sigma_{1,t}} |\gamma(u)_t|^2 \leq \liminf_\epsilon \int_{\Sigma_{1,t}} |\gamma(u^\epsilon)_t|^2$, then we conclude $\gamma(u^\epsilon)_t \to \gamma(u)_t$ $L^2(\Sigma_1)$.

Therefore to have the theorem proved it only remains to prove the claim and we are going to do this in several cases for the function $h$.

ia) If $h^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$, with $f^\epsilon = o(\sqrt{\epsilon})$ in $L^1(0,T,L^2(\Omega))$ and $g^\epsilon \to g$ in $L^2(\Sigma_1)$, then

$$\int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} = \int_0^t < f^\epsilon, u^\epsilon_t > + \int_0^t < g^\epsilon, \gamma(u^\epsilon)_t >$$

Since $f^\epsilon = o(\sqrt{\epsilon})$ in $L^1(0,T,L^2(\Omega))$ and $\sqrt{\epsilon}u^\epsilon_t = O(1)$ in $L^\infty(0,T,L^2(\Omega))$, the first term goes to zero for every $t \in (0,T)$. On the other hand, since $g^\epsilon \to g$ in $L^2(\Sigma_1)$ and $\gamma(u^\epsilon)_t \to \gamma(u)_t$ weakly in $L^2(\Sigma_1)$ then the second term converges to $\int_0^t < g, \gamma(u)_t >$ and $\int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1}$ converges to $\int_0^t < h, u_t >_{-1,1}$ for every $t \in (0,T)$, where $h = g$. Finally, $\left|\int_0^t < f^\epsilon, u^\epsilon_t >\right|$ and $\left|\int_0^t < g^\epsilon, \gamma(u^\epsilon)_t >\right|$ are uniformly bounded in $(0,T)$, and from Lebesgue's dominated convergence theorem we get the result.

ib) If $h^\epsilon$ is as before but $g^\epsilon \to g$ in $W^{1,1}(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$ we have

$$\int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} = \int_0^t < f^\epsilon, u^\epsilon_t > + < g^\epsilon(t), \gamma u^\epsilon(t) >_{-1/2,1/2} -$$
$$- < g^\epsilon(0), \gamma u^\epsilon_0 >_{-1/2,1/2} - \int_0^t < g^\epsilon_t, \gamma(u^\epsilon) >_{-1/2,1/2}$$

The first term is treated as before, while in the second it is clear that we can pass to the limit in the integral term obtaining $-\int_0^t < g_t, \gamma(u) >_{-1/2,1/2}$. Again Lebesgue's theorem implies that $-\int_0^T \int_0^t < g^\epsilon_t, \gamma(u^\epsilon) >_{-1/2,1/2}$ converges to $-\int_0^T \int_0^t < g_t, \gamma(u) >_{-1/2,1/2}$.

For the nonintegral terms, after integrating in $(0,T)$ we get $\int_0^T < g^\epsilon(t), \gamma u^\epsilon(t) >_{-1/2,1/2} \, dt - T < g^\epsilon(0), \gamma u^\epsilon_0 >_{-1/2,1/2}$ and again we can pass to the limit, obtaining $\int_0^T < g(t), \gamma u(t) >_{-1/2,1/2} \, dt - T < g(0), \gamma u_0 >_{-1/2,1/2}$. Therefore the result is proved.

ii) If $h^\epsilon \to h$ in $W^{1,1}(0,T,H^{-1}_{\Gamma_0}(\Omega))$, then

$$\int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} = < h^\epsilon(t), u^\epsilon(t) >_{-1,1} - < h^\epsilon(0), u^\epsilon_0 >_{-1,1} - \int_0^t < h^\epsilon_t, u^\epsilon >_{-1,1}$$

and with a similar argument as above, we have that $\int_0^t < h^\epsilon_t, u^\epsilon >_{-1,1}$ converges to $\int_0^t < h_t, u >_{-1,1}$ and are uniformly bounded on $(0,T)$. Also $\int_0^T < h^\epsilon(t), u^\epsilon(t) >_{-1,1} \, dt - T < h^\epsilon(0), u^\epsilon_0 >_{-1,1}$ converges to $\int_0^T < h(t), u(t) >_{-1,1} \, dt - T < h(0), u_0 >_{-1,1}$ again Lebesgue's theorem concludes.

iii) Finally, if $h^\epsilon = h^\epsilon_1 + h^\epsilon_2$ with $h^\epsilon_1 \to h_1$ in $L^2(\Sigma_1)$ and $h^\epsilon_2 \to h_2$ in $W^{1,1}(0,T,H_0^{-1}(\Omega))$, then

$$\int_0^t < h^\epsilon, u^\epsilon_t >_{-1,1} = \int_{\Sigma_{1,t}} h^\epsilon_1 \gamma(u^\epsilon)_t + < (I - (B\gamma)^*)h^\epsilon_2(t), u^\epsilon(t) >_{-1,1} -$$
$$- < (I - (B\gamma)^*)h^\epsilon_2(0), u^\epsilon_0 >_{-1,1} - \int_0^t < (I - (B\gamma)^*)h^\epsilon_{2t}, u^\epsilon >_{-1,1}$$

and we conclude as above.□

As a consequence, we have

**Corollary 4.1.** *Under the hypotheses of the Theorem, for every* $2 \leq p < \infty$

$$u^\epsilon \to u \quad \textit{in } L^p(0,T,H^1_{\Gamma_0}(\Omega))$$

**Proof** By interpolation, we have for every $2 \leq p < \infty$ and $X = H^1_{\Gamma_0}(\Omega)$, $\|u\|_{L^p(X)} \leq \|u\|^{2/p}_{L^2(X)} \|u\|^{1-2/p}_{L^\infty(X)}$ and applying this inequality to $u^\epsilon - u$ and using the convergence in $L^2(X)$ and the boundedness in $L^\infty(X)$, we get the result.□

**Remark 4.2.** *Note that if $u_0^\epsilon \to u_0$ in $H^1_{\Gamma_0}(\Omega)$ and $u_0$ verifies*

$$-\Delta u_0 + \lambda u_0 = h_2(0)$$

*that is $u_0 = B(\gamma(u_0)) + D_0(h_2(0))$ then the condition on the initial data in the Theorem is verified.*

4.3. **Uniform convergence in time.** We will show now conditions ensuring that the convergence

$$u^\epsilon \to u$$

is uniform in time. For this the assumptions on $h^\epsilon$, $h^\epsilon \to h$, $u_0^\epsilon \to u_0$ weakly in $H^1_{\Gamma_0}(\Omega)$, $E_\epsilon(u_0^\epsilon, v_0^\epsilon) = 0(1)$ of the previous section will be somewhat strengthened.

We first outline the main idea that will drive us in order to show uniform convergence of solutions. Let $u^\epsilon$ be the solution of

$$(\epsilon u_t^\epsilon + \gamma(u^\epsilon))_t + L(u^\epsilon) = h^\epsilon \tag{4.18}$$

with initial data $u^\epsilon(0) = u_0^\epsilon$, $u_t^\epsilon(0) = v_0^\epsilon$ and $u$ the solution of

$$\gamma(u)_t + L(u) = h \tag{4.19}$$

with initial data $\gamma(u(0)) = \gamma(u_0)$. Recall that $u(0) = B(\gamma(u_0)) + D_0(h_2(0))$ and both equations are verified in $H^{-1}_{\Gamma_0}(\Omega)$.

On the one hand, we try the following approach. Denoting $w^\epsilon = u^\epsilon - u$ and assuming $u_{tt}$ exists at least in $H^{-1}_{\Gamma_0}(\Omega)$, we can write the equation for $u$ as $(\epsilon u_t + \gamma(u))_t + L(u) = h + \epsilon u_{tt}$ and therefore, for $w^\epsilon = u^\epsilon - u$, we have

$$(\epsilon w_t^\epsilon + \gamma(w^\epsilon))_t + L(w^\epsilon) = (h^\epsilon - h) - \epsilon u_{tt} \tag{4.20}$$

Now, if the energy of the initial data for that equation and the right hand side are small, we may try to prove that $w^\epsilon$ becomes small in some sense, see (4.4).

On the other hand, since $u_{tt}^\epsilon$ exists at least in $H^{-1}_{\Gamma_0}(\Omega)$, we have

$$\gamma(w^\epsilon)_t + L(w^\epsilon) = (h^\epsilon - h) - \epsilon u_{tt}^\epsilon \tag{4.21}$$

and then if $\epsilon u_{tt}^\epsilon$ is small, see (4.7), and if $h^\epsilon$ converges to $h$ in some sense, we may try to prove that $w^\epsilon$ becomes small.

Note that these two different strategies require different assumptions. The first one will require $h$ to be smooth enough to guarantee that $(u, u_t)$ lies in the energy space $E$. Also some compatibility on the initial data for (4.19) will be needed. On the other hand, the second approach will require $h^\epsilon$ to be smooth and compatibility conditions on the initial data of (4.18) to guarantee that $\epsilon u_{tt}^\epsilon$ is small. Also, in either case, we have to make sure that all the previous formal computations are rigurous.

The following result is an extension of Corollary 3.3 and gives sufficient conditions for $(u, u_t)$ to be in $C([0,T], E)$. As we shall see, when the partition of the boundary is regular, we have to impose less restrictive assumptions on the data.

**Proposition 4.3.** *Assume the partition of $\Gamma$ is regular and $h(t) = h_1(t) + h_2(t) \in H^{-1}_{\Gamma_0}(\Omega)$ is given a.e. $t \in (0,T)$, $h_1 \in L^2(\Sigma_1)$ and $h_2 \in C([0,T], H^{-1}_0(\Omega))$ and let $u$ be a given solution of (3.7) with initial data $u_0 \in H^1_{\Gamma_0}(\Omega)$.*
*i) If*

$$D_0(h_2)_t \in C([0,T], L^2(\Omega)) \tag{4.22}$$

*and*

$$h_{1t} \in L^2(0,T,X^{-1}) \tag{4.23}$$

*and $u_0 \in H^1_{\Gamma_0}(\Omega)$ is such that*

$$-\Delta u_0 + \lambda u_0 = h_2(0) \tag{4.24}$$

*on $\Omega$, i.e. $u_0 = B(\gamma(u_0)) + D_0(h_2(0))$, then*

$$(u, u_t) \in C([0,T], E)$$

*and $\gamma(u) \in L^2(0,T,X^1)$, $\gamma(u)_t \in L^2(\Sigma_1)$, $\gamma(u)_{tt} \in L^2(0,T,X^{-1})$.*

*ii) If moreover*

$$D_0(h_2)_{tt} \in L^1(0,T,L^2(\Omega)) \tag{4.25}$$

*and*

$$h_{1t} \in L^2(0,T,H_{\Gamma_0}^{-1/2}(\Gamma)) \tag{4.26}$$

*and* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *verifies*

$$L(u_0) - h(0) \in L^2_{\Gamma_0}(\Gamma) \tag{4.27}$$

*then*

$$u_{tt} \in L^1(0,T,L^2(\Omega))$$

*and* $\gamma(u)_t \in C([0,T],L^2_{\Gamma_0}(\Gamma)) \cap L^2(0,T,H^{1/2}_{\Gamma_0}(\Gamma))$, $\gamma(u)_{tt} \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$.

*When the partition of the boundary is not regular, we get the same conclusion as in i) provided (4.26), (4.27) hold and* $h_1 \in C([0,T],L^2_{\Gamma_0}(\Gamma))$ *and the same as in ii) provided* $h_1 \in C([0,T],H^{1/2}_{\Gamma_0}(\Gamma))$, $h_{1t} \in L^2(0,T,L^2_{\Gamma_0}(\Gamma))$ *and* $L(u_0) - h(0) \in H^{1/2}_{\Gamma_0}(\Gamma)$.

**Proof** Since the hypotheses imply in particular those of Corollary 3.3, then $u \in C([0,T), H^1_{\Gamma_0}(\Omega))$, $u(0) = u_0$ and $\gamma(u)_t \in L^2(\Sigma_1)$. Recalling (3.9) and (3.10), $u$ verifies $u(t) = D(v(t),h_2(t)) = B(v(t)) + D_0(h_2(t))$ on $\Omega$ and $v(t) = e^{-A_0 t}v_0 + \int_0^t e^{-A_0(t-s)}h_1(s)\,ds$ on $\Gamma$.

Therefore, from the properties of the operator $B$, see Remark 1.2, to have $u_t = B(v_t) + D_0(h_2)_t \in C([0,T],L^2(\Omega))$ and since by hypothesis $D_0(h_2)_t \in C([0,T],L^2(\Omega))$, on $\Gamma$ at least we need $v_t \in C([0,T],H_{\Gamma_0}^{-1/2}$ Note that $h_1$ verifies $h_1 \in L^2(0,T,L^2_{\Gamma_0}(\Gamma))$ and $h_{1t} \in L^2(0,T,X^{-1})$ and then, by interpolation, $h \in C([0,T],X^{-1/2}) = C([0,T],H^{-1/2}_{\Gamma_0}(\Gamma))$, [19]. But, $v_t(0) = h_1(0) - A_0\gamma(u_0) \in H^{-1/2}_{\Gamma_0}(\Gamma)$, this together with $h_{1t} \in L^2(0,T,X^{-1})$ gives, by Theorem 3.2, $v_t \in C([0,T],H^{-1/2}_{\Gamma_0}(\Gamma))$ and $v_{tt} \in L^2(0,T,X^{-1})$. Therefore, i) is proved. Note that on $\Omega$, $u_t(0) = B(v_t(0)) + D_0(h_2)_t(0) \in L^2(\Omega)$.

Moreover, to have $u_{tt} = B(v_{tt}) + D_0(h_2)_{tt} \in L^1(0,T,L^2(\Omega))$ and since $D_0(h_2)_{tt} \in L^1(0,T,L^2(\Omega))$, it will suffice to have on $\Gamma$ $v_{tt} \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$. But, $v_t(0) = h_1(0) - A_0\gamma(u_0)$ and from (4.27) we have $L(u_0) = h(0) + z_0$ for some $z_0 \in L^2_{\Gamma_0}(\Gamma)$ and then from point ii) in Proposition 1.1 we get $A_0\gamma(u_0) = h_1(0) + z_0$, i.e. $v_t(0) \in L^2_{\Gamma_0}(\Gamma)$. This together with $h_{1t} \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$ give us, by Theorem 3.2, $v_t \in C([0,T],L^2_{\Gamma_0}(\Gamma)) \cap L^2(0,T,H^{1/2}_{\Gamma_0}(\Gamma))$ and $v_{tt} \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$.

When the partition of the boundary is not regular we need to ensure that $v_t \in C([0,T],L^2_{\Gamma_0}(\Gamma))$ and this holds provided $h_1 \in C([0,T],L^2_{\Gamma_0}(\Gamma))$, $h_{1t} \in L^2(0,T,H^{-1/2}_{\Gamma_0}(\Gamma))$ and $v_t(0) \in L^2_{\Gamma_0}(\Gamma)$. On the other hand, we can ensure $v_{tt} \in L^2(0,T,L^2_{\Gamma_0}(\Gamma))$ provided $h_1 \in C([0,T],H^{1/2}_{\Gamma_0}(\Gamma))$, $h_{1t} \in L^2(0,T,L^2_{\Gamma_0}(\Gamma))$ and $v_t(0) \in H^{1/2}_{\Gamma_0}(\Gamma)$.□

The following result asserts, in particular, that the computation leading to (4.20) is legitimate and that we have energy estimates for the solution of that equation. Assume $h$ satisfies either condition in Theorem 2.3 or Corollaries 2.4, 2.5. Let $u$ be the unique solution of finite energy of the equation

$$(\epsilon u_t + \gamma(u))_t + L(u) = h \tag{4.28}$$

with initial data $(u_0,v_0) \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. Assume $h = H^1 + H^2$ with $H^i$ verifying some conditions to be precised below. Then, we decompose $u = z + w$, where $z$ is a solution of the parabolic problem

$$\gamma(z)_t + L(z) = H^1 \tag{4.29}$$

and $w$ verifies

$$(\epsilon w_t + \gamma(w))_t + L(w) = H^2 - \epsilon z_{tt} \tag{4.30}$$

Note the coupling term $\epsilon z_{tt}$ in (4.30). Then we have the following result

**Proposition 4.4.** *Assume* $H^1$, *and* $z_0 \in H^1_{\Gamma_0}(\Omega)$ *are such that the conclusions of Proposition 4.3 hold true. Assume also that* $H^2$ *satisfies either condition in Theorem 2.3 or Corollaries 2.4, 2.5.*

*Then, for every* $(u_0,v_0) \in E = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *the solution of (2.12) can be split as the sum of the solutions of (4.29) and (4.30).*

**Proof** The proof is almost straightforward. Note that $z(0) = z_0 = B(p_0) + D_0(H^2)(0)$ and $z_t(0) = B(q_0) + D_0(H^2)_t(0)$ for some $p_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$ and $q_0 \in H^{-1/2}_{\Gamma_0}(\Gamma)$. We now solve (4.30) with initial data in $E$, given by $w(0) = u_0 - z(0)$ and $w_t(0) = v_0 - z_t(0)$. Therefore from the hypothesis on $H^2$ and $z_{tt} \in L^1(0, T, L^2(\Omega))$, we have $W = (w, w_t) \in C([0, T], E)$, $\gamma(w)_t \in L^2(\Sigma_1)$.

It only remains to prove that $u = z+w$ is actually the solution of (4.28). Clearly, $U = (u, u_t) \in C([0, T], E)$ and $U(0) = (u_0, v_0) \in E$. Recalling (2.20), $w$ is characterized by

$$\frac{d}{dt}(< \epsilon w, \psi > + < \epsilon w_t + \gamma(w), \phi >_{-1,1}) - < \epsilon w_t + \gamma(w), \psi >_{-1,1} +$$
$$+ < w, -\Delta\phi + \lambda\phi > = < H^2, \phi >_{-1,1} - \epsilon < z_{tt}, \phi >$$

for every $(\phi, \psi) \in D(A_\epsilon)$.

On the other hand, $z$ verifies $\gamma(z)_t + L(z) = H^1$ in $H^{-1}_{\Gamma_0}(\Omega)$. Therefore, if we take a test function $\phi \in H^1_{\Gamma_0}(\Omega)$ such that $(\phi, \psi) \in D(A_\epsilon)$ for some $\psi$, we get $\frac{d}{dt} < \gamma(z), \phi >_{-1,1} + < z, -\Delta\phi + \lambda\phi > - < \gamma(z), \psi >_{-1,1} = < H^1, \phi >_{-1,1}$. Now, since $z_{tt}(t) \in L^2(\Omega)$, we get $\frac{d}{dt}(< \epsilon z, \psi > + < \epsilon z_t, \phi >_{-1,1}) - < \epsilon z_t, \psi >_{-1,1} = \epsilon < z_{tt}, \phi >$ so, adding up

$$\frac{d}{dt}(< \epsilon z, \psi > + < \epsilon z_t + \gamma(z), \phi >_{-1,1}) - < \epsilon z_t + \gamma(z), \psi >_{-1,1} +$$
$$+ < z, -\Delta\phi + \lambda\phi > = < H^1, \phi >_{-1,1} + \epsilon < z_{tt}, \phi >$$

for every $(\phi, \psi) \in D(A_\epsilon)$. Adding this to the condition on $w$ we get that $u = z + w$ verifies (2.20) and the result is proved.□

**Remark 4.3.** *Note that in general the decomposition is not unique since $p_0 \in H^{1/2}_{\Gamma_0}(\Gamma)$ is a parameter and $q_0$ is determined by $p_0$. If we take two different decomposition $u = z^i + w^i$ then $z^i - z^2$ verifies the homogeneous equation (3.2) and therefore it converges exponentially to zero. The same holds then for $w^1 - w^2$.*

**Corollary 4.2.** *Assume $(u^\epsilon_0, v^\epsilon_0) \in E$ are given, $h$ is such that the conclusions of Proposition 4.3 hold true for (4.19) and $h^\epsilon$ and $h^\epsilon - h$ satisfies either condition in Theorem 2.3 or Corollaries 2.4, 2.5. Then the solution of (4.18) can be split as*

$$u^\epsilon = u + w^\epsilon$$

*where $u$ satisfies (4.19) and $w^\epsilon$ verifies (4.20).*

Now we use energy estimates to obtain a first result of uniform convergence.

**Proposition 4.5.** *Assume the hypotheses of Corollary 4.2 hold true, and*

$$u^\epsilon_0 \to u_0 \text{ in } H^1_{\Gamma_0}(\Omega) \quad \sqrt{\epsilon}v^\epsilon_0 \to 0 \text{ in } L^2(\Omega)$$

*where $u_0$ verifies $u_0 = B(\gamma(u_0)) + D_0(h_2(0)) \in H^1_{\Gamma_0}(\Omega)$, i.e.*

$$-\Delta u_0 + \lambda u_0 = h_2(0)$$

*Moreover, assume*

$$h^\epsilon - h$$

*is small in the sense that it verifies the assumptions in point ii) of Proposition 4.1.*

*Then*

$$w^\epsilon, |\nabla w^\epsilon|, \sqrt{\epsilon}w^\epsilon_t = o(1) \tag{4.31}$$

*in $L^2(\Omega)$, uniformly in $[0, T]$ and $\gamma(w)^\epsilon_t = o(1)$ in $L^2(\Sigma_1)$.*

**Proof** From Proposition 4.1 it just remains to check that $E_\epsilon(w^\epsilon(0), w^\epsilon_t(0)) = o(1)$. For this note that $w^\epsilon(0) = u^\epsilon_0 - u(0)$ and $u(0) = u_0$. Therefore this term goes to zero in $H^1_{\Gamma_0}(\Omega)$. On the other hand $w^\epsilon_t(0) = v^\epsilon_0 - u_t(0)$ and therefore $\epsilon\|w^\epsilon_t(0)\|^2_{L^2(\Omega)}$ also goes to zero.□

Now we will exploit the fact that if $\epsilon u^\epsilon_{tt}$ is small, then in (4.21), $w^\epsilon$ becomes small. Note that now we need stronger regularity assumptions on $h^\epsilon$ and stronger compatibility conditions on the initial data, so we can apply Proposition 4.2. That is we assume $h^\epsilon$ is as regular as in Proposition 4.2 and $(u_0^\epsilon, v_0^\epsilon)$ belong to $H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$, and

$$\frac{1}{\sqrt{\epsilon}} \|\gamma(v_0^\epsilon) + L(u_0^\epsilon) - h^\epsilon(0)\|_{L^2(\Omega)} = O(1)$$

Now, observe that since the solution of (4.18) is in $H^1_{\Gamma_0}(\Omega)$, for fixed $t \geq 0$, using the splitting in Proposition 1.1, we can write

$$u^\epsilon = B(\gamma(u^\epsilon)) + u_2^\epsilon$$

while for the solution of (4.19) we have

$$u = B(v) + D_0(h_2)$$

where, as in Corollary 3.3, we assume $h_1 \in L^2(\Sigma_1)$ and $h_2 \in C([0,T], H_0^{-1}(\Omega))$. With this, for $w^\epsilon$ we have

$$w^\epsilon = u^\epsilon - u = B(\gamma(u^\epsilon) - v) + u_2^\epsilon - D_0(h_2) = B(\gamma(w^\epsilon)) + w_2^\epsilon$$

The following result shows that each term is small.

**Theorem 4.3.**
*i) On $\Omega$ we have*

$$w_2^\epsilon = (u_2^\epsilon - D_0(h_2^\epsilon)) + (D_0(h_2^\epsilon) - D_0(h_2))$$

*and*

$$\sup_{[0,T]} \|u_2^\epsilon - D_0(h_2^\epsilon)\|_{H^2(\Omega)} = O(\sqrt{\epsilon}) \tag{4.32}$$

*ii) On $\Gamma$ we have*

$$\gamma(w^\epsilon) = \gamma(u^\epsilon) - v = e^{-A_0 t}\gamma(w^\epsilon(0)) + \int_0^t e^{-A_0(t-s)} \left(h_1^\epsilon(s) - h_1(s) - B^*(\epsilon u_{tt}^\epsilon)\right) ds$$

*and*

$$B^*(\epsilon u_{tt}^\epsilon) = O(\sqrt{\epsilon}) \ \text{in} \ R(H^{1/2}(\Gamma)) \subset L^2_{\Gamma_0}(\Gamma) \subset H^{-1/2}_{\Gamma_0}(\Gamma)$$

*uniformly on $[0,T]$.*

**Proof** The first formula comes just from adding and subtracting $D_0(h_2^\epsilon)$. From (4.18), and taking test functions in $H^1_0(\Omega)$ in (4.18), we get $< L(u^\epsilon), \phi >_{-1,1} = < L_D(u_2^\epsilon), \phi >_{-1,1}$ and

$$< L_D(u_2^\epsilon), \phi >_{-1,1} = < h_2^\epsilon, \phi >_{-1,1} - < \epsilon u_{tt}^\epsilon, \phi >$$

and since $< L_D(D_0(h_2^\epsilon)), \phi >_{-1,1} = < h_2^\epsilon, \phi >_{-1,1}$ we get $u_2^\epsilon - D_0(h_2^\epsilon) \in H^1_0(\Omega)$ and

$$L_D(u_2^\epsilon - D_0(h_2^\epsilon)) = \epsilon u_{tt}^\epsilon = O(\sqrt{\epsilon}) \text{ in } L^2(\Omega)$$

uniformly on $[0,T]$. From the regularity results for the Dirichlet problem we get i).

Now, taking test functions in $H^{1/2}_{\Gamma_0}(\Gamma)$ in (4.18), we get

$$\gamma(u^\epsilon)_t + A_0\gamma(u^\epsilon) = h_1^\epsilon - B^*(\epsilon u_{tt}^\epsilon)$$

That is, $\gamma(u^\epsilon) = e^{-A_0 t}\gamma(u^\epsilon(0)) + \int_0^t e^{-A_0(t-s)} \left(h_1^\epsilon(s) - B^*(\epsilon u_{tt}^\epsilon)\right) ds$ and the rest is obvious, since $v$ is given on $\Gamma$ by (3.10).

Finally, from the estimate in Proposition 4.2 and the properties of $B^*$ stated in Remark 1.2, we have that $B^*(\epsilon u_{tt}^\epsilon) = O(\sqrt{\epsilon})$, uniformly in $R(H^{1/2}(\Gamma)) \subset L^2_{\Gamma_0}(\Omega)$.□

**Corollary 4.3.** *If under the assumptions above*

$$h_1^\epsilon \to h_1 \ \text{in} \ L^2(\Sigma_1), \quad h_2^\epsilon \to h_2 \ \text{in} \ C([0,T], H_0^{-1}(\Omega))$$

$$\gamma(u^\epsilon) \to \gamma(u_0) \ \text{in} \ H^{1/2}_{\Gamma_0}(\Omega)$$

*then, $u^\epsilon \to u$ uniformly in $C([0,T], H^1_{\Gamma_0}(\Omega))$.*

*Moreover, if the partition of $\Gamma$ is regular and*

$$h_1^\epsilon \to h_1 \ \text{in} \ C([0,T], H^{1/2}_{\Gamma_0}(\Omega)), \quad h_2^\epsilon \to h_2 \ \text{in} \ C([0,T], L^2(\Omega))$$

$$\gamma(u^\epsilon) \to \gamma(u_0) \ \text{in} \ H^{3/2}_{\Gamma_0}(\Omega)$$

*then $u^\epsilon \to u$ uniformly in $C([0,T], H^s(\Omega))$ for every $s < 2$.*

**Proof** First note that if $h_2^\epsilon \to h_2$ uniformly in $H_0^{-1}(\Omega)$ (respectively $L^2(\Omega)$), then $D_0(h_2^\epsilon) \to D_0(h_2)$ uniformly in $H_0^1(\Omega)$ (respectively $H^2(\Omega)\cap H_0^1(\Omega)$). This implies that in the Theorem above, $w_2^\epsilon \to 0$ uniformly in $H_0^1(\Omega)$ (respectively $H^2(\Omega)\cap H_0^1(\Omega)$).

On the other hand, note that $B^*(\epsilon u_{tt}^\epsilon)$ is uniformly small in $R(H^{1/2}(\Gamma)) \subset L^2_{\Gamma_0}(\Omega)$, which coincides with $H^{1/2}_{\Gamma_0}(\Omega)$ if the partition is regular. Therefore, if $h_1^\epsilon \to h_1$ in $L^2(\Sigma_1)$ and $\gamma(u^\epsilon) \to \gamma(u_0)$ in $H^{1/2}_{\Gamma_0}(\Omega)$, from point ii) of Proposition 5.5 of the Appendix, we get $\gamma(w^\epsilon) \to 0$ uniformly in $H^{1/2}_{\Gamma_0}(\Omega)$ and then $B(\gamma(w^\epsilon))$ is uniformly small in $H^1_{\Gamma_0}(\Omega)$.

Finally, if the partition is regular and $h_1^\epsilon \to h_1$ uniformly in $H^{1/2}_{\Gamma_0}(\Omega)$ and $\gamma(u^\epsilon) \to \gamma(u_0)$ in $H^{3/2}_{\Gamma_0}(\Omega)$, then from point i) in Proposition 5.5, we get that $\gamma(w^\epsilon) \to 0$ uniformly in $H^s_{\Gamma_0}(\Omega)$, for any $s < 3/2$, and then $B(\gamma(w^\epsilon))$ is uniformly small in $H^s_{\Gamma_0}(\Omega)$, for any $s < 2$.□

## 5. Appendix: Abstract semigroup results

In this section we state some general abstract results used in previous sections. For a general discussion the reader is referred for example to [4, 12, 13, 23].

**Proposition 5.1.**
*i) Assume $(-A, D(A))$ is the generator of a $C_0$ semigroup $S(t)$, in a reflexive Banach space $E$. Then the adjoint operator $(-A^*, D(A^*))$, generates the $C_0$ semigroup $S^*(t)$ in the dual space of $E$, $E'$.*
*ii) In the situation above, if $u_0 \in E$ and $f \in L^1(0,T,E)$, $T < \infty$, then the Variation of Constants Formula*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds \qquad t \in [0,T] \qquad (VCF)$$

*defines a continuous function in $E$, and $u(0) = u_0$. The mapping $(u_0, f) \to u$ is Lipschitz.*
*iii) If moreover $u_0 \in D(A)$ and either $f \in C([0,T], D(A))$ or $f \in C^1([0,T], E)$, then $u$, as defined above, verifies*
*a) $u \in C([0,T], D(A)) \cap C^1([0,T], E)$, $u_t(0) = -Au_0 + f(0)$*
*b) $u_t + Au = f$, for all $t \in [0,T]$*
*c) In case $f \in C^1([0,T], E)$, the time derivative $u_t$ is given by the (VCF)*

$$u_t(t) = S(t)u_t(0) + \int_0^t S(t-s)f_t(s)\,ds \qquad t \in [0,T]$$

*iv) Even more, if $-Au_0 + f(0) \in D(A)$ and $f \in C^2([0,T], E)$, then*
*a) $u_t \in C([0,T], D(A)) \cap C^1([0,T], E)$, $u_{tt}(0) = -A(-Au_0 + f(0)) + f_t(0)$*
*b) $u_{tt} + Au_t = f_t$, for all $t \in [0,T]$*
*c) The second time derivative, $u_{tt}$ is given by the (VCF)*

$$u_{tt}(t) = S(t)u_{tt}(0) + \int_0^t S(t-s)f_{tt}(s)\,ds \qquad t \in [0,T]$$ □

The following result, due to J.Ball, [1], characterizes in what sense the (VCF), verifies the equation $u_t + Au = f$, when $f$ and $u_0$ are not as regular as in point iii) above.

**Proposition 5.2.** *In the situation above, if $u_0 \in E$ and $f \in L^1(0,T,E)$, then the Variation of Constants Formula, (VCF)*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds \qquad t \in [0,T]$$

*is equivalent to: $u \in C([0,T], E)$, $u(0) = u_0$ and for every $\phi \in D(A^*)$, $< u(t), \phi >$ is absolutely continuous in $[0,T]$ and*

$$\frac{d}{dt} < u(t), \phi > + < u(t), A^*\phi > = < f, \phi > \qquad a.e.\ [0,T]$$

*where here $< \cdot, \cdot >$ denotes the duality pairing between $E$ and its dual.* □

**Proposition 5.3.** *Assume $A$ is a positive, selfadjoint linear operator in a Hilbert space $X$ and let $\{X^\alpha\}_\alpha$ be the fractional power spaces associated to $A$. Consider the problem*

$$u_{tt} + \beta u_t + Au = g \tag{5.1}$$

*Then for $g = 0$ and for any $\alpha$, this equation, written as a first order system for $(u, u_t)$, defines a $C_0$ semigroup in $Z_\alpha = X^\alpha \times X^{\alpha-1/2}$, and the domain of the generator is $Z_{\alpha+1/2}$. In particular Proposition 5.1 applies for $g \neq 0$.*□

**Proposition 5.4.** *Consider the problem*

$$\begin{cases} u_t + Au = g(t) \\ u(0) = u_0 \end{cases} \tag{5.2}$$

*where $A$ is a sectorial operator in a Banach space $X$. Let $\{X^\alpha\}_\alpha$ and $\{e^{-At}\}_t$ be, respectively, the fractional power spaces and the semigroup generated by $A$.*

*If $u_0 \in X^\beta$ and $g \in L^1(0, T, X^\beta)$, then (3.5) has a unique "mild solution" $u \in C([0, T], X^\beta)$, given by*

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}g(s)\, ds \tag{5.3}$$

*If moreover $g$ is locally Hölder continuous, of exponent $\theta$, then the solution is a "strict solution", i.e.*

$$u \in C([0, T], X^\beta) \cap C(0, T, X^{\beta+1}) \quad u_t \in C(0, T, X^\gamma)$$

*for any $\gamma < \beta + \theta$ and $u$ verifies the equation in $X^\beta$.*□

**Proposition 5.5.**
*i) Let $X$ be a Banach space and $(A, D(A))$ a sectorial operator in $X$ and consider problem (5.2) with $g \in L^p(0, T, X^\beta)$, $u_0 \in X^\beta$ and $1 \le p \le \infty$, $T < \infty$.*

*Then, the solution verifies*

*a) $u \in C(0, T, X^\gamma)$ for all $\gamma < \beta + 1/p'$. Moreover, if $u_0 \in X^\gamma$, then $u \in C([0, T], X^\gamma)$.*

*b) The mapping*

$$X^\gamma \times L^p(0, T, X^\beta) \ni (u_0, g) \mapsto u \in C([0, T], X^\gamma)$$

*is Lipschitz. Moreover, if $Re(\sigma(A)) > \delta > 0$ then that holds also for $T = \infty$. In that case $u \in C_b([0, \infty), X^\gamma)$.*
*ii) If $X$ is a Hilbert space, and $A$ is an unbounded, positive, selfadjoint linear operator, then the above results hold true for $p = 2$ and $\gamma = \beta + 1/2$ and $T \le \infty$. Moreover, in this case the mapping*

$$(u_0, g) \longmapsto (u, u_t)$$

*is Lipschitz from $X^{\beta+1/2} \times L^2(0, T, X^\beta)$ into $(C([0, T], X^{\beta+1/2}) \cap L^2(0, T, X^{\beta+1})) \times L^2(0, T, X^\beta)$.*

*Even more, in that case $u$ also verifies*

$$u_t + Au = g$$

*a.e. $t \in (0, T)$.*

**Proof**
i) Take $\gamma \ge \beta$, then $\|u(t)\|_\gamma \le \|e^{-At}u_0\|_\gamma + \int_0^t \|e^{-A(t-s)}\|_{\beta,\gamma}\|g(s)\|_\beta$, were $\|e^{-At}\|_{\beta,\gamma}$ denotes the norm in $\mathcal{L}(X^\beta, X^\gamma)$, and then, since $\|e^{-A(t-s)}\|_{\beta,\gamma} \le M(t-s)^{-(\gamma-\beta)}$, on finite time intervals, we have for $\gamma = \beta$ if $p = 1$ or for $\beta \le \gamma < \beta + 1/p'$ if $1 < p < \infty$, $\|u(t)\|_\gamma \le \|e^{-At}u_0\|_\gamma + c(t)\left(\int_0^t \|g\|_\beta^p\right)^{1/p}$, with $c(t) = M\left(\int_0^t (t-s)^{-p'(\gamma-\beta)}\, ds\right)^{1/p'} \approx t^{1/p'-(\gamma-\beta)}$, so it is bounded on finite intervals and therefore $u(t) \in X^\gamma$ for $t > 0$.

To prove continuity, fix $t > 0$ (or even $t = 0$ if $u_0 \in X^\gamma$) and then

$$\|u(t+h) - u(t)\|_\gamma \le \|(e^{-Ah} - I)u(t)\|_\gamma + \int_t^{t+h} \|e^{-A(t+h-s)}\|_{\beta,\gamma}\|g(s)\|_\beta\, ds$$

The first term on the right hand side above goes to zero with $h$, since the linear semigroup is continuous, while the second can be bounded by $M\left(\int_t^{t+h}(t+h-s)^{-p'(\gamma-\beta)}\, ds\right)^{1/p'}\left(\int_t^{t+h}\|g\|_\beta^p\right)^{1/p} = o(h^{1/p'-(\gamma-\beta)})$ and continuity follows.

Moreover, if $u_0 \in X^\gamma$, we have $\|u\|_{C([0,T],X^\gamma)} \leq c(T)\left(\|u_0\|_\gamma + \|g\|_{L^p(0,T,X^\beta)}\right)$ and this proves the lipschitzness of the mapping $(u_0, g) \mapsto u$.

Finally, if $Re(\sigma(A)) > \delta > 0$ then we have a bound of the type $\|e^{-A(t-s)}\|_{\beta,\gamma} \leq Me^{-\delta(t-s)}(t-s)^{-(\gamma-\beta)}$, for $0 < s < t < \infty$, and the above estimates hold for $T = \infty$, since $c(t)$ remains bounded. In any case note that $c(t) \to 0$ as $t \to 0$.

The proofs for $p = \infty$ follows the same lines, with obvious modifications, and are therefore omited.

ii) Under these hypotheses, $A$ is in particular a sectorial operator and the above applies. Hence, if $g \in L^2(0,T,X^\beta)$ and $u_0 \in X^\gamma$, then $u \in C([0,T],X^\gamma)$ for all $\gamma < \beta + 1/2$.

Now, take $u_0 \in X^{\beta+1/2}$ and $g$ a sufficiently regular function, say $C^1([0,T],X^\beta)$, then from the regularity results for the evolution problem, we get $u \in C([0,T],X^{\beta+1/2}) \cap C(0,T,X^{\beta+1})$ and $u_t \in C(0,T,X^\gamma)$ for all $\gamma < \beta+1$, [6, 12].

Multiplying the equation by $u$ in $X^\beta$, using the inclusion $X^{\beta+1} \subset X^\beta$ and integrating in time we get

$$\|u\|^2_{C([0,T],X^\beta)} + c_1\|u\|^2_{L^2(0,T,X^{\beta+1/2})} \leq \|u_0\|^2_\beta + c_2\|g\|^2_{L^2(0,T,X^\beta)}$$

Now we multiply the equation by $u_t$ in $X^\beta$, use the Cauchy-Schwarz inequality and integrate in time to obtain

$$\|u\|^2_{C([0,T],X^{\beta+1/2})} + \|u_t\|^2_{L^2(0,T,X^\beta)} \leq \|u_0\|^2_{\beta+1/2} + \|g\|^2_{L^2(0,T,X^\beta)}$$

Finally, we multiply the equation by $Au$ in $X^\beta$ and as above, we get

$$\|u\|^2_{C([0,T],X^{\beta+1/2})} + \|u\|^2_{L^2(0,T,X^{\beta+1})} \leq \|u_0\|^2_{\beta+1/2} + \|g\|^2_{L^2(0,T,X^\beta)}$$

therefore, the mapping $(u_0, g) \mapsto (u, u_t)$ is Lipschitz on those spaces. By density, we get the result for an arbitrary $g \in L^2(0,T,X^\beta)$. Note that, by taking subsequences, we can also pass to the limit in the equation $u_t + Au = g$, a.e. $t \in (0,T)$.□

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