# Averaged controllability for Random Evolution Partial Differential Equations

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**Abstract**

We analyze the averaged controllability properties of random evolution Partial Differential Equations.

We mainly consider heat and Schrödinger equations with random parameters, although the problem is also formulated in an abstract frame.

We show that the averages of parabolic equations lead to parabolic-like dynamics that enjoy the null-controllability properties of solutions of heat equations in an arbitrarily short time and from arbitrary measurable sets of positive measure.

In the case of Schrödinger equations we show that, depending on the probability density governing the random parameter, the average may behave either as a conservative or a parabolic-like evolution, leading to controllability properties, in average, of very different kind.

**Key Words**. random evolution Partial Differential Equations, averaged controllability, averaged observability, Schrödinger equation, heat equation.

## 1 Introduction

We analyze the problem of controlling systems with randomly depending coefficients in the context of evolution Partial Differential Equations (PDEs). More precisely, we consider the problem of averaged controllability which consists, roughly, of controlling the averaged dynamics, with respect to the random parameters. This problem was introduced and solved in [48] in the context of finite dimensional systems, where the same issue was also formulated for PDE.

The motivation for considering averaged versions of the classical control theoretical problems is as follows. When the dynamics of the state is governed by a pair of random operators (determining the free dynamics and the control operators, respectively), the effective value of the random parameters is unknown, we aim at choosing a control, independent of the

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unknown parameters, to perform optimally in an averaged sense. This amounts to controlling the mathematical expectation of the solutions, making a robust compromise of all the possible realizations of the system for the various possible values of the random parameters.

In this work we consider mainly the heat and Schrödinger equations, with diffusivity and dispersivity operators depending on a random variable in a multiplicative manner. We show that, while the average of heat equations leads also to a heat-like dynamics, the behavior of the averages for the Schrödinger equations depends in a very sensitive manner on the density of probability of the random variable so that, in some cases, by averaging, this leads to a dynamics of conservative nature, similar to the original Schrödinger equation under consideration and, in others, to a parabolic-like behavior.

Our method of proof combines using Fourier decomposition methods to identify the averaged dynamics to, later, utilizing the existing tools developed for the controllability of parabolic and conservative systems, to deduce the averaged controllability results. When the resulting averaged dynamics is of parabolic nature, averaged null controllability is proved for arbitrarily short time intervals and from measurable sets of positive measure. On the contrary, when the averaged dynamics is of conservative type, averaged exact controllability is proved under suitable geometric conditions on the support of the controls that are by now well known in the context of wave-like and Schrödinger-like equations.

In order to illustrate the effect of averaging and how it may change the dynamics of the original system, let us consider the simplest transport equation

$$\begin{cases} y_t + \alpha \cdot \nabla y = 0 & \text{in } \mathbb{R}^d \times [0,\infty), \\ y(0) = y_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here $y_0 \in L^2(\mathbb{R}^d)$ and $\alpha(\cdot) : \Omega \to \mathbb{R}^d$ is a $d$-dimensional standard normally distributed random variable, with the probability density

$$\rho(\alpha) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|\alpha|^2}{2}} \text{ for } \alpha \in \mathbb{R}^d.$$

The solution to (1.1) reads

$$y(x,t,\omega; y_0) = y_0(x - t\alpha) \text{ for } (x,t) \in \mathbb{R}^d \times [0,\infty).$$

Then, the mathematical expectation or averaged state

$$\begin{aligned} \tilde{y}(x,t) &\triangleq \int_\Omega y(x,t,\omega;y_0) d\mathbb{P}(\omega) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} y_0(x - \alpha t) e^{-\frac{|\alpha|^2}{2}} d\alpha \\ &= \frac{1}{(2\pi)^{\frac{d}{2}} t^d} \int_{\mathbb{R}^d} y_0(z) e^{-\frac{(x-z)^2}{2t^2}} dz \end{aligned}$$

solves the following heat equation

$$\begin{cases} \tilde{y}_t - \frac{1}{t} \Delta \tilde{y} = 0 & \text{in } \mathbb{R}^d \times [0,\infty), \\ \tilde{y}(0) = y_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.2}$$

namely, $\varphi(x,t) = \tilde{y}(x, \sqrt{2t})$ solves

$$\begin{cases} \varphi_t - \Delta\varphi = 0 & \text{in } \mathbb{R}^d \times [0,\infty), \\ \varphi(0) = y_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1.3}$$

The fact that averages of transport-like equations may enjoy enhanced regularity properties, first discovered in [1], is well known in different contexts. Its final form was established in [17] and later employed to study nonlinear transport equations in [4, 9, 10, 32].

**Remark 1.1** *The above computation shows that one can get diffusion processes by averaging a simple random convection process with respect to its velocity. This is also well known from a different perspective, in the context of chaotic and stiff oscillatory systems, that can be regarded as the characteristic systems of transport equations (see [8, 15, 37]).*

This example shows that, by averaging, the solution of a random transport equation may lead to a solution of a heat-like equation and, consequently, that time-reversible systems may become strongly irreversible through averaging. Furthermore, this occurs with the normally distributed random variable which is ubiquitous in nature due to the central limit theorem, which states that the mean of many independent random variables drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution. Accordingly, in the real world, physical quantities that are expected to be the sum of many independent variables, such as measurement errors, often have a distribution very close to the normal distribution (see [7]).

The "smoothing by averaging" effect mentioned above has important consequences from a control theoretical point of view as well. Indeed, while the transport equation (1.1), for a given value of the random variable $\alpha$ (which determines the velocity of propagation of waves), enjoys the property of exact controllability in finite time, proportional to the travel time of characteristics to get to the control set (the boundary or an open subset of the domain), the averaged heat dynamics is controllable to zero (or any other sufficiently smooth target) in an arbitrarily short time and from any subset of the domain where the dynamics evolves, without any geometric condition on the support of the controls, involving the propagation of characteristics. Accordingly, through averaging, we encounter on a single model, with randomly depending coefficients, the classical dichotomy arising in the context of controllability of hyperbolic versus parabolic systems (see [47]).

This paper is devoted to systematically addressing these questions in the context of heat and Schrödinger equations. Our aim here is not, by any means, to systematically address all the possible scenarios but simply to highlight some of the most fundamental phenomena illustrating how, the existing tools for the analysis of the controllability of PDE, can be employed in this averaged context too. It is important to highlight, however, that the averaged states do not obey a PDE, not even a semigroup. The dynamics can however be represented in such a way that its main controllability properties can be identified, by analogy, with some of the main well-known models, and analyzed by similar techniques.

In particular, we shall show that the averages of heat equations lead to heat-like dynamics that enjoy the null-controllability properties of solutions of heat equations in an arbitrarily short time and from arbitrary measurable sets of positive measure. In order to prove these results we employ classical techniques based on Carleman inequalities and the Fourier expansion of solutions on the basis of the eigenfunctions of the Laplacian generating the dynamics.

In the case of Schrödinger equations we show that, depending on the probability density, the average may behave either as a conservative or a heat-like evolution, leading to controllability properties, in average, of very different kind. When the obtained average is of parabolic nature the techniques above, employed to treat the control properties of parabolic averages, can be applied. However, when the average behaves rather in a conservative way we employ specific techniques for the control of wave-like equations.

The paper is organized as follows. In section 2 we present all these problems in an abstract setting in which different relevant PDE models enter naturally. In Section 3, we study the null and approximate averaged controllability problems for a class of random heat equations. In Section 4, we study the null and exact averaged controllability problem for a class of random Schrödinger equations. In Section 5 we give some further comments and open problems.

## 2 An abstract setting

Let $T > 0$ and $E \subset [0, T]$ be a Lebesgue measurable set with positive Lebesgue measure. Let $H$ and $U$ be two Hilbert spaces. Let $V \subset H$ be a Hilbert space which is dense in $H$. Denote by $V'$ the dual space of $V$ with respect to the pivot space $H$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{A(\omega)\}_{\omega\in\Omega}$ be a family of linear operators satisfying the following conditions:

1. $A(\cdot) \in L^2(\Omega; \mathcal{L}(D(A), H))$;
2. $A(\omega) : D(A) \to H$ generates a $C_0$-semigroup $\{S(t, \omega)\}_{t\geq 0}$ on both $H$ and $V$ for all $\omega \in \Omega$;
3. $S(t, \cdot)y \in L^1(\Omega; V)$ for all $y \in V$ and $t \in [0, T]$.

Let $B(\cdot) \in L^2(\Omega; \mathcal{L}(U, V))$.

Consider the following linear control system

$$\begin{cases} y_t(t) = A(\omega)y(t) + \chi_E(t)B(\omega)u(t) & \text{in } (0, T], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $y_0 \in V$ and $u(\cdot) \in L^2(E; U)$ is the control.

In what follows, we denote by $y(\cdot, \omega; y_0)$ the solution to (2.1), which is the state of the system. Although the initial datum $y_0 \in V$ and the control $u(\cdot)$ are independent of the sample point $\omega$, the state $y(t, \omega; y_0)$ of the system depends on $\omega$ nonlinearly.

According to the setting above, for a.e. $\omega \in \Omega$, there is a solution $y(\cdot, \omega; y_0) \in C([0, T]; V)$ and the expectation or averaged state $\int_\Omega y(\cdot, \omega; y_0)d\mathbb{P}(\omega) \in C([0, T]; V)$.

We introduce the following notions of averaged controllability for the system (2.1):

**Definition 2.1** *System* (2.1) *is said to fulfill the property of exact averaged controllability or to be exactly controllable in average in* $E$ *with control cost* $C > 0$ *if given any* $y_0, y_1 \in V$, *there exists a control* $u(\cdot) \in L^2(E; U)$ *such that*

$$|u|_{L^2(E;U)} \leq C(|y_0|_V + |y_1|_V) \tag{2.2}$$

*and the average of solutions to* (2.1) *satisfies*

$$\int_\Omega y(T, \omega; y_0) d\mathbb{P}(\omega) = y_1. \tag{2.3}$$

**Remark 2.1** *The notion of exact averaged controllability was first introduced in [48]. A full characterization was also given in the finite-dimensional setting. In [25], this issue was discussed for systems involving finitely-many linear parametric wave equations.*

**Definition 2.2** *System* (2.1) *fulfills the property of null averaged controllability or is null controllable in average in* $E$ *with control cost* $C$ *if given any initial datum* $y_0 \in V$, *there exists a control* $u \in L^2(E; U)$ *such that*

$$|u|_{L^2(E;U)} \leq C|y_0|_V \tag{2.4}$$

*and the average of the solutions to* (2.1) *satisfies*

$$\int_\Omega y(T, \omega; y_0) d\mathbb{P}(\omega) = 0. \tag{2.5}$$

**Definition 2.3** *System* (2.1) *fulfills the property of approximate averaged controllability or is approximately controllable in average in* $E$ *if given any* $y_0, y_1 \in V$ *and* $\varepsilon > 0$, *there exists a control* $u_\varepsilon \in L^2(E; U)$ *such that the average of solutions to* (2.1) *satisfies*

$$\Big| \int_\Omega y(T, \omega; y_0) d\mathbb{P}(\omega) - y_1 \Big|_V < \varepsilon.$$

**Remark 2.2** *As in the finite dimensional context ([48]), we can also consider the averaged control problem with random initial data, i.e.,* $y_0 \in L^2(\Omega; V)$. *Nevertheless, according to Remark A.1, this does not lead to any essential new difficulty. Thus, for the sake of simplicity of the presentation, we only deal with the case where* $y_0$ *is independent of* $\omega$.

These notions are motivated by the problem of controlling a dynamics governed by a pair of random operators $(A(\omega), B(\omega))$, where the effective value of the parameter $\omega$ is unknown. Then, one aims at choosing a control, independent of the unknown $\omega$, to act optimally in an averaged sense, making a robust compromise of all the possible realizations of the system for the various possible values of the sample point $\omega$. Similar problems can be considered in the case where the initial datum to be controlled depends on $\omega$ too.

We have introduced the notions of exact/null/approximate averaged controllability in the framework of random evolution equations but similar concepts make sense for parametrized evolution equations (see [48] for example). In that context it is sufficient to replace the expectation by a weighted average of the parameter-depending controlled states.

**Remark 2.3** *In the present context of randomly depending operators, the classical subordination properties of some control properties with respect to the others, that are classical for a given system, have to be addressed more carefully. Of course, averaged exact controllability implies the averaged null and approximate controllability properties as well. But, when $A$ and $B$ are independent of $\omega$ and $A$ generates a $C_0$-group, exact controllability is also a consequence of null controllability. However, the later may fail when considering averaged controllability properties, as shown in the example in Remark 4.6 below.*

**Remark 2.4** *For parametric control systems one can also consider the problems of simultaneous controllability and ensemble controllability, which concern the possibility of controlling all states with respect to different parameters simultaneously by one single control. We refer the readers to [30, 31] and [6, 29] for an introduction to the notions of simultaneous controllability and ensemble controllability, respectively. Of course the properties of averaged controllability we consider here are weaker than these other ones because they only deal with the average of the states with respect to those parameters. But, as we shall see, averaged controllability properties may be achieved in situations where simultaneous and ensemble controllability are impossible.*

**Remark 2.5** *In the particular case that $A$ is independent of $\omega$, the averaged controllability problems can be reduced to the classical controllability ones by setting*

$$\bar{B} = \int_\Omega B(\omega) d\mathbb{P}(\omega), \quad \bar{y}(t) = \int_\Omega y(t,\omega;y_0) d\mathbb{P}(\omega).$$

*Then we have that*

$$\begin{cases} \bar{y}_t(t) = A\bar{y}(t) + \chi_E(t)\bar{B}u(t) & \text{in } (0,T], \\ \bar{y}(0) = y_0. \end{cases} \tag{2.6}$$

*The exact (resp. null, approximate) averaged controllability problems of* (2.1) *are equivalent to the exact (resp. null, approximate) controllability problem of* (2.6)*.*

**Remark 2.6** *Random evolution equations can be used to model lots of uncertain physical processes (see [41, 42, 43] for example). Several notions of controllability have been introduced but, as far as we know, all of them concern driving the state to a given destination by a control depending on $\omega$ (see [21, 35, 36] and the references therein). The property of averaged controllability is, however, independent of the specific realization of $\omega$.*

Following the classical approach to deal with controllability problems, we introduce the adjoint system, which also depends on the parameter $\omega$:

$$\begin{cases} -z_t(t) = A^*(\omega)z(t) & \text{in } [0,T), \\ z(T) = z_0, \end{cases} \tag{2.7}$$

where $z_0 \in V'$.

Note that, in this adjoint system the initial value (at time $t = T$) is taken to be independent of $\omega$. This is due to the fact that, although $y(T,\omega;y_0)$ depends on $\omega$, its average, which belongs to $V$, is of course independent of $\omega$. Then, to deal with the averaged state it is sufficient to use as test functions, adjoint states departing from configurations that are independent of $\omega$. This is the reason we choose the final datum of (2.7) to be independent of $\omega$.

As dual notions of the properties of averaged controllability above we introduce the following three concepts of averaged observability.

**Definition 2.4** *System* (2.7) *is exactly averaged observable or exactly observable in average in $E$ if there is a constant $C > 0$ such that for any $z_0 \in V'$,*

$$|z_0|_{V'}^2 \leq C \int_0^T \chi_E(t) \Big| \int_\Omega B(\omega)^* z(t,\omega;z_0) d\mathbb{P}(\omega) \Big|_U^2 dt. \tag{2.8}$$

**Definition 2.5** *System* (2.7) *is null averaged observable or null observable in average in $E$ if there is a constant $C > 0$ such that for any $z_0 \in H$,*

$$\Big| \int_\Omega z(0,\omega;z_0) d\mathbb{P}(\omega) \Big|_{V'}^2 \leq C \int_0^T \chi_E(t) \Big| \int_\Omega B(\omega)^* z(t,\omega;z_0) d\mathbb{P}(\omega) \Big|_U^2 dt. \tag{2.9}$$

**Definition 2.6** *System* (2.7) *is said to satisfy the averaged unique continuation property in $E$ if the fact that $\chi_E \int_\Omega B(\omega)^* z(\cdot,\omega;z_0) d\mathbb{P}(\omega) = 0$ in $L^2(0,T;U)$ implies that $z_0 = 0$.*

The adjoint system (2.7) being exactly (*resp.* null) averaged observable means that one can estimate the norm of the average of the final (*resp.* initial) data of the adjoint system, out of partial measurements done on the averages of the adjoint states, with respect to $\omega$. These concepts have their own interest when dealing with the observation of random systems. The actual realization of the system depending on $\omega$ being unknown, it is natural to address the problem based on the measurements done on averages.

The weakest notion of averaged observability under consideration is averaged unique continuation. System (2.7) satisfies the averaged unique continuation property when its state can be uniquely determined by the partial measurements done on the mathematical expectation. It is a natural generalization of the unique continuation property of evolution equations.

The average of the adjoint state, being represented by $\int_\Omega z(t,\omega;z_0) d\mathbb{P}(\omega)$, does not satisfy the semigroup property and it is not a solution to an evolution equation. Thus, one can not directly employ the existing results on the observability of PDEs to establish the averaged observability of the adjoint system (2.8). However, as we shall see, by carefully analyzing and identifying the dynamics generated by the averages of the adjoint states, we shall be able to apply the existing PDE techniques.

In this paper, we mainly consider the case that $A(\omega) = \alpha(\omega)A$ and $B(\omega) = B$, where $A$ generates a $C_0$-semigroup on $H$, $\alpha(\omega)$ is a random variable and $B \in \mathcal{L}(U,H)$. We will show that the controllability properties of the system (2.1) depend on the choice of the random variable. We only consider the following commonly used ones:

1. *Uniformly distributed random variable*, with probability density function $\rho(\cdot)$ on $[a,b]$, where $0 < a < b$, $\alpha(\cdot)$, given by

$$\rho(\alpha) = \begin{cases} \dfrac{1}{b-a}, & \text{if } \alpha \in [a,b], \\ 0, & \text{if } \alpha \in (-\infty,a) \cup (b,\infty). \end{cases}$$

2. *Exponentially distributed random variable*, with probability density function $\rho(\cdot)$ given by

$$\rho(\alpha) = \begin{cases} e^{-(\alpha-c)}, & \text{if } \alpha \geq c, \\ 0, & \text{if } \alpha < c, \end{cases} \tag{2.10}$$

for a given positive number $c$. In what follows, for simplicity, we choose $c = 1$ in (2.10).

3. *Standard normally distributed random variable*, with probability density function of $\rho(\cdot)$ reads

$$\rho(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} \text{ for } \alpha \in \mathbb{R}.$$

4. *A random variable with standard Laplace distribution*, with density function $\rho(\cdot)$ given by

$$\rho(\alpha) = \frac{1}{2} e^{-|\alpha|} \text{ for } \alpha \in \mathbb{R}.$$

5. *A random variable with standard Chi-squared distribution*, with density function $\rho(\cdot)$ given by

$$\rho(\alpha; k) = \begin{cases} \dfrac{\alpha^{\frac{k}{2}-1} e^{-\frac{\alpha}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})}, & \text{if } \alpha \geq 0, \\ 0, & \text{if } \alpha < 0, \end{cases}$$

where $k \geq 1$ and $\Gamma(\frac{k}{2}) = \int_0^\infty \alpha^{\frac{k}{2}-1} e^{-\alpha} d\alpha$.

6. *A random variable with standard Cauchy distribution*, with density function $\rho(\cdot)$ given by

$$\rho(\alpha) = \frac{1}{\pi(1+\alpha^2)} \quad \text{for } \alpha \in \mathbb{R}.$$

# 3 Averaged controllability for random heat equations

In this section, we study the null averaged controllability problem for random heat equations. Of course, the property of averaged exact controllability has to be excluded because of the very strong regularizing effect of heat equations, that is preserved by averaging. Thus, we focus on the properties of the null and approximate averaged controllability.

Let $T > 0$ and $G \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with $C^2$ boundary $\partial G$ and consider the following controlled random heat equation:

$$\begin{cases} y_t - \alpha \Delta y = \chi_{G_0 \times E} u & \text{in } G \times (0,T), \\ y = 0 & \text{on } \partial G \times (0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{3.1}$$

Here $y_0 \in L^2(G)$, and $G_0$ and $E$ are subsets of $G$ and $[0,T]$ respectively, where the controls are being applied. The constant diffusivity $\alpha : \Omega \to \mathbb{R}^+$ is assumed to be a random variable. In this section, we make the following assumptions on $G_0$ and $E$:

(**A1**) $G_0 \subset G$ is a nonempty open subset.

(**A2**) $E \subset [0,T]$ is a Lebesgue measurable set with positive measure.

We first analyze the dynamics of the mathematical expectation (average) of the solutions of the random heat equations under consideration. This will be done using Fourier series expansions and will give us an intuition for the averaged observability and controllability results to be expected and that will be proved below.

Consider first the following uncontrolled random heat equation:

$$\begin{cases} \hat{y}_t - \alpha\Delta\hat{y} = 0 & \text{in } G\times(0,T), \\ \hat{y} = 0 & \text{on } \partial G\times(0,T), \\ \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases} \tag{3.2}$$

We decompose solutions in Fourier series on the basis of the eigenfunctions of the Dirichlet laplacian. To be more precise, consider the unbounded linear operator $A_\Delta$ on $L^2(G)$ given as

$$\begin{cases} D(A_\Delta) = H^2(G)\cap H_0^1(G), \\ A_\Delta u = -\Delta u, \quad \text{for any } u\in D(A_\Delta). \end{cases}$$

Let us denote by $\{\lambda_j\}_{j=1}^\infty$ (with $0<\lambda_1<\lambda_2\leq\cdots$) the eigenvalues of $A_\Delta$ and let $\{e_j\}_{j=1}^\infty$ be the corresponding eigenfunctions such that $|e_j|_{L^2(G)}=1$ for $j\in\mathbb{N}$.

We assume that the initial datum takes the form $\hat{y}_0=\sum_{j=1}^\infty \hat{y}_{0,j}e_j\in L^2(G)$. The averaged state can also be described in Fourier series as follows, distinguishing the probability densities above.

**Case 1**. If $\alpha(\cdot)$ is a uniformly distributed random variable on $[a,b]$, where $0<a<b$, then,

$$\begin{aligned} \int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \frac{1}{b-a}\int_a^b\sum_{i=1}^\infty \hat{y}_{0,j}e^{-\lambda_j\alpha t}e_j d\alpha \\ &= \frac{1}{b-a}\sum_{j=1}^\infty \frac{1}{\lambda_j t}\hat{y}_{0,j}\big(e^{-\lambda_j a t}-e^{-\lambda_j b t}\big)e_j. \end{aligned} \tag{3.3}$$

**Remark 3.1** *The values of a uniformly distributd random variable, which is a relevant one in practice, are uniformly distributed over an interval, i.e., all points in the interval are equally likely. It models the random phenomenon with "equally possible outcomes". When there is no any a priori knowledge for $\alpha$ other than $\alpha(\omega)\in[a,b]$ for all $\omega\in\Omega$, this is the best possible choice. More details can be found in [20].*

**Case 2.** When $\alpha(\cdot)$ is the exponentially distributed random variable,

$$\begin{aligned} \int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \int_1^\infty e^{-(\alpha-1)}\sum_{j=1}^\infty \hat{y}_{0,j}e^{-\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^\infty \frac{1}{\lambda_j t+1}\hat{y}_{0,j}e^{-\lambda_j t}e_j. \end{aligned} \tag{3.4}$$

**Remark 3.2** *The exponentially distributed random variable is one of the most important random variables that can be used to describe the time between events in a process where they occur continuously and independently at a constant average rate.*

In both cases, the mathematical expectation of the solution of the parameter-depending heat equation evolves according to a heat-like dynamics. The representation of the averages on the basis of eigenfunctions exhibits the exponential decay and smoothing effects that are prototypical of heat-like problems.

Accordingly, the following null averaged controllability result holds.

**Theorem 3.1** *Let (**A1**) and (**A2**) hold. Assume that, either $\alpha(\cdot)$ is a uniformly or exponentially distributed random variable. Then, the system* (3.1) *is null controllable in average with control $u(\cdot) \in L^2(0,T;L^2(G_0))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^2(0,T;L^2(G_0))} \leq C|y_0|_{L^2(G)}. \tag{3.5}$$

**Remark 3.3** *In the context of heat equations, the null controllability result with controls supported in measurable sets is related to the bang-bang property of the time optimal control problems with constrained controls (see [40, 44] for example).*

*More precisely, let us consider the following heat equation:*

$$\begin{cases} \tilde{y}_t - \Delta\tilde{y} = \chi_{G_0}\tilde{u} & \text{in } G \times [0,\infty), \\ \tilde{y} = 0 & \text{on } \partial G \times [0,\infty), \\ \tilde{y}(0) = \tilde{y}_0 & \text{in } G. \end{cases} \tag{3.6}$$

*Here the initial state $\tilde{y}_0 \in L^2(G)$ and the control is assumed to belong to the constrained set of admissible controls*

$$\tilde{u} \in \mathcal{U}_M \stackrel{\triangle}{=} \{\tilde{u} \in L^\infty(0,\infty;L^2(G)) : |\tilde{u}|_{L^2(G)} \leq M \text{ a.e. } t \in [0,\infty)\}$$

*for some $M > 0$. Let $\widetilde{T}^*$ be the $\min\{\tilde{t} : \tilde{y}(\tilde{t};\tilde{u},\tilde{y}_0) = 0,\ \tilde{u} \in \mathcal{U}_M\}$. A control $u^* \in \mathcal{U}_M$ such that $\tilde{y}(\widetilde{T}^*;\tilde{u},\tilde{y}_0) = 0$ is called a time optimal control and satisfies the bang-bang property if $|\tilde{u}(t)|_{L^2(G)} = M$ for a.e. $t \in [0,T^*]$. The proof of this fact requires the null controllability with controls supported in measurable sets (see [44]).*

*The same ban-bang problem can be formulated in the context of averaged null controllability we are considering here. However, the techniques in [44] do not seem to apply because the averages of the heat processes under consideration do not satisfy the semigroup property.*

To prove Theorem 3.1, as in the deterministic frame, we introduce the following adjoint system:

$$\begin{cases} z_t + \alpha\Delta z = 0 & \text{in } G \times (0,T), \\ z = 0 & \text{on } \partial G \times (0,T), \\ z(T) = z_0 & \text{in } G, \end{cases} \tag{3.7}$$

where $z_0 \in L^2(G)$. As mentioned above in the abstract setting, the initial data (at time $t = T$) of the adjoint system are assumed to be independent of the random parameter. According to Theorem A.2, we only need to prove that (3.7) is null observable in average, which is a corollary of the following result.

**Theorem 3.2** *Let (**A1**) and (**A2**) hold. Assume that $\alpha(\cdot)$ is either an uniformly distributed or an exponentially distributed random variable. Then, there exists a constant $C > 0$ such that for any $y_0 \in L^2(G)$ it holds that*

$$\Big|\int_{\Omega} z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(G)} \le C\int_E \Big|\int_{\Omega} z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(G_0)} dt. \tag{3.8}$$

An immediate corollary of Theorem 3.2 is as follows:

**Corollary 3.1** *Let (**A1**) and (**A2**) hold. Assume that $\alpha(\cdot)$ is either an uniformly distributed or an exponentially distributed random variable. Then the system* (3.7) *is null observable in average.*

*Proof*: From Hölder's inequality, we have that

$$\begin{aligned}\int_E \Big|\int_{\Omega} z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(G_0)} dt &\le \Big(\int_E dt\Big)^{\frac{1}{2}}\Big(\int_E \Big|\int_{\Omega} z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_{L^2(G_0)} dt\Big)^{\frac{1}{2}}\\ &\le \sqrt{m(E)}\Big(\int_E \Big|\int_{\Omega} z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_{L^2(G_0)} dt\Big)^{\frac{1}{2}}.\end{aligned}$$

This, together with the inequality (3.8), implies that the system (3.7) is null observable in average. □

To prove Theorem 3.2, we adopt the strategy developed in [38] in the context of the heat equation, using spectral decompositions. The null averaged observability inequality is built in an iterative manner. More precisely, we decompose the set $E$ into an infinite sequence of connective (in time) subsets in which an increasing number of Fourier components of the average of the solution is observed with uniform observability constants. By iteration, the final datum is observed. To apply this strategy, we need to use classical results on how to divide $E$ into an infinite sequence of subsets of positive Lebesgue measure. We also need to know how to observe a finite number of the Fourier components of the average of the solution. These ingredients are given in the following two lemmas.

**Lemma 3.1** *[40, Proposition 2.1] Let $E \subset [0,T]$ be a measurable set of positive Lebesgue measure $m(E)$. Let $\ell$ be a density point of $E$. Then for each $a > 1$, there exists an $\ell_1 \in (\ell, T)$ such that the sequence $\{\ell_k\}_{k=1}^{\infty}$, given by*

$$\ell_{k+1} = \ell + \frac{\ell_1 - \ell}{a^k}, \tag{3.9}$$

*satisfies*

$$m(E \cap (\ell_{k+1}, \ell_k)) \ge \frac{\ell_k - \ell_{k+1}}{3}. \tag{3.10}$$

**Lemma 3.2** *[33, Theorem 1.2] There is a constant $C_1 > 0$ such that for any $r > 0$ and $\{a_j\}_{\lambda_j \le r} \subset \mathbb{C}$,*

$$\Big(\sum_{\lambda_j \le r} |a_j|^2\Big)^{\frac{1}{2}} \le C_1 e^{C_1\sqrt{r}}\Big(\int_{G_0} \Big|\sum_{\lambda_j \le r} a_j e_j(x)\Big|^2 dx\Big)^{\frac{1}{2}}. \tag{3.11}$$

We are now in conditions to proceed to the proof of Theorem 3.2.

*Proof of Theorem 3.2*: We only give a proof for the case where $\alpha(\cdot)$ is an exponentially distributed random variable. The proof for that $\alpha(\cdot)$ is a uniformly distributed random variable is very similar.

Put $\tilde{z}(x,t) = \int_\Omega z(x, T-t, \omega; z_0)d\mathbb{P}(\omega)$. Let $z_{0,j} = \langle z_0, e_j\rangle_{L^2(G)}$. Then, similar to the computation of (3.4), we have that

$$\tilde{z}(x,t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j t + 1} z_{0,j} e^{-\lambda_j t} e_j.$$

We only need to prove that

$$\Big|\sum_{j=1}^{\infty} z_{0,j} e^{-\lambda_j T} e_j\Big|_{L^2(G)} \le C \int_E |\tilde{z}(x,t,\omega)|_{L^2(G_0)} dt. \tag{3.12}$$

Note that, in the right hand side of this inequality, the observation is done in the $L^1(E; L^2(G_0))$-norm. Thus, the result is even stronger than the one we actually need, in which the observation is done in $L^2(E; L^2(G_0))$. As a consequence of the inequality above we shall prove, the controls we shall build will belong to $L^\infty(E; L^2(G_0))$.

Let $X_r = \text{span}\,\{e_j\}_{\lambda_j \le r}$ for each $r > 0$. For any $\xi \in L^2(G)$, we put

$$S(t,\xi) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j t + 1} \xi_j e^{-\lambda_j t} e_j, \tag{3.13}$$

where $\xi_j = \langle \xi, e_j\rangle_{L^2(G)}$. Then we see that

$$\big|S(t,\xi)\big|^2_{L^2(G)} \le \big|S(s,\xi)\big|^2_{L^2(G)}, \qquad \text{for } 0 \le s \le t \le T, \tag{3.14}$$

and

$$\big|S(t,\xi)\big|^2_{L^2(G)} \le e^{-r(t-s)} |S(s,\xi)|_{L^2(G)}, \qquad \text{for all } \xi \in X_r^\perp \text{ and } 0 \le s \le t \le T. \tag{3.15}$$

Let $\ell$ be a density point for $E$. By Lemma 3.1, for a given $a > 1$, there exists a sequence $\{\ell_k\}_{k=1}^\infty$ satisfying (3.9) and (3.10).

We now define a sequence of subsets $\{E_k\}_{k=1}^\infty$ of $(0,T)$ in the following way:

$$E_k \triangleq \Big\{t - \frac{\ell_k - \ell_{k+1}}{6} \,:\, t \in E \cap \Big(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k\Big)\Big\}, \quad \text{for } k \in \mathbb{N}. \tag{3.16}$$

Clearly, $E_k \subset (\ell_{k+1}, \ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1}))$. From (3.10), we have that

$$\begin{aligned} m(E_k) &= m\Big(E \cap \Big(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k\Big)\Big) \\ &= m\Big(E \cap \Big[(\ell_{k+1}, \ell_k) \setminus \Big(\ell_{k+1}, \ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}\Big)\Big]\Big) \\ &\ge m(E \cap (\ell_{k+1}, \ell_k)) - \frac{\ell_k - \ell_{k+1}}{6} \\ &\ge \frac{\ell_k - \ell_{k+1}}{6}. \end{aligned} \tag{3.17}$$

Let $b > a$ be a positive number such that

$$\frac{b}{a} > \frac{C_1 + 6\ln(12C_1 a)}{(a-1)(\ell_1 - \ell)}. \tag{3.18}$$

Set $r_k = b^{2k}$. From (3.14), we have that for any $\xi \in X_{r_k}$,

$$\begin{aligned}
&\int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\Big|S\Big(\ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1}), \xi\Big)\Big|_{L^2(G)} dt \\
\leq &\int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\big|S(t,\xi)\big|_{L^2(G)} dt.
\end{aligned}$$

From (3.11) and (3.17), we find that

$$\begin{aligned}
&\frac{\ell_k - \ell_{k+1}}{6}\Big|S\Big(\ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1}), \xi\Big)\Big|_{L^2(G)} \\
\leq\, & m(E_k)\Big|S\Big(\ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1}), \xi\Big)\Big|_{L^2(G)} \\
\leq &\int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\big|S(t,\xi)\big|_{L^2(G)} dt \\
\leq\, & C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\big|S(t,\xi)\big|_{L^2(G_0)} dt.
\end{aligned} \tag{3.19}$$

Let $z_0 = z_0^1 + z_0^2$, where $z_0^1 \in X_{r_k}$ and $z_0^2 \in X_{r_k}^{\perp}$. Taking $\xi = S\Big(\frac{\ell_k - \ell_{k+1}}{6}, z_0^1\Big)$ in (3.19), we get that

$$\begin{aligned}
&\frac{\ell_k - \ell_{k+1}}{6}|S(\ell_k, z_0^1)|_{L^2(G)} \\
\leq &\int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\Big|S\Big(t + \frac{\ell_k - \ell_{k+1}}{6}, z_0^1\Big)\Big|_{L^2(G)} dt \\
\leq\, & C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{E_k}(t)\Big|S\Big(t + \frac{\ell_k - \ell_{k+1}}{6}, z_0^1\Big)\Big|_{L^2(G_0)} dt \\
\leq\, & C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k} \chi_{E_k}\Big(t - \frac{\ell_k - \ell_{k+1}}{6}\Big)|S(t, z_0^1)|_{L^2(G_0)} dt.
\end{aligned} \tag{3.20}$$

By (3.16), we have that

$$\chi_{E_k}\Big(t - \frac{\ell_k - \ell_{k+1}}{6}\Big) = \chi_E(t), \quad \text{for any } t \in \Big(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k\Big). \tag{3.21}$$

Combining (3.14), (3.15), (3.20) and (3.21), we find that

$$
\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}|S(\ell_k,z_0^1)|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k}\chi_E(t)|S(t,z_0^1)|_{L^2(G_0)}dt\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k}\chi_E(t)\big(|S(t,z_0)|_{L^2(G_0)}+|S(t,z_0^2)|_{L^2(G)}\big)dt\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt\\
&\quad+C_1e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1})\Big|S\Big(\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6},z_0^2\Big)\Big|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt\\
&\quad+C_1e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}\big|S(\ell_{k+1},z_0^2)\big|_{L^2(G)}.
\end{aligned}
\tag{3.22}
$$

Therefore, we obtain that

$$
\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}|S(\ell_k,z_0)|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt+C_1e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}\big|S(\ell_{k+1},z_0^2)\big|_{L^2(G)}\\
&\quad+\frac{\ell_k-\ell_{k+1}}{6}|S(\ell_k,z_0^2)|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt+C_1e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}\big|S(\ell_{k+1},z_0^2)\big|_{L^2(G)}\\
&\quad+\frac{\ell_k-\ell_{k+1}}{6}e^{-r_k(\ell_k-\ell_{k+1})}|S(\ell_{k+1},z_0^2)|_{L^2(G)}.
\end{aligned}
\tag{3.23}
$$

Thus, it holds that

$$
\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}|S(\ell_k,z_0)|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt\\
&\quad+(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}\big(C_1e^{C_1\sqrt{r_k}}+1\big)\big|S(\ell_{k+1},z_0)\big|_{L^2(G)}.
\end{aligned}
\tag{3.24}
$$

This concludes that

$$
\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6C_1e^{C_1\sqrt{r_k}}}|S(\ell_k,z_0)|_{L^2(G)}-\frac{C_1e^{C_1\sqrt{r_k}}+1}{C_1e^{C_1\sqrt{r_k}}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}|S(\ell_{k+1},z_0)|_{L^2(G)}\\
&\le\int_{\ell_{k+1}}^{\ell_k}\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt.
\end{aligned}
\tag{3.25}
$$

By summing the inequality (3.25) from $k=1$ to $k=\infty$, we obtain that

$$\frac{\ell_1-\ell_2}{6C_1e^{C_1\sqrt{r_1}}}|S(\ell_1,z_0)|_{L^2(G)}+\sum_{k=1}^{\infty}f_k|S(\ell_{k+1},z_0)|_{L^2(G)}\le\int_0^T\chi_E(t)|S(t,z_0)|_{L^2(G_0)}dt, \qquad (3.26)$$

where

$$f_k=\frac{\ell_{k+1}-\ell_{k+2}}{6C_1e^{C_1\sqrt{r_{k+1}}}}-\frac{C_1e^{C_1\sqrt{r_k}}+1}{C_1e^{C_1\sqrt{r_k}}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_k}, \quad k=1,2,\cdots$$

From (3.18) and $r_k=b^{2k}$, we have that

$$f_k\ge 0 \;\text{ for any }\; k=1,2,\cdots$$

This, together with (3.26), deduces that

$$|S(\ell_1,z_0)|_{L^2(G)}\le\frac{6C_1e^{C_1\sqrt{r_1}}}{\ell_1-\ell_2}\int_E|S(t,z_0)|_{L^2(G_0)}dt. \qquad (3.27)$$

Since $\ell_1<T$, we can find a constant $C>0$ such that

$$\frac{C}{1+\lambda_j\ell_1}\ge e^{-\lambda_j(T-\ell_1)} \text{ for every } j\in\mathbb{N}. \qquad (3.28)$$

From (3.27) and (3.28), we get that

$$\Big|\sum_{j=1}^{\infty}z_{0,j}e^{-\lambda_jT}e_j\Big|_{L^2(G)}\le C\int_E|S(t,z_0)|_{L^2(G_0)}dt. \qquad (3.29)$$

This completes the proof. □

As an easy corollary of Theorem 3.2, we have the following result.

**Theorem 3.3** *Let (***A1***) and (***A2***) hold. System* (3.1) *is approximately controllable in average, provided that $\alpha(\cdot)$ is a uniformly distributed or an exponentially distributed random variable.*

*Proof*: According to Theorem A.3, we only need to prove that the unique continuation property in $E$ is satisfied.

Assume that $z=0$ in $G_0\times E$. From Theorem 3.2, we obtain that

$$\Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(G)}\le C\int_E\Big|\int_\Omega z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(G_0)}dt.$$

Hence, $\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)=0$. On the other hand, if $z_{0,j}=\int_G z_0e_jdx$,

$$0=\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)=\int_1^\infty e^{-(\alpha-1)}\sum_{j=1}^{\infty}z_{0,j}e^{-\lambda_j\alpha T}e_jd\alpha=\sum_{j=1}^{\infty}\frac{1}{\lambda_jT+1}z_{0,j}e^{-\lambda_jT}e_j. \qquad (3.30)$$

Since $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(G)$, it follows that $\frac{1}{\lambda_jT+1}z_{0,j}e^{-\lambda_jT}=0$ for all $j\in\mathbb{N}$. Thus, we get that $z_{0,j}=0$ for all $j\in\mathbb{N}$, which implies that $z_0=0$. □

# 4 Averaged controllability for random Schrödinger equations

## 4.1 Preliminaries

In this section, we study the null and exact averaged controllability problems for a class of random Schrödinger equations of the form

$$\begin{cases} y_t - i\alpha\Delta y = \chi_{G_0\times E} u & \text{in } G\times(0,T), \\ y = 0 & \text{on } \partial G\times(0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{4.1}$$

Here $\alpha(\cdot):\Omega\to\mathbb{R}$ is a random variable, the initial datum $y_0$ belongs to $L^2(G)$ and $G_0$ is a suitable subdomain of $G$.

In this time-reversible setting the Schrödinger equation is well-posed whatever the sign of $\alpha$ is, contrary to the heat equation. Thus, we have more choices for the random variable $\alpha(\cdot)$.

The average of the solutions to random Schrödinger equations may lead to very different dynamics, depending on the random variable under consideration. To see this, let us first consider the Schrödinger system in the absence of control:

$$\begin{cases} \hat{y}_t - i\alpha\Delta \hat{y} = 0 & \text{in } G\times(0,T), \\ \hat{y} = 0 & \text{on } \partial G\times(0,T), \\ \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases} \tag{4.2}$$

Here $\hat{y}_0 = \sum_{j=1}^{\infty}\hat{y}_{0,j}e_j \in L^2(G)$.

**Case 1**. When $\alpha(\cdot)$ is a uniformly distributed random variable on $[a,b]$, where $a,b\in\mathbb{R}$, then,

$$\begin{aligned} \int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \frac{1}{b-a}\int_a^b \sum_{i=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \frac{1}{b-a}\sum_{j=1}^{\infty}\frac{1}{i\lambda_j t}\hat{y}_{0,j}\big(e^{-i\lambda_j a t} - e^{-i\lambda_j b t}\big)e_j. \end{aligned} \tag{4.3}$$

**Case 2**. When $\alpha(\cdot)$ is an exponentially distributed random variable, then

$$\begin{aligned} \int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \int_1^{\infty} e^{-(\alpha-1)}\sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^{\infty}\frac{1}{i\lambda_j t+1}\hat{y}_{0,j}e^{-i\lambda_j t}e_j. \end{aligned} \tag{4.4}$$

**Case 3**. For the normally distributed random variable $\alpha(\cdot)$ we have,

$$\begin{aligned} \int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}}\sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-\frac{1}{2}\lambda_j^2 t^2}e_j. \end{aligned} \tag{4.5}$$

**Case 4**. When $\alpha(\cdot)$ is a random variable with Laplace distribution we have

$$\begin{aligned}\int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\alpha|}\sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^{\infty}\frac{1}{1+\lambda_j^2t^2}\hat{y}_{0,j}e^{-i\lambda_j t}e_j.\end{aligned} \tag{4.6}$$

**Remark 4.1** *The Laplace distribution can be thought of as two exponential distributions (with an additional location parameter) spliced together back-to-back. It governs the difference of two independent identically distributed exponential random variables and can be regarded as the generalization of the exponential distribution on the whole real line. More details can be found in [23].*

**Case 5**. For the Chi-squared distribution it holds:

$$\begin{aligned}\int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \int_{0}^{\infty} \frac{\alpha^{\frac{k}{2}-1}e^{-\frac{\alpha}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}\sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^{\infty}\frac{1}{(1+2i\lambda_j t)^{\frac{k}{2}}}\hat{y}_{0,j}e^{-i\lambda_j t}e_j.\end{aligned} \tag{4.7}$$

**Remark 4.2** *A Chi-squared distributed random variable is the sum of the squares of $k$ independent standard normally distributed random variables. It is one of the most widely used probability distributions in inferential statistics. We refer the readers to [19] for more details.*

**Case 6**. For the Cauchy distribution we have

$$\begin{aligned}\int_\Omega \hat{y}(x,t,\omega;\hat{y}_0)d\mathbb{P}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\pi(1+\alpha^2)}\sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-i\lambda_j\alpha t}e_j d\alpha \\ &= \sum_{j=1}^{\infty}\hat{y}_{0,j}e^{-\lambda_j t}e_j.\end{aligned} \tag{4.8}$$

**Remark 4.3** *The Cauchy distribution is associated with many processes, including resonance energy distribution, impact and natural spectral and quadratic stark line broadening. It also has important connections with other random variables. For example, when $\gamma_1$ and $\gamma_2$ are two independent standard normally distributed random variables, then the ratio $\gamma_1/\gamma_2$ has the standard Cauchy distribution. More details can be found in [19].*

**Remark 4.4** *From* (4.8)*, we know that when $\alpha$ is a random variable with Cauchy distribution, then the average of the solution to the random Schrödinger equation* (4.2) *becomes a solution of the heat equation. This is another example that after averaging, one enjoy enhanced regularity properties.*

According to the above results, there are essentially two different dynamics for the averages, depending on whether the time-exponentials entering in the Fourier expansion are real or imaginary. In cases 3 and 6, the average has a heat-like behavior. However, in cases 1, 2, 4 and 5, the average has a Schrödinger-like behavior.

### 4.2 Null averaged controllability

We first recall the following assumptions on $G_0$ and $E$:
(**A1**) Let $G_0 \subset G$ be a nonempty open subset.
(**A2**) Let $E \subset [0, T]$ be a Lebesgue measurable set with positive measure.

**Theorem 4.1** *Let (**A1**) and (**A2**) hold. If $\alpha$ is a random variable with normal distribution or Cauchy distribution, then the system* (4.1) *is null controllable in average with control $u(\cdot) \in L^2(0, T; L^2(G_0))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^2(0,T;L^2(G_0))} \leq C|y_0|_{L^2(G)}. \tag{4.9}$$

**Remark 4.5** *Note that, in the present case, the random Schrödinger equations is null controllable in average without any assumption on the support $G_0$ of the control, other than being of positive measure. This is in contrast with the well known results on the null controllability of Schrödinger equations, where $G_0$ is assumed, for instance, to fulfill the classical Geometric Control Condition (GCC)(see [26, 27] for example) or other geometric restrictions associated to multiplier techniques or Carleman inequalities(see [24, 34] for example). In the present case, these restrictions on $G_0$ are not needed since the averages behave in a parabolic fashion.*

To prove Theorem 4.1, we introduce the adjoint system of (4.1) as follows:

$$\begin{cases} z_t + i\alpha\Delta z = 0 & \text{in } G \times (0, T), \\ z = 0 & \text{on } \partial G \times (0, T), \\ z(T) = z_0 & \text{in } G. \end{cases} \tag{4.10}$$

By Theorem A.2, we only need to prove the following result.

**Theorem 4.2** *Under the assumptions of Theorem 4.1 the system* (4.10) *is null observable in average if $\alpha$ is a random variable with normal or Cauchy distribution.*

Indeed, we have the following stronger observability estimates.

**Proposition 4.1** *Let (**A1**) and (**A2**) hold. Assume that $\alpha(\cdot)$ is a random variable with normal distribution or Cauchy distribution. Then there exists a constant $C > 0$ such that for any $z_0 \in L^2(G)$, it holds that*

$$\Big| \int_\Omega z(0, \omega; z_0) d\mathbb{P}(\omega) \Big|_{L^2(G)} \leq C \int_E \Big| \int_\Omega z(x, t, \omega; z_0) d\mathbb{P}(\omega) \Big|_{L^2(G_0)} dt. \tag{4.11}$$

The proof of Proposition 4.1 is very similar to the one for Theorem 3.2. We omit it here.

**Remark 4.6** *If $\alpha(\cdot)$ is a random variable with Cauchy distribution, then we have that*

$$\begin{aligned} \int_\Omega z(x, t, \omega; z_0) d\mathbb{P}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\pi(1 + \alpha^2)} \sum_{j=1}^{\infty} z_{0,j} e^{-i\lambda_j \alpha(T-t)} e_j d\alpha \\ &= \sum_{j=1}^{\infty} z_{0,j} e^{-\lambda_j (T-t)} e_j. \end{aligned} \tag{4.12}$$

*In this case, Proposition 4.1 is an immediate corollary of the observability estimate for heat equations.*

*This is an example of a system that is null but not exactly controllable in average.*

## 4.3 Exact averaged controllability

In this subsection we consider the cases where the averages behave as Schrödinger-like semgroups. We assume that $E = [0, T]$ so that the control is active in any time instant within the time interval $[0, T]$.

Let us first consider the following Schrödinger equation

$$\begin{cases} \varphi_t + \kappa i \Delta \varphi = 0 & \text{in } G \times (0, T], \\ \varphi = 0 & \text{on } \partial G \times (0, T), \\ \varphi(0) = \varphi_0 & \text{in } G, \end{cases} \tag{4.13}$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $\varphi_0 \in L^2(G)$.

We make the following assumption in this subsection on $G_0$:

(**A3**) Whatever $T > 0$ and $k \neq 0$ are, there is a constant $C > 0$ such that for any $\varphi_0 \in L^2(G)$, the solution $\varphi(\cdot, \cdot)$ to (4.13) satisfies

$$|\varphi_0|^2_{L^2(G)} \leq C \int_0^T \int_{G_0} |\varphi|^2 dxdt. \tag{4.14}$$

We refer the readers to [2, 18, 26, 39] for the study of (4.14) under different conditions for $G_0$. In those articles one can find various sufficient conditions on the subset $G_0$ so that the observability inequality above holds, depending of the techniques of proof employed (multipliers, Carleman inequalities, Microlocal analysis). In particular, this observability inequality for the Schrödinger equation holds as soon as it is satisfied for the wave equation in some time horizon. Thus, in particular, it holds under the classical Geometric Control Condition (GCC) guaranteeing, roughly, that all rays of geometric optics enter the observation set $G_0$ in some uniform time.

We have the following observability result for the system (4.13).

**Theorem 4.3** *The following results hold:*

- *Let $\alpha(\cdot)$ be a uniformly distributed random variable on an interval $[a, b]$. Then, the system* (4.1) *is exactly controllable in average with $V = H = L^2(G)$ and $U = H^{-2}(G_0)$.*
- *Let $\alpha(\cdot)$ be an exponentially distributed random variable. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^2(G)$ and $U = L^2(G_0)$.*
- *Let $\alpha(\cdot)$ be a random variable with Laplace distribution. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^4(G)$ and $U = L^2(G_0)$.*
- *Let $\alpha(\cdot)$ be a random variable with standard Chi-squared distribution. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^k(G)$ and $U = L^2(G_0)$.*

*Proof of the first conclusion in Theorem 4.3*: We divide the proof into two steps.

**Step 1**. In this step, we show that the average of the solution is can be represented by the difference of the solutions to two Schrödinger equations. We let $z_0 \in L^2(G)$ in (4.10) and $\tilde{z}(x,t;z_0) = \int_\Omega z(x,T-t,\omega;z_0)d\mathbb{P}(\omega)$. Assume that $z_0 = \sum_{j=1}^\infty z_{0,j}e_j$. Similar to (4.3), we have that

$$\tilde{z}(x,t;z_0) = \frac{1}{b-a}\sum_{j=1}^\infty \frac{1}{i\lambda_j t} z_{0,j}\big(e^{-i\lambda_j at} - e^{-i\lambda_j bt}\big)e_j. \tag{4.15}$$

Let

$$z_a(\cdot,\cdot) = \sum_{j=1}^\infty \frac{z_{0,j}}{i\lambda_j}e^{-i\lambda_j at}, \quad z_b(\cdot,\cdot) = \sum_{j=1}^\infty \frac{z_{0,j}}{i\lambda_j}e^{-i\lambda_j bt}.$$

Clearly, $z_a(\cdot,\cdot)$ and $z_b(\cdot,\cdot)$ solve the following equations respectively:

$$\begin{cases} z_{a,t} + ai\Delta z_a = 0 & \text{in } G\times[0,T],\\ z_a = 0 & \text{on } \partial G\times[0,T],\\ z_a(0) = A_\Delta^{-1}z_0 & \text{in } G, \end{cases} \tag{4.16}$$

$$\begin{cases} z_{b,t} + bi\Delta z_b = 0 & \text{in } G\times[0,T],\\ z_b = 0 & \text{on } \partial G\times[0,T],\\ z_b(0) = A_\Delta^{-1}z_0 & \text{in } G. \end{cases} \tag{4.17}$$

Further,

$$t\tilde{z}(\cdot,\cdot) = (z_a - z_b)(\cdot,\cdot).$$

**Step 2**. In this step, we establish the exactly averaged observability estimate. From (4.16) and (4.17), we have that

$$\begin{aligned} &(i\partial_t + a\Delta)(i\partial_t + b\Delta)(z_a - z_b)\\ &= (i\partial_t + a\Delta)(i\partial_t + b\Delta)z_a - (i\partial_t + a\Delta)(i\partial_t + b\Delta)z_b\\ &= (i\partial_t + b\Delta)(i\partial_t + a\Delta)z_a = 0. \end{aligned}$$

Hence, we know that $(i\partial_t + b\Delta)(z_a - z_b)$ solves

$$\begin{cases} i\varphi_t + a\Delta\varphi = 0 & \text{in } G\times(0,T],\\ \varphi = 0 & \text{on } \partial G\times(0,T),\\ \varphi(0) = (b-a)z_0 & \text{in } G. \end{cases} \tag{4.18}$$

By assumption (**A3**), for any $z_0 \in L^2(G)$, it holds that

$$\begin{aligned} |z_0|^2_{L^2(G)} &\le C\int_0^T\int_{G_0}|\varphi(x,t)|^2dxdt \le C\int_0^T\int_{G_0}|(i\partial_t + b\Delta)(z_a - z_b)(x,t)|^2dxdt\\ &\le C\int_0^T\int_{G_0}\big|(i\partial_t + b\Delta)\big[t\tilde{z}(x,t;z_0)\big]\big|^2dxdt \le C\int_0^T\big|t\tilde{z}(x,t;z_0)\big|^2_{H^2(G_0)}dt\\ &\le C\int_0^T|\tilde{z}(x,t;z_0)|^2_{H^2(G_0)}dt. \end{aligned} \tag{4.19}$$

□

The proofs of the second to the fourth conclusion in Theorem 4.3 are very similar. We only give that for the second one.

*Proof of the second conclusion in Theorem 4.3*: Let $z_0 \in H^{-2}(G)$ in (4.10) and $\tilde{z}(x,t;z_0) = \int_\Omega z(x,T-t,\omega;z_0)d\mathbb{P}(\omega)$. We only need to prove that

$$|z_0|^2_{H^{-2}(G)} \leq C\int_0^T\int_{G_0}|\tilde{z}(x,t;z_0)|^2dxdt. \tag{4.20}$$

We divide the proof into two steps.

**Step 1**. In this step, we prove that a "weak" version of the exact averaged observability, that is, there is a lower order term in the right hand side of the inequality.

Assume that $z_0 = \sum_{j=1}^\infty z_{0,j}e_j \in H^{-2}(G)$. Then,

$$z(x,t,\omega;z_0) = \sum_{j=1}^\infty z_{0,j}e^{-i\alpha\lambda_j(T-t)}e_j.$$

Similar to (4.4), we have that

$$\tilde{z}(x,t;z_0) = \int_\Omega z(x,T-t,\omega;z_0)d\mathbb{P}(\omega) = \sum_{j=1}^\infty \frac{1}{i\lambda_j t+1}z_{0,j}e^{-i\lambda_j t}e_j. \tag{4.21}$$

This implies that for any $\delta > 0$,

$$|\tilde{z}(\cdot,\cdot;z_0)|_{L^2(\delta,T;L^2(G))} \leq C(\delta,T)|z_0|_{H^{-2}(G)}. \tag{4.22}$$

Let

$$v(x,t) = \sum_{j=1}^\infty \frac{1}{i\lambda_j t}z_{0,j}e^{-i\lambda_j t}e_j.$$

From assumption (**A3**), for a fixed $\delta > 0$, we have that

$$\begin{aligned}|z_0|^2_{H^{-2}(G)} &= \Big|\sum_{j=1}^\infty \frac{1}{i\lambda_j}z_{0,j}e^{-i\lambda_j t}e_j\Big|^2_{L^2(G)} \leq C\int_\delta^T\int_{G_0}\Big|\sum_{j=1}^\infty \frac{z_{0,j}}{\lambda_j}e^{-i\lambda_j t}e_j\Big|^2dxdt\\ &= C\int_\delta^T\int_{G_0}|tv(x,t)|^2dxdt.\end{aligned} \tag{4.23}$$

Therefore,

$$\begin{aligned}&|z_0|^2_{H^{-2}(G)}\\ &\leq C\int_\delta^T\int_{G_0}|tv(x,t)|^2dxdt\\ &\leq C\Big[\int_\delta^T\int_{G_0}|t\tilde{z}(x,t;z_0)|^2dxdt + \int_\delta^T\int_{G_0}|tv(x,t)-t\tilde{z}(x,t;z_0)|^2dxdt\Big]\\ &\leq C\int_\delta^T\int_{G_0}|t\tilde{z}(x,t;z_0)|^2dxdt + C\int_\delta^T\int_{G_0}\Big|\sum_{j=1}^\infty\frac{1}{i\lambda_j(i\lambda_j t+1)}z_{0,j}e^{-i\lambda_j t}e_j\Big|^2dxdt\\ &\leq C\int_\delta^T\int_{G_0}|t\tilde{z}(x,t;z_0)|^2dxdt + C\int_\delta^T\int_{G}\Big|\sum_{j=1}^\infty\frac{1}{i\lambda_j(i\lambda_j t+1)}z_{0,j}e^{-i\lambda_j t}e_j\Big|^2dxdt\\ &\leq C\int_\delta^T\int_{G_0}|t\tilde{z}(x,t;z_0)|^2dxdt + C\int_\delta^T\int_{G}\Big|\sum_{j=1}^\infty\frac{1}{\lambda_j^2}z_{0,j}e^{-i\lambda_j t}e_j\Big|^2dxdt\\ &\leq C\int_\delta^T\int_{G_0}|t\tilde{z}(x,t;z_0)|^2dxdt + C|z_0|^2_{H^{-4}(G)}.\end{aligned} \tag{4.24}$$

**Step 2**. In this step, we get rid of the term $|z_0|^2_{H^{-4}(G)}$ in the right hand side of (4.24) by a compactness–uniqueness argument. More precisely, we are going to prove that

$$|z_0|^2_{H^{-2}(G)} \leq C \int_\delta^T \int_{G_0} |t\tilde{z}(x,t;z_0)|^2 dxdt. \tag{4.25}$$

If (4.25) is not true, then we can find a sequence $\{z_0^n\}_{n=1}^\infty \subset L^2(G)$ with $|z_0^n|_{H^{-2}(G)} = 1$ such that

$$\int_\delta^T \int_{G_0} |t\tilde{z}(x,t;z_0^n)|^2 dxdt \leq \frac{1}{n}. \tag{4.26}$$

Since $\{z_0^n\}_{n=1}^\infty$ is bounded in $H^{-2}(G)$, we can find a subsequence $\{z_0^{n_k}\}_{k=1}^\infty \subset \{z_0^n\}_{n=1}^\infty$ such that

$$z_0^{n_k} \text{ converges weakly to some } z_0^* \in H^{-2}(G) \tag{4.27}$$

and

$$z_0^{n_k} \text{ converges strongly to } z_0^* \text{ in } H^{-4}(G). \tag{4.28}$$

According to (4.26) and (4.24), we know that

$$|z_0^{n_k}|^2_{H^{-4}(G)} \geq \frac{1}{C} - \frac{1}{n_k}.$$

This, together with (4.28), implies that there is a positive constant $C > 0$ such that

$$|z_0^*|^2_{H^{-4}(G)} \geq \frac{1}{C}. \tag{4.29}$$

Thus, the limit $z_0^*$ is non trivial.

Further, from (4.21) and (4.22), we know that $\tilde{z}(\cdot,\cdot;z_0^{n_k})$ converges weakly to $\tilde{z}(\cdot,\cdot;z_0^*)$ in $L^2(\delta,T;L^2(G))$. Hence,

$$\int_\delta^T \int_{G_0} |\tilde{z}(x,t;z_0^*)|^2 dxdt \leq \lim_{k\to\infty} \int_\delta^T \int_{G_0} |\tilde{z}(x,t;z_0^{n_k})|^2 dxdt \leq \lim_{k\to\infty} \frac{1}{\delta n_k} = 0.$$

Therefore, we find that

$$\tilde{z}(\cdot,\cdot;z_0^*) = 0 \text{ in } G_0 \times (\delta,T). \tag{4.30}$$

We would like to show that this leads to $z_0^* \equiv 0$ which would then yield to a contradiction. To do this, we introduce the linear subspace

$$\mathcal{E} \overset{\triangle}{=} \big\{ z_0 \in H^{-2}(G) : \text{ the solution to (4.10) with the initial datum } z_0 \text{ fulfills } \tilde{z}(\cdot,\cdot;z_0) = 0 \text{ in } G_0 \times (\delta,T) \big\}.$$

Clearly, $z_0^*$ given in (4.27) belongs to $\mathcal{E}$. We want to prove that $\mathcal{E} = \{0\}$, which would be in contradiction with the fact that $z_0^*$ is nonzero.

**Step 3.** To show the claim that $\mathcal{E} = \{0\}$ and conclude the proof, we proceed in several steps.

**Step 3.1.** We first prove that $\mathcal{E} \subset L^2(G)$.

To do this, given $\varepsilon \in (0, \delta)$ and any solution $\tilde{z}$, we introduce the discrete time-derivative

$$\hat{z}_\varepsilon(x,t;z_0) = \frac{\tilde{z}(x,t+\varepsilon;z_0) - \tilde{z}(x,t;z_0)}{\varepsilon} \quad \text{for } \ t \in [0, T-\delta]. \tag{4.31}$$

Then we have that

$$\begin{aligned}
&\hat{z}_\varepsilon(x,t;z_0)\\
&= \frac{1}{\varepsilon}\sum_{j=1}^{\infty} \frac{1}{i\lambda_j(t+\varepsilon)+1} z_{0,j} e^{-i\lambda_j(t+\varepsilon)} e_j - \frac{1}{\varepsilon}\sum_{j=1}^{\infty} \frac{1}{i\lambda_j t+1} z_{0,j} e^{-i\lambda_j t} e_j\\
&= \frac{1}{\varepsilon}\sum_{j=1}^{\infty} \frac{1}{i\lambda_j(t+\varepsilon)+1} z_{0,j}\big(e^{-i\lambda_j(t+\varepsilon)} - e^{-i\lambda_j t}\big) e_j + \frac{1}{\varepsilon}\sum_{j=1}^{\infty}\Big[\frac{1}{i\lambda_j(t+\varepsilon)+1} - \frac{1}{i\lambda_j t+1}\Big] z_{0,j} e^{-i\lambda_j t} e_j\\
&= \sum_{j=1}^{\infty} \frac{1}{i\lambda_j(t+\varepsilon)+1} z_{0,j} \frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon} e^{-i\lambda_j t} e_j - \sum_{j=1}^{\infty} \frac{i\lambda_j}{[i\lambda_j(t+\varepsilon)+1](i\lambda_j t+1)} z_{0,j} e^{-i\lambda_j t} e_j
\end{aligned} \tag{4.32}$$

and

$$\hat{z}_\varepsilon(x,0;z_0) = \sum_{j=1}^{\infty} \frac{1}{i\lambda_j\varepsilon+1} z_{0,j} \frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon} e_j - \sum_{j=1}^{\infty} \frac{i\lambda_j}{i\lambda_j\varepsilon+1} z_{0,j} e_j. \tag{4.33}$$

Let

$$v_\varepsilon(x,t) = \sum_{j=1}^{\infty} \frac{1}{i\lambda_j(t+\varepsilon)} z_{0,j} \frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon} e^{-i\lambda_j t} e_j - \sum_{j=1}^{\infty} \frac{1}{i\lambda_j(t+\varepsilon)} z_{0,j} e^{-i\lambda_j t} e_j. \tag{4.34}$$

Then, again, by assumption (**A3**), as in the proof of (4.23), we have

$$\Big|\frac{\tilde{z}_\varepsilon(x,\varepsilon;z_0) - \tilde{z}_\varepsilon(x,0;z_0)}{\varepsilon}\Big|^2_{H^{-2}(G)} = \big|\hat{z}_\varepsilon(x,0;z_0)\big|^2_{H^{-2}(G)} \le C\int_\delta^{T-\delta}\int_{G_0} |(t+\varepsilon)v_\varepsilon(x,t)|^2 dxdt,$$

where the constant $C$ is independent of $\varepsilon$. Thus, we obtain that

$$\begin{aligned}
&\Big|\frac{\tilde{z}_\varepsilon(x,\varepsilon;z_0) - \tilde{z}_\varepsilon(x,0;z_0)}{\varepsilon}\Big|^2_{H^{-2}(G)}\\
&\le C\int_\delta^{T-\delta}\int_{G_0} |\hat{z}_\varepsilon(x,t;z_0)|^2 dxdt + C\int_\delta^{T-\delta}\int_{G_0} |\hat{z}_\varepsilon(x,t;z_0) - v_\varepsilon(x,t)|^2 dxdt\\
&\le C\int_\delta^{T-\delta}\int_{G_0} |\hat{z}_\varepsilon(x,t;z_0)|^2 dxdt + C\int_\delta^{T-\delta}\int_{G} |\hat{z}_\varepsilon(x,t;z_0) - v_\varepsilon(x,t)|^2 dxdt.
\end{aligned} \tag{4.35}$$

Let us estimate the second term in the right hand side of (4.35). From (4.32) and (4.34), we have that

$$
\begin{aligned}
&\int_{\delta}^{T-\delta}\int_{G}|\hat{z}_{\varepsilon}(x,t;z_0)-v_{\varepsilon}(x,t)|^2dxdt\\
&\leq 2\int_{\delta}^{T-\delta}\int_{G}\Big|\sum_{j=1}^{\infty}\Big[\frac{1}{i\lambda_j(t+\varepsilon)+1}-\frac{1}{i\lambda_j(t+\varepsilon)}\Big]z_{0,j}\frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon}e^{-i\lambda_j t}e_j\Big|^2dxdt\\
&\quad+2\int_{\delta}^{T-\delta}\int_{G}\Big|\sum_{j=1}^{\infty}\Big[\frac{i\lambda_j}{[i\lambda_j(t+\varepsilon)+1](i\lambda_j t+1)}-\frac{1}{i\lambda_j(t+\varepsilon)}\Big]z_{0,j}e^{-i\lambda_j t}e_j\Big|^2dxdt\\
&\leq C\sum_{j=1}^{\infty}\frac{1}{\lambda_j^4}z_{0,j}^2\Big|\frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon}\Big|^2+C\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}z_{0,j}^2.
\end{aligned}
\tag{4.36}
$$

If $\lambda_j<\frac{1}{\varepsilon}$, then

$$
\Big|\frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon}\Big|=\frac{1}{\varepsilon}\Big|\sum_{k=1}^{\infty}\frac{1}{k!}(-i\lambda_j\varepsilon)^k\Big|=\Big|i\lambda_j\sum_{k=1}^{\infty}\frac{1}{k!}(-i\lambda_j\varepsilon)^{k-1}\Big|\leq\lambda_j\sum_{k=1}^{\infty}\frac{1}{k!}\leq 2\lambda_j. \tag{4.37}
$$

If $\lambda_j>\frac{1}{\varepsilon}$, then

$$
\Big|\frac{e^{-i\lambda_j\varepsilon}-1}{\varepsilon}\Big|=\frac{1}{\varepsilon}\big|e^{-i\lambda_j\varepsilon}-1\big|\leq\lambda_j\big|e^{-i\lambda_j\varepsilon}-1\big|\leq 2\lambda_j. \tag{4.38}
$$

According to (4.36), (4.37) and (4.38), we obtain that

$$
\int_{\delta}^{T-\delta}\int_{G}|\hat{z}_{\varepsilon}(x,t;z_0)-v_{\varepsilon}(x,t)|^2dxdt\leq C\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}z_{0,j}^2. \tag{4.39}
$$

As a result of (4.35) and (4.39), we have that

$$
\Big|\frac{\tilde{z}_{\varepsilon}(x,\varepsilon;z_0)-\tilde{z}_{\varepsilon}(x,0;z_0)}{\varepsilon}\Big|^2_{H^{-2}(G)}\leq C\int_{\delta}^{T-\delta}\int_{G_0}|\hat{z}_{\varepsilon}(x,t;z_0)|^2dxdt+C\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}z_{0,j}^2. \tag{4.40}
$$

Since $z_0\in\mathcal{E}$, we know that

$$
\hat{z}_{\varepsilon}(x,t;z_0)=0\quad\text{ in } G_0\times(\delta,T).
$$

This, together with (4.40), implies that for any $\varepsilon\in(0,\delta)$,

$$
\Big|\frac{\tilde{z}_{\varepsilon}(x,\varepsilon;z_0)-\tilde{z}_{\varepsilon}(x,0;z_0)}{\varepsilon}\Big|^2_{H^{-2}(G)}\leq C\sum_{j=1}^{\infty}\frac{z_{0,j}^2}{\lambda_j^2}\leq C|z_0|^2_{H^{-2}(G)}. \tag{4.41}
$$

Letting $\varepsilon\to 0$ in (4.41), we obtain that

$$
|z_0|^2_{L^2(G)}\leq C\Big|\sum_{j=1}^{\infty}i\lambda_j z_{0,j}e_j\Big|^2_{H^{-2}(G)}\leq C\sum_{j=1}^{\infty}\frac{z_{0,j}^2}{\lambda_j^2}\leq C|z_0|^2_{H^{-2}(G)}. \tag{4.42}
$$

Then, we have that $\mathcal{E}\subset L^2(G)$.

**Step 3.2.** The same estimate deduces that $\mathcal{E}$ is a finite-dimensional subspace of $L^2(G)$, since

$$|z_0|^2_{L^2(G)} \le C|z_0|^2_{H^{-2}(G)}. \tag{4.43}$$

**Step 3.3.** We now claim that $A_\Delta \mathcal{E} \subset \mathcal{E}$.

Utilizing (4.21) again, we have that

$$\tilde{z}(\cdot,\cdot;z_0) \in C([0,T];L^2(G)) \text{ for any } z_0 \in \mathcal{E}.$$

Since $\tilde{z}(\cdot,\cdot;z_0) = 0$ in $G_0 \times (\delta, T)$ for any $\delta > 0$, we see that $z_0 = 0$ in $G_0$.

Thanks to $\mathcal{E} \subset L^2(G)$, we get that $A_\Delta \mathcal{E} \subset H^{-2}(G)$ and $\tilde{z}(x,t;A_\Delta z_0) = A_\Delta \tilde{z}(x,t;z_0) = 0$ in $G_0 \times (\delta, T)$. Therefore, we obtain that $A_\Delta \mathcal{E} \subset \mathcal{E}$.

**Step 3.4.** To conclude, assume that $\mathcal{E} \neq \{0\}$. Then, there would exist a non-trivial eigenfunction $\psi \in \mathcal{E}$ and an eigenvalue $\mu \in \mathbb{R}$ such that

$$\begin{cases} -\Delta\psi = \mu\psi & \text{in } G, \\ \psi = 0 & \text{on } \partial G, \\ \psi = 0 & \text{in } G_0. \end{cases}$$

However, by the classical unique continuation property for elliptic equations, $\psi = 0$ in $G$, which contradicts that $\psi$ is an eigenfunction.

This concludes the proof of the fact that $\mathcal{E} = \{0\}$.

As a consequence, we derive the desired estimate (4.20). □

**Remark 4.7** *We have utilized a compactness-uniqueness argument in the above proof, which has been extensively used in the proof of observability estimates (see [5] for example). Note however that, normally, this is done for solutions of PDE models. The averages under consideration not being solutions of a specific PDE this argument needs to be carefully adapted as we have done above.*

# 5 Further comments and open problems

## 5.1 Further comments

### 5.1.1 Boundary averaged control for random heat equations

In this subsection, for convenience, we assume that $\partial G$ is $C^\infty$ smooth, although most comments and results make sense with different weaker regularity assumptions.

We have solved the internal averaged controllability problems for some particular classes of random heat and Schrödinger equations. The same could be done for boundary control problems.

Let us consider the following heat equation with boundary control and random constant diffusivity:

$$\begin{cases} y_t - \alpha\Delta y = 0 & \text{in } G \times (0,T], \\ y = u & \text{on } \Gamma_0 \times (0,T), \\ y = 0 & \text{on } (\partial G \setminus \Gamma_0) \times (0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5.1}$$

Here $\Gamma_0$ is an open subset of $\partial G$, $\alpha(\cdot)$ is a random variable, $u \in L^2(0,T;L^2(\Gamma_0))$ and $y_0 \in L^2(G)$.

We have the following result.

**Theorem 5.1** *Let $\alpha(\cdot)$ be a uniformly distributed or an exponentially distributed random variable. The system* (5.1) *is null controllable in average with control $u(\cdot) \in L^2(0,T;L^2(\Gamma_0))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^2(0,T;L^2(\Gamma_0))} \leq C|y_0|_{L^2(G)}. \tag{5.2}$$

To prove Theorem 5.1, we consider the adjoint system of (5.1) as follows:

$$\begin{cases} z_t + \alpha \Delta z = 0 & \text{in } G \times (0,T], \\ z = 0 & \text{on } \partial G \times (0,T), \\ z(T) = z_0 & \text{in } G. \end{cases} \tag{5.3}$$

One only need to prove the following result.

**Theorem 5.2** *There exists a constant $C > 0$ such that for any $z_0 \in L^2(G)$, and either $\alpha(\cdot)$ is a uniformly distributed or an exponentially distributed random variable, it holds that*

$$\Big| \int_\Omega z(\cdot,0,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_{L^2(G)} \leq C\int_0^T\int_{\Gamma_0}\Big|\int_\Omega \frac{\partial z(x,t,\omega;z_0)}{\partial\nu}d\mathbb{P}(\omega)\Big|^2 d\Gamma_0 dt. \tag{5.4}$$

The proof is very similar to the one for Theorem 3.2. We only give a sketch here. Let us assume that $\alpha(\cdot)$ is an exponentially distributed random variable. From [28, Page 345], we have the following result:

$$\sum_{\lambda_j\leq r} a_j^2 \leq C_2 e^{C_2\sqrt{r}} e^{\frac{1}{t_2-t_1}} \int_{t_1}^{t_2}\int_{\Gamma_0}\Big|\sum_{\lambda_j\leq r} e^{\sqrt{\lambda_j}t} a_j \frac{\partial e_j}{\partial\nu}\Big|^2 d\Gamma_0 dt \quad \text{for any } 0\leq t_1<t_2\leq T. \tag{5.5}$$

Let $T_k = (1-\frac{1}{2^{k-1}})T$ for $k\in\mathbb{N}$ and $r_k = 2^{2(k+1)}[\ln(6C_2)+C_2]$. Similar to the proof of (3.25), we can obtain that

$$\begin{aligned} &\frac{T_{k+1}-T_k}{6C_2e^{C_2\sqrt{r_k}}}|\tilde z(\cdot,T_k;z_0)|_{L^2(G)} - \frac{C_2e^{C_2\sqrt{r_k}}+1}{C_2e^{C_2\sqrt{r_k}}}(T_{k+1}-T_k)e^{-\frac{T_{k+1}-T_k}{2}r_k}|\tilde z(\cdot,T_{k+1};z_0)|_{L^2(G)} \\ &\leq \int_{T_k}^{T_{k+1}}\int_{\Gamma_0}\Big|\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt. \end{aligned} \tag{5.6}$$

By summarizing the inequality (5.6) from $k=1$ to $k=\infty$, we obtain that

$$\frac{T_2-T_1}{6C_2e^{C_2\sqrt{r_1}}}|z_0|_{L^2(G)} + \sum_{k=1}^{\infty} f_k|\tilde z(\cdot,T_{k+1};z_0)|_{L^2(G)} \leq \int_0^T\int_{\Gamma_0}\Big|\frac{\partial\tilde z(x,t;z_0)}{\partial\nu}\Big|d\Gamma_0 dt, \tag{5.7}$$

where

$$f_k = \frac{T_{k+2}-T_{k+1}}{6C_2e^{C_2\sqrt{r_{k+1}}}} - \frac{C_2e^{C_2\sqrt{r_k}}+1}{C_2e^{C_2\sqrt{r_k}}}(T_{k+1}-T_k)e^{-\frac{T_{k+1}-T_k}{2}r_k}, \quad k=1,2,\cdots$$

From (3.18) and $r_k = 2^{2(k+1)}[\ln(6C_2) + C_2]$, we have that

$$f_k \geq 0 \ \text{ for any } \ k = 1, 2, \cdots$$

This, together with (5.7), deduces that

$$|z_0|^2_{L^2(G)} \leq \frac{6C_2 e^{C_2\sqrt{r_1}}}{T_2 - T_1} \int_0^T \int_{\Gamma_0} \Big|\frac{\partial \tilde{z}(x,t;z_0)}{\partial \nu}\Big|^2 d\Gamma_0 dt. \tag{5.8}$$

This completes the proof. □

### 5.1.2 Boundary averaged control for random Schrödinger equations

Consider the following random Schrödinger equations:

$$\begin{cases} y_t - \alpha i \Delta y = 0 & \text{in } G \times (0,T], \\ y = u & \text{on } \Gamma_0 \times (0,T), \\ y = 0 & \text{on } (\partial G \setminus \Gamma_0) \times (0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5.9}$$

Here $\Gamma_0$ is an open subset of $\partial G$, $\alpha(\cdot)$ is a random variable, $u \in L^2(0,T;L^2(\Gamma_0))$ and $y_0 \in L^2(G)$. We have the following controllability results.

**Theorem 5.3** *System* (5.9) *is null controllable in average if $\alpha$ is a random variable with normal distribution or Cauchy distribution.*

Further, we assume the following condition holds:

(**A4**) Whatever $T > 0$ and $k \neq 0$ are, there are constants $C_3$ and $C_4$ such that for any $\varphi_0 \in H_0^1(G)$, the solution $\varphi(\cdot,\cdot)$ to (4.13) satisfies

$$|\varphi_0|^2_{H_0^1(G)} \leq C_3 \int_0^T \int_{G_0} \Big|\frac{\partial \varphi}{\partial \nu}\Big|^2 d\Gamma_0 dt \leq C_4 |\varphi_0|^2_{H_0^1(G)}. \tag{5.10}$$

We refer the readers to [27] for the conditions on $\Gamma_0$ for which (**A4**) holds.

**Theorem 5.4** *Assume that (**A4**) holds. We have the following results:*

- *Let $\alpha(\cdot)$ be a uniformly distributed random variable on an interval $[a,b]$. Then, the system* (4.1) *is exactly controllable in average with $V = H = L^2(G)$ and $U = H^{-1}(\Gamma_0)$.*
- *Let $\alpha(\cdot)$ be an exponentially distributed random variable. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^1(G)$ and $U = L^2(\Gamma_0)$.*
- *Let $\alpha(\cdot)$ be a random variable with Laplace distribution. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^3(G)$ and $U = L^2(\Gamma_0)$.*
- *Let $\alpha(\cdot)$ be a random variable with standard Chi-squared distribution. Then, the system* (4.1) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^{k-1}(G)$ and $U = L^2(\Gamma_0)$.*

As usual, we introduce the adjoint system of (5.9) as follows:

$$\begin{cases} z_t + i\alpha\Delta z = 0 & \text{in } G \times (0,T), \\ z = 0 & \text{on } \partial G \times (0,T), \\ z(T) = z_0 & \text{in } G. \end{cases} \tag{5.11}$$

We can prove the following results.

**Theorem 5.5** *Let $\alpha(\cdot)$ be a standard normally distributed random variable. There exists a constant $C > 0$ such that for any $z_0 \in L^2(G)$, it holds that*

$$\Big|\sum_{j=1}^{\infty} z_{0,j} e^{-\lambda_j^2 T^2} e_j\Big|^2_{L^2(G)} \le C \int_0^T \int_{\Gamma_0} \Big|\int_\Omega \frac{\partial z(x,t,\omega;z_0)}{\partial \nu} d\mathbb{P}(\omega)\Big|^2 d\Gamma_0 dt. \tag{5.12}$$

**Theorem 5.6** *Let $\alpha(\cdot)$ be a random variable with standard Cauchy distribution. There exists a positive constant $C$ such that for any $z_0 \in L^2(G)$, it holds that*

$$\Big|\sum_{j=1}^{\infty} z_{0,j} e^{-\lambda_j T} e_j\Big|^2_{L^2(G)} \le C \int_0^T \int_{\Gamma_0} \Big|\int_\Omega \frac{\partial z(x,t,\omega;z_0)}{\partial \nu} d\mathbb{P}(\omega)\Big|^2 d\Gamma_0 dt. \tag{5.13}$$

By means of (4.12), Theorem 5.6 is nothing but the boundary observability estimate for heat equations. The proof of Theorem 5.5 is very similar to the one for Theorem 5.2. We omit it here.

Further, the proof of Theorem 5.4 is also very analogous to the proofs for Theorems 4.3. We only give a sketch of the proof of the second conclusion in Theorem 5.4.

Let $z_0 \in H^{-1}(G)$ in (4.10) and $\tilde z(x,t;z_0) = \int_\Omega z(x,T-t,\omega;z_0)d\mathbb{P}(\omega)$. Then, we only need to prove that

$$|z_0|^2_{H^{-1}(G)} \le C \int_0^T \int_{\Gamma_0} \Big|\frac{\partial \tilde z(x,t;z_0)}{\partial \nu}\Big|^2 d\Gamma_0 dt. \tag{5.14}$$

Assume that $z_0 = \sum_{j=1}^\infty z_{0,j} e_j$. We have

$$\tilde z(x,t;z_0) = \sum_{j=1}^{\infty} \frac{1}{i\lambda_j t + 1} z_{0,j} e^{-i\lambda_j t} e_j. \tag{5.15}$$

Let

$$v(x,t) = \sum_{j=1}^{\infty} \frac{1}{i\lambda_j t} z_{0,j} e^{-i\lambda_j t} e_j.$$

From (**A4**), for a fixed $\delta \in (0,T)$, we have that

$$|z_0|^2_{H^{-1}(G)} = \Big|\sum_{j=1}^{\infty} \frac{1}{i\lambda_j} z_{0,j} e^{-i\lambda_j t} e_j\Big|^2_{H_0^1(G)} \le C \int_\delta^T \int_{\Gamma_0} \Big|t\frac{\partial v(x,t)}{\partial \nu}\Big|^2 d\Gamma_0 dt. \tag{5.16}$$

Therefore,

$$
\begin{aligned}
&|z_0|^2_{H^{-1}(G)}\\
&\le C\int_\delta^T\int_{\Gamma_0}\Big|t\frac{\partial v(x,t)}{\partial\nu}\Big|^2 d\Gamma_0 dt\\
&\le C\Big[\int_\delta^T\int_{\Gamma_0}\Big|t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt+\int_\delta^T\int_{\Gamma_0}\Big|t\frac{\partial v(x,t)}{\partial\nu}-t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt\Big]\\
&\le C\int_\delta^T\int_{\Gamma_0}\Big|t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt+C\int_\delta^T\Big|\sum_{j=1}^\infty\frac{1}{i\lambda_j(i\lambda_j t+1)}z_{0,j}e^{-i\lambda_j t}e_j\Big|^2_{H^1(G)}dt\\
&\le C\int_\delta^T\int_{\Gamma_0}\Big|t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt+C|z_0|^2_{H^{-3}(G)}.
\end{aligned}
\tag{5.17}
$$

We claim that

$$
|z_0|^2_{H^{-1}(G)}\le C\int_0^T\int_{\Gamma_0}\Big|t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt. \tag{5.18}
$$

If (5.18) is not true, then we can find a sequence $\{z_0^n\}_{n=1}^\infty\subset L^2(G)$ with $|z_0^n|_{H^{-2}(G)}=1$ such that

$$
\int_0^T\int_{\Gamma_0}\Big|t\frac{\partial \tilde z(x,t;z_0)}{\partial\nu}\Big|^2 d\Gamma_0 dt\le\frac{1}{n}. \tag{5.19}
$$

Since $\{z_0^n\}_{n=1}^\infty$ is bounded in $H^{-1}(G)$, we can find a subsequence $\{z_0^{n_k}\}_{k=1}^\infty\subset\{z_0^n\}_{n=1}^\infty$ such that $z_0^{n_k}$ converges weakly to some $z_0^*\in H^{-1}(G)$. From (5.19), we know that

$$
|z_0^*|^2_{H^{-3}(G)}\ge\frac{1}{C} \tag{5.20}
$$

for a positive constant and

$$
\frac{\partial\tilde z(\cdot,\cdot;z_0^*)}{\partial\nu}=0\ \text{ on }\Gamma_0\times(0,T). \tag{5.21}
$$

Put

$$
\widetilde{\mathcal E}\overset{\triangle}{=}\{z_0\in H^{-1}(G):\ \text{ the solution to (5.11) with the final datum } z_0 \text{ fulfills (5.21)}\}.
$$

Analogous to the proof that $\mathcal E=\{0\}$, we can prove that $\widetilde{\mathcal E}=\{0\}$, which contradicts (5.20). Hence, we know that (5.18) holds. □

### 5.1.3 Averaged control for random heat and Schrödinger equations from measurable sets

We have considered the averaged control problems of random heat and Schrödinger equations for the internal control (*resp.* boundary control) supported in $G_0\times E$ (*resp.* $\Gamma_0\times E$), where $G_0\subset G$ (*resp.* $\Gamma_0\subset\partial G$) is a nonempty open subset. By means of the method in [3], one can consider the case that the internal control (*resp.* the boundary control) is supported in a measurable subset $\mathcal{D} \subset G\times(0,T)$ (*resp.* $\mathcal{G} \subset \partial G\times(0,T)$). For instance, we can consider the following systems:

$$\begin{cases} y_t - \alpha\Delta y = \chi_{\mathcal{D}} u & \text{in } G \times (0,T), \\ y = 0 & \text{on } \partial G \times (0,T), \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{5.22}$$

and

$$\begin{cases} y_t - i\alpha\Delta y = \chi_{\mathcal{D}} u & \text{in } G \times (0,T), \\ y = 0 & \text{on } \partial G \times (0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5.23}$$

Here $y_0 \in L^2(G)$, $\mathcal{D} \subset G \times (0,T)$ is a Lebesgue measurable set with positive Lebesgue measure and $u \in L^\infty(G \times (0,T))$. One can combine the proof of Theorem 3.1 and Corollary 1 in [3] to prove the following results.

**Theorem 5.7** *Assume that, either $\alpha(\cdot)$ is a uniformly or exponentially distributed random variable. Then, the system (5.22) is null controllable in average with control $u(\cdot) \in L^\infty(G \times (0,T))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^\infty(G\times(0,T))} \le C|y_0|_{L^2(G)}.$$

**Theorem 5.8** *If $\alpha$ is a random variable with normal distribution or Cauchy distribution, then the system (5.23) is null controllable in average with control $u(\cdot) \in L^\infty(G \times (0,T))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^\infty(G\times(0,T))} \le C|y_0|_{L^2(G)}.$$

Next, let us consider the following systems:

$$\begin{cases} y_t - \alpha\Delta y = 0 & \text{in } G \times (0,T], \\ y = u & \text{on } \mathcal{G}, \\ y = 0 & \text{on } [\partial G \times (0,T)] \setminus \mathcal{G}, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{5.24}$$

and

$$\begin{cases} y_t - \alpha i\Delta y = 0 & \text{in } G \times (0,T], \\ y = u & \text{on } \mathcal{G}, \\ y = 0 & \text{on } [\partial G \times (0,T)] \setminus \mathcal{G}, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5.25}$$

Here $\mathcal{G}$ is a Lebesgue measurable subset of $\partial G \times (0,T)$ with positive Lebesgue measure, $\alpha(\cdot)$ is a random variable, $u \in L^\infty(\partial G \times (0,T))$ and $y_0 \in L^2(G)$.

One can combine the proof of Theorem 5.1 and Corollary 1 in [3] to prove the following results.

**Theorem 5.9** *Assume that, either $\alpha(\cdot)$ is a uniformly or exponentially distributed random variable. Then, the system* (5.24) *is null controllable in average with control $u(\cdot) \in L^\infty(G \times (0,T))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^\infty(\partial G\times(0,T))} \leq C|y_0|_{L^2(G)}.$$

**Theorem 5.10** *If $\alpha$ is a random variable with normal distribution or Cauchy distribution, then the system* (5.25) *is null controllable in average with control $u(\cdot) \in L^\infty(G \times (0,T))$. Further, there is a constant $C > 0$ such that*

$$|u|_{L^\infty(\partial G\times(0,T))} \leq C|y_0|_{L^2(G)}.$$

### 5.1.4 Internal averaged control for random fractional Schrödinger equations

We have studied the averaged controllability problems for some random heat equations and random Schrödinger equations. In the results proved so far we have obtained averaged controllability for parameter-depending equations that were controllable for each value of the parameter. Here, we give an example of model which is not null controllable when one fixes an $\omega$ but that gains null controllability by the averaging process.

Consider the following equation:

$$\begin{cases} iy_t + \alpha A_\Delta^\gamma y = Bu \quad \text{in } (0,T], \\ y(0) = y_0. \end{cases} \tag{5.26}$$

Here $\gamma \in (\frac{1}{4}, \frac{1}{2})$, $y_0 \in L^2(G)$, $u \in L^2(0,T;L^2(G_0))$ and $Bu = \chi_{G_0} u$.

The adjoint system of (5.26) reads

$$\begin{cases} iz_t - \alpha A_\Delta^\gamma z = 0 \quad \text{in } [0,T), \\ z(T) = z_0, \end{cases} \tag{5.27}$$

where $z_0 \in L^2(G)$. Assume that $z_0 = \sum_{j=1}^\infty z_{0,j} e_j$. If $\alpha(\cdot) : \Omega \to \mathbb{R}$ is a standard normally distributed random variable, then we know that

$$\begin{aligned} \int_\Omega z(x,t,\omega;z_0)d\mathbb{P}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}} \sum_{j=1}^\infty z_{0,j} e^{-i\lambda_j^\gamma \alpha(T-t)} e_j d\alpha \\ &= \sum_{j=1}^\infty z_{0,j} e^{-\lambda_j^{2\gamma}(T-t)^2} e_j. \end{aligned} \tag{5.28}$$

Similar to the proof of Theorem 4.1, we can establish the following result.

**Theorem 5.11** *Let $E \subset [0,T]$ be a measurable set with positive Lebesgue measure $m(E)$. There exists a constant $C > 0$ such that for any $z_0 \in L^2(G)$, it holds that*

$$\Big| \sum_{j=1}^\infty z_{0,j} e^{-\lambda_j^{2\gamma} T^2} e_j \Big|_{L^2(G)} \leq C \int_E \Big| \int_\Omega z(x,t,\omega;z_0) d\mathbb{P}(\omega) \Big|_{L^2(G_0)} dt. \tag{5.29}$$

Theorem 5.11 implies that the system (5.26) is null controllable in average. However, it is not null controllable for any fixed $\alpha \in \mathbb{R}$ even for $d = 1$. For example, let $\alpha = 1$ and $G = (0, 1)$. Then the solution to the system (5.27) reads

$$z(x,t) = \sqrt{2}\sum_{j=1}^{\infty} z_{0,j} e^{-i(j\pi)^{2\gamma} t} \sin(j\pi x).$$

Since

$$\lim_{j\to\infty} \left\{ [(j+1)\pi)]^{2\gamma} - (j\pi)^{2\gamma} \right\} = 0,$$

we know that the following inequality

$$|z(\cdot,T)|^2_{L^2(0,1)} = \sum_{j=1}^{\infty} z_{0,j}^2 \le C \int_0^T \int_{G_0} |z(x,t)|^2 dxdt$$

does not hold for any $C > 0$.

### 5.1.5 Internal averaged control for random heat equations with variable coefficients

We can also consider the approximate averaged controllability problem of some more general random heat equations. We make the following assumptions on the coefficients $a^{jk} : \overline{G} \times \Omega \to \mathbb{R}^{n\times n}$ $(j,k = 1,2,\cdots,n)$:

**(H1)** *$a^{jk}(\cdot,\omega) : \overline{G} \to \mathbb{R}$ is analytic, $\mathbb{P}$-a.s., and $a^{jk} = a^{kj}$.*

**(H2)** *For a.e. $\omega \in \Omega$, there is a constant $C(\omega) > 0$ such that for any multi-index $\eta = (\eta_1,\cdots,\eta_n) \in (\mathbb{N}\cup\{0\})^n$,*

$$\left|\frac{\partial^\eta a^{jk}(x,\omega)}{\partial x^\eta}\right| \le \frac{C(\omega)|\eta|!}{R^{|\eta|}}, \qquad \text{for any } j,k = 1,\cdots,n,$$

*where $R$ is a positive constant larger than $\max_{x\in\overline{G}} |x|$, and $C(\cdot)$ satisfies that*

$$\int_\Omega C(\omega) d\mathbb{P}(\omega) < \infty.$$

**(H3)** *There exists a constant $s_0 > 0$ such that*

$$\sum_{j,k=1}^{n} a^{jk}(\omega,t,x)\xi^j\xi^k \ge s_0|\xi|^2, \quad \forall\, (\omega,x,\xi) \equiv (\omega,x,\xi^1,\cdots,\xi^n) \in \Omega\times G\times\mathbb{R}^n. \tag{5.30}$$

Consider the following random heat equation:

$$\begin{cases} y_t - \displaystyle\sum_{j,k=1}^{n} \left(a^{jk} y_{x_j}\right)_{x_k} = \chi_{G_0} u & \text{in } G\times(0,T), \\ y = 0 & \text{on } \partial G\times(0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5.31}$$

Here the initial datum $y_0 \in L^2(G)$.

We have the following result.

**Theorem 5.12** *Under the assumptions* **(H1)**–**(H3)** *above system* (5.31) *is approximately controllable in average in any time* $T > 0$ *and from any open non-empty subset* $G_0$ *of* $G$.

*Proof*: By Theorem A.3, we only need to prove that the adjoint system of (5.31) satisfies the averaged unique continuation property. Its adjoint system reads

$$\begin{cases} z_t + \sum_{j,k=1}^{n} \left(a^{jk} z_{x_j}\right)_{x_k} = 0 & \text{in } G \times (0,T), \\ z = 0 & \text{on } \partial G \times (0,T), \\ z(T) = z_0 & \text{in } G, \end{cases} \tag{5.32}$$

where the final datum $z_0 \in L^2(G)$. From **(H1)** and **(H3)**, we know that for any $t \in [0,T)$ and a.e. $\omega \in \Omega$, the solution $z(\cdot, t, \omega; z_0)$ is analytic in $G$ (see [14, 22] for example). Further, for any ball $B_r \subset G$ with radius $r$, there is a constant $C > 0$ such that for any multi-index $\eta \in (\mathbb{N} \cup \{0\})^n$,

$$\left|\frac{\partial^\eta z(\cdot, t, \omega; z_0)}{\partial x^\eta}\right| \leq CC(\omega)\frac{|\eta|!}{r^{|\eta|}} \text{ in } B_r.$$

From **(H2)**, we have that

$$\left|\frac{\partial^\eta \int_\Omega z(\cdot, t, \omega) d\mathbb{P}(\omega)}{\partial x^\eta}\right| \leq C\frac{|\eta|!}{r^{|\eta|}}.$$

Then, we know that $\int_\Omega z(\cdot, t, \omega) d\mathbb{P}(\omega)$ is analytic in $B_r$. Hence, it is analytic in $G$. Since $\int_\Omega z(\cdot, \cdot, \omega) d\mathbb{P}(\omega) = 0$ in $G_0 \times (0,T)$, we get that it vanishes everywhere in $G \times (0,T)$. Noting that it is continuous in $L^2(G)$ with respect to $t$, we conclude that $z_0 = 0$ in $G$, which implies that (5.32) satisfies the averaged unique continuation property. □

### 5.1.6 Averaged controllability problems for the random heat and the random Schrödinger equations random initial data

One can also consider the internal and boundary averaged controllability problems for random heat and random Schrödinger equations with random initial data. Let us first consider the following random heat equation:

$$\begin{cases} y_t - \alpha \Delta y = \chi_{G_0 \times E} u & \text{in } G \times (0,T), \\ y = 0 & \text{on } \partial G \times (0,T), \\ y(0,\omega) = y_0(\omega) & \text{in } G, \end{cases} \tag{5.33}$$

Here $y_0(\cdot) \in L^2(\Omega; L^2(G))$, and $G_0$ and $E$ are subsets of $G$ and $[0,T]$ respectively, where the controls are being applied. The constant diffusivity $\alpha : \Omega \to \mathbb{R}^+$ is assumed to be a random variable.

According to Remark A.1, we know that to prove the averaged null controllability of (5.33), we only need to establish the following observability estimate:

$$\left(\int_\Omega \left|z(0,\omega; z_0)\right|^2_{L^2(G)} d\mathbb{P}(\omega)\right)^{\frac{1}{2}} \leq C \int_E \left|\int_\Omega z(t,\omega; z_0) d\mathbb{P}(\omega)\right|_{L^2(G_0)} dt, \tag{5.34}$$

where $z$ solves (3.7) and $C$ is independent of $z_0$.

If $\alpha(\cdot)$ is the exponentially distributed random variable, then

$$\begin{aligned}\int_\Omega \Big|z(0,\omega;z_0)\Big|^2_{L^2(G)} d\mathbb{P}(\omega) &= \int_1^\infty e^{-(\alpha-1)} \sum_{j=1}^\infty z_{0,j}^2 e^{-2\lambda_j \alpha T} d\alpha \\ &= \sum_{j=1}^\infty \frac{1}{2\lambda_j T+1} z_{0,j}^2 e^{-2\lambda_j T}.\end{aligned} \tag{5.35}$$

From (5.35), similar to the proof of the inequality (3.8), one can obtain (5.34). The same thing can be done if $\alpha(\cdot)$ is a uniformly distributed random variable on $[a,b]$ for $0<a<b$.

**Proposition 5.1** *Let* (**A1**) *and* (**A2**) *hold. Assume that, either* $\alpha(\cdot)$ *is a uniformly or exponentially distributed random variable. Then, the system* (5.33) *is null controllable in average with control* $u(\cdot)\in L^2(0,T;L^2(G_0))$*. Further, there is a constant* $C>0$ *such that*

$$|u|_{L^2(0,T;L^2(G_0))} \le C|y_0|_{L^2(\Omega;L^2(G))}. \tag{5.36}$$

Thanks to Remark A.1, we know that in order to show that (5.33) is approximately controllable in average, one just needs to prove that the solution to (3.7) satisfies the averaged unique continuation property, which is obtained in the proof of Theorem 3.3. Hence, we have the following result:

**Proposition 5.2** *Let* (**A1**) *and* (**A2**) *hold. System* (5.33) *is approximately controllable in average, provided that* $\alpha(\cdot)$ *is a uniformly distributed or an exponentially distributed random variable.*

Next, we consider the averaged controllability problem for the following random Schrödinger equation:

$$\begin{cases} y_t - i\alpha\Delta y = \chi_{G_0\times E} u & \text{in } G\times(0,T), \\ y=0 & \text{on } \partial G\times(0,T), \\ y(0)=y_0 & \text{in } G. \end{cases} \tag{5.37}$$

Here $y_0(\cdot)\in L^2(\Omega;V)$, and $G_0$ and $E$ are subsets of $G$ and $[0,T]$ respectively, where the controls are being applied. $\alpha:\Omega\to\mathbb{R}$ is assumed to be a random variable.

In virtue of Remark A.1, we know that to get the averaged exact controllability of the system (5.37), one only need to prove that the solution to the equation (4.10) is exactly averaged observable. As a result of these facts, we know that the conclusions in Theorem 4.3 also hold for the system (5.37). More precisely, we have the following results:

**Proposition 5.3** *The following results hold:*

- *Let* $\alpha(\cdot)$ *be a uniformly distributed random variable on an interval* $[a,b]$*. Then, the system* (5.37) *is exactly controllable in average with* $V=H=L^2(G)$ *and* $U=H^{-2}(G_0)$*.*

- *Let* $\alpha(\cdot)$ *be an exponentially distributed random variable. Then, the system* (5.37) *is exactly controllable in average with* $H=L^2(G)$*,* $V=H_0^2(G)$ *and* $U=L^2(G_0)$*.*

- *Let $\alpha(\cdot)$ be a random variable with Laplace distribution. Then, the system* (5.37) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^4(G)$ and $U = L^2(G_0)$.*

- *Let $\alpha(\cdot)$ be a random variable with standard Chi-squared distribution. Then, the system* (5.37) *is exactly controllable in average with $H = L^2(G)$, $V = H_0^k(G)$ and $U = L^2(G_0)$.*

Further, let us consider the averaged null controllability problem for the system (5.37). Due to Remark A.1, we only need to prove the following observability estimate:

$$\Big( \int_\Omega \Big| z(0,\omega;z_0) \Big|^2_{L^2(G)} d\mathbb{P}(\omega) \Big)^{\frac{1}{2}} \leq C \int_E \Big| \int_\Omega z(x,t,\omega;z_0) d\mathbb{P}(\omega) \Big|_{L^2(G_0)} dt. \tag{5.38}$$

When $\alpha(\cdot)$ is a normally distributed random variable, we have that

$$\begin{aligned} \int_\Omega \Big| z(0,\omega;z_0) \Big|^2_{L^2(G)} d\mathbb{P}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}} \sum_{j=1}^{\infty} z_{0,j}^2 e^{-2i\lambda_j \alpha T} d\alpha \\ &= \sum_{j=1}^{\infty} z_{0,j}^2 e^{-2\lambda_j^2 t^2}. \end{aligned} \tag{5.39}$$

When $\alpha(\cdot)$ is a random variable with the Cauchy distribution, we have that

$$\begin{aligned} \int_\Omega \Big| z(0,\omega;z_0) \Big|^2_{L^2(G)} d\mathbb{P}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\pi(1+\alpha^2)} \sum_{j=1}^{\infty} z_{0,j}^2 e^{-2i\lambda_j \alpha T} d\alpha \\ &= \sum_{j=1}^{\infty} z_{0,j}^2 e^{-2\lambda_j T}. \end{aligned} \tag{5.40}$$

Thanks to (5.39) and (5.39), and similar to the proof of Theorem 3.2, one can show that the inequality (5.38) holds when $\alpha(\cdot)$ is a random variable with normal or Cauchy distribution. Therefore, we have the following result:

**Proposition 5.4** *Under the assumptions of Theorem 4.1 the system* (5.37) *is null controllable in average if $\alpha$ is a random variable with normal or Cauchy distribution.*

## 5.2 Open problems

There are many interesting and important (at least for us) problems in this topic. We present some of them here briefly:

- **Averaged controllability problems for general random heat and Schrödinger equations.**

  We have only studied the averaged controllability problems for some very special classes of random heat and Schrödinger equations, for which the averaged dynamics could be computed explicitly. It would be interesting to investigate some more general classes of parameter dependent systems. For example, is it (3.1) null controllable in average if we take $\alpha$ to be a random variable with Chi-squared distribution?

Furthermore, the method we use to study (3.1) and (4.1) depends on the fact that the eigenfunctions of $-\alpha A_\Delta$ are independent of $\omega$. Thus, more general random heat and Schrödinger equations (as, for instance, (5.31)) where the principal part of the PDE depends on $\omega$ cannot be treated in this way.

The method of proof employed to show the approximate averaged controllability of (5.31) is based on the use of the space-time analyticity properties of solutions (5.32) to derive unique continuation properties, but it does not provide any quantitative information.

- **The relationship between averaged controllability properties and the random variable**.

  We have shown that different random variable $\alpha(\cdot)$ may lead to different controllability property of the system (4.1). It is interesting to give a description of the relationship between the random variable $\alpha(\cdot)$ and the controllability property of the system (4.1). For instance, for what kind of random variables, the system (4.1) is null controllable in average? Is there a random variable such that the system (4.1) is neither exactly controllable in average nor null controllable in average?

- **The null averaged controllability problem for random fractional heat equations**.

  We have proven that the random fractional Schrödinger equations are null controllable in average for $\gamma \in (\frac{1}{2}, \frac{1}{2})$. It is more interesting to study the same problem for random fractional heat equations, which describe the anomalous diffusion process. In particularly, consider the following system:

$$\begin{cases} y_t + \alpha A_\Delta^\gamma y = Bu & \text{in } (0, T], \\ y(0) = y_0. \end{cases} \tag{5.41}$$

  Here $\gamma > 0$, $y_0 \in L^2(G)$, $u \in L^2(0, T; L^2(G_0))$ and $Bu = \chi_{G_0} u$.

  It is clear that when $\gamma \le \frac{1}{2}$ and $\alpha(\cdot)$ is a uniformly or exponentially distributed random variables, the system (5.41) is not null controllable in average. However, is it possible to find some random variable $\alpha(\cdot)$ such that the system (5.41) is null controllable in average for some $\gamma \in (0, \frac{1}{2}]$?

- **Averaged controllability problems for nonlinear random evolution equations**.

  We have studied the averaged controllability problems for linear random evolution equations. The same problem can be considered for nonlinear random evolution equations. A possible method to handle the nonlinear problem is to follow what people do for the classical controllability problems, that is, combining the controllability result and some fixed point theorem or inverse mapping theorem. However, as the average of the solution of linear transport equation with respect to velocity helps people study the nonlinear transport equations, we expect that one can get better result than the ones obtained by applying the method mentioned above directly. For example, can one prove that a random Schrödinger equation with a cubic term is exactly or null controllable in average?

- **Numerical approximation of averaged controls.**

  The numerical approximation of control problems is studied extensively in the literature. We refer the readers to [11, 16, 46] and the rich references therein for this topic. This question also arises in the context of averaged control. This can be done based on the variational characterization of the average controls given in this paper.

# A Appendix

## A.1 Reduction of averaged controllability to averaged observability

We have the following results.

**Theorem A.1** *System* (2.1) *is exactly controllable in average in* $E$ *with the control cost* $C>0$ *if and only if the adjoint system* (2.7) *is exactly observable in average in* $E$.

*Proof of Theorem A.1*: The "if" part. Let us define a linear subspace $\mathcal{X}\subset L^2(E;U)$ as

$$\mathcal{X}\triangleq\Big\{\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega):\ z_0\in V'\Big\}.$$

and a linear functional $F$ on $\mathcal{X}$ as

$$F\Big(\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)\Big)=\langle y_1,z_0\rangle_{V,V'}-\Big\langle y_0,\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}.$$

From (2.8), we know that

$$\begin{aligned}&F\Big(\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)\mathbb{P}(\omega)\Big)\\&\leq C\big(|y_0|_V+|y_1|_V\big)\Big(|z_0|_{V'}+\Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|_{V'}\Big)\\&\leq C\big(|y_0|_V+|y_1|_V\big)\Big|\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)\mathbb{P}(\omega)\Big|_{L^2(0,T;U)}.\end{aligned}$$

Hence, $F$ is a bounded linear functional on $\mathcal{X}$ with norm $|F|_{\mathcal{L}(\mathcal{X},\mathbb{R})}\leq C(|y_0|_V+|y_1|_V)$. Then, it can be extended to a bounded linear functional on $L^2(E;U)$ with the same norm. We still denote by $F$ the extension if there is no confusion. Then, by Riesz Representation Theorem, there is a $u(\cdot)\in L^2(E;U)$ such that for any $v(\cdot)\in L^2(E;U)$,

$$F(v)=\langle v,u\rangle_{L^2(E;U)}$$

and

$$|u|_{L^2(E;U)}=|F|_{\mathcal{L}(\mathcal{X},\mathbb{R})}\leq C(|y_0|_V+|y_1|_V).$$

From the definition of $F(\cdot)$, we know that for any $z_0 \in V'$,

$$\langle y_1, z_0\rangle_{V,V'} - \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'} = \int_0^T \chi_E(t)\Big\langle u(t), \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt. \tag{A.1}$$

We claim that $u(\cdot)$ is the control we need. Indeed, taking the dual product of $V'$ and $V$ of $z = z(t,\omega;z_0)$ with (2.1) and integrating the product with respect to $t$ in $(0,T)$ and $\omega$ in $\Omega$, we obtain that

$$\begin{aligned}
&\int_0^T \chi_E(t)\Big\langle u(t), \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt\\
&= \int_0^T \chi_E(t)\int_\Omega \langle u(t), B^*(\omega)z(t,\omega;z_0)\rangle_U d\mathbb{P}(\omega)dt\\
&= \int_0^T \chi_E(t)\int_\Omega \langle B(\omega)u(t), z(t,\omega;z_0)\rangle_{V,V'} d\mathbb{P}(\omega)dt\\
&= \int_\Omega \langle y(T,\omega;y_0), z_0\rangle_{V,V'}d\mathbb{P}(\omega) - \int_\Omega \langle y_0, z(0,\omega;z_0)\rangle_{V,V'}d\mathbb{P}(\omega)\\
&= \Big\langle \int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega), z_0\Big\rangle_{V,V'} - \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}.
\end{aligned} \tag{A.2}$$

From (A.1) and (A.2), we conclude that

$$\Big\langle \int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega), z_0\Big\rangle_{V,V'} = \langle y_1, z_0\rangle_{V,V'}, \quad \forall\, z_0 \in V',$$

which deduces that $\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = y_1$.

The "only if" part. Let $z_0 \in V'$. We choose $y_0, y_1 \in V'$ which satisfy that

$$\begin{cases} |y_0|_{V'} = |y_1|_{V'} \le 2,\\ -\Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'} = \Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_{V'},\\ \langle y_1, z_0\rangle_{V,V'} = |z_0|^2_{V'}. \end{cases}$$

Let $u(\cdot)$ be the control such that

$$|u|_{L^2(E;U)} \le C(|y_0|_V + |y_1|_V) \le C \tag{A.3}$$

and

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = y_1.$$

Then, from (A.2), we have that

$$|z_0|_{V'} + \Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|_{V'} = \int_0^T \chi_E(t)\Big\langle u(t), \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt.$$

Thus, we find that

$$|z_0|_{V'} + \Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|_{V'} \le |u|_{L^2(E;U)}\Big|\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(E;U)}.$$

This, together with (A.3), implies that

$$|z_0|_{V'} \leq C\Big|\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)\Big|_{L^2(E;U)}.$$

**Theorem A.2** *System* (2.1) *is null controllable in average in* $E$ *with control the cost* $C>0$ *if and only the adjoint system* (2.7) *is null observable in average.*

The proof of Theorem A.2 is very similar to the one for Theorem A.1. We omit it here.

**Theorem A.3** *System* (2.1) *is approximately controllable in average in* $E$ *if and only if the adjoint system* (2.7) *satisfies the averaged unique continuation property in* $E$*.*

*Proof of Theorem A.3*: The "if" part. Since the system (2.1) is linear, we may assume that $y_0 = 0$. Then, we only need to prove the following set

$$\mathcal{A}_T \overset{\triangle}{=} \Big\{\int_\Omega y(T,\omega;0)d\mathbb{P}(\omega) : y \text{ solves (2.1) with some control } u(\cdot)\Big\}$$

is dense in $V$. We do this by contradiction argument. If $\mathcal{A}_T$ was not dense in $V$, then we can find a $\varphi\in V'$ with $|\varphi|_{V'}=1$ such that

$$\langle\psi,\varphi\rangle_{V,V'}=0,\ \forall\,\psi\in\mathcal{A}_T.$$

On the other hand, similar to (A.2), we have that

$$\int_0^T \chi_E(t)\Big\langle u(t),\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt = \Big\langle\int_\Omega y(T,\omega;0)d\mathbb{P}(\omega),z_0\Big\rangle_{V,V'}. \tag{A.4}$$

Let $z_0=\varphi$ in (A.4). We have that

$$\int_0^T \chi_E(t)\Big\langle u(t),\int_\Omega B^*(\omega)z(t,\omega;\varphi)d\mathbb{P}(\omega)\Big\rangle_U dt = 0, \quad \forall\,u(\cdot)\in L^2(0,T;U).$$

Hence, we find that

$$\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;\varphi)d\mathbb{P}(\omega)=0 \quad \text{in } L^2(0,T;U),$$

which implies that $\varphi=0$ and leads to a contradiction.

The "only if" part. We utilize the contradiction argument again. We assume that there is a $z_0\in V'$ with $|z_0|_{V'}=1$, such that

$$\chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)=0 \quad \text{in } L^2(0,T;U).$$

This, together with (A.4), implies that

$$\Big\langle\int_\Omega y(T,\omega;0)d\mathbb{P}(\omega),z_0\Big\rangle_{V,V'}=0, \quad \forall\,u(\cdot)\in L^2(E;U). \tag{A.5}$$

On the other hand, since (2.1) is approximately controllable in average, we can find a $u(\cdot) \in L^2(E;U)$ such that

$$\Big|\int_\Omega y(T,\omega;0)d\mathbb{P}(\omega) - z_0\Big|_{V,V'} < \frac{1}{2}.$$

It is clear that for this $\int_\Omega y(T,\omega;0)d\mathbb{P}(\omega)$,

$$\Big\langle \int_\Omega y(T,\omega;0)d\mathbb{P}(\omega), z_0\Big\rangle_{V,V'} > \frac{1}{2},$$

which contradicts (A.5). □

**Remark A.1** *One can also consider the case where $y_0 \in L^2(\Omega;V)$, i.e. when the datum to be controlled depends on the parameter $\omega$ as well.*

*The proof of Theorem A.1 applies replacing the terms $\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\rangle_{V,V'}$ and $|y_0|_V$ by the terms $\int_\Omega \langle y_0(\omega), z(0,\omega;z_0)\rangle_{V,V'}d\mathbb{P}(\omega)$ and $|y_0(\cdot)|_{L^2(\Omega;V)}$, respectively. The same can be said about Theorem A.3.*

*The situation is different and much more delicate in the context of averaged null controllability. Note that, when considering initial data to be controlled depending on $\omega$, one requires an observability estimate of the form*

$$\int_\Omega \Big|z(0,\omega;z_0)\Big|^2_{V'} d\mathbb{P}(\omega) \le C\int_0^T \chi_E(t)\Big|\int_\Omega B(\omega)^* z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_U dt, \tag{A.6}$$

*which, in principle, is much stronger than the one we have in which, we get observability estimates on*

$$\Big|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big|^2_{V'}.$$

*This difficulty does not arise in the context of exact averaged controllability where, we recover the norm of $z_0$, and this yields also estimates on $z(0,\omega;z_0)$ for all $\omega$ and in particular on $\int_\Omega \Big|z(0,\omega;z_0)\Big|^2_{V'} d\mathbb{P}(\omega)$.*

*But the property of null averaged controllability with initial data independent of $\omega$ does not seem to suffice to derive the same property with initial data that depend on $\omega$. This is an interesting issue for further work.*

## A.2 Variational characterization of the controls of minimal norm

We have shown the existence of the exactly averaged control(*resp.* null averaged control, approximately averaged control), provided that the adjoint system is exactly averaged observable(*resp.* null averaged observable, satisfying averaged unique continuation property). These results allow concluding whether a system is controllable in an averaged sense. In this section, we give variational characterizations of the controls. Such kind of results not only derive characterizations of the controls but also serve for computational purposes.

**Theorem A.4** *If the system* (2.7) *is exactly averaged observable, then the exact averaged control for the system* (2.1) *of minimal $L^2(0,T;U)$-norm is given by*

$$u(t) = \chi_E(t)\int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)\mathbb{P}(\omega), \tag{A.7}$$

*where* $\hat{z}_0 \in V'$ *minimizes the functional*

$$\begin{aligned} J\left(z_0\right) = & \frac{1}{2}\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_U^2 dt - \langle y_1, z_0\rangle_{V,V'} \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}. \end{aligned} \tag{A.8}$$

**Remark A.2** *The control given by* (A.7) *is an average of functions of the form* $B(\cdot)^*\hat{z}(t,\cdot)$. *For each sample point* $\omega$, $B(\omega)^*\hat{z}(t,\omega)$ *can be chosen to be a control. However, generally speaking, it does not steer the initial datum* $y_0$ *to the final one* $y_1$.

*Proof of Theorem A.4*: Define a functional $J(\cdot) : V' \to \mathbb{R}$ as follows:

$$\begin{aligned} J(z_0) = & \frac{1}{2}\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_U^2 dt - \langle y_1, z_0\rangle_{V,V'} \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}. \end{aligned} \tag{A.9}$$

It is easy to see that the functional $J : V' \to \mathbb{R}$ is continuous and convex. From (2.8), we have that $J(\cdot)$ is coercive. Then, we know that $J(\cdot)$ has a unique minimizer $\hat{z}_0$. Let $\hat{z}(\cdot,\cdot)$ be the corresponding solution of the adjoint system. By computing the first variation of $J(\cdot)$, it can be seen that

$$\begin{aligned} \langle y_1, z_0\rangle_{V,V'} = & \int_0^T \chi_E(t)\Big\langle \int_\Omega B^*(\omega)z(t,\omega;\hat{z}_0)d\omega, \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}, \ \forall z_0 \in V'. \end{aligned} \tag{A.10}$$

From (A.10), we know that if we choose the control as (A.7), then

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = y_1.$$

Now we prove that $u(\cdot)$ given by (A.7) is the control with minimal $L^2(0,T;U)$-norm, which drives the mathematical expectation of the solution of the system (2.1) from $y_0$ to $y_1$. Let us choose $z_0 = \hat{z}_0$ in (A.10). We have that

$$\langle y_1, \hat{z}_0\rangle_{V,V'} - \Big\langle y_0, \int_\Omega z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'} = \int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big|_U^2 dt. \tag{A.11}$$

Let $\tilde{u}(\cdot)$ be another control which steers the mathematical expectation of the solution to the system (2.1) from $y_0$ to $y_1$. Then we obtain that

$$\begin{aligned} \langle y_1, \hat{z}_0\rangle_{V,V'} = & \int_0^T \chi_E(t)\Big\langle \tilde{u}(t), \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}, \ \forall z_0 \in H. \end{aligned} \tag{A.12}$$

From (A.11) and (A.12), we get that

$$\begin{aligned}
&\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big|_U^2 dt \\
&= \int_0^T \chi_E(t)\Big\langle \tilde{u}(t), \int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big\rangle_U dt \\
&\le \Big|\chi_E \int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big|_{L^2(0,T;U)} |\chi_E\tilde{u}|_{L^2(0,T;U)},
\end{aligned}$$

which implies that

$$\Big|\chi_E \int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega)\Big|_{L^2(0,T;U)} \le |\chi_E\tilde{u}|_{L^2(0,T;U)}.$$

□

**Remark A.3** *The control is the solution of the adjoint system with the final datum which is the minimizer of the quadratic, convex and coercive functional $J(\cdot)$ in $V'$. One can use numerical methods such as implementing gradient like iterative algorithms to compute it (see [11] for example). However, one will meet the same difficulty as employing this method to compute the control for the exact control problems of PDEs.*

Similar to Theorem A.4, we can prove the following result.

**Theorem A.5** *If the system* (2.7) *is null averaged observable, then the null averaged control of minimal $L^2(0,T;U)$-norm is given by*

$$u(t) = \chi_E(t)\int_\Omega B^*(\omega)z(0,\omega;\hat{z}_0)d\mathbb{P}(\omega), \tag{A.13}$$

*where $z_0 \in V'$ minimizes the functional*

$$J\left(z_0\right) = \frac{1}{2}\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_U^2 dt + \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}. \tag{A.14}$$

**Theorem A.6** *Suppose that the system* (2.7) *satisfies the averaged unique continuation property. Then for any $\varepsilon > 0$, the approximately averaged control is given by*

$$u(t) = \chi_E(t)\int_\Omega B^*(\omega)\hat{z}(t,\omega)d\mathbb{P}(\omega). \tag{A.15}$$

*Here $\hat{z}$ is the solution to the adjoint system* (2.7) *corresponding to the datum $z_0 \in V'$ which minimizes the functional*

$$\begin{aligned}
J_\varepsilon(z_0) = &\frac{1}{2}\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_U^2 dt - \langle y_1, z_0\rangle_{V,V'} \\
&+\Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'} + \varepsilon|z_0|_{V'}.
\end{aligned} \tag{A.16}$$

Borrowing some idea in [45], we can prove the following result stronger than Theorem A.6.

**Theorem A.7** *Suppose that the system* (2.7) *satisfies the averaged unique continuation property. For any* $\varepsilon > 0$ *and any finite dimensional space* $X \subset V$*, the solution to* (2.1) *with the control*

$$u(t) = \chi_E(t) \int_\Omega B^*(\omega)\hat{z}(t,\omega)d\mathbb{P}(\omega) \tag{A.17}$$

*satisfies that*

$$\Big|y_1 - \int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega)\Big|_V < \varepsilon, \quad \Pi_X y_1 = \Pi_X \int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega). \tag{A.18}$$

*Here* $\Pi_X$ *denotes the orthogonal projection from* $V$ *to* $X$ *and* $\hat{z}(\cdot)$ *is the solution to the adjoint system* (2.7) *corresponding to the final datum* $\hat{z}_0 \in V'$ *which minimizes the functional*

$$\begin{aligned} J_\varepsilon(z_0) = & \frac{1}{2}\int_0^T \chi_E(t)\Big|\int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega)\Big|_U^2 dt - \langle y_1, z_0\rangle_{V,V'} \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'} + \varepsilon|(I-\Pi_X^*)z_0|_{V'}. \end{aligned} \tag{A.19}$$

*Proof*: Clearly, $J_\varepsilon(\cdot)$ is continuous and convex. We only need to show that it is coercive. For this, we prove that

$$J_\varepsilon(z_0) \to \infty \text{ as } |z_0|_{V'} \to \infty. \tag{A.20}$$

Let $\{z_0^j\}_{j=1}^\infty \subset V'$ be a sequence such that $|z_0^j|_{V'} \to \infty$ as $j \to \infty$. Put $\check{z}_0^j = |z_0^j|_{V'}^{-1} z_0^j$ for $j \in \mathbb{N}$. Then

$$\begin{aligned} \frac{z_0^j}{|z_0^j|_{V'}} = & \frac{1}{2}|z_0^j|_{V'}\int_0^T \chi_E(t)\Big|\int_\Omega B(\omega)^* z(t,\omega;\check{z}_0^j)d\mathbb{P}(\omega)\Big|_U^2 dt - \langle y_1, \check{z}_0^j\rangle_{V,V'} \\ & + \Big\langle y_0, \int_\Omega z(0,\omega;\check{z}_0^j)d\mathbb{P}(\omega)\Big\rangle_{V,V'} + \varepsilon|(I-\Pi_X^*)\check{z}_0^j|_{V'}. \end{aligned} \tag{A.21}$$

If

$$\varliminf_{j\to\infty} \int_0^T \chi_E(t)\Big|\int_\Omega B(\omega)^* z(t,\omega;\check{z}_0^j)d\mathbb{P}(\omega)\Big|_U^2 dt > 0,$$

then we get from (A.21) that

$$J_\varepsilon(z_0^j) \to \infty \text{ as } j \to \infty.$$

Hence, we only need to consider the case that

$$\varliminf_{j\to\infty} \int_0^T \chi_E(t)\Big|\int_\Omega B(\omega)^* z(t,\omega;\check{z}_0^j)d\mathbb{P}(\omega)\Big|_U^2 dt = 0.$$

Since $\{\check{z}_0^j\}_{j=1}^\infty$ is bounded in $V'$, we can find a subsequence of it, which is still denoted by $\{\check{z}_0^j\}_{j=1}^\infty$ if there is no confusion, such that $\check{z}_0^j$ converges to some $\check{z}_0$ in $V'$ weakly. Thus, we have that $\chi_E(\cdot)\int_\Omega B(\omega)^* z(\cdot,\omega;\tilde{z}_0^j)d\mathbb{P}(\omega)$ converges to $\chi_E(\cdot)\int_\Omega B(\omega)^* z(\cdot,\omega;\check{z}_0)d\mathbb{P}(\omega)$ weakly in $L^2(0,T;U)$. Then, we have that

$$\int_0^T \chi_E(t)\Big|\int_\Omega B(\omega)^* z(t,\omega;\check{z}_0)d\mathbb{P}(\omega)\Big|_U^2 dt \le \varliminf_{j\to\infty}\int_0^T \chi_E(\cdot)\Big|\int_\Omega B(\omega)^* z(t,\omega;\check{z}_0)d\mathbb{P}(\omega)\Big|_U^2 dt = 0,$$

which implies that $\chi_E \int_\Omega B(\omega)^* z(t,\omega;\check{z}_0)d\mathbb{P}(\omega) = 0$ in $L^2(0,T;U)$. Thus, we know that $\check{z}_0 = 0$. Since $X$ is finite dimensional, we have that

$$\lim_{j\to\infty} |(I-\Pi_X^*)\tilde{z}_0^j|_{V'} = 1.$$

Therefore,

$$\varliminf_{j\to\infty}\frac{J_\varepsilon(z_0^j)}{|z_0^j|_{V'}} \ge \varliminf_{j\to\infty}\Big(-\langle y_1,\tilde{z}_0^j\rangle_{V,V'}+\Big\langle y_0,\int_\Omega z(0,\omega;\tilde{z}_0^j)d\mathbb{P}(\omega)\Big\rangle_{V,V'}+\varepsilon\Big) = \varepsilon,$$

which implies that $J_\varepsilon(z_0^j)\to\infty$ as $|z_0^j|_{V'}\to\infty$. By this, we get the coercivity of $J_\varepsilon(\cdot)$. Hence, we know that there is a minimizer $\hat{z}_0$ of $J_\varepsilon(\cdot)$.

For any $\delta>0$ and $z_0\in V'$, we have that

$$0 \le \frac{1}{h}\big[J_\varepsilon(\hat{z}_0+\delta z_0) - J_\varepsilon(\hat{z}_0)\big],$$

which implies that

$$\begin{aligned}-\varepsilon|(I-\Pi_X^*)z_0|_{V'} \le &\int_0^T \chi_E(t)\Big\langle \int_\Omega B(\omega)^* z(t,\omega;\hat{z}_0)d\mathbb{P}(\omega), \int_\Omega B(\omega)^* z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt\\ &-\langle y_1,z_0\rangle_{V,V'} + \Big\langle y_0,\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\varepsilon|(I-\Pi_X^*)z_0|_{V'} \ge &\int_0^T \chi_E(t)\Big\langle \int_\Omega B(\omega)^* z(t,\omega;\hat{z}_0)d\mathbb{P}(\omega), \int_\Omega B(\omega)^* z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt\\ &-\langle y_1,z_0\rangle_{V,V'} + \Big\langle y_0,\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}.\end{aligned}$$

Hence, for any $z_0\in V'$,

$$\begin{aligned}\varepsilon|(I-\Pi_X^*)z_0|_{V'} \ge &\Big|\int_0^T \chi_E(t)\Big\langle \int_\Omega B(\omega)^* z(t,\omega;\hat{z}_0)d\mathbb{P}(\omega), \int_\Omega B(\omega)^* z(t,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_U dt\\ &-\langle y_1,z_0\rangle_{V,V'} + \Big\langle y_0,\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\Big\rangle_{V,V'}\Big|.\end{aligned}$$

This, together with (A.2), implies that for any $z_0\in V'$,

$$\Big|\Big\langle z_0, y_1 - \int_\Omega y(T,\omega;y_0)\Big\rangle_{V,V'}\Big| \le \varepsilon|(I-\Pi_X^*)z_0|_{V'}.$$

Hence, we get (A.18).

## Acknowledgement

This work is supported by the Advanced Grants NUMERIWAVES/FP7-246775 of the European Research Council Executive Agency, FA9550-14-1-0214 of the EOARD-AFOSR, PI2010-04 and the BERC 2014-2017 program of the Basque Government, the MTM2011-29306-C02-00 and SEV-2013-0323 Grants of the MINECO a Humboldt Research Award (University of Erlangen-Nürnberg), and the NSF of China under grant 11101070.

The authors thank the CIMI - Toulouse for the hospitality and support during the preparation of this work in the context of the Excellence Chair in "PDE, Control and Numerics".

The author acknowledges Martin Lazar for helpful comments.


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