Addendum to "Concentration and lack of observability of waves in highly heterogeneous media", [1]

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Abstract

In [1] we introduced a class of 1−d wave equations with rapidly oscillating Hölder continuous coefficients for which the classical boundary observability property fails. We also established that these examples could be used to contradict Strichartz-type inequalities for the wave equation with low regularity coefficients. The object of this Note is to further analyze this issue. As we will see, the argument in [1] only provides sharp counterexamples to the Strichartz estimates when the coefficient ρ belongs to L^∞. We carefully analyze this counterexamples for Hölder continuous coefficients. We also give a new application of our construction showing that some eigenfunction estimates for elliptic operators due to Sogge can fail when coefficients are not smooth enough.

1. Introduction

In [1] we introduced a counterexample to the boundary observability property of the wave equation with a density $\rho \in C^{0,s}$ for all $0 < s < 1$. Moreover, we observed that this construction could be adapted to obtain counterexamples to the Strichartz estimates for the wave equation with Hölder continuous coefficients. However, the proof in [1] is only valid when $\rho \in L^\infty$.

The aim of this addendum is twofold. In section 2 we present a complete and rigorous statement on this matter complementing [1] and, in section 3, we give a new application of our construction that allows obtaining sharp limits to the Sogge's estimates for the eigenfunctions of second order elliptic operators.

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2. Strichartz estimates

We consider the following wave equation in $\mathbb{R}^d$ with a variable coefficient $\rho(x)$:

$$\begin{cases} \rho(x)u_{tt} - \Delta u = 0, & x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

The coefficient ρ is assumed to be measurable and bounded above and below by finite, positive constants, i.e.

$$0 < \rho_0 \leq \rho(x) \leq \rho_1 < \infty \text{ a.e. } x \in \mathbb{R}^d. \tag{2.2}$$

We say that (p,q) is an admissible pair if it satisfies

$$\frac{1}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}, \quad 2 \leq p, q \leq \infty. \tag{2.3}$$

For ρ constant and $d \geq 2$, the following Strichartz-type estimates hold

$$\|u\|_{L_t^p([0,1];L_x^q(\mathbb{R}^d))} \leq c \left(\|u_0\|_{H^r(\mathbb{R}^d)} + \|u_1\|_{H^{r-1}(\mathbb{R}^d)} \right), \tag{2.4}$$

provided that the pair (p,q) is admissible, $(d,p,q) \neq (3,2,\infty)$ and r is given by

$$r = d\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p}. \tag{2.5}$$

For variable coefficients, $\rho \in C^s$ with $0 \leq s \leq 2$, there exist weakened versions of estimates (2.4) (see [8]).

Note that the above estimates cannot be obtained by classical Sobolev embeddings and energy methods.

In [1] we stated that for coefficients ρ in the class $C^{0,s}$, with $0 < s < 1$, (2.4) may not hold (even locally) except of course for the pairs (p,q) corresponding to the integrability properties that Sobolev's embeddings and energy estimates provide [1](Th.8, p.66). However, the proof in [1] is only valid when $\rho \in L^\infty$. Our construction yields a weaker result for coefficients $\rho \in C^{0,s}$. More precisely, the correct statement of Theorem 8 in [1] should be the following:

Theorem 1. *Given any point $x_{sg} \in \mathbb{R}^d$, there exist density functions $\rho \in L^\infty(\mathbb{R}^d)$ satisfying (2.2), and a sequence of solutions u_j of (2.1) for which*

$$\lim_{j\to\infty} \frac{\left(\int_{I^d} |u_j(\cdot,t)|^q dx\right)^{1/q}}{\|u_j(\cdot,0)\|_{H^r(\mathbb{R}^d)} + \|\partial_t u_j(\cdot,0)\|_{H^{r-1}(\mathbb{R}^d)}} = \infty \tag{2.6}$$

for any $q > \frac{2d}{d-2r}$, $t \in \mathbb{R}$ and for all d-dimensional cube $I^d = [\, x_{sg}, x_{sg} + \delta]^d$ with $\delta > 0$.

Moreover, the same holds if $q > \frac{2d}{d(1-s)-2r}$ for a suitable $\rho \in C^{0,s}(\mathbb{R}^d)$ with $0 < s < 1$.

Theorem 1 establishes that when $\rho \in L^\infty$ one cannot guarantee any further integrability property of the solution, other than that implied by energy estimates and Sobolev embeddings. In the class of coefficients $\rho \in C^{0,s}(\mathbb{R}^d)$ we also show that inequality (2.4) may fail for admissible pairs (p,q) with $q > 2d/[d(1-s)-2r]$. According to this, we have the following:

Corollary 1. *Assume that for any $\rho \in L^\infty$ there exists a constant $c > 0$ such that (2.4) holds. Then*

$$r \geq d\left(\frac{1}{2}-\frac{1}{q}\right) \qquad \left(\text{or, equivalently, } q \leq \frac{2d}{d-2r}\right). \tag{2.7}$$

Moreover, if (2.4) is assumed to hold in the class of densities $\rho \in C^{0,s}$, $0 < s < 1$, then

$$r \geq d\left(\frac{1}{2}-\frac{1}{q}\right) - \frac{ds}{2}. \tag{2.8}$$

Remark 1. 1. The bound for r in (2.8) is greater than the value in (2.5) if $s < 2/(pd)$. This shows in particular that one may not guarantee the Strichartz estimate (2.4) for all coefficients ρ in the class $C^{0,s}$ unless $s \geq 2/(pd)$.

2. The results in [8] show that if $\rho \in C^s$ with $0 \leq s \leq 2$ (L^∞ if $s = 0$, $C^s = C^{0,s}$ if $0 < s < 1$, Lipschitz if $s = 1$ and $C^s = C^{1,s-1}$ if $1 < s < 2$) then the estimates (2.4) hold, with a constant c depending only on the C^s norm of ρ, when

$$r = d\left(\frac{1}{2}-\frac{1}{q}\right) + \frac{\sigma-1}{p}, \quad \sigma = \frac{2-s}{2+s}. \tag{2.9}$$

Moreover, in [7] it is proved that this result is sharp in the sense that if the inequalities (2.4) hold with a constant $c > 0$ depending only on the C^s-norm of the coefficients then r must be greater or equal than the value in (2.9).

When $s = 0$ the value of r in (2.9) coincides with the bound (2.7) and then the result of Corollary 1 is sharp. Note also that, in this case, the result in Corollary 1 is somehow stronger than the one in [7]. Indeed, in [7] it is proved that the constant $c > 0$ in (2.4) can not be chosen depending only on the $L^\infty-$norm of ρ, while Corollary 1 shows that, in fact, there are particular coefficients $\rho \in L^\infty$ for which the constant c in (2.4) does not even exist.

If $s > 0$, the optimal value of r in (2.9) is strictly greater than the bound we get in (2.8). Therefore Corollary 1 is not sharp in this case.

The counterexamples in [7] are based on a construction of a sequence of coefficients, bounded in the C^s-norm, for which there are solutions concentrated along characteristics. Then, the constant in (2.4) becomes unbounded unless r satisfies (2.9). Our examples are of different nature. We construct particular coefficients $\rho \in C^{0,s}$ for which there exists a sequence of solutions, arbitrarily concentrated near a given point (instead of a characteristic curve), for all time. Since our construction is based on eigenfunctions for the elliptic problem in the whole space it is also useful to analyze Strichartz-like estimates for other models too as, for example, for the Schrödinger equation.

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Proof of Theorem 1: For completeness we sketch the proof given in [1] paying special attention to computing the regularity s of the coefficient making the Strichartz estimate (2.4) fail. We first consider the case $\rho \in C^{0,s}$ with $s > 0$. The case $\rho \in L^\infty$ will be treated separately at the end of the proof.

Assume, without loss of generality, that $x_{sg} = 0$ and $K = [0, 1]$. In [1] (formula (6.6)) we constructed a positive density function $\rho \in L^\infty(K^d)$ for which there exists a sequence of eigenpairs (h_j^2, φ_j) satisfying

$$\Delta\varphi_j + h_j^2\rho(x)\varphi_j = 0,\ x \in K^d, \tag{2.10}$$

with

$$h_j \to \infty, \tag{2.11}$$

and such that φ_j is strongly concentrated around $x_{sg} = 0$.

Note that no boundary conditions are imposed on (2.10). Actually, the sequence φ_j is constituted by local eigenfunctions whose main property is the concentration around $x_{sg} = 0$. The growth of the sequence h_j, the regularity of the coefficient ρ and the degree of concentration of the sequence φ_j are intimately related. In order to get the bounds stated in the Theorem, we need to further analyze the properties of h_j, ρ and φ_j.

The density ρ is constructed in separated variables. Thus, the key point is the understanding of its behavior in one space dimension. It is a positive function that oscillates more and more rapidly along a sequence I_j of disjoint intervals of K. Roughly, h_j is the frequency of the oscillation of ρ over I_j. The intervals I_j are of length $l_j \to 0$ and their centers m_j converge to the singular point $x_{sg} = 0$.

We now give a more precise description.

According to (3.11) and (6.13) in [1] the density ρ and the eigenpairs (h_j^2, φ_j) may be built so that

$$|\rho|_{C^{0,s}(K^d)} \leq M \sup_j \epsilon_j h_j^s, \tag{2.12}$$

for a suitable $M > 0$, $\epsilon_j \to 0$ and $0 < s < 1$.

Furthermore, for suitable constants C and $C_p > 0$:

$$\int_{(I_j^-)^d} |\varphi_j(x)|^2 dx \geq C h_j^{-3d}, \tag{2.13}$$

$$|\varphi_j(x)|^2 + |\nabla\varphi_j(x)|^2 \leq C_p h_j^{-p}, \qquad \forall p > 0, \quad x \in K^d \backslash (I_j^-)^d, \tag{2.14}$$

where

$$I_j^- = (m_j - \frac{l_j}{2}, m_j + \frac{l_j}{2}], \quad m_j = \frac{l_j}{2} + \sum_{k=j+1}^{\infty} r_k. \tag{2.15}$$

This may be done provided that the sequences l_j, h_j and ϵ_j satisfy the following conditions

$$\sum_{j=1}^{\infty} l_j = 1, \quad \frac{h_j l_j}{2} \text{ is an integer,}$$
$$\epsilon_k \leq \frac{1}{2M}, \quad 4M \sum_{j=1}^{k-1} \epsilon_j h_j r_j \leq \epsilon_k h_k r_k, \quad 2M \sum_{j=k+1}^{\infty} \epsilon_j r_j \leq \epsilon_k r_k,$$
$$h_j^p e^{-\epsilon_j h_j l_j} \to 0 \text{ as } j \to \infty \text{ for all } p > 0. \tag{2.16}$$

The following choice guarantees that all these conditions are fulfilled:

$$l_j = 2^{-j}, \quad h_j = 2^{2^{Nj}}, \quad \epsilon_j = h_j^{-s}, \tag{2.17}$$

for a sufficiently large integer fixed N. Note that with this choice, we have $\rho \in C^{0,s}$, in view of (2.12).

Remark 2. In [1] we assumed $\epsilon_j = h_j^{-1}(\log h_j)^2$ instead of $\epsilon_j = h_j^{-s}$ in (2.17). This allowed us to prove that the coefficient ρ belongs to $C^{0,s}$ for all $0 < s < 1$. But the construction above is better adapted to deal with coefficients in a specific class $C^{0,s}$.

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As mentioned above, the functions $\varphi_j(x)$ and $\rho(x)$ are defined in separated variables over K^d (see (6.6) and (6.8) in [1]). In fact, we have

$$\varphi_j(x) = \widehat{\varphi}_j(x_1)...\widehat{\varphi}_j(x_d),$$
$$\rho(x) = \widehat{\rho}(x_1) + ... + \widehat{\rho}(x_d), \tag{2.18}$$

where $\widehat{\rho} \in C(K)$ is a positive function which takes the value $4\pi^2$ on the boundary of K and $\widehat{\varphi}_j$ solves

$$(\widehat{\varphi}_j)''(y) + h_j^2 \widehat{\rho}(y)\widehat{\varphi}_j(y) = 0, \qquad \text{for } y \in K. \tag{2.19}$$

In particular, over each I_j^- we have

$$\varphi_j(x) = \widehat{\varphi}_j(x_1)...\widehat{\varphi}_j(x_d) = w_{\epsilon_j}(h_j(x_1 - m_j^-))...w_{\epsilon_j}(h_j(x_d - m_j^-)), \tag{2.20}$$

where w_ϵ is of the form,

$$w_\epsilon(x) = p_\epsilon(x) e^{-\epsilon|x|} \tag{2.21}$$

for some function $p_\epsilon(x)$ 1−periodic on $x > 0$ and $x < 0$ (see (2.5) in [1]), and satisfying

$$|w_\epsilon| + |w'_\epsilon| + |w''_\epsilon| \leq C, \qquad \int_0^1 |w_\epsilon(x)|^2 dx \geq \gamma, \tag{2.22}$$

for some $C, \gamma > 0$, independent of ϵ (see Lemma 1 in [1], p. 42).

In fact, we take as w_{ϵ_j} the explicit example given in [1] (pag. 44) for which the following bound holds for the derivatives of any order r:

$$|w_\epsilon^{r)}(x)| \leq C_r, \qquad \forall x \in \mathbb{R},\ \forall \epsilon > 0. \tag{2.23}$$

Once the eigenpair (h_j, φ_j) is built we construct the solutions of the wave equation by separation of variables:

$$u_j(x,t) = e^{ih_j t}\varphi_j(x). \tag{2.24}$$

This constitutes a sequence of solutions of (2.1) over $K^d \times [0,T]$ that we extend to solutions of (2.1) over $\mathbb{R}^d \times [0,T]$. To this end we first extend the coefficient ρ from K^d to $\mathbb{R}^d$. This extension is defined in separated variables, as in (2.18), by extending $\widehat{\rho}$ from K to $\mathbb{R}$ by the constant value $4\pi^2$. The exponent in the Hölder continuous regularity property of ρ in $\mathbb{R}^d$ is the same as the one of ρ in K^d since $\widehat{\rho} = 4\pi^2$ on the boundary of K. Then, we extend φ_j in separated variables, as in (2.18), by extending $\widehat{\varphi_j}$ in such a way that it satisfies (2.19) in $\mathbb{R}$ (with $\hat{\rho}$ extended as before). This is done by solving the ODE in the exterior domain $\mathbb{R}\backslash K$ with the Cauchy data given by $\widehat{\varphi_j}(y)$ and $\widehat{\varphi_j'}(y)$ at the extremes of K.

In this way, u_j, defined by (2.24), is extended to a solution of (2.1) in $\mathbb{R}^d \times [0,T]$. To simplify the notation we denote the extension simply by u_j. Note however that we can not guarantee that $u_j(0) = \varphi_j \in H^r(\mathbb{R}^d)$ since our extension of φ_j does not decay as $x \to \infty$, since it solves a constant coefficients second order differential equation away of K.

To avoid this difficulty we rather define the extension of u_j as the solution of (2.1) with truncated initial data. More precisely, $(u(0), u_t(0)) = (\varphi_j(x)\chi(x), ih_j\varphi_j(x)\chi(x))$, where $\chi(x)$ is a cut-off function that satisfies

$$\begin{aligned}&\chi(x) \in C^\infty(\mathbb{R}^d), \quad |\chi| \leq 1,\\ &\chi(x) = 1, \quad \text{if } |x| < R \text{ and } \chi(x) = 0, \text{ if } |x| > R+1,\end{aligned}$$

and R is a sufficiently large number that we chose in order to guarantee that the solutions of (2.1) with the above truncated initial data coincide with (2.24) over $K^d \times [0,1]$. Note that, by the finite velocity of propagation, this holds if R is sufficiently large. Due to the estimates (2.14) and the fact that $\widehat{\varphi_j}$ in $\mathbb{R}\backslash K$ is a linear combination of sinusoidal functions which oscillate with frequency h_j we get

$$\begin{aligned}&\|(u_j(x,0), \partial_t u_j(x,0))\|_{H^r(\mathbb{R}^d)\times H^{r-1}(\mathbb{R}^d)} = \|(\chi\varphi_j, ih_j\chi\varphi_j)\|_{H^r(\mathbb{R}^d)\times H^{r-1}(\mathbb{R}^d)}\\ &= \|(\varphi_j, ih_j\varphi_j)\|_{H^r(K^d)\times H^{r-1}(K^d)} + \mathcal{O}(h_j^{-p}).\end{aligned} \tag{2.25}$$

Observe that $|u_j(\cdot,t)| = |\varphi_j(\cdot)|$ for all t and $x \in K^d$. Therefore, taking (2.25) into account, the limit in (2.6) coincides with

$$\lim_{j\to\infty} \frac{\left(\int_{I^d} |\varphi_j(x)|^q dx\right)^{1/q}}{\|\varphi_j\|_{H^r(K^d)} + h_j\,\|\varphi_j\|_{H^{r-1}(K^d)} + \mathcal{O}(h_j^{-p})}. \tag{2.26}$$

The change of variables $y_\alpha = h_j(x_\alpha - m_j^-)$, $\alpha = 1, ..., d$, in (2.26) and the estimates (2.14) provide

$$\frac{\left(\int_{I^d} |\varphi_j(x)|^q dx\right)^{1/q}}{\|\varphi_j\|_{H^r(K^d)} + h_j \|\varphi_j\|_{H^{r-1}(K^d)} + \mathcal{O}(h_j^{-p})} \geq h_j^{-\frac{d}{q}+\frac{d}{2}-r} \frac{\left(\int_{\overline{I_j}} |w_{\epsilon_j}(y)|^q dy\right)^{d/q}}{\left\|w_{\epsilon_j}\right\|^d_{H^r(\overline{I_j})} + h_j \left\|w_{\epsilon_j}\right\|^d_{H^{r-1}(\overline{I_j})} + \mathcal{O}(h_j^{-p})}, \tag{2.27}$$

where $\overline{I}_j$ is the interval $\overline{I}_j = h_j(I_j^- - m_j^-)$. This holds for all $p \geq 0$.

For j sufficiently large, $[0,1] \subset \overline{I}_j$ and the numerator in (2.27), in view of (2.22), can be bounded below by a constant $C(d,p)$ which does not depend on ϵ_j, i.e.

$$\left(\int_{\overline{I_j}} |w_{\epsilon_j}(y)|^q dy\right)^{d/q} \geq \left(\int_0^1 |w_{\epsilon_j}(y)|^q dy\right)^{d/q} \geq C(d,q) > 0.$$

Concerning the denominator in (2.27) we have

$$\begin{aligned} \|w_{\epsilon_j}\|^2_{H^r(\overline{I_j})} &\leq C \int_{\overline{I_j}} \left|\frac{d^r}{dx^r}(e^{-\epsilon_j|x|}p_{\epsilon_j}(x))\right|^2 dx \leq C' \int_{\overline{I_j}} e^{-2\epsilon_j|x|}dx \\ &\leq C''\epsilon_j^{-1} = C'' h_j^s, \end{aligned} \tag{2.28}$$

since all the derivatives of p_{ϵ_j} must be uniformly bounded (with respect to ϵ_j) in $x \in R$ in view of (2.22)-(2.23) and the 1−periodicity of p_ϵ.

Thus, the limit in (2.27) is unbounded if

$$-\frac{d}{q} + \frac{d}{2} - r - \frac{sd}{2} > 0,$$

which proves the result for $s > 0$.

Now we consider the case $s = 0$ in which we only assume $\rho \in L^\infty$. The main difference is that it is not necessary to consider $\epsilon_j \to 0$. Instead of (2.17) we make the following choice

$$l_j = 2^{-Lj}(2^L - 1), \quad h_j = 2^{2^{Nj}}, \quad \epsilon_j = \epsilon_1, \tag{2.29}$$

for some fixed sufficiently large integers N and L, and a sufficiently small ϵ_1 so that all conditions in (2.16) are fulfilled.

Following the previous argument we obtain a uniform bound in (2.28) independent of h_j. Thus, the limit in (2.27) is unbounded if

$$-\frac{d}{q} + \frac{d}{2} - r > 0,$$

which proves the result.

3. Eigenfunction estimates

Let T^d be the d-dimensional torus. We consider the following eigenvalue problem:

$$-\Delta\psi + \lambda^2\rho(x)\psi = 0, \text{ on } T^d. \tag{3.1}$$

The coefficient ρ is assumed to be measurable and bounded above and below by finite, positive constants, i.e. it satisfies (2.2).

Let $(\lambda_k, \psi_k)_{k\in N}$ be a sequence of eigenpairs such that $(\psi_k)_{k\in N}$ constitutes an orthonormal basis for $L^2(T^d)$.

For $\lambda \in I\!R$, let $\Pi_\lambda f$ denote the orthogonal projection of a function f onto the subspace generated by the eigenfunctions with frequencies in the range $[\lambda, \lambda+1)$, i.e.

$$\Pi_\lambda f = \sum_{\lambda_j\in[\lambda,\lambda+1)} (\psi_j, f)\psi_j,$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(T^d)$.

We are interested in the following class of estimates

$$\|\Pi_\lambda f\|_{L^q(T^d)} \le C\lambda^\gamma \|f\|_{L^2(T^d)}, \qquad q \ge 2,\ d \ge 1, \tag{3.2}$$

for some $\gamma = \gamma(d,q)$, that may possibly depend on d and q (but not on ρ). We denote by $\gamma_{\min}(d,q)$ the minimum value of γ for which (3.2) holds.

These estimates were proved for the first time by Sogge for elliptic operators with smooth coefficients on smooth compact manifolds without boundary ([5]). More precisely, in [5] it is proved that (3.2) holds with

$$\gamma = h(d,q), \qquad q_d \le q \le \infty \tag{3.3}$$

and

$$h(d,q) = d\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}, \qquad q_d = \frac{2(d+1)}{d-1}. \tag{3.4}$$

This means in particular that

$$\gamma_{\min}(d,q) \le h(d,q), \qquad q_d \le q \le \infty \tag{3.5}$$

In [4], H. Smith proved that the same is true if both the coefficients of the underlying operator and the metric g are in the class $C^{1,1}$.

Similar estimates hold also true for $2 \le q \le q_d$ with different values of the exponent γ.

When considering elliptic operators with low regularity coefficients in (3.1) the estimates (3.2) can fail in the ranges (3.3)-(3.4). For example, in [6] it is proved that, for each $1 \le s < 2$, there exist coefficients in the class $C^s(T^d)$, if $1 < s < 2$, and $Lip\ (T^d)$ if $s = 1$ for which, for (3.2) to hold, one needs

$$\gamma_{\min}(d,q) > l(d,q)\left(1 + \frac{2-s}{2+s}\right), \qquad l(d,q) = \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right). \tag{3.6}$$

Note that this is incompatible with (3.5) when

$$q_d \le q < \frac{2(d+2s^{-1})}{d-1}, \tag{3.7}$$

since

$$l(d,q)\left(1+\frac{2-s}{2+s}\right) > h(d,q),$$

in this range of values of q.

This counterexample has been extended to all $0 \le s < 1$ in [7].

On the other hand, some weakened versions of estimates (3.2) were recently obtained by Smith in [3], when the coefficients of the underlying elliptic operator belong to C^s for $1 < s < 2$, or they are Lipschitz (which corresponds to $s = 1$ below). More precisely, in [3] it is shown that (3.2) holds with

$$\gamma_{\min}(d,q) \le h(d,q) + \frac{1}{q}\frac{2-s}{2+s}, \qquad q_d \le q \le \infty \text{ and } h(d,q) + \frac{1}{q}\frac{2-s}{2+s} \le 1. \tag{3.8}$$

Note that the counterexamples in [7] do not contradict this estimate, even if (3.8) would hold for $0 \le s < 1$, a fact that is open so far.

The problem is worse understood for q large. In [2] the inequality (3.2) is proved for L^∞ coefficients and $q = \infty$ with

$$\gamma_{\min}(d,\infty) \le h(d,\infty) + \frac{1}{2} = \frac{d}{2}. \tag{3.9}$$

On the other hand, a sequence of smooth coefficients with uniform upper and lower bounds is built so that the constant C in (3.2) is not uniformly bounded for $d = 2$, $q = \infty$ and $\gamma = h(2,\infty)$.

Here we prove the following:

Theorem 2. *There exists $\rho \in C^{0,s}(T^d)$ for $0 < s < 1$ and $\rho \in L^\infty(T^d)$ for $s = 0$ such that estimate (3.2) does not hold for*

$$\gamma < d\left(\frac{1}{2}-\frac{1}{q}\right) - \frac{sd}{2}$$

when $q > q_d$. In other words, for $q > q_d$, necessarily

$$\gamma_{\min}(d,q) \ge h(d,q) + \frac{1-sd}{2}, \qquad q > q_d. \tag{3.10}$$

Remark 3. 1. This result extends the counterexample in [2] to all $q > q_d$, all $0 < s < 1$ and all dimensions d.

2. Theorem 2 shows that one may not expect the estimate (3.8), proved in [3] for C^s coefficients with $s > 1$, to be true for $0 \le s < 1$ and all q. Indeed, if $0 \le sd < 1$,

$$h(d,q) + \frac{1-sd}{2} > h(d,q) + \frac{1}{q}\frac{2-s}{2+s},$$

whenever

$$q > \frac{2(2-s)}{(1-sd)(2+s)}.$$

3. This result shows the optimality of the estimate in [2] guaranteeing that (3.2) holds for $s = 0$ and $q = \infty$ with γ as in (3.9).
4. The result in Theorem 2 is also true if the domain is any compact manifold with boundary. Indeed, the counterexamples given in the proof are of local nature and they can be easily adapted to consider any boundary conditions. We only have to use suitable cut-off functions in the step 1 of the proof below.

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Proof: Our proof is inspired in the proof of Theorem 2 in ([6], p. 736). We divide it in 3 steps: first we construct a sequence of quasi-eigenpairs $(h_j^2, \theta_j(x))$ that satisfy (3.1) up to a small rest r_j. Then we prove that the projection of θ_j in the set of eigenfunctions with eigenvalues λ_j satisfying $|h_j^2 - \lambda_j| > 2$ is small. Finally, in the third step, we use this fact to prove the result.

Step 1: Construction of θ_j. We take $x \in (-1,1]^d$ as coordinates on T^d. We may assume, without loss of generality, that $x_{sg} = 0$. Let $K = [-1,1]$ and consider the density ρ defined on the compact set K^d as in [1] (formula (6.6) in p. 57). The density ρ is periodic in K^d since it is defined in separate variables by (2.18) and each one of the functions $\hat{\rho}(x_i)$ (which depends on the only variable x_i and therefore it is trivially periodic in the other variables) takes the value $4\pi^2$ in a neighborhood of the boundary of $x_i \in [-1,1]$. Therefore, $\rho(x)$ can be viewed as a smooth function defined in the torus T^d with coordinates $x \in (-1,1]^d$.

For this ρ, there exists a sequence (h_j^2, φ_j) of eigenpairs of the associated eigenvalue problem concentrated around $x = 0$, i.e. solutions of (2.10) satisfying (2.13)-(2.22).

Note that, in general, φ_j does not satisfy necessarily the periodicity conditions at the boundary of $K^d = [-1,1]^d$. To compensate this fact we introduce a cutoff function $\eta(x) = \widehat{\eta}(x_1)...\widehat{\eta}(x_d)$ where

$$\begin{gathered}\widehat{\eta} \in C^\infty(-1,1), \qquad 0 \le \widehat{\eta} \le 1,\\ \widehat{\eta}(x) = 1 \text{ if } x \in (-1,-2/3) \cup (2/3,1),\\ \widehat{\eta}(x) = 0 \text{ if } x \in (-1/3,1/3).\end{gathered}$$

Clearly $\theta_j = \phi_j - \eta\phi_j$ is a smooth function which vanishes in a neighborhood of the boundary of $[-1,1]^d$ and then we can take it as a smooth function in the torus T^d. On the other hand, it satisfies

$$\Delta\theta_j + h_j^2 \rho(x)\theta_j = r_j$$

where $r_j(x)$ is a small function in the sense that for any index $\gamma = (\gamma_1, \gamma_2, ..., \gamma_m)$, and for any $N > 0$ there exists a constant $C_{\gamma,N}$ that only depends on γ and N, such that

$$|\partial^\gamma r_j| \leq C_{\gamma,N} h_j^{-N}, \tag{3.11}$$

due to (2.14) and the fact that the support of r_j is included in a region where φ_j (and so θ_j) satisfies (2.14).

Step 2. We show that for all $M > 0$ and $N > 0$ there exists a constant $C_{N,M}$ such that if $|\lambda - h_j^2| > 2$ then

$$\|\Pi_\lambda \theta_j\|_{L^2(T^n)} \leq C_{N,M} \lambda^{-M} h_j^{-N}. \tag{3.12}$$

Indeed, if ψ_k is an eigenfunction with eigenvalue λ_k,

$$\int_{T^d} \rho\, \theta_j \psi_k = \frac{1}{\lambda_k - h_j^2} \int_{T^d} \rho\, r_j \psi_k = \frac{\lambda_k^{-M}}{\lambda_k - h_j^2} \int_{T_d} \left(\frac{-1}{\rho(x)} \Delta \right)^M \rho\, r_j \psi_k. \tag{3.13}$$

Note that the integration by parts is valid since r_j is supported in the region where ρ is smooth. Now, using Minkowsky inequality, the orthogonality of the eigenfunctions ψ_k and (3.13) we obtain

$$\begin{aligned}
\|\Pi_\lambda \theta_j\|_{L^2(T^d)} &= \left\| \sum_{\lambda_k \in [\lambda, \lambda+1)} \left(\int_{T^d} \rho\, \theta_j \psi_k \right) \psi_k \right\|_{L^2(T^d)} \\
&\leq \sum_{\lambda_k \in [\lambda, \lambda+1)} \left| \int_{T^d} \rho\, \theta_j \psi_k \right| = \sum_{\lambda_k \in [\lambda, \lambda+1)} \frac{\lambda_k^{-M}}{\lambda_k - h_j^2} \left| \int_{T^d} \left(\frac{-1}{\rho(x)} \Delta \right)^M \rho\, r_j \psi_k \right| \\
&\leq C \sum_{\lambda_k \in [\lambda, \lambda+1)} \frac{\lambda_k^{-M}}{\lambda_k - h_j^2} \left\| \left(\frac{-1}{\rho(x)} \Delta \right)^M r_j \right\|_{L^2(T^d)}.
\end{aligned} \tag{3.14}$$

In this finite sum all the terms satisfy that $\lambda_k \geq \lambda$ and that $\lambda_k - h_j^2$ is bounded from below. Thus, taking (3.11) into account we can estimate the right hand side in (3.14) by (3.12) for all M and N.

Step 3: Conclusion. Recall that $\varphi_j(x)$ is defined in separated variables over each I_j^- as follows

$$\varphi_j(x) = \widehat{\phi}_j(x_1)...\widehat{\phi}_j(x_d) = w_{\epsilon_j}(h_j(x_1 - m_j^-))...w_{\epsilon_j}(h_j(x_d - m_j^-)).$$

Therefore, with the change of variables $y_a = h_j(x_\alpha - m_j^-)$ and taking $\overline{I}_j = h_j(I_j^- - m_j^-)$, we obtain

$$\left(\int_{I^d} |\phi_j(x)|^p dx \right)^{1/p} = h_j^{-\frac{d}{p}} \left(\int_{\overline{I_j}} |w_{\epsilon_j}(y)|^p dx \right)^{d/p} \geq C h_j^{-\frac{d}{p}}, \tag{3.15}$$

in view of (2.22). On the other hand, from (2.21), and (2.17)

$$\left(\int_{T^d} |\phi_j(x)|^2 dx\right)^{1/2} \leq h_j^{-\frac{d}{2}} \left(\int_{\mathbb{R}} |w_{\epsilon_j}(y)|^2 dx\right)^{d/2} \leq C h_j^{-\frac{d}{2}} \epsilon_j^{-\frac{d}{2}} \leq C h_j^{-\frac{d(1-s)}{2}}. \tag{3.16}$$

Both estimates (3.15) and (3.16) are also true if we change ϕ_j by θ_j since $\phi_j = \theta_j$ on I^d and $|\theta_j| \leq |\phi_j|$ for all $x \in T^d$, i.e.

$$\|\theta_j\|_{L^p(I^d)} \leq C h_j^{-\frac{d}{p}}, \quad \|\theta_j\|_{L^2(T^d)} \leq C h_j^{-\frac{d(1-s)}{2}}. \tag{3.17}$$

Therefore, using (3.17) and Minkowsky inequality,

$$C h_j^{-\frac{d}{q}} \leq \left\| \sum_{k=1}^{\infty} \Pi_k \theta_j \right\|_{L^q(I^d)} \leq \sum_{k=1}^{\infty} \|\Pi_k \theta_j\|_{L^q(I^d)}. \tag{3.18}$$

On the other hand, if (3.2) holds, then by step 2 above,

$$\begin{aligned} \sum_{k=1}^{\infty} \|\Pi_k \theta_j\|_{L^q(I^d)} &\leq \sum_{k=1}^{\infty} C k^{\alpha} \|\Pi_k \theta_j\|_{L^2(T^d)} \leq C h_j^{\alpha} \|\theta_j\|_{L^2(T^d)} \\ &\leq C h_j^{\alpha - \frac{d(1-s)}{2}} \end{aligned}$$

which contradicts (3.18) if $\alpha < d\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{sd}{2}$. This concludes the proof when $\rho \in C^{0,s}$ with $s > 0$.

Now we consider the particular case in which we only assume $\rho \in L^{\infty}$. Instead of (2.17) we consider the choice (2.29) for some fixed sufficiently large integers N and L, and a sufficiently small ϵ_1 so that (2.16) holds. Note that, with this choice, ϵ_j does not converge to zero as $j \to \infty$.

Following the previous argument we obtain estimate (3.16) with $s = 0$. The rest of the proof is the same but with $s = 0$.

Conclusion. According to the previous discussion and the result of Theorem 2 the state of the art on inequalities of the form (3.2) is the following:

1. For C^s coefficients with $s \geq 2$, (3.2) is known to hold for γ as in (3.3) and the estimate is known to be sharp.
2. For C^s coefficients with $1 \leq s < 2$ the results in [3] and [6] provide both positive results and counterexamples but the sharp exponents are still unknown in some ranges of d, s and q.
3. For C^s coefficients with $0 \leq s < 1$ Theorem 2 establishes some lower bounds on the exponent γ but there are very few results of positive nature, except for [2] which only address the case $s = 0$.

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