Advances in Differential Equations Volume 6, Number 8, August 2001, Pages 987–1023

GENERIC SIMPLICITY OF THE EIGENVALUES OF THE STOKES SYSTEM IN TWO SPACE DIMENSIONS

Jaime H. Ortega[1]
Universidad del Bío-Bío, Facultad de Ciencias, Departamento de Matemática
Casilla 5-C, Concepción, Chile

Enrique Zuazua[2]
Universidad Complutense de Madrid, Departamento de Matemática Aplicada
28040 Madrid, Spain

(Submitted by: J.L. Bona)

Abstract. We analyze the multiplicity of the eigenvalues for the Stokes operator in a bounded domain of $\mathbb{R}^2$ with Dirichlet boundary conditions. We prove that, generically with respect to the domain, all the eigenvalues are simple. In other words, given a bounded domain of $\mathbb{R}^2$, we prove the existence of arbitrarily small deformations of its boundary such that the spectrum of the Stokes operator in the deformed domain is simple. We prove that this can be achieved by means of deformations which leave invariant an arbitrarily large subset of the boundary. The proof combines Baire's lemma and shape differentiation. However, in contrast with the situation one encounters when dealing with scalar elliptic eigenvalue problems, the problem is reduced to a unique continuation question that may not be solved by means of Holmgrem's uniqueness theorem. We show however that this unique continuation property holds generically with respect to the domain and that this fact suffices to prove the generic simplicity of the spectrum.

1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open domain with boundary of class C^2. We consider the following eigenvalue problem for the Stokes system:

$$(P) \qquad \begin{cases} -\triangle v + \nabla p &= \lambda v, & \text{in } \Omega \\ \nabla \cdot v &= 0, & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega. \end{cases}$$

[1]Supported by the Fellowship Programs of the "Instituto de Cooperación Iberoamericana" (ICI) (1993–1996) and "Presidente de la República de Chile" (1996–1997).

[2]Supported by grant PB96-0663 of the DGES and FMRX-CT960033 of the European Union.

Accepted for publication October 2000.

AMS Subject Classifications: 35P99, 35Q30, 47A75, 49J20, 93C20, 93B60.

In (P), $v = (v_1, v_2)$ denotes the velocity field, while the scalar function p is the pressure. By $\cdot$ we denote the scalar product in $\mathbb{R}^2$. Thus, $\nabla\cdot$ denotes the divergence operator. It is well known that problem (P) admits a sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ tending to infinity as $n \to \infty$. The eigenfunctions $\{v_n\}_{n\geq 1} \subset (H_0^1(\Omega))^2$ and the eigenpressures $\{p_n\}_{n\geq 1} \subset L^2(\Omega)$ may be taken so that $\{v_n\}_{n\geq 1}$ constitutes an orthonormal basis of $H(\Omega) = \{v \in (L^2(\Omega))^2 : \nabla \cdot v = 0 \text{ in } \Omega, v \cdot n = 0 \text{ on } \partial\Omega\}$. The pressure p is determined up to an additive constant. We shall normalize it by the condition $\int_\Omega p(x)\, dx = 0$.

It is also well known that the spectrum of the Stokes operator is not always simple. This is for instance the case when Ω is a ball as pointed out in G. Watson [38], pp. 124–127.

The problem we address in this paper is roughly the following: *If Ω is such that the spectrum of (P) is not simple, can we find arbitrarily small deformations of Ω such that the spectrum becomes simple in the new deformed domain?*

The problem of the simplicity of the spectrum arises in many contexts. This is for instance the case when analyzing stabilizability and controllability issues for evolution systems. When the spectrum is simple one can often reduce these problems to the analysis of suitable properties of eigenfunctions, which is an easier problem to deal with because of the lack of dependence with respect to time. At the end of this paper we present an application to the pointwise controllability of the evolutionary Stokes system.

In this paper we give a positive answer to the problem above showing that the Stokes spectrum is simple generically with respect to the domain. Our proof is inspired by the work of J. Albert (see [1]), where the generic simplicity of the Dirichlet Laplacian is proved. We also refer to A. Micheletti [20] and K. Uhlenbeck [36] for other similar results in this direction. In [1] the problem of the generic simplicity is reduced to the obtaining of a suitable unique continuation property for the eigenfunctions by applying Baire's lemma. This unique continuation property turns out to be a consequence of Holmgrem's uniqueness theorem. However, in the context of the Stokes system, the unique continuation property one needs for the eigenfunctions may not be understood as a noncharacteristic Cauchy problem and it may not be solved by means of Holmgrem's uniqueness theorem. Therefore, additional developments are required to obtain such unique continuation properties. We proceed in two different ways:

a) By means of Pohožaev's identities;

b) Proving generic unique continuation results.

The first approach consists of obtaining simple global identities for the eigenfunctions of the Stokes system, in the spirit of Pohožaev's identities for the scalar elliptic equations (see for instance M. Struwe [32]). This allows us to show the desired unique continuation property and, consequently, the generic simplicity of the spectrum. The drawback of this approach is that we need to deform rather large subsets of the boundary of the reference domain Ω in order to guarantee the simplicity of the spectrum. The second approach is based on the key observation that the unique continuation property is not needed to hold for all domains Ω but only generically with respect to the domain. Using classical tools of shape differentiation we show that this generic unique continuation property problem may be reduced to a question that may be solved applying Holmgrem's uniqueness theorem. In this case we require, for technical reasons, that the set Ω has boundary of class C^3. The main tool used in this approach is *differentiation with respect to the domain.* In Section 2.3 we give some basic definitions and results; we refer to the works of Simon [29] and the works of Murat and Simon [21]–[23], for details.

In order to state the main result of this paper some basic notation is needed. Given a bounded domain Ω of class C^2 of $\mathbb{R}^2$ and a deformation $u \in W^{3,\infty}(\Omega;\mathbb{R}^2)$ we introduce the following deformed domain:

$$\Omega + u = \left\{ z \in \mathbb{R}^2 : z = x + u(x), x \in \Omega \right\}.$$

We then consider the Stokes system in the deformed domain $\Omega + u$:

$$(P_u) \qquad \left\{ \begin{array}{rcll} -\triangle v + \nabla p & = & \lambda v, & \text{in } \Omega + u \\ \nabla \cdot v & = & 0, & \text{in } \Omega + u \\ v & = & 0 & \text{on } \partial(\Omega + u). \end{array} \right.$$

The main result of this paper is as follows:

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{R}^2$ of class C^2. Then the set $A = \{u \in W^{3,\infty}(\Omega;\mathbb{R}^2) :$ all the eigenvalues of (P_u) are simple$\}$ is residual on $W^{3,\infty}(\Omega;\mathbb{R}^2)$. In other words, it is a countable intersection of dense open sets on $W^{3,\infty}(\Omega;\mathbb{R}^2)$. In particular, A is dense in $W^{3,\infty}(\Omega;\mathbb{R}^2)$. Moreover, if Ω has boundary of class C^3 and given any open nonempty subset Γ_0 of $\partial\Omega$, the same holds if the deformations are restricted to satisfy the condition*

$$u = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0.$$

Theorem 1.1 guarantees the generic simplicity of the spectrum of the Stokes system with respect to perturbations of the domain Ω. The last statement of the theorem indicates that this may be achieved by means of deformations that leave invariant most of the boundary of the domain Ω.

As indicated above, the proof of the last statement of Theorem 1.1 requires a suitable unique continuation property for the eigenfunctions of the system, namely,

$$(SUC) \qquad \begin{cases} \text{If } v \text{ solves } (P_u) \text{ and } \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_0, \\ \text{can we guarantee that } v \equiv 0? \end{cases}$$

The unique continuation problem (SUC) does not fit in the framework of Holmgrem's uniqueness theorem, which needs conditions on the quantity $\frac{\partial v}{\partial n} + pn$ rather than on $\frac{\partial v}{\partial n}$. Our second main result shows that (SUC) holds generically with respect to the domain:

Theorem 1.2. *Let Ω be a bounded domain of $\mathbb{R}^2$ with boundary of class C^3. Let Γ_0 be a nonempty open subset of $\partial\Omega$. We set*

$$B = \left\{ u \in W^{4,\infty}(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial\Omega \setminus \Gamma_0 \text{ such that the unique continuation property } (SUC) \text{ holds in } \Omega + u \text{ with } \Gamma_0 \text{ replaced by } \Gamma_0 + u \right\}.$$

Then, B is residual in $\{u \in W^{4,\infty}(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}$.

Note that Theorem 1.2 remains true in three space dimensions. However, the result of Theorem 1.1 is still an open problem in 3-d. We explain in detail the difficulty we encountered when dealing with the 3-d case in Section 6.

The rest of this paper is organized as follows. In Section 2 we present some preliminary results about the Stokes system. We also recall some basic tools from shape differentiation.

In Section 3 we analyze the regularity of eigenvalues and eigenfunctions with respect to the perturbation u of the domain. In Section 4 we prove Theorem 1.1 in the particular case where $\Gamma_0 = \partial\Omega$ in which the unique continuation holds as a consequence of a Pohožaev identity. In Section 5 we prove the generic unique continuation result Theorem 1.2 and complete the proof of Theorem 1.1. In Section 6 we discuss the technical difficulty which arises when trying to extend Theorem 1.1 to three space dimensions. Finally, in Section 7 we apply Theorem 1.1 to deduce the generic pointwise controllability of the Stokes system as an example of application.

2. Some preliminary results

2.1. **Baire's lemma.** Firstly we remember Baire's lemma, which will be a useful tool.

Lemma 2.1. (Baire's lemma) *Let X be a complete metric space and A_n be an open dense subset of X for all $n \in \mathbb{N}$. Then $\cap_{n\in\mathbb{N}} A_n$ is dense in X.*

A direct consequence of the Baire's lemma is the following result.

Lemma 2.2. *Let X be a complete metric space and $\{A_n\}_{n\geq 0}$ be a sequence of open subsets of X such that*
1. $A_0 = X$.
2. A_{n+1} *is a dense subset of* A_n, *for all* $n \geq 0$.

Then $\cap_{n=1}^{\infty} A_n$ *is dense in* X.

2.2. **Some classical results on the Stokes system.** We start with a version of the classical theorem of G. de Rham due to J. Simon [30].

Theorem 2.3. [30] *Let $\Omega \subset \mathbb{R}^d$ be an open set and $q \in (\mathcal{D}'(\Omega))^d$ such that*

$$\langle q, \psi\rangle_{(\mathcal{D}'(\Omega))^d \times (\mathcal{D}(\Omega))^d} = 0, \quad \forall \psi \in (\mathcal{D}(\Omega))^d \quad \text{such that} \;\; \nabla \cdot \psi = 0. \tag{2.1}$$

Then there exists a function $p \in \mathcal{D}'(\Omega)$, such that $q = \nabla p$.

Let us consider the space $J_0(\Omega) = \{\varphi \in (H_0^1(\Omega))^d : \nabla \cdot \varphi = 0 \text{ in } \Omega\}$, with the scalar product

$$\langle \phi, \varphi\rangle = \sum_{i,j=1}^{d} \int_{\Omega} \frac{\partial \phi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j}\, dx. \tag{2.2}$$

Let $H(\Omega) = \{\varphi \in (L^2(\Omega))^d : \nabla \cdot \varphi = 0 \text{ in } \Omega,\ \varphi \cdot n = 0 \text{ on } \partial\Omega\}$. We note that if $\varphi \in L^2(\Omega)$ and $\nabla \cdot \varphi \in L^2(\Omega)$, then the trace of $\varphi \cdot n$ exists and belongs to $H^{-\frac{1}{2}}(\partial\Omega)$. Thus $H(\Omega)$ is well defined.

We define on $H(\Omega)$ the linear unbounded operator A :

$$(A\phi, \varphi)_H = \langle \phi, \varphi\rangle = \sum_{i,j=1}^{d} \int_{\Omega} \frac{\partial \phi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j}\, dx, \quad \forall \phi\,, \varphi \in J_0(\Omega). \tag{2.3}$$

The domain of A in $H(\Omega)$ is denoted by $D(A)$. We can see that A is a positive self-adjoint operator on $H(\Omega)$. Furthermore, A is an isomorphism from $D(A)$ to $H(\Omega)$, and we have that $D(A) = (H^2(\Omega))^d \cap J_0(\Omega)$. Let P be the orthogonal projection from $(L^2(\Omega))^d$ into $H(\Omega)$. We have that

$$A\varphi = -P\triangle\varphi, \quad \forall \varphi \in D(A). \tag{2.4}$$

Therefore, for each $v \in D(A)$, $f \in H(\Omega)$, the equation $Av = f$ is equivalent to the system

$$\left\{ \begin{array}{rll} -\triangle v + \nabla p = & f, & \text{in } \Omega \\ \nabla \cdot v = & 0, & \text{in } \Omega \\ v = & 0, & \text{on } \partial\Omega. \end{array} \right. \tag{2.5}$$

Concerning the solutions of (2.5) the following is known:

Theorem 2.4. [13, 33] *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2 with $d \geq 2$. If $f \in (L^r(\Omega))^d$ with $1 < r < \infty$, then the solution of* (2.5) *satisfies that $(v,p) \in (W^{2,r}(\Omega))^d \times W^{1,r}(\Omega)$ and*

$$\|v\|_{(W^{2,r}(\Omega))^d} + \|\nabla p\|_{(L^r(\Omega))^d} \leq C \|f\|_{(L^r(\Omega))^d}, \tag{2.6}$$

where the constant $C = C(r,\Omega)$ does not depend on f. Moreover, if Ω is of class C^{m+2} and $f \in (H^m(\Omega))^d$, then $(v,p) \in (H^{m+2}(\Omega))^d \times H^{m+1}(\Omega)$.

The operator A^{-1} is continuous from $H(\Omega)$ to $D(A)$; $A^{-1} : H(\Omega) \longrightarrow H(\Omega)$ is also self-adjoint. On the other hand, provided that the embedding from $H^1(\Omega)$ to $L^2(\Omega)$ is compact, the embedding from $J_0(\Omega)$ to $H(\Omega)$ is compact. Therefore A^{-1} is a self-adjoint compact operator on $H(\Omega)$. Then, by classical spectral theory (see [12]) we have that the operator A has a countable discrete spectrum; that is, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$.

Furthermore, each eigenvalue, λ_i, has finite multiplicity, and the eigenfunctions are orthogonal on $H(\Omega)$.

In two space dimensions it will often be useful to use the *stream function* associated to the velocity field.

Lemma 2.5. [34, Lemma 2.5, p. 38] *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set of class C^2. Let Γ_0 be the exterior boundary of Ω and Γ_i, $i = 1, \ldots, r$, be the other connected components of $\partial\Omega$ (see Figure* 1*). Let $w \in (H_0^1(\Omega))^2$ be a function such that* $\operatorname{div} w = 0$ *on Ω. Then there exists a unique function $\rho \in H^2(\Omega)$ such that*

$$\rho = 0, \quad \text{on } \Gamma_0, \quad \rho = \text{constant on } \Gamma_i, i = 1, \ldots, r,$$
$$\frac{\partial \rho}{\partial n} = 0, \quad \text{on } \Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_r,$$

which satisfies $w = \left(-\frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial x}\right)$, in Ω.

2.3. **Shape differentiation.** We now present some basic results related to shape differentiation. Given $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ we consider the new domain $\Omega + u$, defined as:

$$\Omega + u = \{z \in \mathbb{R}^d : z = x + u(x),\ x \in \Omega\}. \tag{2.7}$$

Consider the Stokes eigenvalue problem in the perturbed domain $\Omega + u$:

$$(P_u) \qquad \left\{ \begin{array}{rcll} -\triangle v(u) + \nabla p(u) & = & \lambda(u) v(u), & \text{on } \Omega + u \\ \nabla \cdot v(u) & = & 0, & \text{on } \Omega + u \\ v(u) & \in & (H_0^1(\Omega + u))^d & \\ p(u) & \in & L^2(\Omega + u) & \\ \displaystyle\int_{\Omega + u} p(u) & = & 0. & \end{array} \right.$$

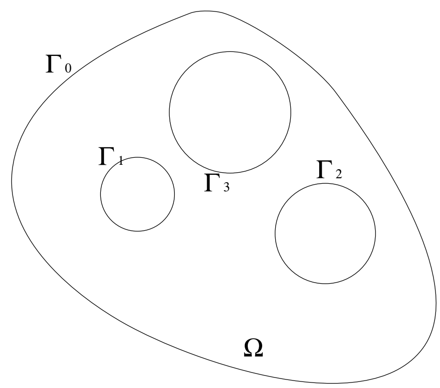

FIGURE 1

We consider perturbations u in the space $W^{k,\infty}(\Omega,\mathbb{R}^d)$ with norm

$$\|u\|_{k,\infty} = \sup_{0\le|\alpha|\le k,\, x\in\Omega} \text{ess} \; |D^\alpha u(x)| \,.$$

Lemma 2.6. [31] *Let $u \in W^{k,\infty}(\Omega,\mathbb{R}^d)$ with $k \ge 1$, such that $\|u\|_{k,\infty} \le \frac{1}{2}$. Then $(I+u) : \Omega \to \Omega + u$ is invertible.*

Furthermore, there exists $w \in W^{k,\infty}(\Omega,\mathbb{R}^d)$ such that

$$(I+u)^{-1} = I + w \qquad \text{and} \qquad \|w\|_{k,\infty} \le C_k \|u\|_{k,\infty} \,,$$

where C_k is a constant independent of u.

Remark 2.1. According to this result, if Ω is of class C^j we can choose $k = j+1$ (and therefore the perturbation spaces $W^{k,\infty}(\Omega,\mathbb{R}^d)$) such that our new domain $\Omega + u$ is also of class C^j. In particular, if Ω is of class C^2, then $\Omega + u$ is also of class C^2, and the solutions of (P_u) satisfy $(v(u), p(u)) \in (H^2(\Omega))^d \times (H^1(\Omega) \cap L_0^2(\Omega))$ for every $u \in W^{3,\infty}(\Omega,\mathbb{R}^d)$ small enough.

Lemma 2.7. [31] *Let $k \ge 1$ and consider the function*

$$\begin{array}{rrcl} \gamma: & W^{k,\infty}(\Omega,\mathbb{R}^d) & \to & W^{k-1,\infty}(\Omega,\mathbb{R}) \\ & u & \to & \gamma(u) = Jac(I+u) = |\det[\partial_j (I+u)_i]| \,. \end{array}$$

This function γ is differentiable at $u = 0$. Furthermore, the directional derivative in the direction w at the point $u = 0$ is $\operatorname{div} w$, that is,

$$\langle D\gamma(0), w\rangle = \operatorname{div} w \,, \quad \forall\, w \in W^{k,\infty}(\Omega,\mathbb{R}^d).$$

Lemma 2.8. [31] *Let $k \ge 1$. The map*

$$\begin{array}{rrcl} \beta: & \mathcal{W} \subset W^{k,\infty}(\Omega,\mathbb{R}^d) & \to & \mathcal{M}_{d\times d}(W^{k-1,\infty}(\Omega,\mathbb{R})) \\ & u & \to & {}^t[\partial_j (I+u)_i]^{-1} \,, \end{array}$$

where $\mathcal{W}$ is a neighborhood of $u = 0$ on $W^{k,\infty}(\Omega, \mathbb{R}^d)$, is differentiable on $u = 0$. Its directional derivative on $u = 0$ in the direction w is given by the matrix $-^t[\partial_j w_i]$, i.e., the adjoint of $[\partial_j w_i]$,

$$^t[\partial_j (I+u)_i]^{-1} = [I] -^t [\partial_j u_i] + \theta(u),$$

where the matrix $\theta(u)$ satisfies

$$\frac{\|\theta(u)\|_{k-1,\infty}}{\|u\|_{k,\infty}} \to 0, \qquad \textit{as} \;\; \|u\|_{k,\infty} \to 0.$$

We now give some useful properties of general maps:

$$\begin{array}{rccl} v: & W^{k,\infty}(\Omega, \mathbb{R}^d) & \to & W^{m,r}(\Omega + u) \\ & u & \to & v(u), \end{array}$$

where $m \leq k$ is an integer. In practice, $v(u)$ will be the solution of a suitable problem which depends on the perturbation function u (for instance, a solution of our eigenvalue problem (P_u)).

Definition 2.1. [31] Let $k \geq m \geq 1$. We say that the function $v(u)$ has a *first-order local variation* at $u = 0$ on $W^{m-1,r}_{loc}(\Omega)$, if $v(u) \in W^{m,r}(\Omega + u)$ for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ and there exists a linear map $v'(\Omega; u)$ defined from $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ to $W^{m-1,r}_{loc}(\Omega)$ such that, for all open sets $\omega \subset\subset \Omega$,

$$v(u) = v(0) + v'(\Omega; u) + \theta(u), \text{ in } \omega,$$

when $\|u\|_{k,\infty}$ is small enough and

$$\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{ in } W^{m-1,r}(\omega) \quad \text{ as } \|u\|_{k,\infty} \to 0.$$

Definition 2.2. [31] Let $k \geq m \geq 1$. We say that the function $v(u)$ has a *first total variation* at $u = 0$ in $W^{m,r}(\Omega)$, if $v(u) \circ (I + u) \in W^{m,r}(\Omega)$ and there exists a linear map $\dot{v}(\Omega; u)$ defined from $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ into $W^{m,r}(\Omega)$, such that

$$v(u) \circ (I + u) = v(0) + \dot{v}(\Omega; u) + \theta(u), \text{ in } \Omega,$$

for $\|u\|_{k,\infty}$ small enough and

$$\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{ in } W^{m,r}(\Omega) \quad \text{ as } \|u\|_{k,\infty} \to 0.$$

Remark 2.2. Let

$$\begin{array}{rccl} v: & W^{k,\infty}(\Omega, \mathbb{R}^d) & \to & W^{m,r}(\Omega + u) \\ & u & \to & v(u), \end{array}$$

be a function of the perturbation parameter u. Then if $k \geq m$, from Definition 2.1 it follows that the *first local variation* can be defined as

$$v'(\Omega; u) = \lim_{t \to 0} \frac{v(tu)|_\omega - v(0)|_\omega}{t} \quad \text{in } \omega, \tag{2.8}$$

where $\omega \subset\subset \Omega$ and $v(tu)|_\omega$, $v(0)|_\omega$ are the restrictions of the functions $v(tu), v(0)$ to ω.

On the other hand, from Definition 2.2 we note that the *first-order total variation* can be defined as

$$\dot{v}(\Omega; u) = \lim_{t \to 0} \frac{v(tu) \circ (I + tu) - v(0)}{t} \quad \text{in } \Omega \tag{2.9}$$

where $V(u) = v(tu) \circ (I + tu)$.

It is important to observe that the total variation $\dot{v}(\Omega; u)$ is a function defined on the whole of Ω while the local variation is a function defined "locally" in subsets $\omega \subset\subset \Omega$.

Remark 2.3. In what follows, to simplify the notation, we will write

$$\dot{v}(u) = \dot{v}(\Omega; u); \quad v'(u) = v'(\Omega; u). \tag{2.10}$$

The following theorem provides a sufficient condition for the existence of local variations. Furthermore, it allows us to relate the concepts of local and total variation by means of the formula

$$v'(u) = \dot{v}(u) - u \cdot \nabla v(0).$$

Note that $v(0) \in W^{m,r}(\Omega)$ and $\dot{v}(u) \in W^{m,r}(\Omega)$, with $m \leq k$; thus, $v'(u) \in W^{m-1,r}_{loc}(\Omega)$. This formula explains the difference of regularity between the total derivative and the local derivative.

Theorem 2.9. [31] *Consider a map $u \to v(u) \in W^{m,r}(\Omega + u)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$, with $k \geq m \geq 1$ and $1 \leq r < \infty$. Let us assume that there exists a linear, continuous map $u \to \dot{v}(u)$ defined on $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m,r}(\Omega)$, such that*

$$v(u) \circ (I + u) = v(0) + \dot{v}(u) + \theta(u), \ \text{ in } W^{m,r}(\Omega),$$

for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough, where

$$\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{on } W^{m,r}(\Omega) \quad \text{as } \|u\|_{k,\infty} \to 0.$$

Then

$$v'(u) = \dot{v}(u) - u \cdot \nabla v(0), \tag{2.11}$$

for each $\omega \subset\subset \Omega$ which satisfies that

$$v' : W^{k,\infty}(\Omega, \mathbb{R}^d) \to W^{m-1,r}(\omega) \tag{2.12}$$

is linear and continuous and

$$v(u) = v(0) + v'(u) + \theta(u), \qquad in\ \omega, \tag{2.13}$$

where $\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0$ *on* $W^{m-1,r}(\omega)$ *as* $\|u\|_{k,\infty} \to 0$.

The following theorem provides sufficient conditions for the existence of the first local variation for functions which depend on the deformation u. Furthermore, it provides an expression for the local variation on the boundary in terms of the normal derivative of $v(0)$.

Theorem 2.10. [31] *Let Ω be a $C^{0,1}$ domain. Suppose that the hypotheses of Theorem 2.9 hold for $m \geq 2$ and $1 \leq r < \infty$. Furthermore, assume that for each $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough, $v(u) = 0$ on $\partial(\Omega + u)$. Then, for each $\omega \subset\subset \Omega$, the function $u \to v_\omega(u) = v(u)|_\omega$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m-1,r}(\omega)$ is differentiable at $u = 0$. Moreover, the map $u \to v(u)$ has a local derivative at $u = 0$ (see Definition 2.1) and the local derivative at $u = 0$, in the direction u, denoted by $v'(u)$, satisfies $v'(u) \in W^{m-1,r}(\Omega)$ and*

$$v'(u) = -(u \cdot n)\frac{\partial v(0)}{\partial n} \qquad on \qquad \partial\Omega,$$

where n is the unit outward normal vector to Ω.

In what follows we will use the notation

$$\mathcal{W} = \{u \in W^{k,\infty}(\Omega, \mathbb{R}^d) : \|u\|_{k,\infty} < c_\Omega\},$$

where $k \geq 1$ and $c_\Omega < 1/2$ is small enough such that all the previous results hold.

Lemma 2.11. [3, Lemma 9] *Let $u \in \mathcal{W}$. If $f \in H_0^1(\Omega + u)$, then there exists a unique $g \in H_0^1(\Omega)$, such that $f \circ (I + u) = g$. Moreover,*

$$\left(\frac{\partial f}{\partial z_i}\right) \circ (I + u) = \sum_j M_{ij}(u)\frac{\partial g}{\partial x_j} = D_i(u)g, \tag{2.14}$$

where the matrix $M(u)$ is defined as

$$M(u) = \left[M_{i,j}(u)\right] =^t \left[\frac{\partial}{\partial x_j}(I + u)_i\right]^{-1} = \left[\frac{\partial}{\partial x_i}(I + u)_j\right]^{-1},$$

and $z_i = x_i + u_i(x)$, $\forall\, x \in \Omega$.

3. Regularity of the eigenvalues and eigenfunctions

3.1. **Reformulation of the Stokes eigenvalue problem.** In order to study the regularity of eigenvalues and eigenfunctions with respect to the perturbation u, we have to write the problem (P_u) in a suitable equivalent way. We consider the space

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega f = 0 \right\},$$

with the norm induced by $L^2(\Omega)$.

Lemma 3.1. *Let $u \in \mathcal{W}$. Consider the map that to each $k \in L^2(\Omega)$ with*

$$\int_\Omega k \; Jac(I+u) = 0$$

associates

$$K = k - \frac{1}{|\Omega|} \int_\Omega k \in L_0^2(\Omega).$$

This map is bijective.

Remark 3.1. Note that if $p(u) \in L_0^2(\Omega+u)$ we have that

$$\int_{\Omega+u} p(u) = \int_\Omega p(u) \circ (I+u) Jac(I+u) = 0.$$

Therefore, from Lemma 3.1, the map

$$p(u) \in L_0^2(\Omega) \longrightarrow P(u) = p(u) \circ (I+u) - \frac{1}{|\Omega|} \int_\Omega p(u) \circ (I+u)$$

is bijective.

Now, let $\{v(u), p(u), \lambda(u)\}$ be a solution of (P_u). We define

$$V(u) = v(u) \circ (I+u), \qquad V = (V_1, \ldots, V_d)$$

and

$$P(u) = p(u) \circ (I+u) - \frac{1}{|\Omega|} \int_\Omega p(u) \circ (I+u),$$

the image of $p(u)$ by the map of Lemma 3.1.

The following system is equivalent to (P_u) :

$$\begin{cases} -\displaystyle\sum_{ij} \partial_j (M_{ij}(u) Jac(I+u) D_i(u) V_k(u)) \\ +\displaystyle\sum_j \partial_j (M_{kj}(u) P(u) Jac(I+u)) = \lambda(u) V_k(u) Jac(I+u), \quad \text{in } \Omega \\ D(u) \cdot V(u) = 0, \quad \text{in } \Omega \\ V(u) \in (H_0^1(\Omega))^d, \quad P(u) \in L_0^2(\Omega), \end{cases} \tag{3.1}$$

where $1 \leq k \leq 2$, with

$$D_i(u)g = (\frac{\partial f}{\partial z_i}) \circ (I+u) = \sum_j M_{ij}(u) \frac{\partial g}{\partial x_j},$$

for $g = f \circ (I+u)$, with $M(u) = (M_{ij}(u))_{i,j=1}^d$ as in Lemma 2.5.

On the other hand, note that the condition

$$D(u) \cdot Y = 0, \quad Y \in (H_0^1(\Omega))^d,$$

with $Y = y \circ (I+u)$ and $y \in (H_0^1(\Omega+u))^d$, is equivalent to

$$D(u) \cdot Y - \frac{1}{|\Omega|} \int_\Omega D(u) \cdot Y = 0, \quad Y \in (H_0^1(\Omega))^d.$$

Indeed, if

$$D(u) \cdot Y = \frac{1}{|\Omega|} \int_\Omega D(u) \cdot Y = c,$$

since

$$\int_\Omega D(u) \cdot Y Jac(I+u) = \int_{\Omega+u} \operatorname{div}\, y = \int_{\partial(\Omega+u)} y \cdot n = 0,$$

with $y \circ (I+u) = Y$, we have

$$0 = \int_\Omega D(u) \cdot Y Jac(I+u) = c \int_\Omega Jac(I+u) = c\, |\Omega+u| \Leftrightarrow c = 0.$$

That is, $D(u) \cdot Y = 0$ in Ω.

Thus, the following problem is equivalent to (3.1) and therefore also equivalent to (P_u): *To find* $\lambda(u) \in \mathbb{R}$ *and* $V(u) \in J_u(\Omega)$, *where*

$$J_u = \left\{ \varphi \in (H^2(\Omega) \cap H_0^1(\Omega))^2 : D\,(u) \cdot V = 0, \ \textit{in } \Omega \right\} \tag{3.2}$$

such that

$$\begin{cases} -\sum_{ij} \partial_j (M_{ij}(u) Jac(I+u) D_i(u) V_k(u)) + \\ \sum_j \partial_j (M_{kj}(u) P(u) Jac(I+u)) = \lambda(u) V_k(u) Jac(I+u), \quad \textit{in } \Omega \\ D(u) \cdot V(u) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega D(u) \cdot V(u) = 0, \quad \textit{in } \Omega \\ V(u) \in (H_0^1(\Omega))^d, \quad P(u) \in L_0^2(\Omega). \end{cases} \tag{3.3}$$

3.2. **Some preliminary results of spectral theory.** To prove the existence and regularity of the eigenvalues and eigenfunctions of the Stokes system with respect to the perturbation parameter u we will use the Lyapunov-Schmidt method (see [37], [5, p. 30]).

Lemma 3.2. [5, Lemma 4.1, p. 31] *Suppose that X and Z are Hilbert spaces and $A : X \longrightarrow Z$ is a continuous linear operator. Let $U : X \longrightarrow N(A)$, $E : Z \longrightarrow R(A)$ be the orthogonal projection from X and Z on the kernel and range of A respectively. Then, there exists a bounded linear operator $K : R(A) \longrightarrow N(A)^{\perp}$ called the right inverse of A such that*

$$AK = I : R(A) \longrightarrow R(A), \qquad KA = I - U : Z \longrightarrow N(A)^{\perp}.$$

Let Λ be a closed subset of a Banach space, such that $Int\Lambda \neq \emptyset$. If $N : \Lambda \times X \longrightarrow Z$ is a continuous operator, then the problem

$$Ax - N(x, \lambda) = 0 \tag{3.4}$$

is equivalent to the equations

$$z - KEN(y + z, \lambda) = 0 \tag{3.5}$$

$$(I - E)N(y + z, \lambda) = 0, \tag{3.6}$$

where $x = y + z$, $y \in N(A)$ and $z \in N(A)^{\perp}$.

Assume that the operator N satisfies that

$$N(0, 0) = 0, \qquad \frac{\partial N}{\partial x}(0, 0) = 0,$$

and consider the equation (3.5), for (x, λ) in a neighborhood of $(0, 0)$ in $X \times \Lambda$. Applying the implicit function theorem to (3.5), we deduce the existence of a neighborhood $V \subset N(A) \times \Lambda$ of $(0, 0)$ and a function $z^* : V \longrightarrow N(A)^{\perp}$ with the same regularity as N providing the solution of (3.5). Therefore, if $\{y_1, \ldots, y_h\}$ is an orthonormal basis of $N(A)$, the solution $x(\lambda)$ of (3.5) satisfies

$$x(\lambda) = \sum_{i=1}^{h} c_i(\lambda) y_i + z^*\Big(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda\Big) = 0, \tag{3.7}$$

for suitable coefficients $c_1, \ldots, c_h$. Then, $(x, \lambda) \in V$ satisfy (3.4) if and only if

$$(I - E)N\Big(\sum_{i=1}^{h} c_i(\lambda) y_i + z^*\Big(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda\Big), \lambda\Big) = 0, \tag{3.8}$$

which is a finite-dimensional system of equations on the constants $c_1, \ldots, c_h$.

3.3. **Regularity of the eigenvalues and eigenfunctions with respect to the perturbation parameter.** Now, we apply the Lyapunov-Schmidt method to our problem.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2.*

Then the map

$$S : W^{3,\infty}(\Omega, \mathbb{R}^d) \longrightarrow \mathcal{L}((H_0^1(\Omega))^d \times L_0^2(\Omega); (H^{-1}(\Omega))^d \times L_0^2(\Omega)),$$

such that

$$S(u)(\varphi, \pi) = \begin{pmatrix} -D_i(u)(Jac(I+u)D_i(u)\varphi) + D(u)(\pi Jac(I+u)) \\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega D(u)\cdot\varphi - D(u)\cdot\varphi \end{pmatrix} \tag{3.9}$$

is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$.

Proof. $Jac(I + u)$ is a polynomial on the first partial derivatives of u. Then the map $u \longrightarrow Jac(I + u)$ is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$. On the other hand,

$$M_{ij}(u) = \frac{1}{Jac(I+u)}(\delta_{ij} + a_{ij}),$$

where $a_{ij}(u)$ is the minor of the matrix $M^{-1}(u)$ associated to its ij-th element which is also a polynomial of the first partial derivatives of u. Moreover, for u small enough $Jac(I + u) > 0$. Therefore, $u \longrightarrow M(u)$ is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ as well.

Note also that $S(u)$ can be rewritten as

$$S(u)(\varphi, \pi) = \begin{pmatrix} -M(u)\nabla\cdot(Jac(I+u)M(u)\nabla\varphi) + M(u)\nabla(\pi Jac(I+u)) \\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega M(u)\nabla\cdot\varphi - M(u)\nabla\cdot\varphi \end{pmatrix} \tag{3.10}$$

since $D(u)\varphi = M(u)\nabla\varphi$. From the analyticity of the functions $u \longrightarrow Jac(I + u)$ and $u \longrightarrow M(u)$ we obtain that the map $S(u)$ is analytic in a neighborhood of $u = 0$ on $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with values in

$$\mathcal{L}((H_0^1(\Omega))^d \times L_0^2(\Omega), (H^{-1}(\Omega))^d \times L_0^2(\Omega)),$$

and the proof is complete. $\square$

Remark 3.2. Given any $\lambda > 0$ the map

$$A : (H_0^1(\Omega))^d \times L_0^2(\Omega) \longrightarrow (H^{-1}(\Omega))^d \times L_0^2(\Omega) \tag{3.11}$$

defined as

$$A(\varphi,\pi)=\begin{pmatrix}-\triangle\varphi+\nabla\pi-\lambda\varphi\\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega\nabla\cdot\varphi-\nabla\cdot\varphi\end{pmatrix}=\begin{pmatrix}-\triangle\varphi+\nabla\pi-\lambda\varphi\\ \\ -\nabla\cdot\varphi\end{pmatrix} \tag{3.12}$$

is self-adjoint.

Furthermore, if λ is an eigenvalue of multiplicity h of the Stokes system on the domain Ω and $v_1,\dots,v_h$ are the orthonormal eigenfunctions associated to λ and $p_1,\dots,p_h$ the eigenpressures, we have

$$N(A)=span\left\{(v_i,p_i):i=1,\dots,h\right\}. \tag{3.13}$$

Now we have the following result, which is a slight variation of a theorem due to J. Albert [2] on the scalar eigenvalue problem for the Laplacian.

Theorem 3.4. *Let $\Omega\subset\mathbb{R}^d$ be a bounded open set of class C^2. Assume that λ is an eigenvalue of multiplicity h of the Stokes system on the domain Ω and $v_1,\dots,v_h$ are the orthonormal eigenfunctions associated to λ with associated pressures $p_1,\dots,p_h$.*

Then there exist h analytic functions defined in a neighborhood of $u=0$ in $W^{3,\infty}(\Omega,\mathbb{R}^d)$ with values in $\mathbb{R}, u\to\lambda_i(u)$, and h analytic functions $u\to(\phi_i(u),\rho_i(u))$, with values in $J_u\times L^2_0(\Omega)$, $i=1,\dots,h$, defined in a neighborhood of $u=0$ in $W^{3,\infty}(\Omega,\mathbb{R}^d)$, such that the following hold:

(1) $\lambda_j(0)=\lambda,\qquad j=1,\dots,h.$
(2) *For all u small enough, $(\lambda_j(u),\phi_j(u),\rho_j(u))$ is a solution of* (3.3)*; that is, the function $v_i(u)=\phi_i(u)\circ(I+u)^{-1}\in J_0(\Omega+u)$ is an eigenfunction of (P_u) associated to the eigenvalue $\lambda_j(u)$.*
(3) *For all u small enough the set $\{v_1(u),\dots,v_h(u)\}$ is orthonormal on $H(\Omega+u)$.*
(4) *For each interval $I\subset\mathbb{R}$ such that $\overline{I}$ contains only the eigenvalue λ of (P), there exists a neighborhood U of $u=0$ such that there are exactly h eigenvalues (counting the multiplicity) $\lambda_1(u),\dots,\lambda_h(u)$ of (P_u) contained on I.*

Remark 3.3. To prove Theorem 3.4 we prove first that if λ is an eigenvalue of multiplicity h we can find functions $u\to\lambda(u)\in\mathbb{R}$ and $u\longrightarrow(v(u),p(u))\in J_0(\Omega+u)\times L^2_0(\Omega+u)$ such that $v(u)$ is an eigenfunction of (P_u) associated to the eigenvalue $\lambda(u)$, with pressure $p(u)$. To find the other $h-1$ branches of eigenvalues and eigenfunctions, we apply in an iterative form the method described in the following proposition.

Proposition 3.5. *Let $\Omega\subset\mathbb{R}^d$ be a bounded open set of class C^2. Assume that λ is an eigenvalue of multiplicity h of the Stokes system on the domain*

Ω and $v_1, \ldots, v_h$ are the orthonormal eigenfunctions associated to λ with associated pressures $p_1, \ldots, p_h$.

Then there exists at least a function $u \longrightarrow (\lambda(u), \phi(u), \rho(u)) \in \mathbb{R} \times J_u \times L_0^2(\Omega)$ which is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ such that

(1) $\lambda(0) = \lambda$,

(2) *$v(u)$ defined as $v(u) \circ (I + u) = \phi(u)$ is an eigenfunction of the Stokes system in the domain $\Omega + u$, associated to the eigenvalue $\lambda(u)$, with pressure $p(u)$ where $p(u) \circ (I + u) = \rho(u)$.*

Proof. Let λ be an eigenvalue of multiplicity h of the Stokes system on the domain Ω and $v_1, \ldots, v_h$ be the orthonormal eigenfunctions associated to λ with associated pressures $p_1, \ldots, p_h$.

We apply the Lyapunov-Schmidt method to problem (P_u). Let us consider the map

$$\begin{array}{rcl} T : (H_0^1(\Omega))^d \times L_0^2(\Omega) & \longrightarrow & (H^{-1}(\Omega))^d \times L_0^2(\Omega) \\ (\varphi, \pi) & \longrightarrow & T(\varphi, \pi) = (\varphi, 0). \end{array} \tag{3.14}$$

Thus, in view of (3.3) and the definition (3.10) of $S(u)$, we can write problem (P_u) as

$$S(u)(\phi(u), \rho(u)) - \lambda(u) Jac(I + u) T(\phi(u), \rho(u)) = 0, \tag{3.15}$$

or, equivalently, as

$$\begin{aligned} &[S(0) - \lambda T](\phi(u), \rho(u)) \\ &= [(S(0) - S(u)) - \lambda T + \lambda(u) Jac(I + u) T](\phi(u), \rho(u)) \\ &= [(S(0) - S(u)) + (\lambda(u) - \lambda) Jac(I + u) T + \lambda(Jac(I + u) - 1) T](\phi(u), \rho(u)). \end{aligned} \tag{3.16}$$

If $R(u) = S(0) - S(u) + \lambda(Jac(I + u) - 1)T$, then

$$[S(0) - \lambda T] (\phi(u), \rho(u)) = [R(u) + (\lambda(u) - \lambda) Jac(I + u) T] (\phi(u), \rho(u)). \tag{3.17}$$

From Lemma 3.2 we have that the map $A = S(0) - \lambda T$ has a right inverse operator K. Thus, in view of (3.17) we obtain that

$$\begin{array}{rl} (\phi(u), \rho(u)) = & [K(R(u) + (\lambda(u) - \lambda) Jac(I + u) T)] (\phi(u), \rho(u)) \\ + & (\psi(u), \pi(u)), \end{array} \tag{3.18}$$

where $(\psi(u), \pi(u)) \in N(A)$; that is,

$$(\psi(u), \pi(u)) = \sum_{i=1}^{h} c_i(u) (v_i, p_i). \tag{3.19}$$

On the other hand, from (3.17)

$$[R(u) + (\lambda(u) - \lambda)Jac(I+u)T]\,(\phi(u), \rho(u)) \in R(A) = N(A)^{\perp}.$$

Thus, if

$$Q(u) = R(u) + (\lambda(u) - \lambda)Jac(I+u)T, \tag{3.20}$$

we have that

$$\begin{aligned} 0 &= \langle Q(u)(\phi(u), \rho(u)), (v_i, p_i)\rangle \\ &= \left\langle Q(u)\,[I - KQ(u)]^{-1}\,(\psi(u), \pi(u)), (v_i, p_i)\right\rangle \\ &= \left\langle Q(u)\,[I - KQ(u)]^{-1} \sum_{j=1}^{h} c_j(u)(v_j, p_j), (v_i, p_i)\right\rangle \\ &= \sum_{j=1}^{h} c_j(u) \left\langle Q(u)\,[I - KQ(u)]^{-1}\,(v_j, p_j), (v_i, p_i)\right\rangle, \end{aligned} \tag{3.21}$$

for all $i = 1, \dots, h$ which is a linear system of equations on the unknowns $c_j(u)$. This system has a nontrivial solution if and only if

$$\det(\left\langle Q(u)\,[I - KQ(u)]^{-1}\,(v_j, p_j), (v_i, p_i)\right\rangle) = 0. \tag{3.22}$$

We replace $\lambda(u) - \lambda$ by α and we define $\widehat{R}(u, \alpha) = R(u) + \alpha Jac(I+u)T$,

$$f_{ij}(\alpha, u) = \left\langle \left[\widehat{R}(u, \alpha)\right] \left[I - K(\widehat{R}(u, \alpha))\right]^{-1} (v_j, p_j), (v_i, p_i)\right\rangle, \tag{3.23}$$

and

$$F(\alpha, u) = \det(f_{ij}(\alpha, u)). \tag{3.24}$$

For u small enough, the map $u \longrightarrow [I - K\widehat{R}(u, \alpha)]^{-1}$ is well defined. Indeed for $\alpha = 0$ and $u = 0$ we have that $[I - K\widehat{R}(0, 0)] = I$ and the map is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$. On the other hand, as we mentioned above, if $F(\alpha, u) = 0$, system (3.21) has a nontrivial solution $c_1(u), \dots, c_h(u)$, and then

$$\lambda(u) = \lambda + \alpha \tag{3.25}$$

is an eigenvalue of (P_u). Moreover, from (3.18) and (3.19) we deduce that

$$(\phi(u), \rho(u)) = \sum_{j=1}^{h} c_j(u)\,[I - K(R(u) + (\lambda(u) - \lambda)Jac(I+u)T)]^{-1}\,(v_j, p_j) \tag{3.26}$$

is an eigenfunction of (P_u) associated to the eigenvalue $\lambda(u)$.

According to our previous discussion, for these values of $\alpha(u)$ and setting $\lambda(u) = \lambda + \alpha(u)$, system (3.21) admits a solution $c_1(u), \dots, c_h(u)$, not all the components being zero. We have that

$$f_{ij}(\alpha, 0) = \left\langle [\alpha T] \left[I - \alpha KT\right]^{-1} (v_j, p_j), (v_i, p_i) \right\rangle . \tag{3.27}$$

For α sufficiently small, the operator $I - \alpha KT$ is invertible and, moreover,

$$[I - \alpha KT]^{-1} = \sum_{n \geq 0} (\alpha KT)^n = I + \sum_{n \geq 1} \alpha^n (KT)^n.$$

Therefore, for all $i, j = 1, \dots, h$, we have

$$\begin{aligned} f_{ij}(\alpha, 0) &= \left\langle [\alpha T] \left[I - \alpha KT\right]^{-1} (v_j, p_j), (v_i, p_i) \right\rangle \\ &= \alpha \left\langle T(v_j, p_j), (v_i, p_i) \right\rangle + \alpha \sum_{n \geq 1} \alpha^n \left\langle T(KT)^n (v_j, p_j), (v_i, p_i) \right\rangle \\ &= \alpha \delta_{ij} + \sum_{n \geq 1} \alpha^{n+1} \left\langle T(KT)^n (v_j, p_j), (v_i, p_i) \right\rangle . \end{aligned} \tag{3.28}$$

Therefore,

$$F(\alpha, 0) = \alpha^h + \sum_{n \geq 1} s_n \alpha^{n+h}$$

for suitable coefficients s_n, and F satisfies

$$\frac{\partial^r F}{\partial \alpha^r}(0,0) = 0, \; r = 0, \dots, h-1; \quad \frac{\partial^h F}{\partial \alpha^h}(0,0) \neq 0.$$

Applying the Weierstrass preparation theorem we deduce that

$$F(\alpha, u) = (\alpha^h + a_1(u)\alpha^{h-1} + \cdots + a_h(u)) E(\alpha, u)$$

with $E(\alpha, u) \neq 0$ in a neighborhood of $(0, 0)$. Then for (α, u) small enough we have that $E(\alpha, u) \neq 0$, and the functions $a_j(u)$ are analytic in a neighborhood of $u = 0$. Then, $F(\alpha, u) = 0$ if and only if

$$\alpha^h + a_1(u)\alpha^{h-1} + \cdots + a_h(u) = 0. \tag{3.29}$$

Let $\alpha_j(u)$, $j = 1, \dots, h$, be the complex roots of (3.29). Then there exist constants $c_1(u), \dots, c_h(u)$, not all vanishing simultaneously, a solution of system (3.21). Thus, from (3.26) we obtain that

$$(\phi(u), \rho(u)) = \sum_{j=1}^{h} c_j(u) \left[I + K(R(u) + (\lambda(u) - \lambda) Jac(I + u)T)\right]^{-1} (v_j, p_j),$$

and $\lambda(u) = \lambda + \alpha_1(u)$ constitutes an eigenpair.

Note that if $c_j(u)$ is complex, it is enough to consider the real part $\mathcal{R}c_j(u)$ to get a real eigenfunction. Since the Stokes operator is self-adjoint we have that $\alpha_j(u)$ is real, which completes the proof of Proposition 3.5. $\square$

Remark 3.4. Proposition 3.5 provides the existence of one branch of eigenpairs associated to the root $\alpha(u)$ of (3.29). We do not use the eigenpairs associated to the other roots α_j by now since, so far, we do not know whether they coincide or not with the eigenpair associated to $\alpha_1(u)$.

Now, we prove Theorem 3.4.

Proof of Theorem 3.4. By using an iterative argument on h we prove the existence of the h analytic functions $u \to (\lambda_i(u), v_i(u))$, the eigenvalues and eigenfunctions of the Stokes system in the domain $\Omega + u$.

From Proposition 3.5 we have that there exists an analytic function $u \to (\lambda_1(u), v_1(u), p_1(u))$ defined in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ with values in $\mathbb{R} \times J_0(\Omega + u) \times L^2(\Omega + u)$, $\lambda_1(u)$ being an eigenvalue of the Stokes system, $v_1(u)$ and $p_1(u)$ being the corresponding eigenfunction and eigenpressure.

Therefore, Theorem 3.4 holds for $h = 1$. We must prove it for $h \geq 2$. Let $\Pi_1(u) : J_0(\Omega + u) \longrightarrow J_0(\Omega + u)$ be the orthogonal projection on the eigenspace generated by $v_1(u)$. Then we define the map

$$B(u) = P(u) - \Pi_1(u), \tag{3.30}$$

where $P(u)$ is, as in Section 2.1, the composition of the Laplacian with the projection operator from $(L^2(\Omega + u))^d$ into $H(\Omega + u)$. Then

$$B(0)v_j = (P(0) - \Pi_1(0))v_j = \lambda v_j - \delta_{1j} v_j; \tag{3.31}$$

that is, $B(0)v_j = \lambda v_j$, $j = 2, \ldots, h$, and $B(0)v_1 = (\lambda - 1)v_1$.

Then, λ is an eigenvalue of multiplicity $h - 1$ of the operator $B = B(0)$, with eigenfunctions $v_2, \ldots, v_h$.

Note that other linearly independent eigenfunctions of B associated to λ do not exist. Indeed, if v is another eigenfunction of B associated to the eigenvalue λ such that $\langle v, v_j \rangle = 0$, $j = 2, \ldots, h$, then $\langle v, v_1 \rangle = 0$ (because v_1 is an eigenfunction associated to the eigenvalue $\lambda - 1$) and $Bv = \lambda v$. Then

$$Pv = Bv + \Pi_1 v = Bv + \langle v, v_1 \rangle v_1 = \lambda v;$$

that is, v is an eigenfunction of P associated to λ, and thus λ is an eigenvalue of multiplicity $h + 1$, which is impossible because the multiplicity of λ is h.

It is not difficult to see that $B(u)$ satisfies the same conditions of the Stokes operator $P(u)$ to apply the Lyapunov-Schmidt method used in the proof of Proposition 3.5. Applying this method in an iterative form we obtain $h - 1$ analytic functions in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$,

$u \longrightarrow \lambda_i(u)$ and $u \longrightarrow (v_i(u), p_i(u))$, with $i = 1, \ldots, h$ such that $B(u)v_i(u) = \lambda_i(u)v_i(u)$. Moreover, the functions $v_2(u), \ldots, v_h(u)$ form an orthonormal set in $H(\Omega + u)$. This shows us the existence of the h branches of eigenpairs.

Now we prove the last part of the theorem. Since the eigenvalues $u \rightarrow \lambda_i(u)$ are analytic in a neighborhood of $u = 0$, there exist constants c_i such that

$$|\lambda_i(u) - \lambda_i(v)| \leq c_i \|u - v\|_{3,\infty}.$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \cdots$ be the eigenvalues of the Stokes system in the domain Ω and assume that $\cdots \leq \lambda_{n-1} < \lambda = \lambda_n = \cdots = \lambda_{n+h-1} < \lambda_{n+h} \leq \cdots$. Let $I \subset \mathbb{R}$ be an interval such that λ is the unique eigenvalue contained in I. Then there exists $\delta > 0$ such that $I \subset (\lambda_{n-1} + \delta, \lambda_{n+h} - \delta)$. Let $u \in B(0, \frac{\delta}{c})$, with $c = \max\{c_i : i = 1, \ldots, n+h\}$. Then

$$|\lambda_{n-1}(u) - \lambda_{n-1}| \leq c_{n-1}\|u\|_{3,\infty} < c_{n-1}\frac{\delta}{c} \leq \delta,$$

and

$$|\lambda_{n+h}(u) - \lambda_{n+h}| \leq c_{n+h}\|u\|_{3,\infty} < c_{n+h}\frac{\delta}{c} \leq \delta.$$

Therefore $\lambda_{n-1}(u) \notin \overline{I}$ and $\lambda_{n+h}(u) \notin \overline{I}$; that is, $P(u)$ has at most h eigenvalues contained in $\overline{I}$ counting multiplicity. This completes the proof of Theorem 3.4. $\square$

3.4. **Local variations of the eigenvalues and eigenfunctions.** Let $v_i(u) \in J_0(\Omega + u)$, $i = 1, \ldots, h$, be the eigenfunctions of (P_u) associated to the eigenvalue $\lambda_i(u)$, where

$$\lambda = \lambda_i(0), \quad v_i(0) = v_i, \quad i = 1, \ldots, h.$$

According to the results of the previous section, the branches of the eigenvalues $u \longrightarrow \lambda_i(u) \in \mathbb{R}$ and the eigenfunctions $u \longrightarrow v_i(u)$ are analytic with respect to the perturbation parameter u in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$. The first local variation of the branches solves the system

$$\left\{\begin{array}{rcll} -\triangle v_i'(u) + \nabla p_i'(u) & = & \lambda_i'(u)v_i + \lambda v_i'(u), & \text{in } \Omega \\ \nabla \cdot v_i'(u) & = & 0, & \text{in } \Omega \\ v_i'(u) + (u \cdot \nabla)v_i & \in & (H_0^1(\Omega))^d & \\ p_i'(u) + \text{div } (up_i(0)) & \in & L_0^2(\Omega). & \end{array}\right. \tag{3.32}$$

The following identity is easy to prove:

Lemma 3.6. *Let* $\varphi \in (C^1(\overline{\Omega}))^d \cap (H_0^1(\Omega))^d$ *and* $\Omega \subseteq \mathbb{R}^d$ *be a domain of* C^2*-class. Then*

$$\frac{\partial \varphi}{\partial n} \cdot n = \sum_{i,j=1}^{d} \frac{\partial \varphi_i}{\partial n} n_i \, n_j = \nabla \cdot \varphi, \qquad \text{on } \partial\Omega. \tag{3.33}$$

Proof. Since $\varphi \in (H_0^1(\Omega))^d \cap (C^1(\overline{\Omega}))^d$, we have that, for each $x_0 \in \partial\Omega$ such that $\nabla\varphi(x_0) \neq 0$,

$$n(x_0) = c_i \frac{\nabla \varphi_i(x_0)}{|\nabla \varphi_i(x_0)|},$$

for each $i = 1, \ldots, d$ such that $|\nabla\varphi_i(x_0)| \neq 0$ where $c_i = \pm 1$ depends on the direction in which $\nabla\varphi_i(x_0)$ is oriented. Thus,

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{n_j}{c_i} \left| \nabla \varphi_i \right|, \quad i, j = 1, \ldots, d. \tag{3.34}$$

We have

$$\frac{\partial \varphi}{\partial n} \cdot n = \sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial n} n_i = \sum_{i=1}^{d} (\nabla \varphi_i \cdot n) n_i = \sum_{i,j=1}^{d} \frac{\partial \varphi_i}{\partial x_j} n_i n_j.$$

From (3.34), we obtain that

$$\begin{aligned} \frac{\partial \varphi}{\partial n} \cdot n &= \sum_{i,j=1}^{d} \frac{n_j}{c_i} \left| \nabla \varphi_i \right| n_i n_j = \sum_{i,j=1}^{d} \frac{1}{c_i} \left| \nabla \varphi_i \right| n_i (n_j)^2 \\ &= \sum_{j=1}^{d} (n_j)^2 \Big[\sum_{i=1}^{d} \frac{n_i}{c_i} \left| \nabla \varphi_i \right| \Big] = \sum_{j=1}^{d} (n_j)^2 \Big[\sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial x_i} \Big] = |n|^2 \left(\nabla \cdot \varphi \right) = (\nabla \cdot \varphi). \end{aligned}$$

This completes the proof of the lemma. □

The following lemma provides a useful identity on the derivative of the eigenvalues.

Lemma 3.7. *Let* λ *be an eigenvalue of the Stokes system in* Ω *with multiplicity* h *and let* $v_1, \ldots, v_h$ *be the eigenfunctions associated to* λ*. Let* $u \in W^{3,\infty}(\Omega, \mathbb{R}^d) \longrightarrow \lambda_i(u) \in \mathbb{R}$, $i = 1, \ldots, h$, *be the branches of eigenvalues for* u *small enough.*

Then the first local derivatives of the branches satisfy

$$\delta_{ij} \lambda_i'(u) = -\int_{\Gamma} (u \cdot n) \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n}, \quad \forall i, j = 1, \ldots, h. \tag{3.35}$$

Note that here and in the sequel $\lambda_i'(u)$ denotes the derivative of λ_i at $u = 0$ in the direction u.

Proof. Let $w \in J_0(\Omega)$. Then from (3.32) we have that

$$\int_\Omega \nabla v_i'(u) \cdot \nabla w = \lambda_i'(u) \int_\Omega v_i \cdot w + \lambda \int_\Omega v_i'(u) \cdot w. \tag{3.36}$$

Taking $w = v_j$ in (3.36) we deduce that

$$\int_\Omega \nabla v_i'(u) \cdot \nabla v_j = \lambda_i'(u) \int_\Omega v_i \cdot v_j + \lambda \int_\Omega v_i'(u) \cdot v_j.$$

On the other hand, since Ω is a C^2 domain, we have that $v_j \in (H^2(\Omega))^d$ (see Theorem 2.4). Integrating by parts we deduce that

$$-\int_\Omega (\triangle v_j) \cdot v_i'(u) + \int_\Gamma v_i'(u) \cdot \frac{\partial v_j}{\partial n} = \delta_{ij}\lambda_i'(u) + \int_\Omega v_i'(u) \cdot (-\triangle v_j + \nabla p_j).$$

That is,

$$\begin{aligned} \delta_{ij}\lambda_i'(u) &= \int_\Gamma v_i'(u) \cdot \frac{\partial v_j}{\partial n} - \int_\Omega v_i'(u) \cdot \nabla p_j \\ &= \int_\Gamma v_i'(u) \cdot \frac{\partial v_j}{\partial n} + \int_\Omega p_j (\nabla \cdot v_i'(u)) - \int_\Gamma v_i'(u) \cdot (p_j n) \\ &= \int_\Gamma v_i'(u) \cdot \frac{\partial v_j}{\partial n} - \int_\Gamma v_i'(u) \cdot (p_j n) = -\int_\Gamma (u \cdot n) \frac{\partial v_i}{\partial n} \cdot \Big[\frac{\partial v_j}{\partial n} - p_j n\Big]. \end{aligned} \tag{3.37}$$

From Lemma 3.6 we have that

$$\frac{\partial v_j}{\partial n} \cdot n = \nabla \cdot v_j = 0, \quad \text{in } \Omega.$$

The proof of the lemma is now complete. $\square$

From (3.35) we deduce that

$$\lambda_i'(u) = -\int_\Gamma (u \cdot n) \Big| \frac{\partial v_i}{\partial n} \Big|^2. \tag{3.38}$$

The next lemma provides an identity for the Stokes system in the spirit of the classical Pohožaev's identity for scalar elliptic equations.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain of class C^2 and $x_0 \in \mathbb{R}^d$. Let (λ, φ, p) and (λ, ϕ, q) be solutions of the eigenvalue problem (P). Then*

$$\sum_{i=1}^d \int_\Omega \nabla \varphi_i \cdot \nabla \phi_i = \frac{1}{2} \int_\Gamma ((x - x_0) \cdot n) \frac{\partial \varphi}{\partial n} \cdot \frac{\partial \phi}{\partial n}. \tag{3.39}$$

This lemma may be proved by multiplying the equation satisfied by φ (respectively ϕ) by $(x - x_0) \cdot \nabla \phi$ (respectively $(x - x_0) \cdot \nabla \varphi$) and integrating by parts. We refer to [25] for the details of the proof.

Corollary 3.9. *Under the hypotheses of Lemma 3.8, for each* (λ, v, p) *a solution of the eigenvalue problem* (P), *we have that*

$$\int_\Omega |\nabla v|^2 = \frac{1}{2}\int_\Gamma ((x - x_0)\cdot n)\left|\frac{\partial v}{\partial n}\right|^2. \tag{3.40}$$

Remark 3.5. Identity (3.39) provides a unique continuation result for the eigenfunctions of the Stokes system. Indeed, if v is a eigenfunction of (P) and

$$\frac{\partial v}{\partial n} = 0, \qquad \text{on } \Gamma,$$

according to (3.39) we have that

$$\int_\Omega |\nabla v|^2 = 0 \Leftrightarrow v = 0.$$

The same holds when $\dfrac{\partial v}{\partial n}$ vanishes on a subset of the boundary of the form

$$\Gamma(x_0) = \{x \in \partial\Omega : (x - x_0)\cdot n(x) > 0\}.$$

Note that these results are not a consequence of the classical Holmgrem's uniqueness theorem, which requires the quantity $\dfrac{\partial v}{\partial n} + np$ to vanish in an open subset of $\partial\Omega$.

Using the variants introduced by A. Osses [26] of the classical multiplier $(x - x_0)\cdot\nabla\varphi$ one can show the same unique continuation result for other subsets of the boundary. However, the subsets of the boundary we obtain by multiplier methods are always rather large.

4. Proof of Theorem 1.1: Deformations of large subsets of the boundary

Our proof of the generic simplicity of the eigenvalues of the Stokes problem is inspired in the work of J. Albert (see [1]) for the Laplace operator.

Proof of Theorem 1.1. We define the sets

$$A_0 = W^{3,\infty}(\Omega, \mathbb{R}^2), \tag{4.1}$$

and for $n \geq 1$,

$$A_n = \big\{u \in W^{3,\infty}(\Omega, \mathbb{R}^2) : \quad \begin{array}{l}\text{the first } n \text{ branches } \lambda_1(u), \dots, \lambda_n(u)\\ \text{of eigenvalues of } (P_u) \text{ are simple}\big\}.\end{array} \tag{4.2}$$

Note that the branches $u \longrightarrow \lambda_i(u)$ are those obtained in Theorem 3.4, which are analytic. Therefore the eigenvalues λ_i are not necessarily ordered in an increasing way in the definition of A_n.

The proof of the generic simplicity is based on the application of Baire's lemma (see Lemma 2.1 and Lemma 2.2) on the sets $A_n \subset W^{3,\infty}(\Omega, \mathbb{R}^2)$. We claim that it is sufficient to show that the sets A_n are open and that the set A_{n+1} is dense in A_n for all $n \geq 0$. Indeed, then applying Baire's lemma to the sets A_n in $W^{3,\infty}(\Omega, \mathbb{R}^2)$ we deduce that $A = \bigcap_n A_n$ is dense in $W^{3,\infty}(\Omega, \mathbb{R}^2)$. On the other hand,

$$A = \left\{u \in W^{3,\infty}(\Omega, \mathbb{R}^2)\colon \text{ all the eigenvalues of } (P_u) \text{ are simple}\right\}.$$

Thus, it is sufficient to check these two facts:

a) A_n is open in $W^{3,\infty}(\Omega, \mathbb{R}^2)$. It is clear that A_0 is an open set. Now we analyze the case $n \geq 1$.

Let $u \in A_n$. Then, by definition, the first n branches of the eigenvalues of (P_u), $\lambda_i(u)$ with $i = 1, \ldots, n$, are simple. Let

$$\delta = \min\{|\lambda_i(u) - \lambda_j(u)| : i, j = 1, \ldots, n+1, i \neq j\} > 0.$$

We consider $w \in U$, where $U = \{w \in \mathcal{W} : \|w\|_{3,\infty} < \dfrac{\delta}{2c}\}$, c being the maximum of the Lipschitz constants for the functions $w \to \lambda_i(w)$, $i = 1, \ldots, n+1$, defined from $W^{3,\infty}(\Omega, \mathbb{R}^2)$ with values in $\mathbb{R}$. Then

$$\begin{aligned}
\delta &\leq |\lambda_i(u) - \lambda_j(u)| \\
&\leq |\lambda_i(u+w) - \lambda_i(u)| + |\lambda_i(u+w) - \lambda_j(u+w)| + |\lambda_j(u+w) - \lambda_j(u)| \\
&\leq c\,|(u+w) - u| + |\lambda_i(u+w) - \lambda_j(u+w)| + c\,|(u+w) - u| \\
&< \delta + |\lambda_i(u+w) - \lambda_j(u+w)|.
\end{aligned}$$

Therefore, $|\lambda_i(u+w) - \lambda_j(u+w)| > 0$, and then $u + w \in A_n$, which proves that A_n is an open set.

b) A_{n+1} is dense in A_n. We will prove that A_{n+1} is dense in A_n, for all $n \geq 0$; in particular, for $n = 0$ we will prove that A_1 is dense in $A_0 = W^{3,\infty}(\Omega, \mathbb{R}^2)$. We proceed by contradiction. Assume that A_{n+1} is not dense in A_n; then there exists $u \in A_n \setminus A_{n+1}$ and a neighborhood W of u such that $W \subset A_n \setminus A_{n+1}$.

Since $u \in A_n \setminus A_{n+1}$, we have that the first n branches of the eigenvalues of (P_u) are simple; that is, $\lambda_i(u) \neq \lambda_j(u)$, $\forall i, j = 1, \ldots, n+1$, if $i \neq j$.

Assume that λ_{n+1} is an eigenvalue with multiplicity $h \geq 2$; that is, $\lambda = \lambda_{n+1}(u) = \cdots = \lambda_{n+h}(u)$ (in the case of $n = 0$, we have that the first eigenvalue has multiplicity $h \geq 2$; that is, $\lambda = \lambda_1(u) = \cdots = \lambda_h(u)$). Let $v_1,\ v_2, \ldots,\ v_h$ be the eigenfunctions associated to λ. Let w be such that $u + w \in \mathcal{W}$. We consider the eigenvalues $\lambda_{n+i}(u+w)$, with $i = 1, \ldots, h$, and the corresponding eigenfunctions $v_i(u+w)$, with $i = 1, \ldots, h$, of (P_{u+w}).

Then, from (3.35) and Lemma 3.7 we have

$$\lambda'_{n+i}(\Omega+u;w)\delta_{ij} = -\int_{\widetilde{\Gamma}}(w\cdot n)\frac{\partial v_i}{\partial n}\cdot\frac{\partial v_j}{\partial n}, \tag{4.3}$$

where $\widetilde{\Gamma} = \partial(\Omega+u)$. We want to prove that there exists $w \in \mathcal{W}$, such that $\lambda'_i(\Omega+u;w) \neq \lambda'_j(\Omega+u;w)$, for each $i \neq j$, $i,j = 1,\dots,n+h$. That will immediately imply that

$$\lambda_i(u+\varepsilon w) \neq \lambda_j(u+\varepsilon w),\ \forall i,j = 1,\dots,n+h+1, \tag{4.4}$$

for $\varepsilon > 0$ small enough. Thus $u+\varepsilon w \in A_{n+1}$.

We proceed by contradiction. If there exist $i \neq j$, $i,j = 1,\dots,n+h$ such that $\lambda_i(u+w) = \lambda_j(u+w)$, for all w in a neighborhood of $w = 0$, we have that

$$\lambda'_i(\Omega+u;w) = \lambda'_j(\Omega+u;w). \tag{4.5}$$

Thus, from (3.35) we have that

$$\int_{\widetilde{\Gamma}}(w\cdot n)(\frac{\partial v_i}{\partial n}\cdot\frac{\partial v_j}{\partial n}) = 0, \tag{4.6}$$

for all w in a neighborhood of $w = 0$, where $\widetilde{\Gamma} = \partial(\Omega+u)$.

Moreover, from the equality (4.5) and Lemma 3.7 we obtain that

$$\int_{\widetilde{\Gamma}}(w\cdot n)\Big(\Big|\frac{\partial v_i}{\partial n}\Big|^2 - \Big|\frac{\partial v_j}{\partial n}\Big|^2\Big) = 0, \tag{4.7}$$

for all w in a neighborhood of $w = 0$, where $\widetilde{\Gamma} = \partial(\Omega+u)$.

Therefore, we have that

$$\frac{\partial v_i}{\partial n}(x)\cdot\frac{\partial v_j}{\partial n}(x) = 0, \quad \text{ on } \widetilde{\Gamma} \tag{4.8}$$

and

$$\left|\frac{\partial v_1}{\partial n}\right| = \left|\frac{\partial v_2}{\partial n}\right|, \quad \text{ on } \widetilde{\Gamma}. \tag{4.9}$$

Since

$$\text{div } v_i = \frac{\partial v_i}{\partial n}\cdot n = 0 = \frac{\partial v_j}{\partial n}\cdot n = \text{div } v_j, \quad \text{ on } \widetilde{\Gamma}$$

we have that

$$\frac{\partial v_i}{\partial n} = a(x)(-n_2,n_1), \qquad \frac{\partial v_j}{\partial n} = b(x)(-n_2,n_1). \tag{4.10}$$

From (4.9) we obtain that $|a(x)| = |b(x)|$.

On the other hand, (4.8) implies that

$$a(x)b(x) = 0, \quad \text{on } \widetilde{\Gamma},$$

and then $a(x) = b(x) = 0$, on $\widetilde{\Gamma}$. That is,

$$\frac{\partial v_i}{\partial n} = \frac{\partial v_j}{\partial n} = 0, \qquad \text{on } \partial(\Omega + u).$$

Using Pohožaev's identity (3.39) we have that $v_i = v_j = 0$, in $\Omega + u$. This is a contradiction of the fact that the functions v_i are eigenfunctions of the Stokes system. This completes the proof of Theorem 1.1. □

Remark 4.1. Using the results of M. Dauge (see [7, 8]), one can show that Theorem 1.1 remains true in convex polygonal domains.

Indeed, the $(H^2(\Omega) \cap H_0^1(\Omega))^2 \times H^1(\Omega)$ regularity of the eigenfunctions (v, p) of (P) holds in this case, too. This regularity result suffices to justify all the arguments above in which the normal derivative of the eigenfunctions needs to be in $(L^2(\partial\Omega))^2$.

5. Deformation of an arbitrarily small subset of the boundary

In this section, we show that the main result remains true if we consider deformations of an arbitrarily small subset of the boundary. However, in this case we need a nonstandard unique continuation property.

Our result is the following:

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain of class C^3 and $\Gamma_0 \subset \partial\Omega$ be a nonempty open set. Then, there exists a perturbation $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$, with $u = 0$ on $\Gamma \setminus \Gamma_0$ and $\|u\|_{4,\infty}$ arbitrarily small, such that the spectrum of the Stokes system in $\Omega + u$ is simple.*

Remark 5.1. As we said in the Introduction, we need a generic unique continuation result to replace, in the proof of Theorem 5.1, the one we got by means of multipliers. To prove it the following proposition is needed.

Proposition 5.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain of class C^3, with $d \geq 2$ and $\Gamma_0 \subset \Gamma$ an open, nonempty set. Let $v \in J_0(\Omega) \cap (H^3(\Omega))^d$ be a solution of*

$$\begin{cases} -\Delta v + \nabla p = \lambda v & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma \\ \frac{\partial v}{\partial n} = \frac{\partial^2 v}{\partial n^2} = 0 & \text{on } \Gamma_0 \\ \int_\Omega p = 0. & \end{cases}$$

Then $v = 0$, in Ω.

Proof. Let $\tau^1, \dots, \tau^{d-1}$ be the tangent vectors to $\partial\Omega$ which form an orthogonal basis of the tangent hyperplane. If $f \in H^2(\Omega)$,

$$\partial_i f = \sum_{j=1}^{d-1} \frac{\partial f}{\partial \tau^j}\tau_i^j + \frac{\partial f}{\partial n} n_i, \qquad \text{on } \partial\Omega.$$

On Γ_0 we have that

$$\begin{aligned} \Delta v &= \sum_i \partial_i^2 v = \sum_i \frac{\partial}{\partial n}(\partial_i v) n_i + \sum_{i,j} \frac{\partial}{\partial \tau^j}(\partial_i v)\tau_i^j \\ &= \sum_i \nabla(\partial_i v)\cdot n n_i = \sum_{ij} \partial_i(\partial_j v) n_i n_j = \frac{\partial^2 v}{\partial n^2} = 0. \end{aligned} \tag{5.1}$$

Thus,

$$\nabla p = (\lambda v + \triangle v) = 0, \qquad \text{on } \Gamma_0. \tag{5.2}$$

Therefore, $\dfrac{\partial p}{\partial n} = 0$ on Γ_0, and moreover p is constant in each connected component of Γ_0. Since $\triangle p = 0$ in Ω, we deduce that p is constant in Ω.

This implies that each component v_i of v is a solution of

$$\begin{cases} -\Delta v_i &= \lambda v_i & \text{in } \Omega \\ v_i &= 0 & \text{on } \Gamma \\ \dfrac{\partial v_i}{\partial n} &= 0 & \text{on } \Gamma_{0,} \end{cases}$$

and we conclude that $v_i = 0$, in Ω. This completes the proof. $\square$

We are now in a condition to prove the following result of generic unique continuation:

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^3, with $d \geq 2$ and $\Gamma_0 \subset \Gamma$ an open, nonempty, bounded set. Then there exists a perturbation $u \in W^{4,\infty}(\Omega, \mathbb{R}^d)$, with $u = 0$ on $\Gamma \setminus \Gamma_0$, arbitrarily small, such that the unique continuation property holds for all the eigenfunctions in the deformed domain $\Omega + u$. In other words, if v is an eigenfunction of the Stokes system in $\Omega + u$, then $\partial v/\partial n \neq 0$ in $\Gamma_0 + u$.*

Remark 5.2. Theorem 5.3 holds with perturbations of the boundary with support in a part of the boundary Γ_0 where the normal derivative of the solution vanishes. This fact is essential in the proof. The heuristic motivation of this fact is as follows. Assume that a branch of eigenfunctions $v(u)$ is such that $\partial v/\partial n = 0$ on $\Gamma_0 + u$ for all u. Then, by taking derivatives with respect to the perturbation parameter u, one deduces that $\partial^2 v/\partial n^2 = 0$ on $\Gamma_0 + u$. But then we are under the conditions of Proposition 5.2.

Proof of Theorem 5.3. Let $A = \{u \in W^{4,\infty}(\Omega, \mathbb{R}^d) : (SUC)$ holds for all the eigenvalues of (P) in $\Omega + u\}$. Here by (SUC) we mean the unique continuation property for the eigenfunctions stated in Theorem 5.2. Obviously, A is the countable intersection of the sets A_n, where $A_0 = W^{4,\infty}(\Omega, \mathbb{R}^d)$, and

$$A_n = \Big\{ u \in W^{4,\infty}(\Omega, \mathbb{R}^d) : \quad (SUC) \text{ holds for the first } n \text{ branches} \\ \text{of eigenvalues of } (P) \text{ in } \Omega + u \subset \mathbb{R}^d \Big\}.$$

It is sufficient to show that each set A_n is open in $W^{4,\infty}(\Omega, \mathbb{R}^d)$ and that A_{n+1} is dense in A_n for all $n \geq 0$. Then, using Baire's lemma (see Lemma 2.1 and Lemma 2.2) we have that the set A is dense on $W^{4,\infty}(\Omega)$.

i) A_n **is open.** Let us suppose that A_n^c is not closed. Then there exists $u \in A_n$ and a sequence of deformations $\{u_k\}_k \subset W^{4,\infty}(\Omega, \mathbb{R}^d)$ such that $u_k \notin A_n$ and $u_k \to u$ in $W^{4,\infty}(\Omega, \mathbb{R}^d)$. Since $u_k \notin A_n$, for every k we have that

$$\left\{ \begin{array}{rcll} -\triangle v(u_k) + \nabla p(u_k) & = & \lambda(u_k) v(u_k), & \text{in } \Omega + u_k \\ \nabla \cdot v(u_k) & = & 0, & \text{in } \Omega + u_k \\ v(u_k) & = & 0, & \text{on } \Gamma + u_k \\ \dfrac{\partial v(u_k)}{\partial n} & = & 0, & \text{on } \Gamma_0 + u_k \\ \|v(u_k)\|_{L^2(\Omega + u_k)} & = & 1, & \end{array} \right.$$

where $\lambda(u_k)$ is an eigenvalue belonging to one of the first n branches.

Given that the branches $u \to \lambda_i(u)$ and $u \to V_i(u) = v_i(u) \circ (I + u)$, $i = 1, \ldots, n$, are continuous from $W^{4,\infty}(\Omega, \mathbb{R}^d)$ to $\mathbb{R}$ and $H^2(\Omega)$ respectively, we have that for at least one of the eigenvalues $\lambda(u)$ of the first n branches the corresponding eigenfunction $v(u)$ is such that

$$\left\{ \begin{array}{rcll} -\triangle v(u) + \nabla p(u) & = & \lambda(u) v(u), & \text{in } \Omega + u \\ \nabla \cdot v(u) & = & 0, & \text{in } \Omega + u \\ v(u) & = & 0, & \text{on } \Gamma + u \\ \dfrac{\partial v(u)}{\partial n} & = & 0, & \text{on } \Gamma_0 + u \\ \|v(u)\|_{L^2(\Omega + u)} & = & 1, & \end{array} \right.$$

u being the limit of u_k.

But this is impossible since the property (SUC) holds in $\Omega + u$ for the first n branches since $u \in A_n$. This shows us that A_n is an open set.

ii) A_{n+1} **is dense on** A_n**.** Let $u \in A_n \setminus A_{n+1}$. We consider two cases:

a) $\lambda = \lambda_{n+1}(u)$ is a simple eigenvalue of (P).

b) $\lambda = \lambda_{n+1}(u)$ is a multiple eigenvalue of (P).

a) λ is a simple eigenvalue. Without loss of generality we may suppose that $u = 0$. Assume that the property (SUC) does not hold in a neighborhood of $u = 0$. Then the problem

$$\begin{cases} -\Delta v(u) + \nabla p(u) &= \lambda(u)v(u) & \text{in } \Omega + u \\ \nabla \cdot v(u) &= 0 & \text{in } \Omega + u \\ v(u) &= 0 & \text{on } \Gamma + u \\ \dfrac{\partial v(u)}{\partial n} &= 0 & \text{on } \Gamma_0 + u, \end{cases} \tag{5.3}$$

has a nontrivial solution $(\lambda(u), v(u), p(u))$, for all u small enough.

The local derivatives satisfy

$$\begin{cases} -\Delta v'(u) + \nabla p'(u) &= \lambda'(u)v + \lambda v'(u) & \text{in } \Omega \\ \nabla \cdot v'(u) &= 0 & \text{in } \Omega \\ v'(u) &= -(u \cdot n)\dfrac{\partial v}{\partial n} & \text{on } \Gamma \\ \dfrac{\partial v'(u)}{\partial n} &= -(u \cdot n)\dfrac{\partial^2 v}{\partial n^2} & \text{on } \Gamma_0 \end{cases}$$

where $v'(u)$ and $\lambda'(u)$ denote the derivative of v and λ in the direction u at $u = 0$. Thus, if we consider perturbations u such that $u = 0$ on $\Gamma \backslash \Gamma_0$, then

$$\lambda'(u) = -\int_\Gamma (u \cdot n)\left|\frac{\partial v}{\partial n}\right|^2 = -\int_{\Gamma_0} (u \cdot n)\left|\frac{\partial v}{\partial n}\right|^2 - \int_{\Gamma \backslash \Gamma_0} (u \cdot n)\left|\frac{\partial v}{\partial n}\right|^2 = 0, \tag{5.4}$$

and moreover $v'(u) = 0, \text{ on } \Gamma$. Therefore,

$$\begin{cases} -\Delta v'(u) + \nabla p'(u) &= \lambda v'(u) & \text{in } \Omega \\ \nabla \cdot v'(u) &= 0 & \text{in } \Omega \\ v'(u) &= 0 & \text{on } \Gamma \\ \dfrac{\partial v'(u)}{\partial n} &= -(u \cdot n)\dfrac{\partial^2 v}{\partial n^2} & \text{on } \Gamma_0; \end{cases}$$

that is, $(\lambda(u), v'(u), p'(u))$ is a solution of (P) in Ω.

Now, taking into account that λ is a simple eigenvalue of the Stokes system, we deduce that there exists a constant α_u such that $v'(u) = \alpha_u v$, where v is an eigenfunction of unit norm associated with λ. Thus,

$$\frac{\partial v'}{\partial n} = \alpha_u \frac{\partial v}{\partial n} = 0, \quad \text{on } \Gamma_0.$$

Since

$$\frac{\partial v'}{\partial n} = -(u \cdot n)\frac{\partial^2 v}{\partial n^2}, \quad \text{on } \Gamma_0,$$

we obtain

$$\frac{\partial^2 v}{\partial n^2} = 0, \text{ on } \Gamma_0.$$

Proposition 5.2 implies then that $v \equiv 0$, which is impossible because v is an eigenfunction of the Stokes system.

b) λ is a multiple eigenvalue. First, we consider the case where the multiplicity of λ is two and let (v_1, p_1), (v_2, p_2) be the eigenfunctions and eigenpressures associated to λ. Moreover, assume that $(v_1(u), p_1(u))$, solve (5.3) in a neighborhood of $u = 0$.

Proceeding as in a), if we consider perturbations u such that

$$u = 0, \quad \text{on } \Gamma \setminus \Gamma_0,$$

and if v_1 satisfies

$$\frac{\partial v_1}{\partial n} = 0, \quad \text{on } \Gamma_0,$$

then $v_1'(u)$ is an eigenfunction of (P) associated to λ. Thus,

$$v_1'(u) = \alpha_u v_1 + \beta_u v_2, \quad \text{in } \Omega.$$

Then

$$\frac{\partial v_1'(u)}{\partial n} = \beta_u \frac{\partial v_2}{\partial n}, \text{ in } \Omega.$$

That is,

$$\frac{\partial v_1'(u)}{\partial n} = -(u \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_u \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0.$$

We distinguish two cases. First, if $\partial v_2 / \partial n = 0$ on Γ_0 as well, we then have $\partial^2 v_1 / \partial n^2 = 0$ on Γ_0, and by Proposition 5.1, $v_1 \equiv 0$. This is in contradiction with the fact that v_1 is a nontrivial eigenfunction. Therefore we can assume that $\partial v_2 / \partial n \neq 0$.

Let u_1, u_2 be two perturbations of the domain such that the quotient $(u_1 \cdot n)/(u_2 \cdot n)$ is well defined and is not constant in any open subset of Γ_0. Then

$$-(u_1 \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_{u_1} \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0 \tag{5.5}$$

$$-(u_2 \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_{u_2} \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0. \tag{5.6}$$

If $\dfrac{\partial^2 v_1}{\partial n^2}(x) \neq 0$, for $x \in \Gamma_0$, we divide (5.5) by (5.6), and we have that

$$\frac{(u_1(x) \cdot n(x))}{(u_2(x) \cdot n(x))} = \frac{\beta_{u_1}}{\beta_{u_2}} = \text{ constant},$$

which is impossible because of the choice of the functions u_i. With this we prove that

$$\frac{\partial^2 v_1}{\partial n^2}(x) = 0, \quad \text{for all } x \in \Gamma_0.$$

Thus, we can apply Lemma 5.2, obtaining that $v_1 = 0$ in Ω, which is a contradiction with the fact that v_1 is an eigenfunction of (P).

Thus, there exists a perturbation $u \in W^{4,\infty}(\Omega, \mathbb{R}^d)$ such that the unique continuation property holds for v_1. Obviously, the fact that $\partial v_2/\partial n \neq 0$ on $\Gamma_0 + u$ is kept under small perturbations u.

Now, we assume that λ is an eigenvalue of multiplicity $h > 2$ and $v_1, \ldots, v_h$ are the eigenfunctions associated to λ normalized on $L^2(\Omega)$. Let $\lambda_i(u)$ with $i = 1, \ldots, h$, be the eigenvalues of the Stokes system for the domain $\Omega + u$, such that $\lambda_i(0) = \lambda$ for $i = 1, \ldots, h$. Moreover, we suppose that the multiplicity of $\lambda(u)$ is h in a neighborhood M of $u = 0$ and that

$$\frac{\partial v_1(u)}{\partial n(\Omega + u)} = 0, \qquad \text{on } \Gamma_0 + u. \tag{5.7}$$

Taking into account that the multiplicity does not change in the neighborhood M, we have that for every $u \in M$

$$\lambda_1'(u) = \cdots = \lambda_h'(u). \tag{5.8}$$

If we consider perturbations u such that $u = 0$ on $\Gamma \setminus \Gamma_0$, from (5.4) we have that

$$\left|\frac{\partial v_1}{\partial n}\right| = \cdots = \left|\frac{\partial v_h}{\partial n}\right|, \qquad \text{on } \Gamma_0. \tag{5.9}$$

Therefore, from (5.7) we have that

$$\frac{\partial v_i}{\partial n} = 0, \qquad \text{on } \Gamma_0, \quad i = 1, \ldots, h. \tag{5.10}$$

On the other hand, using the argument of the case of double eigenvalues, we have that the local variation of the first eigenfunction $v_1'(u)$ is an eigenfunction of the Stokes system associated to the eigenvalue λ, and then, for each perturbation $u \in M$ with $u = 0$ on $\Gamma \setminus \Gamma_0$, there exist constants $c_1(u), \ldots, c_h(u)$ such that

$$v_1'(u) = c_1(u)v_1 + c_2(u)v_2 + \cdots + c_h(u)v_h. \tag{5.11}$$

Combining (5.10) with the fact that

$$\frac{\partial v_1'(u)}{\partial n} = -(u \cdot n)\frac{\partial^2 v_1}{\partial n^2} = 0, \qquad \text{on } \Gamma_0, \tag{5.12}$$

we deduce that

$$\frac{\partial^2 v_1}{\partial n^2} = 0, \qquad \text{on } \Gamma_0. \tag{5.13}$$

Now, from Proposition 5.2 we deduce that $v_1 \equiv 0$, which is impossible because v_1 is an eigenfunction. Then, there exists a perturbation of the domain

u such that the multiplicity of the eigenvalue decreases to $h-1$ or the unique continuation property holds.

If the unique continuation property does not hold, we can apply the same argument to obtain a small-enough deformation u such that the unique continuation property holds or the eigenvalue has multiplicity $h-2$.

Applying this argument in an iterative way, we can obtain a deformation u' as small as we want such that the unique continuation property holds in $\Omega + u'$ or the eigenvalue has multiplicity two. If the multiplicity is two we apply the argument above. This shows that A_{n+1} is dense in A_n. Now, we can apply Baire's lemma to complete the proof of Theorem 5.3. □

Remark 5.3. The proof of Theorem 5.1 is analogous to the proof of Theorem 1.1. We use however the generic unique continuation property of Theorem 5.3 instead of Pohožaev's identity.

Remark 5.4. We can see that our results give us a generic unique continuation property. The restriction to generic domains is of technical nature, but as far as we know, the existence of counterexamples or the positive answer for the unique continuation property for every domain is an open problem.

An interesting problem is the classification of the domains in which the property holds.

6. The three-dimensional case

The argument described in the proof of Theorem 1.1 applies only in dimension $d = 2$. When $d \geq 3$ the results of existence and regularity of the branches for the eigenvalues and eigenfunctions of the Stokes system are true. The computation of the local variations and the Pohožaev's identity remain also valid. However, the argument used to prove the generic simplicity of the eigenvalues fails at the following point:

In the proof of the density of the set A_{n+1} on A_n we have that

$$\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \qquad \text{on } \widetilde{\Gamma}.$$

This is true also in $d \geq 3$.

Later, we saw that if v_1, v_2 are two eigenfunctions of the Stokes system associated to the same eigenvalue and such that

$$\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \quad \left|\frac{\partial v_1}{\partial n}\right| = \left|\frac{\partial v_2}{\partial n}\right|, \qquad \text{on } \widetilde{\Gamma},$$

using the fact

$$\frac{\partial v_j}{\partial n} \cdot n = 0, \quad j = 1, 2 \ \text{ on } \widetilde{\Gamma},$$

then

$$\frac{\partial v_j}{\partial n} = 0, \quad j = 1, 2 \text{ on } \widetilde{\Gamma},$$

and therefore $v_1 = v_2 = 0$, in $\Omega + u$. The same problem arises in dimensions $d \geq 3$ as well. In other words, we must exclude the existence of a couple of eigenfunctions such that

$$\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \quad \left|\frac{\partial v_1}{\partial n}\right| = \left|\frac{\partial v_2}{\partial n}\right|, \quad \frac{\partial v_j}{\partial n} \cdot n = 0 \qquad \text{on } \widetilde{\Gamma},\ j = 1, 2.$$

These conditions give us that $\dfrac{\partial v_j}{\partial n}$ are tangent vectors to $\partial\Omega$, orthogonal to each other and with the same modulus. But these facts do not immediately imply a contradiction since the tangent plane has dimension $d - 1 \geq 2$.

The recent developments made by A. Osses in [26] may be of some use to solve this problem at least in the cases where a large-enough subset of the boundary of Ω is deformed.

7. An application to pointwise control

We have proved that the eigenvalues of the two-dimensional Stokes system are simple for almost every domain. In this section we show briefly an application of this result to the pointwise controllability of the Stokes system.

Assume that Ω is a bounded, regular domain of $\mathbb{R}^2$ such that all the eigenvalues of the Stokes system are simple. From our previous results we know that this holds generically with respect to the domain Ω. Let

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty \tag{7.1}$$

$$w_1\,,\, w_2\,,\, \ldots\,,\, w_n\,,\, \ldots \tag{7.2}$$

be the eigenvalues and the eigenfunctions of the system respectively. The eigenfunctions are assumed to constitute an orthonormal basis of $H(\Omega)$.

Let $x_0 \in \Omega$ and δ_{x_0} be the Dirac delta measure at the point x_0. Given $T > 0$ we consider the following controllability problem:

$$\begin{cases} v_t - \triangle v &= \nabla p + \omega(t)\delta_{x_0}, & \text{in } \Omega \times (0,T) \\ v &= 0, & \text{on } \partial\Omega \times (0,T) \\ \nabla \cdot v &= 0, & \text{in } \Omega \times (0,T) \\ v(x,0) &= v_0(x), & \text{in } \Omega, \end{cases} \tag{7.3}$$

with $v_0 \in J_0(\Omega)$ and $\omega \in L^2(0,T;\mathbb{R}^2)$.

In (7.3), the vector function $\omega \in L^2(0,T;\mathbb{R}^2)$ is the control. It is easy to see that the solution of (7.3) satisfies $v \in C([0,T]\,;(H^{-s}(\Omega))^2)$ for every $s > 0$, and moreover $v \in L^2([0,T]\,;(L^2(\Omega))^2)$ (see [17] for a similar result for the heat equation).

The approximate controllability problem for system (7.3) can be formulated in the following form: *Let* $s > 0$. *Is the set of reachable states*

$$R(T; v_0) = \left\{ v(T) \,:\, v \textit{ is a solution of (7.3) with } \omega \in L^2(0,T;\mathbb{R}^2) \right\} \tag{7.4}$$

dense in H^{-s}, *the completion of* $H(\Omega)$ *with respect to the norm of* $(H^{-s}(\Omega))^2$, *for all* $v_0 \in J_0(\Omega)$?

Using Hahn-Banach's theorem, it is easy to see that system (7.3) is approximately controllable if and only if the following unique continuation property holds for the adjoint system: If φ is a solution of

$$\begin{cases} -\varphi_t - \triangle\varphi &= \nabla\pi, & \text{in } \Omega \times (0,T) \\ \varphi &= 0, & \text{on } \partial\Omega \times (0,T) \\ \nabla \cdot \varphi &= 0, & \text{in } \Omega \times (0,T) \\ \varphi(x,T) &= \varphi_0(x), & \text{in } \Omega, \end{cases} \tag{7.5}$$

with $\varphi_0 \in J_0(\Omega) \cap (H^s(\Omega))^2$ such that

$$\varphi(x_0, t) = 0 \quad \text{in } (0,T), \tag{7.6}$$

can we guarantee that

$$\varphi \equiv 0? \tag{7.7}$$

It is easy to check that the answer to this uniqueness problem is negative if one of the eigenvalues of the Stokes system is not simple and two of the corresponding eigenfunctions coincide on x_0.

Since Ω is such that the spectrum is simple, we can see that the property holds if x_0 is a *"strategic point"* of Ω; that is,

$$w(x_0) \neq 0, \tag{7.8}$$

for all eigenfunction w of the Stokes system. In fact, the solution φ of (7.5) can be developed in Fourier series:

$$\varphi(x,t) = \sum_{k\geq 1} c_k e^{-\lambda_k (T-t)} w_k(x). \tag{7.9}$$

From (7.6), the analyticity in time of the solutions of the backwards Stokes system and the simplicity of the spectrum we deduce that

$$c_k w_k(x_0) = 0, \quad \forall k \geq 1, \tag{7.10}$$

and (7.8) implies $c_k = 0$, for all $k \geq 1$; that is, $\varphi \equiv 0$.

On the other hand, the set of strategic points x_0 is dense on Ω. To see this we define the sets

$$A_k = \{ x \in \Omega \,:\, w_i(x) \neq 0\,,\, i = 1, \dots, k \}\,. \tag{7.11}$$

Then, for all $k \in \mathbb{N}$, A_k is an open subset of Ω; furthermore, A_{k+1} is dense on A_k, and applying Baire's lemma, we conclude the density of the strategic points on Ω.

In this way we prove the following:

Theorem 7.1. *Let Ω be an open domain of class C^2 such that the spectrum of the Stokes operator is simple. Then the system* (7.3) *is approximately controllable for any time T for any strategic point $x_0 \in \Omega$. Moreover, the set of strategic points is dense in Ω.*

We can also prove the following result:

Lemma 7.2. *The set of the points $x_0 \in \Omega \subset \mathbb{R}^2$ such that*

$$w_k^1(x_0) \neq 0, \quad \forall k \geq 1, \tag{7.12}$$

where w_k^1 denotes the first component of the eigenfunction w_k, is dense in Ω.

Proof. Let us consider the set

$$A_n = \left\{ x \in \Omega \,:\, w_k^1(x) \neq 0\,,\, k = 1, \ldots, n \right\} \tag{7.13}$$

for each $n \in \mathbb{N}$.

It is easy to check that the set A_n is open. Let us prove that A_{n+1} is dense in A_n. If A_{n+1} is not dense in A_n, there exists an open set $U \subset A_n \cap A_{n+1}^c$; that is,

$$w_k^1(x) \neq 0,\ x \in U,\ k = 1, \ldots n, \qquad \text{and } w_{n+1}^1(x) = 0,\ x \in U. \tag{7.14}$$

Taking into account that div $w_{n+1} = 0$ we deduce that $\partial_2 w_{n+1}^2 = 0$ in U. Applying Holmgrem's uniqueness theorem to $\partial_2 w$ we deduce that $\partial_2 w_{n+1} = 0$ in Ω. Taking into account that w_{n+1} vanishes on the boundary we deduce that $w_{n+1} \equiv 0$, and this is a contradiction.

Therefore, the set A_{n+1} is dense in A_n. Now applying Baire's lemma to the set $A = \cap_{n \geq 1} A_n$ we complete the proof of Lemma 7.1. $\square$

As a consequence of Lemma 7.1 and proceeding as in the proof of Theorem 7.1 we obtain the following result:

Theorem 7.3. *Let Ω be an open domain of class C^2 such that the spectrum of the Stokes operator is simple. Then system* (7.3) *is approximately controllable for any strategic point $x_0 \in \Omega$ in the sense of Lemma* 7.1 *with controls of the form $\omega(t) = (\omega^1(t), 0)$.*

References

[1] J.H. Albert, *Genericity of simple eigenvalues for elliptic pde's,* Proc. Amer. Math. Soc., 48 (1975), 413–418.

[2] J.H. Albert, *Topology of the nodal and critical points sets for eigenfunctions of elliptic operators,* Ph. D. Thesis, M. I. T., Boston MA, 1971.

[3] J.A. Bello, E. Fernández-Cara, J. Lemoine, and J. Simon, *The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes flow,* SIAM J. Control Optim., 35 (1997), 626–640.

[4] H. Brezis, "Analyse Fonctionnelle. Théorie et Applications," Masson, Paris, 1983.

[5] S. Chow and J. Hale, "Methods of Bifurcation Theory," Springer-Verlag, New York, 1982.

[6] R. Courant and D. Hilbert, "Methods of Mathematical Physics," Interscience Publishers, Inc., New York, Vol. I, 1953.

[7] M. Dauge, "Elliptic Boundary Value Problems on Corner Domains," Lecture Notes in Mathematics, **1341**, Springer-Verlag, Berlin, Heidelberg, 1988.

[8] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. Part I: linearized equations,* SIAM J. Math. Anal., 20 (1989), 74–97.

[9] C. Fabre, J.P. Puel, and E. Zuazua, *Approximate controllability of the semilinear heat equation,* Proc. Roy. Soc. Edinburgh, 125-A (1995), 31–61.

[10] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, Heidelberg, 2nd ed., 1983.

[11] P. Grisvard, "Elliptic Problems in Nonsmooth Domains," Monographs and Studies in Mathematics, **24**, Pitman Advanced Publishing Program, Boston, 1985.

[12] T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin, Heidelberg, 1980.

[13] O. Ladyzhenskaya, "The Mathemathical Theory of Viscous Incompressible Flows," Gordon and Breach, New York, 1968.

[14] H. Lamb, "Hydrodynamics," Cambridge University Press, New York, 6th ed., 1993.

[15] J. Lemoine, *Ph.D. Thesis,* Université Blaise Pascal, Clermont Ferrand, France, 1995.

[16] J.L. Lions, "Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués," Tome 1. RMA Collection, Masson, 8, 1988.

[17] J.L. Lions, *Pointwise control for distributed systems,* in "Control and Estimation in Distributed Parameter Systems," SIAM Publications series *Frontier in applied mathematics*, 11, H.T. Banks ed., 1992.

[18] J.L. Lions and E. Magenes, "Non-homogeneous Boundary Value Problems and Applications, Volume I," Springer-Verlag, Berlin, Heidelberg, 1972.

[19] J.L. Lions and E. Zuazua, *A generic uniqueness result for the Stokes system and its control theoretical consequences,* in "Partial Differential Equations," P. Marcellini, G. Talenti, E. Visentini eds., Marcel Dekker Inc., LNPAS **177** (1996), 221–235.

[20] A.M. Micheletti, *Perturbazione dello spettro di un operatore ellittico di tipo variazionale, in relazione ad una variazione del campo,* Ann. Scuola Norm. Sup. Pisa, 26 (1972), 151–169.

[21] F. Murat and J. Simon, *Quelques résultats sur le contrôle par un domaine géométrique,* Rapport du L. A. 189, Univ. Paris VI (1974).

[22] F. Murat and J. Simon, *Sur le contrôle par un domaine géométrique,* Rapport du L. A. 189, Univ. Paris VI (1976).

[23] F. Murat and J. Simon, *Etude de problèmes d'optimal design.* Proc. of the 7th IFIP Conf. on Optimization Techniques, J. Cea Ed., Lect. Notes in Comp. Sci., **41**, Springer-Verlag (1976), 54–62.

[24] J. Nečas, "Les Méthodes Directes en Théorie des Equations Elliptiques," Masson, Paris, 1967.

[25] J.H. Ortega, *Comportamiento asintótico, control y estabilización de algunos sistemas parabólicos y de placas,* Ph.D. Thesis, Universidad Complutense de Madrid, Madrid, Spain, 1997.

[26] A. Osses, *Une nouvelle famille de multiplicateurs et ses applications à la contrôlabilité exacte des ondes*, C. R. Acad. Sci. Paris, 326 (1998), 1099–1104.

[27] P.A. Raviart and J.M. Thomas, "Introduction à l'Analyse Numérique des Équations aux Dérivées Partielles," Masson, Paris, 1983.

[28] F. Rellich, "Perturbation Theory of Eigenvalue Problems," Gordon & Breach Science Publishers Inc., New York, 1969.

[29] J. Simon, "Diferenciación con Respecto al Dominio," Lecture Notes, Universidad de Sevilla, 1989.

[30] J. Simon, *Démonstration constructive d'un théorème de G. de Rham,* C. R. Acad. Sci. Paris, Série I, **316** (1993), 1167–1172.

[31] J. Simon, *Differentiation with respect to the domain in boundary value problems,* Numer. Func. Anal. Optim., 2 (1980), 649–687.

[32] M. Struwe, "Variational Methods," Springer-Verlag, Berlin, Heidelberg, 1990.

[33] R. Temam, "On the Theory and Numerical Analysis of the Navier-Stokes Equations," Lecture Notes **9** , University of Maryland, 1973.

[34] R. Temam, "Navier-Stokes Equations, Theory and Numerical Analysis," Elsevier Science Publishers B. V., Amsterdam, 1984.

[35] R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, Inc., 1988.

[36] K. Uhlenbeck, *Generic properties of eigenfunctions,* American Journal of Mathematics, **98** (4) (1976), 1059–1078.

[37] M.M. Vainberg and V.A. Trenogin, "Theory of Branching of Solutions of Nonlinear Equations," Noordhoff International Publishing, Leyden, 1974.

[38] G.N. Watson, "A Treatise on the Theory of Bessel Functions," Cambridge University Press, New York, 1948.

[39] H. Weinberger, "A First Course in Partial Differential Equations with Complex Variables and Transform Methods," Blaisdell Publishing Company, New York, 1965.

[40] E. Zuazua, "Controlabilidad Exacta y Estabilización de la Ecuación de Ondas," Textos de Métodos Matemáticos **23**, Instituto de Matemática, Universidad Federal do Rio de Janeiro, 1990.

[41] E. Zuazua, *Finite dimensional null controllability for the semilinear heat equation,* J. Math. Pures Appl., 76 (1997), 237–264.