GENERIC SIMPLICITY OF THE EIGENVALUES OF THE STOKES SYSTEM IN TWO SPACE DIMENSIONS

JAIME H. ORTEGA
Universidad del Bío-Bío, Facultad de Ciencias, Departamento de Matemática
Casilla 5-C, Concepción, Chile

ENRIQUE ZUAZUA
Universidad Complutense de Madrid, Departamento de Matemática Aplicada
28040 Madrid, Spain

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Abstract. We analyze the multiplicity of the eigenvalues for the Stokes operator in a bounded domain of $\mathbb{R}^2$ with Dirichlet boundary conditions. We prove that, generically with respect to the domain, all the eigenvalues are simple. In other words, given a bounded domain of $\mathbb{R}^2$, we prove the existence of arbitrarily small deformations of its boundary such that the spectrum of the Stokes operator in the deformed domain is simple. We prove that this can be achieved by means of deformations which leave invariant an arbitrarily large subset of the boundary. The proof combines Baire’s lemma and shape differentiation. However, in contrast with the situation one encounters when dealing with scalar elliptic eigenvalue problems, the problem is reduced to a unique continuation question that may not be solved by means of Holmgren’s uniqueness theorem. We show however that this unique continuation property holds generically with respect to the domain and that this fact suffices to prove the generic simplicity of the spectrum.

1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open domain with boundary of class $C^2$. We consider the following eigenvalue problem for the Stokes system:

\[ (P) \begin{cases} 
-\Delta v + \nabla p = \lambda v, & \text{in } \Omega \\
\nabla \cdot v = 0, & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases} \]

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In (P), \( v = (v_1, v_2) \) denotes the velocity field, while the scalar function \( p \) is the pressure. By \( \cdot \) we denote the scalar product in \( \mathbb{R}^2 \). Thus, \( \nabla \cdot \) denotes the divergence operator. It is well known that problem (P) admits a sequence of positive eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \) tending to infinity as \( n \to \infty \). The eigenfunctions \( \{v_n\}_{n \geq 1} \subset (H_0^1(\Omega))^2 \) and the eigenpressures \( \{p_n\}_{n \geq 1} \subset L^2(\Omega) \) may be taken so that \( \{v_n\}_{n \geq 1} \) constitutes an orthonormal basis of \( H(\Omega) = \{ v \in (L^2(\Omega))^2 : \nabla \cdot v = 0 \text{ in } \Omega, v \cdot n = 0 \text{ on } \partial \Omega \} \). The pressure \( p \) is determined up to an additive constant. We shall normalize it by the condition \( \int_{\Omega} p(x) \, dx = 0 \).

It is also well known that the spectrum of the Stokes operator is not always simple. This is for instance the case when \( \Omega \) is a ball as pointed out in G. Watson [38], pp. 124–127.

The problem we address in this paper is roughly the following: If \( \Omega \) is such that the spectrum of (P) is not simple, can we find arbitrarily small deformations of \( \Omega \) such that the spectrum becomes simple in the new deformed domain?

The problem of the simplicity of the spectrum arises in many contexts. This is for instance the case when analyzing stabilizability and controllability issues for evolution systems. When the spectrum is simple one can often reduce these problems to the analysis of suitable properties of eigenfunctions, which is an easier problem to deal with because of the lack of dependence with respect to time. At the end of this paper we present an application to the pointwise controllability of the evolutionary Stokes system.

In this paper we give a positive answer to the problem above showing that the Stokes spectrum is simple generically with respect to the domain. Our proof is inspired by the work of J. Albert (see [1]), where the generic simplicity of the Dirichlet Laplacian is proved. We also refer to A. Micheletti [20] and K. Uhlenbeck [36] for other similar results in this direction. In [1] the problem of the generic simplicity is reduced to the obtaining of a suitable unique continuation property for the eigenfunctions by applying Baire’s lemma. This unique continuation property turns out to be a consequence of Holmgrem’s uniqueness theorem. However, in the context of the Stokes system, the unique continuation property one needs for the eigenfunctions may not be understood as a noncharacteristic Cauchy problem and it may not be solved by means of Holmgrem’s uniqueness theorem. Therefore, additional developments are required to obtain such unique continuation properties.

We proceed in two different ways:

a) By means of Pohozaev’s identities;

b) Proving generic unique continuation results.
The first approach consists of obtaining simple global identities for the eigenfunctions of the Stokes system, in the spirit of Pohožaev’s identities for the scalar elliptic equations (see for instance M. Struwe [32]). This allows us to show the desired unique continuation property and, consequently, the generic simplicity of the spectrum. The drawback of this approach is that we need to deform rather large subsets of the boundary of the reference domain \( \Omega \) in order to guarantee the simplicity of the spectrum. The second approach is based on the key observation that the unique continuation property is not needed to hold for all domains \( \Omega \) but only generically with respect to the domain. Using classical tools of shape differentiation we show that this generic unique continuation property problem may be reduced to a question that may be solved applying Holmgren's uniqueness theorem. In this case we require, for technical reasons, that the set \( \Omega \) has boundary of class \( C^3 \). The main tool used in this approach is differentation with respect to the domain. In Section 2.3 we give some basic definitions and results; we refer to the works of Simon [29] and the works of Murat and Simon [21]–[23], for details.

In order to state the main result of this paper some basic notation is needed. Given a bounded domain \( \Omega \) of class \( C^2 \) of \( \mathbb{R}^2 \) and a deformation \( u \in W^{3,\infty}(\Omega; \mathbb{R}^2) \) we introduce the following deformed domain:

\[
\Omega + u = \{ z \in \mathbb{R}^2 : z = x + u(x), x \in \Omega \}.
\]

We then consider the Stokes system in the deformed domain \( \Omega + u \):

\[
(P_u) \begin{cases}
-\Delta v + \nabla p &= \lambda v, \quad \text{in } \Omega + u \\
\nabla \cdot v &= 0, \quad \text{in } \Omega + u \\
v &= 0 \quad \text{on } \partial(\Omega + u).
\end{cases}
\]

The main result of this paper is as follows:

**Theorem 1.1.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) of class \( C^2 \). Then the set

\[
A = \{ u \in W^{3,\infty}(\Omega; \mathbb{R}^2) : \text{all the eigenvalues of } (P_u) \text{ are simple} \}
\]

is residual on \( W^{3,\infty}(\Omega; \mathbb{R}^2) \). In other words, it is a countable intersection of dense open sets on \( W^{3,\infty}(\Omega; \mathbb{R}^2) \). In particular, \( A \) is dense in \( W^{3,\infty}(\Omega; \mathbb{R}^2) \). Moreover, if \( \Omega \) has boundary of class \( C^3 \) and given any open nonempty subset \( \Gamma_0 \) of \( \partial\Omega \), the same holds if the deformations are restricted to satisfy the condition

\[
u = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0.
\]

Theorem 1.1 guarantees the generic simplicity of the spectrum of the Stokes system with respect to perturbations of the domain \( \Omega \). The last statement of the theorem indicates that this may be achieved by means of deformations that leave invariant most of the boundary of the domain \( \Omega \).
As indicated above, the proof of the last statement of Theorem 1.1 requires a suitable unique continuation property for the eigenfunctions of the system, namely,

\[ (SUC) \quad \begin{cases} 
  \text{If } v \text{ solves } (P_u) 
  \text{ and } \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_0, \\
  \text{can we guarantee that } v \equiv 0? 
\end{cases} \]

The unique continuation problem \((SUC)\) does not fit in the framework of Holmgren’s uniqueness theorem, which needs conditions on the quantity \(\frac{\partial v}{\partial n} + pn\) rather than on \(\frac{\partial v}{\partial n}\). Our second main result shows that \((SUC)\) holds generically with respect to the domain:

**Theorem 1.2.** Let \(\Omega\) be a bounded domain of \(\mathbb{R}^2\) with boundary of class \(C^3\). Let \(\Gamma_0\) be a nonempty open subset of \(\partial \Omega\). We set

\[
B = \{ u \in W^{4,\infty}(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial \Omega \setminus \Gamma_0 \text{ such that the unique continuation property (SUC) holds in } \Omega + u \text{ with } \Gamma_0 \text{ replaced by } \Gamma_0 + u \} .
\]

Then, \(B\) is residual in \(\{ u \in W^{4,\infty}(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial \Omega \setminus \Gamma_0 \}\).

Note that Theorem 1.2 remains true in three space dimensions. However, the result of Theorem 1.1 is still an open problem in 3-d. We explain in detail the difficulty we encountered when dealing with the 3-d case in Section 6.

The rest of this paper is organized as follows. In Section 2 we present some preliminary results about the Stokes system. We also recall some basic tools from shape differentiation.

In Section 3 we analyze the regularity of eigenvalues and eigenfunctions with respect to the perturbation \(u\) of the domain. In Section 4 we prove Theorem 1.1 in the particular case where \(\Gamma_0 = \partial \Omega\) in which the unique continuation holds as a consequence of a Pohožaev identity. In Section 5 we prove the generic unique continuation result Theorem 1.2 and complete the proof of Theorem 1.1. In Section 6 we discuss the technical difficulty which arises when trying to extend Theorem 1.1 to three space dimensions. Finally, in Section 7 we apply Theorem 1.1 to deduce the generic pointwise controllability of the Stokes system as an example of application.

## 2. SOME PRELIMINARY RESULTS

### 2.1. Baire’s lemma

Firstly we remember Baire’s lemma, which will be a useful tool.

**Lemma 2.1.** (Baire’s lemma) Let \(X\) be a complete metric space and \(A_n\) be an open dense subset of \(X\) for all \(n \in \mathbb{N}\). Then \(\cap_{n \in \mathbb{N}} A_n\) is dense in \(X\).

A direct consequence of the Baire’s lemma is the following result.
**Lemma 2.2.** Let $X$ be a complete metric space and $\{A_n\}_{n \geq 0}$ be a sequence of open subsets of $X$ such that

1. $A_0 = X$.
2. $A_{n+1}$ is a dense subset of $A_n$, for all $n \geq 0$.

Then $\cap_{n=1}^{\infty} A_n$ is dense in $X$.

2.2. Some classical results on the Stokes system. We start with a version of the classical theorem of G. de Rham due to J. Simon [30].

**Theorem 2.3.** [30] Let $\Omega \subset \mathbb{R}^d$ be an open set and $q \in (D'(\Omega))^d$ such that

$$\langle q, \psi \rangle_{(D'(\Omega))^d \times (D(\Omega))^d} = 0, \quad \forall \psi \in (D(\Omega))^d \text{ such that } \nabla \cdot \psi = 0. \quad (2.1)$$

Then there exists a function $p \in D'(\Omega)$, such that $q = \nabla p$.

Let us consider the space $J_0(\Omega) = \{ \varphi \in (H^1_0(\Omega))^d : \nabla \cdot \varphi = 0 \text{ in } \Omega \}$, with the scalar product

$$\langle \phi, \varphi \rangle = \sum_{i,j=1}^{d} \int_{\Omega} \frac{\partial \phi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} \, dx. \quad (2.2)$$

Let $H(\Omega) = \{ \varphi \in (L^2(\Omega))^d : \nabla \cdot \varphi = 0 \text{ in } \Omega, \varphi \cdot n = 0 \text{ on } \partial \Omega \}$. We note that if $\varphi \in L^2(\Omega)$ and $\nabla \cdot \varphi \in L^2(\Omega)$, then the trace of $\varphi \cdot n$ exists and belongs to $H^{-\frac{1}{2}}(\partial \Omega)$. Thus $H(\Omega)$ is well defined.

We define on $H(\Omega)$ the linear unbounded operator $A$:

$$(A\phi, \varphi)_H = \langle \phi, \varphi \rangle = \sum_{i,j=1}^{d} \int_{\Omega} \frac{\partial \phi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} \, dx, \quad \forall \phi, \varphi \in J_0(\Omega). \quad (2.3)$$

The domain of $A$ in $H(\Omega)$ is denoted by $D(A)$. We can see that $A$ is a positive self-adjoint operator on $H(\Omega)$. Furthermore, $A$ is an isomorphism from $D(A)$ to $H(\Omega)$, and we have that $D(A) = (H^2(\Omega))^d \cap J_0(\Omega)$. Let $P$ be the orthogonal projection from $(L^2(\Omega))^d$ into $H(\Omega)$. We have that

$$A\varphi = -P\Delta \varphi, \quad \forall \varphi \in D(A). \quad (2.4)$$

Therefore, for each $v \in D(A)$, $f \in H(\Omega)$, the equation $Av = f$ is equivalent to the system

$$\begin{cases}
-\Delta v + \nabla p = f, & \text{in } \Omega \\
\nabla \cdot v = 0, & \text{in } \Omega \\
v = 0, & \text{on } \partial \Omega.
\end{cases} \quad (2.5)$$

Concerning the solutions of (2.5) the following is known:
Theorem 2.4. [13, 33] Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class $C^2$ with $d \geq 2$. If $f \in (L^r(\Omega))^d$ with $1 < r < \infty$, then the solution of (2.5) satisfies that $(v, p) \in (W^{2,r}(\Omega))^d \times W^{1,r}(\Omega)$ and

$$
\|v\|_{(W^{2,r}(\Omega))^d} + \|\nabla p\|_{(L^r(\Omega))^d} \leq C \|f\|_{(L^r(\Omega))^d},
$$

where the constant $C = C(r, \Omega)$ does not depend on $f$. Moreover, if $\Omega$ is of class $C^{m+2}$ and $f \in (H^m(\Omega))^d$, then $(v, p) \in (H^{m+2}(\Omega))^d \times H^{m+1}(\Omega)$.

The operator $A^{-1}$ is continuous from $H(\Omega)$ to $D(A)$; $A^{-1} : H(\Omega) \rightarrow H(\Omega)$ is also self-adjoint. On the other hand, provided that the embedding from $H^1(\Omega)$ to $L^2(\Omega)$ is compact, the embedding from $J_0(\Omega)$ to $H(\Omega)$ is compact. Therefore $A^{-1}$ is a self-adjoint compact operator on $H(\Omega)$. Then, by classical spectral theory (see [12]) we have that the operator $A$ has a countable discrete spectrum; that is, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow +\infty$.

Furthermore, each eigenvalue, $\lambda_i$, has finite multiplicity, and the eigenfunctions are orthogonal on $H(\Omega)$.

In two space dimensions it will often be useful to use the stream function associated to the velocity field.

Lemma 2.5. [34, Lemma 2.5, p. 38] Let $\Omega \subset \mathbb{R}^2$ be a bounded open set of class $C^2$. Let $\Gamma_0$ be the exterior boundary of $\Omega$ and $\Gamma_i$, $i = 1, \ldots, r$, be the other connected components of $\partial \Omega$ (see Figure 1). Let $w \in (H^1_0(\Omega))^2$ be a function such that $\text{div} w = 0$ on $\Omega$. Then there exists a unique function $\rho \in H^2(\Omega)$ such that

$$
\rho = 0, \quad \text{on } \Gamma_0, \quad \rho = \text{constant on } \Gamma_i, \quad i = 1, \ldots, r,
$$

$$
\frac{\partial \rho}{\partial n} = 0, \quad \text{on } \Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_r,
$$

which satisfies $w = (-\frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial x})$, in $\Omega$.

2.3. Shape differentiation. We now present some basic results related to shape differentiation. Given $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ we consider the new domain $\Omega + u$, defined as:

$$
\Omega + u = \{ z \in \mathbb{R}^d : z = x + u(x), \ x \in \Omega \}.
$$

Consider the Stokes eigenvalue problem in the perturbed domain $\Omega + u$:

$$
(P_u) \quad \begin{cases}
-\Delta v(u) + \nabla p(u) &= \lambda(u)v(u), & \text{on } \Omega + u \\
\nabla \cdot v(u) &= 0, & \text{on } \Omega + u \\
v(u) &\in (H^1_0(\Omega + u))^d \\
p(u) &\in L^2(\Omega + u) \\
\int_{\Omega + u} p(u) &= 0.
\end{cases}
$$
We consider perturbations $u$ in the space $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with norm
\[
\|u\|_{k,\infty} = \sup_{0 \leq |\alpha| \leq k, x \in \Omega} |D^\alpha u(x)|.
\]

Lemma 2.6. \cite{31} Let $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ with $k \geq 1$, such that $\|u\|_{k,\infty} \leq \frac{1}{2}$. Then $(I + u) : \Omega \to \Omega + u$ is invertible.

Furthermore, there exists $w \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ such that
\[
(I + u)^{-1} = I + w \quad \text{and} \quad \|w\|_{k,\infty} \leq C_k \|u\|_{k,\infty},
\]
where $C_k$ is a constant independent of $u$.

Remark 2.1. According to this result, if $\Omega$ is of class $C^j$ we can choose $k = j + 1$ (and therefore the perturbation spaces $W^{k,\infty}(\Omega, \mathbb{R}^d)$) such that our new domain $\Omega + u$ is also of class $C^j$. In particular, if $\Omega$ is of class $C^2$, then $\Omega + u$ is also of class $C^2$, and the solutions of $(P_u)$ satisfy $(v(u), p(u)) \in (H^2(\Omega))^d \times (H^1(\Omega) \cap L_0^2(\Omega))$ for every $u \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ small enough.

Lemma 2.7. \cite{31} Let $k \geq 1$ and consider the function
\[
\gamma : W^{k,\infty}(\Omega, \mathbb{R}^d) \to W^{k-1,\infty}(\Omega, \mathbb{R})
\]
\[
u \mapsto \gamma(u) = \text{Jac}(I + u) = |\text{det} [\partial_j(I + u)_i]|.
\]
This function $\gamma$ is differentiable at $u = 0$. Furthermore, the directional derivative in the direction $w$ at the point $u = 0$ is $\text{div } w$, that is,
\[
\langle D\gamma(0), w \rangle = \text{div } w, \quad \forall \ w \in W^{k,\infty}(\Omega, \mathbb{R}^d).
\]

Lemma 2.8. \cite{31} Let $k \geq 1$. The map
\[
\beta : W \subset W^{k,\infty}(\Omega, \mathbb{R}^d) \to \mathcal{M}_{d \times d}(W^{k-1,\infty}(\Omega, \mathbb{R}))
\]
\[
u \mapsto t [\partial_j(I + u)_i]^{-1},
\]
where \( \mathcal{W} \) is a neighborhood of \( u = 0 \) on \( W^{k,\infty}(\Omega, \mathbb{R}^d) \), is differentiable on \( u = 0 \). Its directional derivative on \( u = 0 \) in the direction \( w \) is given by the matrix \(-t[\partial_j w]_i\), i.e., the adjoint of \([\partial_j w]_i\),

\[
\begin{bmatrix}
\partial_j (I + u) & \theta(u)
\end{bmatrix}_i = -t[\partial_j w]_i + \theta(u),
\]

where the matrix \( \theta(u) \) satisfies

\[
\|\theta(u)\|_{k-1,\infty} \rightarrow 0, \quad \text{as} \quad \|u\|_{k,\infty} \rightarrow 0.
\]

We now give some useful properties of general maps:

\[
v : \ W^{k,\infty}(\Omega, \mathbb{R}^d) \rightarrow W^{m,r}(\Omega + u)
\]

\[
u(u) \rightarrow v(u),
\]

where \( m \leq k \) is an integer. In practice, \( v(u) \) will be the solution of a suitable problem which depends on the perturbation function \( u \) (for instance, a solution of our eigenvalue problem \((P_u)\)).

**Definition 2.1.** [31] Let \( k \geq m \geq 1 \). We say that the function \( v(u) \) has a **first-order local variation** at \( u = 0 \) on \( W^{k-1,\infty}_\text{loc}(\Omega) \), if \( v(u) \in W^{m,r}(\Omega + u) \) for all \( u \in W^{k,\infty}(\Omega, \mathbb{R}^d) \) and there exists a linear map \( \dot{v}(\Omega; u) \) defined from \( u \in W^{k,\infty}(\Omega, \mathbb{R}^d) \) to \( W^{m-1,\infty}_\text{loc}(\Omega) \) such that, for all open sets \( \omega \subset \Omega \),

\[
v(u) = v(0) + \dot{v}(\Omega; u) + \theta(u), \quad \text{in} \ \omega,
\]

when \( \|u\|_{k,\infty} \) is small enough and

\[
\frac{\theta(u)}{\|u\|_{k,\infty}} \rightarrow 0 \quad \text{in} \ W^{m-1,r}(\omega) \quad \text{as} \quad \|u\|_{k,\infty} \rightarrow 0.
\]

**Definition 2.2.** [31] Let \( k \geq m \geq 1 \). We say that the function \( v(u) \) has a **first total variation** at \( u = 0 \) in \( W^{m,r}(\Omega) \), if \( v(u) \circ (I + u) \in W^{m,r}(\Omega) \) and there exists a linear map \( \dot{v}(\Omega; u) \) defined from \( u \in W^{k,\infty}(\Omega, \mathbb{R}^d) \) into \( W^{m,r}(\Omega) \), such that

\[
v(u) \circ (I + u) = v(0) + \dot{v}(\Omega; u) + \theta(u), \quad \text{in} \ \Omega,
\]

for \( \|u\|_{k,\infty} \) small enough and

\[
\frac{\theta(u)}{\|u\|_{k,\infty}} \rightarrow 0 \quad \text{in} \ W^{m,r}(\Omega) \quad \text{as} \quad \|u\|_{k,\infty} \rightarrow 0.
\]

**Remark 2.2.** Let

\[
v : \ W^{k,\infty}(\Omega, \mathbb{R}^d) \rightarrow W^{m,r}(\Omega + u)
\]

\[
u(u) \rightarrow v(u),
\]
be a function of the perturbation parameter $u$. Then if $k \geq m$, from Definition 2.1 it follows that the first local variation can be defined as

$$v'(\Omega; u) = \lim_{t \to 0} \frac{v(tu) - v(0)}{t} \quad \text{in} \quad \omega,$$

(2.8)

where $\omega \subset \subset \Omega$ and $v(tu)|_\omega, v(0)|_\omega$ are the restrictions of the functions $v(tu), v(0)$ to $\omega$.

On the other hand, from Definition 2.2 we note that the first-order total variation can be defined as

$$\dot{v}(\Omega; u) = \lim_{t \to 0} \frac{v(tu) \circ (I + tu) - v(0)}{t} \quad \text{in} \quad \Omega$$

(2.9)

where $V(u) = v(tu) \circ (I + tu)$.

It is important to observe that the total variation $\dot{v}(\Omega; u)$ is a function defined on the whole of $\Omega$ while the local variation is a function defined “locally” in subsets $\omega \subset \subset \Omega$.

**Remark 2.3.** In what follows, to simplify the notation, we will write

$$\dot{v}(u) = \dot{v}(\Omega; u); \quad v'(u) = v'(\Omega; u).$$

(2.10)

The following theorem provides a sufficient condition for the existence of local variations. Furthermore, it allows us to relate the concepts of local and total variation by means of the formula

$$v'(u) = \dot{v}(u) - u \cdot \nabla v(0).$$

Note that $v(0) \in W^{m,r}(\Omega)$ and $\dot{v}(u) \in W^{m,r}(\Omega)$, with $m \leq k$; thus, $v'(u) \in W^{m-1,r}_\text{loc}(\Omega)$. This formula explains the difference of regularity between the total derivative and the local derivative.

**Theorem 2.9.** [31] Consider a map $u \to v(u) \in W^{m,r}(\Omega + u)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$, with $k \geq m \geq 1$ and $1 \leq r < \infty$. Let us assume that there exists a linear, continuous map $u \to \dot{v}(u)$ defined on $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m,r}(\Omega)$, such that

$$v(u) \circ (I + u) = v(0) + \dot{v}(u) + \theta(u), \quad \text{in} \quad W^{m,r}(\Omega),$$

for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough, where

$$\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{on} \quad W^{m,r}(\Omega) \quad \text{as} \quad \|u\|_{k,\infty} \to 0.$$

Then

$$v'(u) = \dot{v}(u) - u \cdot \nabla v(0),$$

(2.11)

for each $\omega \subset \subset \Omega$ which satisfies that

$$v' : W^{k,\infty}(\Omega, \mathbb{R}^d) \to W^{m-1,r}_\text{loc}(\omega)$$

(2.12)
is linear and continuous and
\[ v(u) = v(0) + v'(u) + \theta(u), \quad \text{in } \omega, \tag{2.13} \]
where \( \frac{\theta(\omega)}{\|u\|_{k,\infty}} \to 0 \) on \( W^{m-1,r}(\omega) \) as \( \|u\|_{k,\infty} \to 0 \).

The following theorem provides sufficient conditions for the existence of the first local variation for functions which depend on the deformation \( u \). Furthermore, it provides an expression for the local variation on the boundary in terms of the normal derivative of \( v(0) \).

**Theorem 2.10.** [31] Let \( \Omega \) be a \( C^{0,1} \) domain. Suppose that the hypotheses of Theorem 2.9 hold for \( m \geq 2 \) and \( 1 \leq r < \infty \). Furthermore, assume that for each \( u \in W^{k,\infty}(\Omega, \mathbb{R}^d) \) small enough, \( v(u) = 0 \) on \( \partial(\Omega + u) \). Then, for each \( \omega \subset \subset \Omega \), the function \( u \to v_\omega(u) = v(u)|_\omega \) defined on a neighborhood of \( u = 0 \) in \( W^{k,\infty}(\Omega, \mathbb{R}^d) \) with values in \( W^{m-1,r}(\omega) \) is differentiable at \( u = 0 \). Moreover, the map \( u \to v(u) \) has a local derivative at \( u = 0 \) (see Definition 2.1) and the local derivative at \( u = 0 \), in the direction \( u \), denoted by \( v'(u) \), satisfies \( v'(u) \in W^{m-1,r}(\Omega) \) and
\[ v'(u) = -(u \cdot n) \frac{\partial v(0)}{\partial n} \quad \text{on} \quad \partial \Omega, \]
where \( n \) is the unit outward normal vector to \( \Omega \).

In what follows we will use the notation
\[ \mathcal{W} = \{ u \in W^{k,\infty}(\Omega, \mathbb{R}^d) : \|u\|_{k,\infty} < c_\Omega \}, \]
where \( k \geq 1 \) and \( c_\Omega < 1/2 \) is small enough such that all the previous results hold.

**Lemma 2.11.** [3, Lemma 9] Let \( u \in \mathcal{W} \). If \( f \in H^1_0(\Omega + u) \), then there exists a unique \( g \in H^1_0(\Omega) \), such that \( f \circ (I + u) = g \). Moreover,
\[ \left( \frac{\partial f}{\partial z_i} \right) \circ (I + u) = \sum_j M_{ij}(u) \frac{\partial g}{\partial x_j} = D_i(u)g, \tag{2.14} \]
where the matrix \( M(u) \) is defined as
\[ M(u) = \left[ M_{i,j}(u) \right] = \left[ \frac{\partial}{\partial x_j} (I + u)_i \right]^{-1} = \left[ \frac{\partial}{\partial x_i} (I + u)_j \right]^{-1}, \]
and \( z_i = x_i + u_i(x), \forall x \in \Omega. \)
3. Regularity of the eigenvalues and eigenfunctions

3.1. Reformulation of the Stokes eigenvalue problem. In order to study the regularity of eigenvalues and eigenfunctions with respect to the perturbation \( u \), we have to write the problem \((P_u)\) in a suitable equivalent way. We consider the space

\[ L^2_0(\Omega) = \left\{ f \in L^2(\Omega) : \int_{\Omega} f = 0 \right\}, \]

with the norm induced by \( L^2(\Omega) \).

**Lemma 3.1.** Let \( u \in \mathcal{W} \). Consider the map that to each \( k \in L^2(\Omega) \) with

\[ \int_{\Omega} k \text{Jac}(I + u) = 0 \]

associates

\[ K = k - \frac{1}{|\Omega|} \int_{\Omega} k \in L^2_0(\Omega). \]

This map is bijective.

**Remark 3.1.** Note that if \( p(u) \in L^2_0(\Omega + u) \) we have that

\[ \int_{\Omega + u} p(u) = \int_{\Omega} p(u) \circ (I + u)\text{Jac}(I + u) = 0. \]

Therefore, from Lemma 3.1, the map

\[ p(u) \in L^2_0(\Omega) \longrightarrow P(u) = p(u) \circ (I + u) - \frac{1}{|\Omega|} \int_{\Omega} p(u) \circ (I + u) \]

is bijective.

Now, let \( \{v(u), p(u), \lambda(u)\} \) be a solution of \((P_u)\). We define

\[ V(u) = v(u) \circ (I + u), \quad V = (V_1, \ldots, V_d) \]

and

\[ P(u) = p(u) \circ (I + u) - \frac{1}{|\Omega|} \int_{\Omega} p(u) \circ (I + u), \]

the image of \( p(u) \) by the map of Lemma 3.1.

The following system is equivalent to \((P_u)\):

\[
\begin{cases}
- \sum_{ij} \partial_j (M_{ij}(u) \text{Jac}(I + u) D_i(u) V_k(u)) \\
+ \sum_j \partial_j (M_{kj}(u) P(u) \text{Jac}(I + u)) = \lambda(u) V_k(u) \text{Jac}(I + u), & \text{in } \Omega \\
D(u) \cdot V(u) = 0, & \text{in } \Omega \\
V(u) \in (H^1_0(\Omega))^d, & P(u) \in L^2_0(\Omega),
\end{cases}
\]

(3.1)
where \(1 \leq k \leq 2\), with
\[
D_i(u)g = \left(\frac{\partial f}{\partial z_i}\right) \circ (I + u) = \sum_j M_{ij}(u) \frac{\partial g}{\partial x_j},
\]
for \(g = f \circ (I + u)\), with \(M(u) = (M_{ij}(u))_{i,j=1}^d\) as in Lemma 2.5.

On the other hand, note that the condition
\[
D(u) \cdot Y = 0, \quad Y \in (H^1_0(\Omega))^d,
\]
with \(Y = y \circ (I + u)\) and \(y \in (H^1_0(\Omega + u))^d\), is equivalent to
\[
D(u) \cdot Y - \frac{1}{|\Omega|} \int_{\Omega} D(u) \cdot Y = 0, \quad Y \in (H^1_0(\Omega))^d.
\]
Indeed, if
\[
D(u) \cdot Y = \frac{1}{|\Omega|} \int_{\Omega} D(u) \cdot Y = c,
\]
since
\[
\int_{\Omega} D(u) \cdot Y \text{Jac}(I + u) = \int_{\Omega + u} \text{div } y = \int_{\partial(\Omega + u)} y \cdot n = 0,
\]
with \(y \circ (I + u) = Y\), we have
\[
0 = \int_{\Omega} D(u) \cdot Y \text{Jac}(I + u) = c \int_{\Omega} \text{Jac}(I + u) = c |\Omega + u| \iff c = 0.
\]
That is, \(D(u) \cdot Y = 0\) in \(\Omega\).

Thus, the following problem is equivalent to (3.1) and therefore also equivalent to \((P_u)\): To find \(\lambda(u) \in \mathbb{R}\) and \(V(u) \in J_u(\Omega)\), where
\[
J_u = \left\{ \varphi \in (H^2(\Omega) \cap H^1_0(\Omega))^2 : D(u) \cdot V = 0, \text{ in } \Omega \right\}
\]
such that
\[
\begin{cases}
-\sum_{i,j} \partial_j (M_{ij}(u) \text{Jac}(I + u) D_i(u) V_k(u)) + \\
\sum_j \partial_j (M_{kj}(u) P(u) \text{Jac}(I + u)) = \lambda(u) V_k(u) \text{Jac}(I + u), \text{ in } \Omega \\
D(u) \cdot V(u) - \frac{1}{|\Omega|} \int_{\Omega} D(u) \cdot V(u) = 0, \text{ in } \Omega \\
V(u) \in (H^1_0(\Omega))^d, \quad P(u) \in L^2_0(\Omega).
\end{cases}
\]
3.2. Some preliminary results of spectral theory. To prove the existence and regularity of the eigenvalues and eigenfunctions of the Stokes system with respect to the perturbation parameter $u$ we will use the Lyapunov-Schmidt method (see [37], [5, p. 30]).

**Lemma 3.2.** [5, Lemma 4.1, p. 31] Suppose that $X$ and $Z$ are Hilbert spaces and $A : X \to Z$ is a continuous linear operator. Let $U : X \to N(A)$, $E : Z \to R(A)$ be the orthogonal projection from $X$ and $Z$ on the kernel and range of $A$ respectively. Then, there exists a bounded linear operator $K : R(A) \to N(A)\perp$ called the right inverse of $A$ such that

$$AK = I : R(A) \to R(A), \quad KA = I - U : Z \to N(A)\perp.$$ 

Let $\Lambda$ be a closed subset of a Banach space, such that $\text{Int}\Lambda \neq \emptyset$. If $N : \Lambda \times X \to Z$ is a continuous operator, then the problem

$$Ax - N(x, \lambda) = 0 \quad (3.4)$$

is equivalent to the equations

$$z - KEN(y + z, \lambda) = 0 \quad (3.5)$$

$$(I - E)N(y + z, \lambda) = 0, \quad (3.6)$$

where $x = y + z$, $y \in N(A)$ and $z \in N(A)\perp$.

Assume that the operator $N$ satisfies that

$$N(0, 0) = 0, \quad \frac{\partial N}{\partial x}(0, 0) = 0,$$

and consider the equation (3.5), for $(x, \lambda)$ in a neighborhood of $(0, 0)$ in $X \times \Lambda$. Applying the implicit function theorem to (3.5), we deduce the existence of a neighborhood $V \subset N(A) \times \Lambda$ of $(0, 0)$ and a function $z^* : V \to N(A)\perp$ with the same regularity as $N$ providing the solution of (3.5). Therefore, if $\{y_1, \ldots, y_h\}$ is an orthonormal basis of $N(A)$, the solution $x(\lambda)$ of (3.5) satisfies

$$x(\lambda) = \sum_{i=1}^{h} c_i(\lambda)y_i + z^*\left(\sum_{i=1}^{h} c_i(\lambda)y_i, \lambda\right) = 0, \quad (3.7)$$

for suitable coefficients $c_1, \ldots, c_h$. Then, $(x, \lambda) \in V$ satisfy (3.4) if and only if

$$(I - E)N\left(\sum_{i=1}^{h} c_i(\lambda)y_i + z^*\left(\sum_{i=1}^{h} c_i(\lambda)y_i, \lambda\right), \lambda\right) = 0, \quad (3.8)$$

which is a finite-dimensional system of equations on the constants $c_1, \ldots, c_h$. 
3.3. Regularity of the eigenvalues and eigenfunctions with respect to the perturbation parameter. Now, we apply the Lyapunov-Schmidt method to our problem.

Lemma 3.3. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class $C^2$. Then the map

$$S : W^{3,\infty}(\Omega, \mathbb{R}^d) \longrightarrow \mathcal{L}((H^1_0(\Omega))^d \times L^2_0(\Omega); (H^{-1}(\Omega))^d \times L^2_0(\Omega)),$$

such that

$$S(u)(\varphi, \pi) = \begin{pmatrix} -D_i(u)(\text{Jac}(I+u)D_i(\varphi)) + D(u)(\text{Jac}(I+u)) - D_i(u)\varphi & + D(u)\varphi \end{pmatrix}$$

is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$.

Proof. $\text{Jac}(I + u)$ is a polynomial on the first partial derivatives of $u$. Then the map $u \longrightarrow \text{Jac}(I + u)$ is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$. On the other hand,

$$M_{ij}(u) = \frac{1}{\text{Jac}(I + u)}(\delta_{ij} + a_{ij}),$$

where $a_{ij}(u)$ is the minor of the matrix $M^{-1}(u)$ associated to its $ij$-th element which is also a polynomial of the first partial derivatives of $u$. Moreover, for $u$ small enough $\text{Jac}(I + u) > 0$. Therefore, $u \longrightarrow M(u)$ is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ as well.

Note also that $S(u)$ can be rewritten as

$$S(u)(\varphi, \pi) = \begin{pmatrix} -M(u)\nabla \cdot (\text{Jac}(I+u)M(u)\nabla \varphi) + M(u)\nabla(\text{Jac}(I+u)\pi) & + M(u)\nabla \cdot \varphi \end{pmatrix}$$

since $D(u)\varphi = M(u)\nabla \varphi$. From the analyticity of the functions $u \longrightarrow \text{Jac}(I + u)$ and $u \longrightarrow M(u)$ we obtain that the map $S(u)$ is analytic in a neighborhood of $u = 0$ on $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with values in

$$\mathcal{L}((H^1_0(\Omega))^d \times L^2_0(\Omega), (H^{-1}(\Omega))^d \times L^2_0(\Omega)),$$

and the proof is complete. □

Remark 3.2. Given any $\lambda > 0$ the map

$$A : (H^1_0(\Omega))^d \times L^2_0(\Omega) \longrightarrow (H^{-1}(\Omega))^d \times L^2_0(\Omega)$$

(3.11)
defined as
\[
A(\varphi, \pi) = \begin{pmatrix}
-\Delta \varphi + \nabla \pi - \lambda \varphi \\
\frac{1}{|\Omega|} \int_{\Omega} \nabla \cdot \varphi - \nabla \cdot \varphi
\end{pmatrix} = \begin{pmatrix}
-\Delta \varphi + \nabla \pi - \lambda \varphi \\
-\nabla \cdot \varphi
\end{pmatrix}
\] (3.12)

is self-adjoint.

Furthermore, if \(\lambda\) is an eigenvalue of multiplicity \(h\) of the Stokes system on the domain \(\Omega\) and \(v_1, \ldots, v_h\) are the orthonormal eigenfunctions associated to \(\lambda\) and \(p_1, \ldots, p_h\) the eigenpressures, we have
\[
N(A) = \text{span} \{ (v_i, p_i) : i = 1, \ldots, h \}. \tag{3.13}
\]

Now we have the following result, which is a slight variation of a theorem due to J. Albert [2] on the scalar eigenvalue problem for the Laplacian.

**Theorem 3.4.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded open set of class \(C^2\). Assume that \(\lambda\) is an eigenvalue of multiplicity \(h\) of the Stokes system on the domain \(\Omega\) and \(v_1, \ldots, v_h\) are the orthonormal eigenfunctions associated to \(\lambda\) with associated pressures \(p_1, \ldots, p_h\).

Then there exist \(h\) analytic functions defined in a neighborhood of \(u = 0\) in \(W^{3,\infty}(\Omega, \mathbb{R}^d)\) with values in \(\mathbb{R}\), \(u \rightarrow \lambda_i(u)\), and \(h\) analytic functions \(u \rightarrow (\phi_i(u), p_i(u))\), with values in \(J_u \times L^2_0(\Omega)\), \(i = 1, \ldots, h\), defined in a neighborhood of \(u = 0\) in \(W^{3,\infty}(\Omega, \mathbb{R}^d)\), such that the following hold:

1. \(\lambda_j(0) = \lambda, \quad j = 1, \ldots, h.\)
2. For all \(u\) small enough, \((\lambda_j(u), \phi_j(u), p_j(u))\) is a solution of (3.3); that is, the function \(v_i(u) = \phi_i(u) \circ (I + u)^{-1} \in J_0(\Omega + u)\) is an eigenfunction of \((P_u)\) associated to the eigenvalue \(\lambda_j(u)\).
3. For all \(u\) small enough the set \(\{v_1(u), \ldots, v_h(u)\}\) is orthonormal on \(H(\Omega + u)\).
4. For each interval \(I \subset \mathbb{R}\) such that \(I\) contains only the eigenvalue \(\lambda\) of \((P)\), there exists a neighborhood \(U\) of \(u = 0\) such that there are exactly \(h\) eigenvalues (counting the multiplicity) \(\lambda_1(u), \ldots, \lambda_h(u)\) of \((P_u)\) contained on \(I\).

**Remark 3.3.** To prove Theorem 3.4 we prove first that if \(\lambda\) is an eigenvalue of multiplicity \(h\) we can find functions \(u \rightarrow \lambda(u) \in \mathbb{R}\) and \(u \rightarrow (v(u), p(u)) \in J_0(\Omega + u) \times L^2_0(\Omega + u)\) such that \(v(u)\) is an eigenfunction of \((P_u)\) associated to the eigenvalue \(\lambda(u)\), with pressure \(p(u)\). To find the other \(h - 1\) branches of eigenvalues and eigenfunctions, we apply in an iterative form the method described in the following proposition.

**Proposition 3.5.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded open set of class \(C^2\). Assume that \(\lambda\) is an eigenvalue of multiplicity \(h\) of the Stokes system on the domain
\( \Omega \) and \( v_1, \ldots, v_h \) are the orthonormal eigenfunctions associated to \( \lambda \) with associated pressures \( p_1, \ldots, p_h \).

Then there exists at least a function \( u \mapsto (\lambda(u), \phi(u), \rho(u)) \in \mathbb{R} \times J_u \times L^2_0(\Omega) \) which is analytic in a neighborhood of \( u = 0 \) in \( W^{3,\infty}(\Omega, \mathbb{R}^d) \) such that

\[ (1) \quad \lambda(0) = \lambda, \]
\[ (2) \quad \nu(u) \text{ defined as } \nu(u) \circ (I + u) = \phi(u) \text{ is an eigenfunction of the Stokes system in the domain } \Omega + u, \text{ associated to the eigenvalue } \lambda(u), \text{ with pressure } p(u) \text{ where } p(u) \circ (I + u) = \rho(u). \]

**Proof.** Let \( \lambda \) be an eigenvalue of multiplicity \( h \) of the Stokes system on the domain \( \Omega \) and \( v_1, \ldots, v_h \) be the orthonormal eigenfunctions associated to \( \lambda \) with associated pressures \( p_1, \ldots, p_h \).

We apply the Lyapunov-Schmidt method to problem \( (P_u) \). Let us consider the map

\[ T : (H^1_0(\Omega))^d \times L^2_0(\Omega) \rightarrow (H^{-1}(\Omega))^d \times L^2_0(\Omega) \]
\[ \langle \phi, \pi \rangle \rightarrow T(\phi, \pi) = (\varphi, 0). \]  \hspace{1cm} (3.14)

Thus, in view of (3.3) and the definition (3.10) of \( S(u) \), we can write problem \( (P_u) \) as

\[ S(u)(\phi(u), \rho(u)) - \lambda(u) \text{Jac}(I + u)T(\phi(u), \rho(u)) = 0, \]  \hspace{1cm} (3.15)

or, equivalently, as

\[ [S(0) - \lambda T](\phi(u), \rho(u)) \]
\[ = [(S(0) - S(u)) - \lambda T + \lambda(u) \text{Jac}(I + u)T](\phi(u), \rho(u)) \]  \hspace{1cm} (3.16)
\[ = [(S(0) - S(u)) + (\lambda(u) - \lambda) \text{Jac}(I + u)T + \lambda(\text{Jac}(I + u) - 1)T](\phi(u), \rho(u)). \]

If \( R(u) = S(0) - S(u) + \lambda \text{Jac}(I + u) - 1)T \), then

\[ [S(0) - \lambda T](\phi(u), \rho(u)) = [R(u) + (\lambda(u) - \lambda) \text{Jac}(I + u)T](\phi(u), \rho(u)). \]  \hspace{1cm} (3.17)

From Lemma 3.2 we have that the map \( A = S(0) - \lambda T \) has a right inverse operator \( K \). Thus, in view of (3.17) we obtain that

\[ (\phi(u), \rho(u)) = [K(R(u) + (\lambda(u) - \lambda) \text{Jac}(I + u)T)](\phi(u), \rho(u)) \]
\[ + (\psi(u), \pi(u)), \]  \hspace{1cm} (3.18)

where \( (\psi(u), \pi(u)) \in N(A) \); that is,

\[ (\psi(u), \pi(u)) = \sum_{i=1}^h c_i(u)(v_i, p_i). \]  \hspace{1cm} (3.19)
On the other hand, from (3.17)
\[ R(u) + (\lambda(u) - \lambda)Jac(I + u)T] (\phi(u), \rho(u)) \in R(A) = N(A)^{\perp}. \]
Thus, if
\[ Q(u) = R(u) + (\lambda(u) - \lambda)Jac(I + u)T, \] (3.20)
we have that
\[
0 = \langle Q(u) (\phi(u), \rho(u)), (v_i, p_i) \rangle \\
= \langle Q(u) [I - KQ(u)]^{-1} (\psi(u), \pi(u)), (v_i, p_i) \rangle \\
= \langle Q(u) [I - KQ(u)]^{-1} \sum_{j=1}^{h} c_j(u)(v_j, p_j), (v_i, p_i) \rangle \\
= \sum_{j=1}^{h} c_j(u) \langle Q(u) [I - KQ(u)]^{-1} (v_j, p_j), (v_i, p_i) \rangle,
\] (3.21)
for all \( i = 1, \ldots, h \) which is a linear system of equations on the unknowns \( c_j(u) \). This system has a nontrivial solution if and only if
\[
\text{det} \left( \langle Q(u) [I - KQ(u)]^{-1} (v_j, p_j), (v_i, p_i) \rangle \right) = 0.
\] (3.22)
We replace \( \lambda(u) - \lambda \) by \( \alpha \) and we define \( \hat{R}(u, \alpha) = R(u) + \alpha Jac(I + u)T, \)
\[ f_{ij}(\alpha, u) = \left\langle \left[ \hat{R}(u, \alpha) \right] [I - K(\hat{R}(u, \alpha))]^{-1} (v_j, p_j), (v_i, p_i) \right\rangle, \] (3.23)
and
\[ F(\alpha, u) = \text{det} (f_{ij}(\alpha, u)). \] (3.24)
For \( u \) small enough, the map \( u \mapsto [I - K\hat{R}(u, \alpha)]^{-1} \) is well defined. Indeed for \( \alpha = 0 \) and \( u = 0 \) we have that \( [I - K\hat{R}(0, 0)] = I \) and the map is analytic in a neighborhood of \( u = 0 \) in \( W^{3, \infty}(\Omega, \mathbb{R}^d) \). On the other hand, as we mentioned above, if \( F(\alpha, u) = 0 \), system (3.21) has a nontrivial solution \( c_1(u), \ldots, c_h(u) \), and then
\[
\lambda(u) = \lambda + \alpha
\] (3.25)
is an eigenvalue of \( (P_u) \). Moreover, from (3.18) and (3.19) we deduce that
\[ (\phi(u), \rho(u)) = \sum_{j=1}^{h} c_j(u) [I - K(R(u) + (\lambda(u) - \lambda)Jac(I + u)T)]^{-1} (v_j, p_j) \] (3.26)
is an eigenfunction of \( (P_u) \) associated to the eigenvalue \( \lambda(u) \).
According to our previous discussion, for these values of $\alpha(u)$ and setting $\lambda(u) = \lambda + \alpha(u)$, system (3.21) admits a solution $c_1(u), \ldots, c_h(u)$, not all the components being zero. We have that

$$f_{ij}(\alpha, 0) = \langle \alpha T \left[ I - \alpha KT \right]^{-1} (v_j, p_j), (v_i, p_i) \rangle . \quad (3.27)$$

For $\alpha$ sufficiently small, the operator $I - \alpha KT$ is invertible and, moreover,

$$[I - \alpha KT]^{-1} = \sum_{n \geq 0} (\alpha KT)^n = I + \sum_{n \geq 1} \alpha^n (KT)^n .$$

Therefore, for all $i, j = 1, \ldots, h$, we have

$$f_{ij}(\alpha, 0) = \alpha \langle T(v_j, p_j), (v_i, p_i) \rangle + \sum_{n \geq 1} \alpha^n \langle T(KT)^n (v_j, p_j), (v_i, p_i) \rangle$$

$$= \alpha \delta_{ij} + \alpha^{n+1} \langle T(KT)^n (v_j, p_j), (v_i, p_i) \rangle . \quad (3.28)$$

Therefore,

$$F(\alpha, 0) = \alpha^h + \sum_{n \geq 1} s_n \alpha^{n+h}$$

for suitable coefficients $s_n$, and $F$ satisfies

$$\frac{\partial^r F}{\partial \alpha^r}(0, 0) = 0, \quad r = 0, \ldots, h - 1; \quad \frac{\partial^h F}{\partial \alpha^h}(0, 0) \neq 0 .$$

Applying the Weierstrass preparation theorem we deduce that

$$F(\alpha, u) = (\alpha^h + a_1(u)\alpha^{h-1} + \cdots + a_h(u))E(\alpha, u)$$

with $E(\alpha, u) \neq 0$ in a neighborhood of $(0, 0)$. Then for $(\alpha, u)$ small enough we have that $E(\alpha, u) \neq 0$, and the functions $a_j(u)$ are analytic in a neighborhood of $u = 0$. Then, $F(\alpha, u) = 0$ if and only if

$$\alpha^h + a_1(u)\alpha^{h-1} + \cdots + a_h(u) = 0 . \quad (3.29)$$

Let $\alpha_j(u), j = 1, \ldots, h$, be the complex roots of (3.29). Then there exist constants $c_1(u), \ldots, c_h(u)$, not all vanishing simultaneously, a solution of system (3.21). Thus, from (3.26) we obtain that

$$(\phi(u), \rho(u)) = \sum_{j=1}^h c_j(u) \left[ I + K(R(u) + (\lambda(u) - \lambda)Jac(I + u)T) \right]^{-1} (v_j, p_j),$$

and $\lambda(u) = \lambda + \alpha_1(u)$ constitutes an eigenpair.
Note that if \( c_j(u) \) is complex, it is enough to consider the real part \( \Re c_j(u) \) to get a real eigenfunction. Since the Stokes operator is self-adjoint we have that \( \alpha_j(u) \) is real, which completes the proof of Proposition 3.5. \( \square \)

**Remark 3.4.** Proposition 3.5 provides the existence of one branch of eigenpairs associated to the root \( \alpha(u) \) of (3.29). We do not use the eigenpairs associated to the other roots \( \alpha_j \) by now since, so far, we do not know whether they coincide or not with the eigenpair associated to \( \alpha_1(u) \).

Now, we prove Theorem 3.4.

**Proof of Theorem 3.4.** By using an iterative argument on \( h \) we prove the existence of the \( h \) analytic functions \( u \to (\lambda_i(u), v_i(u)) \), the eigenvalues and eigenfunctions of the Stokes system in the domain \( \Omega + u \).

From Proposition 3.5 we have that there exists an analytic function \( u \to (\lambda_1(u), v_1(u), p_1(u)) \) defined in a neighborhood of \( u = 0 \) in \( W^{3,\infty}(\Omega, \mathbb{R}^d) \) with values in \( \mathbb{R} \times J_0(\Omega + u) \times L^2(\Omega + u) \), \( \lambda_1(u) \) being an eigenvalue of the Stokes system, \( v_1(u) \) and \( p_1(u) \) being the corresponding eigenfunction and eigenpressure.

Therefore, Theorem 3.4 holds for \( h = 1 \). We must prove it for \( h \geq 2 \).

Let \( \Pi_1(u) : J_0(\Omega + u) \longrightarrow J_0(\Omega + u) \) be the orthogonal projection on the eigenspace generated by \( v_1(u) \). Then we define the map

\[
B(u) = P(u) - \Pi_1(u),
\]

(3.30)

where \( P(u) \) is, as in Section 2.1, the composition of the Laplacian with the projection operator from \( (L^2(\Omega + u))^d \) into \( H(\Omega + u) \). Then

\[
B(0)v_j = (P(0) - \Pi_1(0))v_j = \lambda v_j - \delta_{1j}v_j;
\]

(3.31)

that is, \( B(0)v_j = \lambda v_j, j = 2, \ldots, h, \) and \( B(0)v_1 = (\lambda - 1)v_1 \).

Then, \( \lambda \) is an eigenvalue of multiplicity \( h - 1 \) of the operator \( B = B(0) \), with eigenfunctions \( v_2, \ldots, v_h \).

Note that other linearly independent eigenfunctions of \( B \) associated to \( \lambda \) do not exist. Indeed, if \( v \) is another eigenfunction of \( B \) associated to the eigenvalue \( \lambda \) such that \( \langle v, v_j \rangle = 0, j = 2, \ldots, h, \) then \( \langle v, v_1 \rangle = 0 \) (because \( v_1 \) is an eigenfunction associated to the eigenvalue \( \lambda - 1 \)) and \( Bv = \lambda v \).

Then

\[
Pv = Bv + \Pi_1v = Bv + \langle v, v_1 \rangle v_1 = \lambda v;
\]

that is, \( v \) is an eigenfunction of \( P \) associated to \( \lambda \), and thus \( \lambda \) is an eigenvalue of multiplicity \( h + 1 \), which is impossible because the multiplicity of \( \lambda \) is \( h \).

It is not difficult to see that \( B(u) \) satisfies the same conditions of the Stokes operator \( P(u) \) to apply the Lyapunov-Schmidt method used in the proof of Proposition 3.5. Applying this method in an iterative form we obtain \( h - 1 \) analytic functions in a neighborhood of \( u = 0 \) in \( W^{3,\infty}(\Omega, \mathbb{R}^d) \),
with respect to the perturbation parameter \( u \) that \( \in \). Moreover, the functions \( v_2(u), \ldots, v_h(u) \) form an orthonormal set in \( H(\Omega + u) \). This shows us the existence of the \( h \) branches of eigenpairs.

Now we prove the last part of the theorem. Since the eigenvalues \( u \rightarrow \lambda_i(u) \) are analytic in a neighborhood of \( u = 0 \), there exist constants \( c_i \) such that

\[
|\lambda_i(u) - \lambda_i(v)| \leq c_i \| u - v \|_{3,\infty}.
\]

Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) be the eigenvalues of the Stokes system in the domain \( \Omega \) and assume that \( \cdots \leq \lambda_{n-1} < \lambda = \lambda_n = \cdots = \lambda_{n+h-1} < \lambda_{n+h} \leq \cdots \). Let \( I \subset \mathbb{R} \) be an interval such that \( \lambda \) is the unique eigenvalue contained in \( I \). Then there exists \( \delta > 0 \) such that \( I \subset (\lambda_{n-1} + \delta, \lambda_{n+h} - \delta) \).

Let \( u \in B(0, \frac{\delta}{c}) \), with \( c = \max \{ c_i : i = 1, \ldots, n + h \} \). Then

\[
|\lambda_{n-1}(u) - \lambda_{n-1}| \leq c_{n-1} \| u \|_{3,\infty} < c_{n-1} \frac{\delta}{c} \leq \delta,
\]

and

\[
|\lambda_{n+h}(u) - \lambda_{n+h}| \leq c_{n+h} \| u \|_{3,\infty} < c_{n+h} \frac{\delta}{c} \leq \delta.
\]

Therefore \( \lambda_{n-1}(u) \notin \mathcal{I} \) and \( \lambda_{n+h}(u) \notin \mathcal{I} \); that is, \( P(u) \) has at most \( h \) eigenvalues contained in \( \mathcal{I} \) counting multiplicity. This completes the proof of Theorem 3.4. \( \square \)

3.4. Local variations of the eigenvalues and eigenfunctions. Let \( v_i(u) \in \mathcal{J}_0(\Omega + u), i = 1, \ldots, h, \) be the eigenfunctions of \( P(u) \) associated to the eigenvalue \( \lambda_i(u) \), where

\[
\lambda = \lambda_i(0), \quad v_i(0) = v_i, \quad i = 1, \ldots, h.
\]

According to the results of the previous section, the branches of the eigenvalues \( u \rightarrow \lambda_i(u) \in \mathbb{R} \) and the eigenfunctions \( u \rightarrow v_i(u) \) are analytic with respect to the perturbation parameter \( u \) in a neighborhood of \( u = 0 \) in \( W^{3,\infty}(\Omega, \mathbb{R}^d) \). The first local variation of the branches solves the system

\[
\begin{cases}
-\Delta v_i'(u) + \nabla p_i'(u) = \lambda_i'(u)v_i + \lambda v_i'(u), & \text{in } \Omega \\
\nabla \cdot v_i'(u) = 0, & \text{in } \Omega \\
v_i'(u) + (u \cdot \nabla)v_i \in (H^1_0(\Omega))^d \\
p_i'(u) + \text{div } (up_i(0)) \in L^2_0(\Omega).
\end{cases}
\]

(3.32)

The following identity is easy to prove:
Lemma 3.6. Let \( \varphi \in (C^1(\overline{\Omega}))^d \cap (H_0^1(\Omega))^d \) and \( \Omega \subseteq \mathbb{R}^d \) be a domain of \( C^2 \)-class. Then

\[
\frac{\partial \varphi}{\partial n} \cdot n = \sum_{i,j=1}^{d} \frac{\partial \varphi_i}{\partial n} n_i n_j = \nabla \cdot \varphi, \quad \text{on } \partial \Omega. \tag{3.33}
\]

Proof. Since \( \varphi \in (H_0^1(\Omega))^d \cap (C^1(\overline{\Omega}))^d \), we have that, for each \( x_0 \in \partial \Omega \) such that \( \nabla \varphi(x_0) \neq 0 \),

\[
n(x_0) = c_i \left| \nabla \varphi_i(x_0) \right|,
\]

for each \( i = 1, \ldots, d \) such that \( |\nabla \varphi_i(x_0)| \neq 0 \) where \( c_i = \pm 1 \) depends on the direction in which \( \nabla \varphi_i(x_0) \) is oriented. Thus,

\[
\frac{\partial \varphi_i}{\partial x_j} = \frac{n_j}{c_i} |\nabla \varphi_i|, \quad i, j = 1, \ldots, d. \tag{3.34}
\]

We have

\[
\frac{\partial \varphi}{\partial n} \cdot n = \sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial n} n_i = \sum_{i=1}^{d} (\nabla \varphi_i \cdot n)n_i = \sum_{i,j=1}^{d} \frac{\partial \varphi_i}{\partial x_j} n_i n_j.
\]

From (3.34), we obtain that

\[
\frac{\partial \varphi}{\partial n} \cdot n = \sum_{i,j=1}^{d} \frac{n_j}{c_i} |\nabla \varphi_i| n_i n_j = \sum_{i,j=1}^{d} \frac{1}{c_i} |\nabla \varphi_i| n_i (n_j)^2 = \sum_{j=1}^{d} (n_j)^2 \left[ \sum_{i=1}^{d} \frac{n_i}{c_i} |\nabla \varphi_i| \right] = \sum_{j=1}^{d} (n_j)^2 \left[ \sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial x_j} \right] = |n|^2 (\nabla \cdot \varphi) = (\nabla \cdot \varphi).
\]

This completes the proof of the lemma. \( \square \)

The following lemma provides a useful identity on the derivative of the eigenvalues.

Lemma 3.7. Let \( \lambda \) be an eigenvalue of the Stokes system in \( \Omega \) with multiplicity \( h \) and let \( v_1, \ldots, v_h \) be the eigenfunctions associated to \( \lambda \). Let \( u \in W^{3,\infty}(\Omega, \mathbb{R}^d) \rightarrow \lambda_i(u) \in \mathbb{R}, i = 1, \ldots, h, \) be the branches of eigenvalues for \( u \) small enough.

Then the first local derivatives of the branches satisfy

\[
\delta_{ij} \lambda_i'(u) = -\int_{\Gamma} (u \cdot n) \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n}, \quad \forall i, j = 1, \ldots, h. \tag{3.35}
\]

Note that here and in the sequel \( \lambda_i'(u) \) denotes the derivative of \( \lambda_i \) at \( u = 0 \) in the direction \( u \).
Proof. Let $w \in J_0(\Omega)$. Then from (3.32) we have that

$$\int_{\Omega} \nabla v'_i(u) \cdot \nabla w = \lambda'_i(u) \int_{\Omega} v_i \cdot w + \lambda \int_{\Omega} v'_i(u) \cdot w.$$  \hspace{1cm} (3.36)

Taking $w = v_j$ in (3.36) we deduce that

$$\int_{\Omega} \nabla v'_i(u) \cdot \nabla v_j = \lambda'_i(u) \int_{\Omega} v_i \cdot v_j + \lambda \int_{\Omega} v'_i(u) \cdot v_j.$$  \hspace{1cm} (3.36)

On the other hand, since $\Omega$ is a $C^2$ domain, we have that $v_j \in (H^2(\Omega))^d$ (see Theorem 2.4). Integrating by parts we deduce that

$$-\int_{\Omega} (\nabla v_j) \cdot v'_i(u) + \int_{\Gamma} v'_i(u) \cdot \frac{\partial v_j}{\partial n} = \delta_{ij} \lambda'_i(u) + \int_{\Omega} v'_i(u) \cdot (-\Delta v_j + \nabla p_j).$$

That is,

$$\delta_{ij} \lambda'_i(u) = \int_{\Gamma} v'_i(u) \cdot \frac{\partial v_j}{\partial n} - \int_{\Omega} v'_i(u) \cdot \nabla p_j$$

$$= \int_{\Gamma} v'_i(u) \cdot \frac{\partial v_j}{\partial n} + \int_{\Omega} p_j (\nabla \cdot v'_i(u)) - \int_{\Gamma} v'_i(u) \cdot (p_j n)$$

$$= \int_{\Omega} v'_i(u) \cdot \frac{\partial v_j}{\partial n} - \int_{\Omega} v'_i(u) \cdot (p_j n) = -\int_{\Gamma} (u \cdot n) \frac{\partial v_i}{\partial n} \left[ \frac{\partial v_j}{\partial n} - p_j n \right].$$

From Lemma 3.6 we have that

$$\frac{\partial v_j}{\partial n} \cdot n = \nabla \cdot v_j = 0, \text{ in } \Omega.$$  \hspace{1cm} (3.37)

The proof of the lemma is now complete. \hfill \Box

From (3.35) we deduce that

$$\lambda'_i(u) = -\int_{\Gamma} (u \cdot n) \left| \frac{\partial v_i}{\partial n} \right|^2.$$  \hspace{1cm} (3.38)

The next lemma provides an identity for the Stokes system in the spirit of the classical Pohožaev’s identity for scalar elliptic equations.

Lemma 3.8. Let $\Omega \subset \mathbb{R}^n$ be a domain of class $C^2$ and $x_0 \in \mathbb{R}^d$. Let $(\lambda, \phi, p)$ and $(\lambda, \phi, q)$ be solutions of the eigenvalue problem (P). Then

$$\sum_{i=1}^d \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_i = \frac{1}{2} \int_{\Gamma} ((x - x_0) \cdot n) \frac{\partial \phi}{\partial n} \cdot \frac{\partial \phi}{\partial n}.$$  \hspace{1cm} (3.39)

This lemma may be proved by multiplying the equation satisfied by $\phi$ (respectively $\phi$) by $(x - x_0) \cdot \nabla \phi$ (respectively $(x - x_0) \cdot \nabla \phi$) and integrating by parts. We refer to [25] for the details of the proof.
Corollary 3.9. Under the hypotheses of Lemma 3.8, for each \((\lambda, v, p)\) a solution of the eigenvalue problem \((P)\), we have that
\[
\int_{\Omega} |\nabla v|^2 = \frac{1}{2} \int_{\Gamma} ((x - x_0) \cdot n) \left| \frac{\partial v}{\partial n} \right|^2.
\] (3.40)

Remark 3.5. Identity (3.39) provides a unique continuation result for the eigenfunctions of the Stokes system. Indeed, if \(v\) is a eigenfunction of \((P)\) and
\[
\frac{\partial v}{\partial n} = 0, \quad \text{on } \Gamma,
\]
according to (3.39) we have that
\[
\int_{\Omega} |\nabla v|^2 = 0 \iff v = 0.
\]
The same holds when \(\frac{\partial v}{\partial n}\) vanishes on a subset of the boundary of the form
\[
\Gamma(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot n(x) > 0 \}.
\]
Note that these results are not a consequence of the classical Holmgren’s uniqueness theorem, which requires the quantity \(\frac{\partial v}{\partial n} + np\) to vanish in an open subset of \(\partial \Omega\).

Using the variants introduced by A. Osses [26] of the classical multiplier \((x - x_0) \cdot \nabla \varphi\) one can show the same unique continuation result for other subsets of the boundary. However, the subsets of the boundary we obtain by multiplier methods are always rather large.

4. Proof of Theorem 1.1: Deformations of large subsets of the boundary

Our proof of the generic simplicity of the eigenvalues of the Stokes problem is inspired in the work of J. Albert (see [1]) for the Laplace operator.

Proof of Theorem 1.1. We define the sets
\[
A_0 = W^{3,\infty}(\Omega, \mathbb{R}^2),
\] (4.1)
and for \(n \geq 1\),
\[
A_n = \{ u \in W^{3,\infty}(\Omega, \mathbb{R}^2) : \text{the first } n \text{ branches } \lambda_1(u), \ldots, \lambda_n(u) \text{ of eigenvalues of } (P_u) \text{ are simple} \}.
\] (4.2)
Note that the branches \(u \rightarrow \lambda_i(u)\) are those obtained in Theorem 3.4, which are analytic. Therefore the eigenvalues \(\lambda_i\) are not necessarily ordered in an increasing way in the definition of \(A_n\).
The proof of the generic simplicity is based on the application of Baire’s lemma (see Lemma 2.1 and Lemma 2.2) on the sets $A_n \subset W^{3,\infty}(\Omega, \mathbb{R}^2)$. We claim that it is sufficient to show that the sets $A_n$ are open and that the set $A_{n+1}$ is dense in $A_n$ for all $n \geq 0$. Indeed, then applying Baire’s lemma to the sets $A_n$ in $W^{3,\infty}(\Omega, \mathbb{R}^2)$ we deduce that $A = \bigcap_n A_n$ is dense in $W^{3,\infty}(\Omega, \mathbb{R}^2)$.

On the other hand,

$$A = \{ u \in W^{3,\infty}(\Omega, \mathbb{R}^2) : \text{all the eigenvalues of } (P_u) \text{ are simple} \}.$$ 

Thus, it is sufficient to check these two facts:

**a)** $A_n$ is open in $W^{3,\infty}(\Omega, \mathbb{R}^2)$. It is clear that $A_0$ is an open set. Now we analyze the case $n \geq 1$.

Let $u \in A_n$. Then, by definition, the first $n$ branches of the eigenvalues of $(P_u)$, $\lambda_i(u)$ with $i = 1, \ldots, n$, are simple. Let

$$\delta = \min \{ |\lambda_i(u) - \lambda_j(u)| : i, j = 1, \ldots, n + 1, i \neq j \} > 0.$$ 

We consider $w \in U$, where $U = \{ w \in W : \|w\|_{3,\infty} < \frac{\delta}{2c}\}$, $c$ being the maximum of the Lipschitz constants for the functions $w \mapsto \lambda_i(w)$, $i = 1, \ldots, n + 1$, defined from $W^{3,\infty}(\Omega, \mathbb{R}^2)$ with values in $\mathbb{R}$. Then

$$\delta \leq |\lambda_i(u) - \lambda_j(u)|$$
$$\leq |\lambda_i(u + w) - \lambda_i(u)| + |\lambda_i(u + w) - \lambda_j(u + w)| + |\lambda_j(u + w) - \lambda_j(u)|$$
$$\leq c|(u + w) - u| + |\lambda_i(u + w) - \lambda_j(u + w)| + c|(u + w) - u|$$
$$< \delta + |\lambda_i(u + w) - \lambda_j(u + w)|.$$ 

Therefore, $|\lambda_i(u + w) - \lambda_j(u + w)| > 0$, and then $u + w \in A_n$, which proves that $A_n$ is an open set.

**b)** $A_{n+1}$ is dense in $A_n$. We will prove that $A_{n+1}$ is dense in $A_n$, for all $n \geq 0$; in particular, for $n = 0$ we will prove that $A_1$ is dense in $A_0 = W^{3,\infty}(\Omega, \mathbb{R}^2)$. We proceed by contradiction. Assume that $A_{n+1}$ is not dense in $A_n$; then there exists $u \in A_n \setminus A_{n+1}$ and a neighborhood $W$ of $u$ such that $W \subset A_n \setminus A_{n+1}$.

Since $u \in A_n \setminus A_{n+1}$, we have that the first $n$ branches of the eigenvalues of $(P_u)$ are simple; that is, $\lambda_i(u) \neq \lambda_j(u)$, $\forall i, j = 1, \ldots, n + 1$, if $i \neq j$.

Assume that $\lambda_{n+1}$ is an eigenvalue with multiplicity $h \geq 2$; that is, $\lambda = \lambda_{n+1}(u) = \cdots = \lambda_{n+h}(u)$ (in the case of $n = 0$, we have that the first eigenvalue has multiplicity $h \geq 2$; that is, $\lambda = \lambda_1(u) = \cdots = \lambda_h(u)$). Let $v_1, v_2, \ldots, v_h$ be the eigenfunctions associated to $\lambda$. Let $w$ be such that $u + w \in W$. We consider the eigenvalues $\lambda_{n+i}(u + w)$, with $i = 1, \ldots, h$, and the corresponding eigenfunctions $v_i(u + w)$, with $i = 1, \ldots, h$, of $(P_{u+w})$. 

Then, from (3.35) and Lemma 3.7 we have
\[
\lambda'_{n+i}(\Omega + u; w)\delta_{ij} = - \int_{\Gamma}(w \cdot n) \frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n},
\] (4.3)
where $\Gamma = \partial(\Omega + u)$. We want to prove that there exists $w \in \mathcal{W}$, such that \(\lambda'_i(\Omega + u; w) \neq \lambda'_j(\Omega + u; w)\), for each \(i \neq j, \ i, j = 1, \ldots, n + h\). That will immediately imply that
\[
\lambda_i(u + \varepsilon w) \neq \lambda_j(u + \varepsilon w), \ \forall i, j = 1, \ldots, n + h + 1,
\] (4.4)
for $\varepsilon > 0$ small enough. Thus $u + \varepsilon w \in A_{n+1}$.

We proceed by contradiction. If there exist $i \neq j, \ i, j = 1, \ldots, n + h$ such that $\lambda_i(u + w) = \lambda_j(u + w)$, for all $w$ in a neighborhood of $w = 0$, we have that
\[
\lambda'_i(\Omega + w; w) = \lambda'_j(\Omega + w; w).
\] (4.5)
Thus, from (3.35) we have that
\[
\int_{\Gamma}(w \cdot n)(\frac{\partial v_i}{\partial n} \cdot \frac{\partial v_j}{\partial n}) = 0,
\] (4.6)
for all $w$ in a neighborhood of $w = 0$, where $\Gamma = \partial(\Omega + u)$.

Moreover, from the equality (4.5) and Lemma 3.7 we obtain that
\[
\int_{\Gamma}(w \cdot n)(\frac{\partial v_i}{\partial n})^2 - \frac{\partial v_j}{\partial n})^2 = 0,
\] (4.7)
for all $w$ in a neighborhood of $w = 0$, where $\Gamma = \partial(\Omega + u)$.

Therefore, we have that
\[
\frac{\partial v_i}{\partial n}(x) \cdot \frac{\partial v_j}{\partial n}(x) = 0, \ \text{on } \Gamma
\] (4.8)
and
\[
\left| \frac{\partial v_i}{\partial n} \right| = \left| \frac{\partial v_j}{\partial n} \right|, \ \text{on } \Gamma.
\] (4.9)

Since\[ \text{div } v_i = \frac{\partial v_i}{\partial n} \cdot n = 0 = \frac{\partial v_j}{\partial n} \cdot n = \text{div } v_j, \ \text{on } \Gamma \]
we have that
\[
\frac{\partial v_i}{\partial n} = a(x)(-n_2, n_1), \quad \frac{\partial v_j}{\partial n} = b(x)(-n_2, n_1).
\] (4.10)
From (4.9) we obtain that $|a(x)| = |b(x)|$.

On the other hand, (4.8) implies that
\[
a(x) b(x) = 0, \ \text{on } \Gamma,
\]
and then \( a(x) = b(x) = 0 \), on \( \overline{\Gamma} \). That is,
\[
\frac{\partial v_i}{\partial n} = \frac{\partial v_j}{\partial n} = 0, \quad \text{on } \partial(\Omega + u).
\]
Using Pohožaev's identity (3.39) we have that \( v_i = v_j = 0 \), in \( \Omega + u \). This is a contradiction of the fact that the functions \( v_i \) are eigenfunctions of the Stokes system. This completes the proof of Theorem 1.1. \( \square \)

**Remark 4.1.** Using the results of M. Dauge (see [7, 8]), one can show that Theorem 1.1 remains true in convex polygonal domains.

Indeed, the \((H^2(\Omega) \cap H^1_0(\Omega))^2 \times H^1(\Omega)\) regularity of the eigenfunctions \((v, p)\) of \((P)\) holds in this case, too. This regularity result suffices to justify all the arguments above in which the normal derivative of the eigenfunctions needs to be in \((L^2(\partial\Omega))^2\).

5. **Deformation of an Arbitrarily Small Subset of the Boundary**

In this section, we show that the main result remains true if we consider deformations of an arbitrarily small subset of the boundary. However, in this case we need a nonstandard unique continuation property.

Our result is the following:

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^2 \) be an open, bounded domain of class \( C^3 \) and \( \Gamma_0 \subset \partial\Omega \) be a nonempty open set. Then, there exists a perturbation \( u \in W^{4,\infty}(\Omega, \mathbb{R}^2) \), with \( u = 0 \) on \( \Gamma \setminus \Gamma_0 \) and \( \|u\|_{4,\infty} \) arbitrarily small, such that the spectrum of the Stokes system in \( \Omega + u \) is simple.

**Remark 5.1.** As we said in the Introduction, we need a generic unique continuation result to replace, in the proof of Theorem 5.1, the one we got by means of multipliers. To prove it the following proposition is needed.

**Proposition 5.2.** Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded domain of class \( C^3 \), with \( d \geq 2 \) and \( \Gamma_0 \subset \Gamma \) an open, nonempty set. Let \( v \in J_0(\Omega) \cap (H^3(\Omega))^d \) be a solution of
\[
\begin{align*}
-\Delta v + \nabla p &= \lambda v & \text{in } \Omega \\
\nabla \cdot v &= 0 & \text{in } \Omega \\
v &= 0 & \text{on } \Gamma \\
\frac{\partial v}{\partial n} &= \frac{\partial^2 v}{\partial n^2} = 0 & \text{on } \Gamma_0 \\
\int_\Omega p &= 0.
\end{align*}
\]
Then \( v = 0 \), in \( \Omega \).
Proof. Let $\tau^1, \ldots, \tau^{d-1}$ be the tangent vectors to $\partial\Omega$ which form an orthogonal basis of the tangent hyperplane. If $f \in H^2(\Omega)$,

$$
\partial_i f = \sum_{j=1}^{d-1} \frac{\partial f}{\partial \tau^j} \tau_i^j + \frac{\partial f}{\partial n} n_i, \quad \text{on } \partial \Omega.
$$

On $\Gamma_0$ we have that

\begin{align}
\Delta v &= \sum_i \partial_i^2 v = \sum_i \frac{\partial}{\partial n} (\partial_i v) n_i + \sum_{i,j} \frac{\partial}{\partial \tau^j} (\partial_i v) \tau_i^j \\
&= \sum_i \nabla (\partial_i v) \cdot n n_i = \sum_{i,j} \partial_i (\partial_j v) n_i n_j = \frac{\partial^2 v}{\partial n^2} = 0.
\end{align}

Thus,

$$
\nabla p = (\lambda v + \Delta v) = 0, \quad \text{on } \Gamma_0.
$$

Therefore, $\frac{\partial p}{\partial n} = 0$ on $\Gamma_0$, and moreover $p$ is constant in each connected component of $\Gamma_0$. Since $\Delta p = 0$ in $\Omega$, we deduce that $p$ is constant in $\Omega$.

This implies that each component $v_i$ of $v$ is a solution of

$$
\begin{cases}
-\Delta v_i &= \lambda v_i & \text{in } \Omega \\
v_i &= 0 & \text{on } \Gamma \\
\frac{\partial v_i}{\partial n} &= 0 & \text{on } \Gamma_0,
\end{cases}
$$

and we conclude that $v_i = 0$, in $\Omega$. This completes the proof. \qed

We are now in a condition to prove the following result of generic unique continuation:

**Theorem 5.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^3$, with $d \geq 2$ and $\Gamma_0 \subset \Gamma$ an open, nonempty, bounded set. Then there exists a perturbation $u \in W^{4,\infty}(\Omega, \mathbb{R}^d)$, with $u = 0$ on $\Gamma \setminus \Gamma_0$, arbitrarily small, such that the unique continuation property holds for all the eigenfunctions in the deformed domain $\Omega + u$. In other words, if $v$ is an eigenfunction of the Stokes system in $\Omega + u$, then $\frac{\partial v}{\partial n} \neq 0$ in $\Gamma_0 + u$.

**Remark 5.2.** Theorem 5.3 holds with perturbations of the boundary with support in a part of the boundary $\Gamma_0$ where the normal derivative of the solution vanishes. This fact is essential in the proof. The heuristic motivation of this fact is as follows. Assume that a branch of eigenfunctions $v(u)$ is such that $\frac{\partial v}{\partial n} = 0$ on $\Gamma_0 + u$ for all $u$. Then, by taking derivatives with respect to the perturbation parameter $u$, one deduces that $\frac{\partial^2 v}{\partial n^2} = 0$ on $\Gamma_0 + u$. But then we are under the conditions of Proposition 5.2.
Proof of Theorem 5.3. Let $A = \{ u \in W^{4,\infty}(\Omega, \mathbb{R}^d) : (SUC) \text{ holds for all the eigenvalues of } (P) \text{ in } \Omega + u \}$. Here by (SUC) we mean the unique continuation property for the eigenfunctions stated in Theorem 5.2. Obviously, $A$ is the countable intersection of the sets $A_n$, where $A_0 = W^{4,\infty}(\Omega, \mathbb{R}^d)$, and

$$A_n = \left\{ u \in W^{4,\infty}(\Omega, \mathbb{R}^d) : \text{ (SUC) holds for the first } n \text{ branches of eigenvalues of } (P) \text{ in } \Omega + u \subset \mathbb{R}^d \right\}.$$ 

It is sufficient to show that each set $A_n$ is open in $W^{4,\infty}(\Omega, \mathbb{R}^d)$ and that $A_{n+1}$ is dense in $A_n$ for all $n \geq 0$. Then, using Baire’s lemma (see Lemma 2.1 and Lemma 2.2) we have that the set $A$ is dense on $W^{4,\infty}(\Omega)$.

i) $A_n$ is open. Let us suppose that $A_{n+1}^c$ is not closed. Then there exists $u \in A_n$ and a sequence of deformations $\{u_k\}_k \subset W^{4,\infty}(\Omega, \mathbb{R}^d)$ such that $u_k \notin A_n$ and $u_k \to u$ in $W^{4,\infty}(\Omega, \mathbb{R}^d)$. Since $u_k \notin A_n$, for every $k$ we have that

$$\begin{cases}
-\triangle v(u_k) + \nabla p(u_k) = \lambda(u_k)v(u_k), & \text{ in } \Omega + u_k \\
\nabla \cdot v(u_k) = 0, & \text{ in } \Omega + u_k \\
v(u_k) = 0, & \text{ on } \Gamma + u_k \\
\frac{\partial v(u_k)}{\partial n} = 0, & \text{ on } \Gamma_0 + u_k \\
\|v(u_k)\|_{L^2(\Omega+u_k)} = 1,
\end{cases}$$

where $\lambda(u_k)$ is an eigenvalue belonging to one of the first $n$ branches.

Given that the branches $u \to \lambda_i(u)$ and $u \to V_i(u) = v_i(u) \circ (I + u)$, $i = 1, \ldots, n$, are continuous from $W^{4,\infty}(\Omega, \mathbb{R}^d)$ to $\mathbb{R}$ and $H^2(\Omega)$ respectively, we have that for at least one of the eigenvalues $\lambda(u)$ of the first $n$ branches the corresponding eigenfunction $v(u)$ is such that

$$\begin{cases}
-\triangle v(u) + \nabla p(u) = \lambda(u)v(u), & \text{ in } \Omega + u \\
\nabla \cdot v(u) = 0, & \text{ in } \Omega + u \\
v(u) = 0, & \text{ on } \Gamma + u \\
\frac{\partial v(u)}{\partial n} = 0, & \text{ on } \Gamma_0 + u \\
\|v(u)\|_{L^2(\Omega+u)} = 1,
\end{cases}$$

$u$ being the limit of $u_k$.

But this is impossible since the property (SUC) holds in $\Omega + u$ for the first $n$ branches since $u \in A_n$. This shows us that $A_n$ is an open set.

ii) $A_{n+1}$ is dense on $A_n$. Let $u \in A_n \setminus A_{n+1}$. We consider two cases:

a) $\lambda = \lambda_{n+1}(u)$ is a simple eigenvalue of $(P)$.

b) $\lambda = \lambda_{n+1}(u)$ is a multiple eigenvalue of $(P)$.
a) \( \lambda \) is a simple eigenvalue. Without loss of generality we may suppose that \( u = 0 \). Assume that the property \( (SUC) \) does not hold in a neighborhood of \( u = 0 \). Then the problem

\[
\begin{cases}
-\Delta v(u) + \nabla p(u) = \lambda(u)v(u) & \text{in } \Omega + u \\
\nabla \cdot v(u) = 0 & \text{in } \Omega + u \\
v(u) = 0 & \text{on } \Gamma + u \\
\frac{\partial v(u)}{\partial n} = 0 & \text{on } \Gamma_0 + u,
\end{cases}
\]

(5.3)

has a nontrivial solution \((\lambda(u), v(u), p(u))\), for all \( u \) small enough.

The local derivatives satisfy

\[
\begin{cases}
-\Delta v'(u) + \nabla p'(u) = \lambda'(u)v + \lambda v'(u) & \text{in } \Omega \\
\nabla \cdot v'(u) = 0 & \text{in } \Omega \\
v'(u) = -(u \cdot n) \frac{\partial v}{\partial n} & \text{on } \Gamma \\
\frac{\partial v'(u)}{\partial n} = -(u \cdot n) \frac{\partial^2 v}{\partial n^2} & \text{on } \Gamma_0
\end{cases}
\]

where \( v'(u) \) and \( \lambda'(u) \) denote the derivative of \( v \) and \( \lambda \) in the direction \( u \) at \( u = 0 \). Thus, if we consider perturbations \( u \) such that \( u = 0 \) on \( \Gamma \setminus \Gamma_0 \), then

\[
\lambda'(u) = -\int_{\Gamma} (u \cdot n) \left| \frac{\partial v}{\partial n} \right|^2 = -\int_{\Gamma_0} (u \cdot n) \left| \frac{\partial v}{\partial n} \right|^2 - \int_{\Gamma \setminus \Gamma_0} (u \cdot n) \left| \frac{\partial v}{\partial n} \right|^2 = 0,
\]

(5.4)

and moreover \( v'(u) = 0 \), on \( \Gamma \). Therefore,

\[
\begin{cases}
-\Delta v'(u) + \nabla p'(u) = \lambda v'(u) & \text{in } \Omega \\
\nabla \cdot v'(u) = 0 & \text{in } \Omega \\
v'(u) = 0 & \text{on } \Gamma \\
\frac{\partial v'(u)}{\partial n} = -(u \cdot n) \frac{\partial^2 v}{\partial n^2} & \text{on } \Gamma_0;
\end{cases}
\]

that is, \((\lambda(u), v'(u), p'(u))\) is a solution of \((P)\) in \( \Omega \).

Now, taking into account that \( \lambda \) is a simple eigenvalue of the Stokes system, we deduce that there exists a constant \( \alpha_u \) such that \( v'(u) = \alpha_u v \), where \( v \) is an eigenfunction of unit norm associated with \( \lambda \). Thus,

\[
\frac{\partial v'}{\partial n} = \alpha_u \frac{\partial v}{\partial n} = 0, \text{ on } \Gamma_0.
\]

Since

\[
\frac{\partial v'}{\partial n} = -(u \cdot n) \frac{\partial^2 v}{\partial n^2}, \text{ on } \Gamma_0,
\]

we obtain

\[
\frac{\partial^2 v}{\partial n^2} = 0, \text{ on } \Gamma_0.
\]
Proposition 5.2 implies then that \( v \equiv 0 \), which is impossible because \( v \) is an eigenfunction of the Stokes system.

\textbf{b) } \lambda \text{ is a multiple eigenvalue.} First, we consider the case where the multiplicity of \( \lambda \) is two and let \( (v_1,p_1) \), \( (v_2,p_2) \) be the eigenfunctions and eigenpressures associated to \( \lambda \). Moreover, assume that \( (v_1(u),p_1(u)) \), solve (5.3) in a neighborhood of \( u = 0 \).

Proceeding as in a), if we consider perturbations \( u \) such that
\[
u_1(u) = \alpha_u v_1 + \beta_u v_2, \quad \text{in } \Omega.
\]

Then
\[
\frac{\partial v_1'}{\partial n} = \beta_u \frac{\partial v_2}{\partial n}, \quad \text{in } \Omega.
\]

That is,
\[
\frac{\partial v_1'}{\partial n} = -(u \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_u \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0.
\]

We distinguish two cases. First, if \( \partial v_2/\partial n = 0 \) on \( \Gamma_0 \) as well, we then have \( \partial^2 v_1/\partial n^2 = 0 \) on \( \Gamma_0 \), and by Proposition 5.1, \( v_1 \equiv 0 \). This is in contradiction with the fact that \( v_1 \) is a nontrivial eigenfunction. Therefore we can assume that \( \partial v_2/\partial n \neq 0 \).

Let \( u_1, u_2 \) be two perturbations of the domain such that the quotient \( (u_1 \cdot n)/(u_2 \cdot n) \) is well defined and is not constant in any open subset of \( \Gamma_0 \). Then
\[
-(u_1 \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_{u_1} \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0, \quad (5.5)
\]
\[
-(u_2 \cdot n) \frac{\partial^2 v_1}{\partial n^2} = \beta_{u_2} \frac{\partial v_2}{\partial n}, \quad \text{on } \Gamma_0. \quad (5.6)
\]

If \( \frac{\partial^2 v_1}{\partial n^2}(x) \neq 0 \), for \( x \in \Gamma_0 \), we divide (5.5) by (5.6), and we have that
\[
\frac{(u_1(x) \cdot n(x))}{(u_2(x) \cdot n(x))} = \frac{\beta_{u_1}}{\beta_{u_2}} = \text{constant},
\]

which is impossible because of the choice of the functions \( u_i \). With this we prove that
\[
\frac{\partial^2 v_1}{\partial n^2}(x) = 0, \quad \text{for all } x \in \Gamma_0.
\]
Thus, we can apply Lemma 5.2, obtaining that \( v_1 = 0 \) in \( \Omega \), which is a contradiction with the fact that \( v_1 \) is an eigenfunction of \((P)\).

Thus, there exists a perturbation \( u \in W^{4,\infty}(\Omega, \mathbb{R}^d) \) such that the unique continuation property holds for \( v_1 \). Obviously, the fact that \( \partial v_2 / \partial n \neq 0 \) on \( \Gamma_0 + u \) is kept under small perturbations \( u \).

Now, we assume that \( \lambda \) is an eigenvalue of multiplicity \( h>2 \) and \( v_1, \ldots, v_h \) are the eigenfunctions associated to \( \lambda \) normalized on \( L^2(\Omega) \). Let \( \lambda_i(u) \) with \( i = 1, \ldots, h \), be the eigenvalues of the Stokes system for the domain \( \Omega + u \), such that \( \lambda_i(0) = \lambda \) for \( i = 1, \ldots, h \). Moreover, we suppose that the multiplicity of \( \lambda(u) \) is \( h \) in a neighborhood \( M \) of \( u = 0 \) and that

\[
\frac{\partial v_1(u)}{\partial n(\Omega + u)} = 0, \quad \text{on } \Gamma_0 + u. \tag{5.7}
\]

Taking into account that the multiplicity does not change in the neighborhood \( M \), we have that for every \( u \in M \)

\[
\lambda_i'(u) = \cdots = \lambda_h'(u). \tag{5.8}
\]

If we consider perturbations \( u \) such that \( u = 0 \) on \( \Gamma \setminus \Gamma_0 \), from (5.4) we have that

\[
\left| \frac{\partial v_1}{\partial n} \right| = \cdots = \left| \frac{\partial v_h}{\partial n} \right|, \quad \text{on } \Gamma_0. \tag{5.9}
\]

Therefore, from (5.7) we have that

\[
\frac{\partial v_i}{\partial n} = 0, \quad \text{on } \Gamma_0, \quad i = 1, \ldots, h. \tag{5.10}
\]

On the other hand, using the argument of the case of double eigenvalues, we have that the local variation of the first eigenfunction \( v_1'(u) \) is an eigenfunction of the Stokes system associated to the eigenvalue \( \lambda \), and then, for each perturbation \( u \in M \) with \( u = 0 \) on \( \Gamma \setminus \Gamma_0 \), there exist constants \( c_1(u), \ldots, c_h(u) \) such that

\[
v_1'(u) = c_1(u)v_1 + c_2(u)v_2 + \cdots + c_h(u)v_h. \tag{5.11}
\]

Combining (5.10) with the fact that

\[
\frac{\partial v_1'(u)}{\partial n} = -(u \cdot n) \frac{\partial^2 v_1}{\partial n^2} = 0, \quad \text{on } \Gamma_0, \tag{5.12}
\]

we deduce that

\[
\frac{\partial^2 v_1}{\partial n^2} = 0, \quad \text{on } \Gamma_0. \tag{5.13}
\]

Now, from Proposition 5.2 we deduce that \( v_1 \equiv 0 \), which is impossible because \( v_1 \) is an eigenfunction. Then, there exists a perturbation of the domain
such that the multiplicity of the eigenvalue decreases to $h - 1$ or the unique continuation property holds.

If the unique continuation property does not hold, we can apply the same argument to obtain a small-enough deformation $u$ such that the unique continuation property holds or the eigenvalue has multiplicity $h - 2$.

Applying this argument in an iterative way, we can obtain a deformation $u'$ as small as we want such that the unique continuation property holds in $\Omega + u'$ or the eigenvalue has multiplicity two. If the multiplicity is two we apply the argument above. This shows that $A_{n+1}$ is dense in $A_n$. Now, we can apply Baire's lemma to complete the proof of Theorem 5.3. □

**Remark 5.3.** The proof of Theorem 5.1 is analogous to the proof of Theorem 1.1. We use however the generic unique continuation property of Theorem 5.3 instead of Pohožaev's identity.

**Remark 5.4.** We can see that our results give us a generic unique continuation property. The restriction to generic domains is of technical nature, but as far as we know, the existence of counterexamples or the positive answer for the unique continuation property for every domain is an open problem.

An interesting problem is the classification of the domains in which the property holds.

### 6. The Three-Dimensional Case

The argument described in the proof of Theorem 1.1 applies only in dimension $d = 2$. When $d \geq 3$ the results of existence and regularity of the branches for the eigenvalues and eigenfunctions of the Stokes system are true. The computation of the local variations and the Pohožaev’s identity remain also valid. However, the argument used to prove the generic simplicity of the eigenvalues fails at the following point:

In the proof of the density of the set $A_{n+1}$ on $A_n$ we have that

$$\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \quad \text{on } \bar{\Gamma}.$$  

This is true also in $d \geq 3$.

Later, we saw that if $v_1, v_2$ are two eigenfunctions of the Stokes system associated to the same eigenvalue and such that

$$\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \quad \left| \frac{\partial v_1}{\partial n} \right| = \left| \frac{\partial v_2}{\partial n} \right|, \quad \text{on } \bar{\Gamma},$$

using the fact

$$\frac{\partial v_j}{\partial n} \cdot n = 0, \quad j = 1, 2 \quad \text{on } \bar{\Gamma},$$
then \[ \frac{\partial v_j}{\partial n} = 0, \quad j = 1, 2 \] on \( \Gamma \), and therefore \( v_1 = v_2 = 0 \) in \( \Omega + u \). The same problem arises in dimensions \( d \geq 3 \) as well. In other words, we must exclude the existence of a couple of eigenfunctions such that

\[
\frac{\partial v_1}{\partial n} \cdot \frac{\partial v_2}{\partial n} = 0, \quad \left| \frac{\partial v_1}{\partial n} \right| = \left| \frac{\partial v_2}{\partial n} \right|, \quad \frac{\partial v_j}{\partial n} \cdot n = 0 \quad \text{on} \ \tilde{\Gamma}, \ j = 1, 2.
\]

These conditions give us that \( \frac{\partial v_j}{\partial n} \) are tangent vectors to \( \partial \Omega \), orthogonal to each other and with the same modulus. But these facts do not immediately imply a contradiction since the tangent plane has dimension \( d - 1 \geq 2 \).

The recent developments made by A. Osses in [26] may be of some use to solve this problem at least in the cases where a large-enough subset of the boundary of \( \Omega \) is deformed.

7. An application to pointwise control

We have proved that the eigenvalues of the two-dimensional Stokes system are simple for almost every domain. In this section we show briefly an application of this result to the pointwise controllability of the Stokes system.

Assume that \( \Omega \) is a bounded, regular domain of \( \mathbb{R}^2 \) such that all the eigenvalues of the Stokes system are simple. From our previous results we know that this holds generically with respect to the domain \( \Omega \). Let

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \rightarrow \infty \quad (7.1)
\]

\[
w_1, w_2, \ldots, w_n, \ldots \quad (7.2)
\]

be the eigenvalues and the eigenfunctions of the system respectively. The eigenfunctions are assumed to constitute an orthonormal basis of \( H(\Omega) \).

Let \( x_0 \in \Omega \) and \( \delta_{x_0} \) be the Dirac delta measure at the point \( x_0 \). Given \( T > 0 \) we consider the following controllability problem:

\[
\begin{cases}
  v_t - \nabla p + \omega(t)\delta_{x_0}, & \text{in} \ \Omega \times (0, T) \\
  v = 0, & \text{on} \ \partial \Omega \times (0, T) \\
  \nabla \cdot v = 0, & \text{in} \ \Omega \times (0, T) \\
  v(x, 0) = v_0(x), & \text{in} \ \Omega,
\end{cases} \quad (7.3)
\]

with \( v_0 \in J_0(\Omega) \) and \( \omega \in L^2(0, T; \mathbb{R}^2) \).

In (7.3), the vector function \( \omega \in L^2(0, T; \mathbb{R}^2) \) is the control. It is easy to see that the solution of (7.3) satisfies \( v \in C([0, T]; (H^{-s}(\Omega))^2) \) for every \( s > 0 \), and moreover \( v \in L^2([0, T]; (L^2(\Omega))^2) \) (see [17] for a similar result for the heat equation).
The approximate controllability problem for system (7.3) can be formulated in the following form: Let \( s > 0 \). Is the set of reachable states
\[
R(T; v_0) = \{ v(T) : v \text{ is a solution of (7.3) with } \omega \in L^2(0, T; \mathbb{R}^2) \} \tag{7.4}
\]
dense in \( H^{-s} \), the completion of \( H(\Omega) \) with respect to the norm of \( (H^{-s}(\Omega))^2 \), for all \( v_0 \in J_0(\Omega) \)?

Using Hahn-Banach’s theorem, it is easy to see that system (7.3) is approximately controllable if and only if the following unique continuation property holds for the adjoint system: If \( \varphi \) is a solution of

\[
\begin{cases}
-\varphi_t - \Delta \varphi = \nabla \pi, & \text{in } \Omega \times (0, T) \\
\varphi = 0, & \text{on } \partial \Omega \times (0, T) \\
\nabla \cdot \varphi = 0, & \text{in } \Omega \times (0, T) \\
\varphi(x, T) = \varphi_0(x), & \text{in } \Omega,
\end{cases}
\tag{7.5}
\]

with \( \varphi_0 \in J_0(\Omega) \cap (H^s(\Omega))^2 \) such that

\[
\varphi(x_0, t) = 0 \quad \text{in } (0, T),
\tag{7.6}
\]

can we guarantee that

\[
\varphi \equiv 0? \tag{7.7}
\]

It is easy to check that the answer to this uniqueness problem is negative if one of the eigenvalues of the Stokes system is not simple and two of the corresponding eigenfunctions coincide on \( x_0 \).

Since \( \Omega \) is such that the spectrum is simple, we can see that the property holds if \( x_0 \) is a “strategic point” of \( \Omega \); that is,

\[
w(x_0) \neq 0, \tag{7.8}
\]

for all eigenfunction \( w \) of the Stokes system. In fact, the solution \( \varphi \) of (7.5) can be developed in Fourier series:

\[
\varphi(x, t) = \sum_{k \geq 1} c_k e^{-\lambda_k(T-t)} w_k(x). \tag{7.9}
\]

From (7.6), the analyticity in time of the solutions of the backwards Stokes system and the simplicity of the spectrum we deduce that

\[
c_k w_k(x_0) = 0, \quad \forall k \geq 1, \tag{7.10}
\]

and (7.8) implies \( c_k = 0 \), for all \( k \geq 1 \); that is, \( \varphi \equiv 0 \).

On the other hand, the set of strategic points \( x_0 \) is dense on \( \Omega \). To see this we define the sets

\[
A_k = \{ x \in \Omega : w_i(x) \neq 0, i = 1, \ldots, k \}. \tag{7.11}
\]
Then, for all \( k \in \mathbb{N} \), \( A_k \) is an open subset of \( \Omega \); furthermore, \( A_{k+1} \) is dense on \( A_k \), and applying Baire’s lemma, we conclude the density of the strategic points on \( \Omega \).

In this way we prove the following:

**Theorem 7.1.** Let \( \Omega \) be an open domain of class \( C^2 \) such that the spectrum of the Stokes operator is simple. Then the system \((7.3)\) is approximately controllable for any time \( T \) for any strategic point \( x_0 \in \Omega \). Moreover, the set of strategic points is dense in \( \Omega \).

We can also prove the following result:

**Lemma 7.2.** The set of the points \( x_0 \in \Omega \subset \mathbb{R}^2 \) such that

\[
  w^1_k(x_0) \neq 0, \quad \forall k \geq 1, \tag{7.12}
\]

where \( w_k \) denotes the first component of the eigenfunction \( w_k \), is dense in \( \Omega \).

**Proof.** Let us consider the set

\[
  A_n = \{ x \in \Omega : w^1_k(x) \neq 0, \quad k = 1, \ldots, n \} \tag{7.13}
\]

for each \( n \in \mathbb{N} \).

It is easy to check that the set \( A_n \) is open. Let us prove that \( A_{n+1} \) is dense in \( A_n \). If \( A_{n+1} \) is not dense in \( A_n \), there exists an open set \( U \subset A_n \cap A_{n+1}^c \); that is,

\[
  w^1_k(x) \neq 0, \quad x \in U, \quad k = 1, \ldots, n, \quad \text{and} \quad w^1_{n+1}(x) = 0, \quad x \in U. \tag{7.14}
\]

Taking into account that \( \text{div} \; w_{n+1} = 0 \) we deduce that \( \partial_2 w_{n+1}^2 = 0 \) in \( U \). Applying Holmgren’s uniqueness theorem to \( \partial_2 w \) we deduce that \( \partial_2 w_{n+1} = 0 \) in \( \Omega \). Taking into account that \( w_{n+1} \) vanishes on the boundary we deduce that \( w_{n+1} \equiv 0 \), and this is a contradiction.

Therefore, the set \( A_{n+1} \) is dense in \( A_n \). Now applying Baire’s lemma to the set \( A = \cap_{n \geq 1} A_n \) we complete the proof of Lemma 7.1.

As a consequence of Lemma 7.1 and proceeding as in the proof of Theorem 7.1 we obtain the following result:

**Theorem 7.3.** Let \( \Omega \) be an open domain of class \( C^2 \) such that the spectrum of the Stokes operator is simple. Then system \((7.3)\) is approximately controllable for any strategic point \( x_0 \in \Omega \) in the sense of Lemma 7.1 with controls of the form \( \omega(t) = (\omega^1(t), 0) \).

**References**


