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TIME DISCRETE WAVE EQUATIONS: BOUNDARY OBSERVABILITY AND CONTROL

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Dedicated to Professor Ta-Tsien Li on the occasion of his 70th birthday

Abstract. In this paper we study the exact boundary controllability of a trapezoidal time discrete wave equation in a bounded domain. We prove that the projection of the solution in an appropriate filtered space is exactly controllable with uniformly bounded cost with respect to the time-step. In this way, the well-known exact-controllability property of the wave equation can be reproduced as the limit, as the time step $h \to 0$, of the controllability of projections of the time-discrete one. By duality these results are equivalent to deriving uniform observability estimates (with respect to $h \to 0$) within a class of solutions of the time-discrete problem in which the high frequency components have been filtered. The later is established by means of a time-discrete version of the classical multiplier technique. The optimality of the order of the filtering parameter is also established, although a careful analysis of the expected velocity of propagation of time-discrete waves indicates that its actual value could be improved.

1. Introduction.

Let Ω be a nonempty open bounded domain in $\mathbb{R}^d$ $(d \in \mathbb{N})$ with C^2 boundary Γ, Γ_0 be a nonempty open subset of Γ, and $T > 0$ be a given time duration.

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We consider the following controlled (time continuous) wave equation with a controller acting on the subset Γ_0 of the boundary:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad (0,T)\times\Omega \\ y = u\chi_{\Gamma_0} & \text{on} \quad (0,T)\times\Gamma \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here and henceforth, χ_{Γ_0} is the characteristic function of the set Γ_0 and Δ is the Laplacian in the space variable $x \in \Omega$. In (1.1), $(y(t,\cdot), y_t(t,\cdot))$ is the *state* and $u(t,\cdot)$ is the *control*. The *state and control spaces* of system (1.1) are chosen to be $L^2(\Omega)\times H^{-1}(\Omega)$ and $L^2((0,T)\times\Gamma_0)$, respectively.

The property of exact (boundary) controllability of (1.1) is defined as follows: *For any $(y_0,y_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, there exists a control $u\in L^2((0,T)\times\Gamma_0)$ such that the solution $y\in C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$ of (1.1), defined by the classical transposition method ([9]), satisfies:*

$$y(T) = y_t(T) = 0 \qquad \text{in } \Omega. \tag{1.2}$$

This property holds under suitable geometric restrictions on the subset Γ_0 of the boundary where the control acts and provided that the controllability time T is large enough.

By classical duality arguments ([9]), the above controllability property is equivalent to a (boundary) observability estimate of the following uncontrolled wave equation:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\Gamma \\ \varphi(T) = \varphi_0, \quad \varphi_t(T) = \varphi_1, & \text{in } \Omega. \end{cases} \tag{1.3}$$

The observability inequality reads as follows:

$$E(0) \le C\int_0^T\int_{\Gamma_0}\Big|\frac{\partial\varphi}{\partial\nu}\Big|^2 d\Gamma_0 dt, \qquad \forall\ (\varphi_0,\varphi_1)\in H_0^1(\Omega)\times L^2(\Omega). \tag{1.4}$$

Here and thereafter, we will use C to denote a generic positive constant (depending only on T, Ω and Γ_0) which may vary from line to line. On the other hand, $E(0)$ stands for the energy $E(t)$ of (1.3) at $t=0$, with

$$E(t) = \frac{1}{2}\int_\Omega \Big[|\varphi_t(t,x)|^2 + |\nabla\varphi(t,x)|^2\Big]dx, \tag{1.5}$$

which remains constant in time, i.e.

$$E(t) = E(0), \qquad \forall\, t\in[0,T]. \tag{1.6}$$

Inequality (1.4) asserts that the total energy of any solution of (1.3) can be observed in terms of the energy concentrated on Γ_0 in the time interval $(0,T)$. It is well-known that there are typically two classes of conditions on (T,Ω,Γ_0) guaranteeing (1.4).

i) The first one is given by the *classical multiplier condition*. Fix some $x_0\in\mathbb{R}^d$, put

$$\Gamma_0 \stackrel{\triangle}{=} \big\{x\in\Gamma \ \big|\ (x-x_0)\cdot\nu(x) > 0\big\}, \qquad R \stackrel{\triangle}{=} \max_{x\in\Omega}|x-x_0|, \tag{1.7}$$

where $\nu(x)$ is the unit outward normal vector of Ω at $x\in\Gamma$. Then (1.4) holds for Γ_0 as in (1.7) provided $T>2R$. This is the typical situation one encounters when applying the multiplier technique ([9]), and Carleman inequalities (e.g. [14]) to deduce (1.4), which can also be applied to many other models.

ii) The second one is when (T, Ω, Γ_0) satisfies the *Geometric Control Condition* (GCC, for short) introduced in [1], which asserts that all rays of geometric optics in Ω intersect the subset of the boundary Γ_0 in a uniform time T. In this case, (1.4) is established by means of tools from micro-local analysis ([1]). This condition is optimal.

In this paper, we are interested in the time semi-discretization of systems (1.1) and (1.3). We are thus replacing the continuous dynamics (1.1) and (1.3) by time-discrete ones and analyze their controllability/observability properties. Here we take the point of view of numerical analysis and, therefore, we analyze the limit behavior as the time-step tends to zero.

For this purpose, we set the time step h by $h = T/K$, where $K > 1$ is a given integer. Denote by y^k and u^k respectively the approximations of the solution y and the control u of (1.1) at time $t_k = kh$ for any $k = 0, \cdots, K$. We then introduce the following trapezoidal time semi-discretization of (1.1):

$$\begin{cases} \dfrac{y^{k+1}+y^{k-1}-2y^k}{h^2} - \Delta\left(\dfrac{y^{k+1}+y^{k-1}}{2}\right) = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1, \cdots, K-1 & \\ \dfrac{y^{k+1}+y^{k-1}}{2} = u^k \chi_{\Gamma_0}, \qquad \text{on } \Gamma, \quad k = 1, \cdots, K-1 & \\ y^0 = y_0, \;\; y^1 = y_0 + hy_1, \qquad \text{in } \Omega. & \end{cases} \tag{1.8}$$

Here $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ are the data given in system (1.1) that allow determining the initial data for the time-discrete system too. We refer to Theorem 4.4 below for the well-posedness of system (1.8) by means of the transposition method.

The controllability problem for system (1.8) may be formulated as follows: *For any $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, to find a control $\{u^k \in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$ such that the solution $\{y^k\}_{k=0,\cdots,K}$ of (1.8) satisfies:*

$$y^{K-1} = y^K = 0 \qquad \text{in } \Omega. \tag{1.9}$$

Note that (1.9) is equivalent to the condition $y^{K-1} = (y^K - y^{K-1})/h = 0$, which is a natural discrete version of (1.2).

As in the context of the above continuous wave equation, we also consider the uncontrolled system

$$\begin{cases} \dfrac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2} - \Delta\left(\dfrac{\varphi^{k+1}+\varphi^{k-1}}{2}\right) = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1, \cdots, K-1 & \\ \varphi^k = 0, \qquad\qquad\qquad\qquad\qquad \text{on } \Gamma, \quad k = 0, \cdots, K & \\ \varphi^K = \varphi_0^h + h\varphi_1^h, \quad \varphi^{K-1} = \varphi_0^h, \qquad \text{in } \Omega, & \end{cases} \tag{1.10}$$

where $(\varphi_0^h, \varphi_1^h) \in (H_0^1(\Omega))^2$. In particular, to guarantee the convergence of the solutions of (1.10) towards those of (1.3) one considers convergent data such that

$$\begin{cases} \varphi_0^h \to \varphi_0 \text{ strongly in } H_0^1(\Omega), \\ \varphi_1^h \to \varphi_1 \text{ strongly in } L^2(\Omega), \end{cases} \qquad \text{as } K \to \infty \text{ (or } h \to 0), \tag{1.11}$$

with $h\varphi_1^h$ being bounded in $H_0^1(\Omega)$. Obviously because of the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ this choice is always possible.

Remark 1. Note that the choice of the values of φ^K and φ^{K-1} in (1.10) is motivated by the transposition arguments that are needed to define the solution of the time-discrete non-homogenous problem (1.8), and especially the process to analyze the convergence of the (time-discrete) controls in terms of the duality argument, as we shall see in Section 8.

The energy of system (1.10) is given by

$$E_h^k \triangleq \frac{1}{2}\int_\Omega \left(\left| \frac{\varphi^{k+1}-\varphi^k}{h} \right|^2 + \frac{|\nabla\varphi^{k+1}|^2 + |\nabla\varphi^k|^2}{2} \right) dx, \qquad k = 0, \cdots, K-1, \tag{1.12}$$

which is a discrete counterpart of the continuous energy $E(t)$ in (1.5). Multiplying the first equation of system (1.10) by $(\varphi^{k+1}-\varphi^{k-1})/2$ and integrating it in Ω, using integration by parts, it is easy to show the following property of conservation of energy:

$$E_h^k = E_h^0, \qquad k = 0, \cdots, K-1. \tag{1.13}$$

Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting (in particular in the finite-energy space $H_0^1(\Omega)\times L^2(\Omega)$, under the condition (1.11)).

By means of classical duality arguments, it is easy to show that the above controllability property (1.9) is equivalent to the following boundary observability property for solutions $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.10):

$$E_h^0 \le Ch\sum_{k=1}^{K-1}\int_{\Gamma_0}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2 d\Gamma_0, \quad \forall\, (\varphi_0^h, \varphi_1^h) \in (H_0^1(\Omega))^2. \tag{1.14}$$

The analysis of controllability and/or observability properties of numerical approximation schemes for the wave equation has been the object of intensive studies. However most analytical results concern the case of space semi-discretizations (see [17] and the references cited therein). In practical applications, fully discrete schemes need to be used. The most typical example is the classical central scheme which converges under a suitable CFL condition ([5, 6, 12]). However, in the present setting in which the Laplacian Δ is kept continuous, without discretizing it, this scheme is unsuitable since it is unstable. To see this, we choose $\{\mu_j^2\}_{j\ge1}$ to be the eigenvalues of the Dirichlet Laplacian and $\{\Phi_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ the corresponding eigenvectors (constituting an orthonormal basis of $L^2(\Omega)$), i.e.,

$$\begin{cases} -\Delta\Phi_j = \mu_j^2\Phi_j, & \text{in} \quad \Omega \\ \Phi_j = 0, & \text{on} \quad \Gamma. \end{cases} \tag{1.15}$$

Since $\{\mu_j^2\}_{j\ge1}$ tends to infinity, it is easy to check that the central scheme

$$\frac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2} - \Delta\varphi^k = 0 \tag{1.16}$$

is unstable. Indeed, the stability of (1.16) would be equivalent to the stability of the scheme

$$\frac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2} + \mu_j^2\varphi^k = 0$$

for all values of μ_j^2, $j\ge1$. But this stability property fails clearly, regardless how small h is, when μ_j^2 is large enough. Hence, we choose the trapezoidal scheme (1.10) for the time-discrete problem, which is stable (due to the property of conservation of energy), as mentioned before.

The first result of this paper is of negative nature. Indeed, as we shall see in Theorem 5.1, the observability inequality (1.14) (*resp.* the controllability property (1.9)) fails for system (1.10) (*resp.* (1.8)) without filtering. From the proof of Theorem 5.1, it will be obvious that these negative results of observability and controllability are related to the fact that the spaces in which the solutions evolve are infinite dimensional; while the number of time-steps is finite. Accordingly, to make the observability inequality possible one has to restrict the class of solutions of the adjoint system (1.10) under consideration by filtering the high frequency components. Similarly, since the property of exact controllability of system (1.8) fails, the final requirement (1.9) has to be relaxed by considering only low frequency projections of the solutions. Controlling such a projection can be viewed as a *partial* controllability problem. This filtering method has been applied successfully in the context of controllability of time discrete heat equations ([15]) and space semi-discrete schemes for wave equations ([2, 8, 16, 17]).

As far as we know, the subject of control and observation of the time-discrete wave equation under consideration has not been addressed before. In this paper we shall develop a discrete version of the classical multiplier approach which allows to view the time discrete wave equation as an evolution process with its own dynamics.

As in the continuous case, the multiplier technique we use here applies mainly to the case when the controller/observer Γ_0 is given in (1.7) and some variants ([11]), but does not work when (T, Ω, Γ_0) is assumed to satisfy the GCC. As we shall see, the main advantage of our multiplier approach is that the filtering parameter we use has the optimal scaling in what concerns the frequency of observed/controlled solutions with respect to h.

It is important to note that this kind of results can not be obtained by standard perturbation arguments that rely simply on measuring the distance between solutions of the time-discrete and continuous wave equations. Indeed, when proceeding that way, one needs much stronger filtering requirements. In other words, the optimal filtering can only be obtained by a careful analysis of the time evolution of the system under consideration. This is already well-known in the context of space semi-discretizations (see [17]). Our discrete multiplier approach can also be extended to other PDEs of conservative nature, and in particular to the Schrödinger, plate, Maxwell's equations, among others.

After this paper was finished the same subject was treated in [4] in an abstract setting of conservative semigroups generated by skew-adjoint operators that contains, in particular, the wave equation under consideration. There, using resolvent characterizations of the controllability and observability properties, a general result was proved showing that, whenever the continuous dynamics is observable, then the time-discrete one is uniformly observable in a suitable class of filtered initial data. The results in [4], although they do not yield the optimal time when applied in the present setting, have the advantage that they do not require the filtering parameter to be small, which is agreement with the expected result in view of behavior of the dispersion diagram.

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results which are useful in what follows. In Section 3, we present two fundamental identities by means of discrete multipliers, which will play an important role in the sequel. In Section 4 we discuss the hidden regularity property of solutions of (1.10) and the uniform well-posedness property of system (1.8). Section 5 is devoted to show the lack of controllability/observability of systems (1.8) and (1.10) without filtering. The uniform observability result for (1.10) is presented in Section 6. In Section 7 we show the optimality of the filtering parameter in the uniform observability result. Moreover, we give a heuristic explanation of the necessity of the filtering in terms of the group velocity of propagation of waves. Section 8 is devoted to the uniform controllability of system (1.8) and the convergence of the controls and solutions.

2. Preliminaries.

In this section, we collect some preliminary results that will be used in the sequel.

First of all, for any given $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$ and $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, suppose $\{\theta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$ solves the system

$$\begin{cases} \dfrac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2} - \Delta\left(\dfrac{\theta^{k+1}+\theta^{k-1}}{2}\right) = f_k + \dfrac{g^{k+1}-g^k}{h}, \\ \qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1,\cdots,K-1 \\ \theta^k = 0, \qquad\qquad\quad \text{on } \Gamma, \quad k = 0,\cdots,K. \end{cases} \tag{2.1}$$

We define the energy of system (2.1) by

$$\mathcal{E}_h^k \triangleq \frac{1}{2}\int_\Omega \left(\left| \frac{\theta^{k+1}-\theta^k}{h} \right|^2 + \frac{|\nabla\theta^{k+1}|^2 + |\nabla\theta^k|^2}{2} \right) dx. \tag{2.2}$$

We establish the following discrete version of the energy estimate:

Lemma 2.1. *For any $h > 0$, it holds*

$$\begin{aligned} \max_{0\le k\le K-1} \mathcal{E}_h^k \le \quad & C\left\{ \min\left(\mathcal{E}_h^0, \mathcal{E}_h^{K-1}\right) + \left[h\sum_{k=1}^{K-1} \left(|f^k|_{L^2(\Omega)} + |g^k|_{H_0^1(\Omega)} \right) \right]^2 \right. \\ & \left. + h\sum_{k=1}^{K-1} \int_\Omega \left| f^k \frac{g^{k+1}+g^k}{2} \right| dx \right\}. \end{aligned} \tag{2.3}$$

Proof of Lemma 2.1: Fix any $\ell \in \{1,\cdots K-1\}$. Multiplying both sides of (2.1) by $(\theta^{k+1}-\theta^{k-1})/2h$, integrating it in Ω and summing it for $k = 1,\cdots,\ell$, we obtain:

$$\begin{aligned} & h\sum_{k=1}^{\ell} \int_\Omega \left[\frac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2} - \Delta\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right) \right] \frac{\theta^{k+1}-\theta^{k-1}}{2h} dx \\ & = h\sum_{k=1}^{\ell} \int_\Omega \left(f^k + \frac{g^{k+1}-g^k}{h} \right) \frac{\theta^{k+1}-\theta^{k-1}}{2h} dx. \end{aligned} \tag{2.4}$$

From the definition of the energy $\mathcal{E}_h^k$ in (2.2), the left hand side term of (2.4) can be written as

$$\text{LHS of (2.4)} = \sum_{k=1}^{\ell} \left(\mathcal{E}_h^k - \mathcal{E}_h^{k-1}\right) = \mathcal{E}_h^\ell - \mathcal{E}_h^0. \tag{2.5}$$

We now analyze the right hand side (2.4). It is clear that

$$\left| h\sum_{k=1}^{\ell} \int_\Omega f^k \frac{\theta^{k+1}-\theta^{k-1}}{2h} dx \right| \le Ch\sum_{k=1}^{\ell} |f^k|_{L^2(\Omega)} \left(\sqrt{\mathcal{E}_h^k} + \sqrt{\mathcal{E}_h^{k-1}} \right). \tag{2.6}$$

On the other hand, since $g^1=0$, it follows

$$
\begin{aligned}
& h\sum_{k=1}^{\ell}\int_\Omega \frac{g^{k+1}-g^k}{h}\frac{\theta^{k+1}-\theta^{k-1}}{2h}dx\\
&=\sum_{k=1}^{\ell}\int_\Omega (g^{k+1}-g^k)\left(\frac{\theta^{k+1}-\theta^k}{2h}+\frac{\theta^k-\theta^{k-1}}{2h}\right)dx\\
&=\sum_{k=1}^{\ell}\int_\Omega g^{k+1}\frac{\theta^{k+1}-\theta^k}{2h}dx-\sum_{k=1}^{\ell}\int_\Omega g^k\frac{\theta^k-\theta^{k-1}}{2h}dx\\
&\quad+\sum_{k=1}^{\ell}\int_\Omega g^{k+1}\frac{\theta^k-\theta^{k-1}}{2h}dx-\sum_{k=1}^{\ell}\int_\Omega g^k\frac{\theta^{k+1}-\theta^k}{2h}dx\\
&=\int_\Omega g^{\ell+1}\frac{\theta^{\ell+1}-\theta^\ell}{2h}dx+\sum_{k=2}^{\ell}\int_\Omega g^k\frac{\theta^k-\theta^{k-1}}{2h}dx\\
&\quad-\sum_{k=0}^{\ell-1}\int_\Omega g^{k+1}\frac{\theta^{k+1}-\theta^k}{2h}dx+\int_\Omega g^{\ell+1}\frac{\theta^\ell-\theta^{\ell-1}}{2h}dx\\
&\quad+\sum_{k=1}^{\ell-1}\int_\Omega g^{k+1}\frac{\theta^k-\theta^{k-1}}{2h}dx-\sum_{k=1}^{\ell}\int_\Omega g^k\frac{\theta^{k+1}-\theta^k}{2h}dx\\
&=\int_\Omega g^{\ell+1}\frac{\theta^{\ell+1}-\theta^\ell}{h}dx-h\sum_{k=1}^{\ell}\int_\Omega \frac{g^{k+1}+g^k}{2}\frac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2}dx.
\end{aligned}
\tag{2.7}
$$

It is obvious that

$$
\left|\int_\Omega g^{\ell+1}\frac{\theta^{\ell+1}-\theta^\ell}{h}dx\right|\le C|g^{\ell+1}|_{L^2(\Omega)}\sqrt{\mathcal{E}_h^\ell}\le C|g^{\ell+1}|_{H_0^1(\Omega)}\sqrt{\mathcal{E}_h^\ell}. \tag{2.8}
$$

In view of (2.1) and noting again $g^1=0$, it holds

$$
\begin{aligned}
&-h\sum_{k=1}^{\ell}\int_\Omega \frac{g^{k+1}+g^k}{2}\frac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2}dx\\
&=-h\sum_{k=1}^{\ell}\int_\Omega \frac{g^{k+1}+g^k}{2}\left[\Delta\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)+f^k+\frac{g^{k+1}-g^k}{h}\right]dx\\
&=h\sum_{k=1}^{\ell}\int_\Omega \left[\nabla\left(\frac{g^{k+1}+g^k}{2}\right)\cdot\nabla\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)+f^k\frac{g^{k+1}+g^k}{2}\right]dx\\
&\quad-\int_\Omega\frac{\left|g^{\ell+1}\right|^2-\left|g^1\right|^2}{2}dx\\
&\le Ch\sum_{k=1}^{\ell}\left(|g^{k+1}|_{H_0^1(\Omega)}+|g^k|_{H_0^1(\Omega)}\right)\left(|\theta^{k+1}|_{H_0^1(\Omega)}+|\theta^{k-1}|_{H_0^1(\Omega)}\right)\\
&\quad+h\sum_{k=1}^{\ell}\int_\Omega f^k\frac{g^{k+1}+g^k}{2}dx\\
&\le Ch\sum_{k=1}^{\ell}\left(|g^{k+1}|_{H_0^1(\Omega)}+|g^k|_{H_0^1(\Omega)}\right)\left(\sqrt{\mathcal{E}_h^k}+\sqrt{\mathcal{E}_h^{k-1}}\right)\\
&\quad+h\sum_{k=1}^{\ell}\int_\Omega f^k\frac{g^{k+1}+g^k}{2}dx.
\end{aligned}
\tag{2.9}
$$

Combining (2.4)–(2.9) , we conclude that

$$
\begin{aligned}
\mathcal{E}_h^\ell \leq\ & Ch\sum_{k=1}^{\ell}\Big(|f^k|_{L^2(\Omega)}+|g^{k+1}|_{H_0^1(\Omega)}+|g^k|_{H_0^1(\Omega)}\Big)\Big(\sqrt{\mathcal{E}_h^k}+\sqrt{\mathcal{E}_h^{k-1}}\Big)\\
&+h\sum_{k=1}^{\ell}\int_\Omega\left|f^k\frac{g^{k+1}+g^k}{2}\right|dx+\mathcal{E}_h^0.
\end{aligned}\tag{2.10}
$$

Put

$$
F_h^\ell \triangleq \max_{0\leq k\leq \ell}\mathcal{E}_h^k.\tag{2.11}
$$

Since (2.10) holds for all $\ell=1,\cdots,K-1$, it is still true if $\mathcal{E}_h^\ell$ is replaced by F_h^ℓ. Hence, from (2.10) and recalling $g^1=g^K=0$, we obtain

$$
\begin{aligned}
F_h^\ell &\leq Ch\sum_{k=1}^{K-1}\Big(|f^k|_{L^2(\Omega)}+|g^k|_{H_0^1(\Omega)}\Big)\sqrt{F_h^\ell}\\
&\quad +h\sum_{k=1}^{\ell}\int_\Omega\left|f^k\frac{g^{k+1}+g^k}{2}\right|dx+\mathcal{E}_h^0\\
&\leq C\left[h\sum_{k=1}^{K-1}\Big(|f^k|_{L^2(\Omega)}+|g^k|_{H_0^1(\Omega)}\Big)\right]^2\\
&\quad +\frac{F_h^\ell}{2}+h\sum_{k=1}^{K-1}\int_\Omega\left|f^k\frac{g^{k+1}+g^k}{2}\right|dx+\mathcal{E}_h^0.
\end{aligned}\tag{2.12}
$$

Now, combining (2.11) and (2.12), it follows

$$
\begin{aligned}
\max_{1\leq k\leq K-1}\mathcal{E}_h^k \leq C\Bigg\{&\mathcal{E}_h^0+\left[h\sum_{k=1}^{K-1}\Big(|f^k|_{L^2(\Omega)}+|g^k|_{H_0^1(\Omega)}\Big)\right]^2\\
&+h\sum_{k=1}^{K-1}\int_\Omega\left|f^k\frac{g^{k+1}+g^k}{2}\right|dx\Bigg\}.
\end{aligned}\tag{2.13}
$$

Noting the "time reversibility" of system (2.1), similar to (2.13), we have

$$
\begin{aligned}
\max_{1\leq k\leq K-1}\mathcal{E}_h^k \leq C\Bigg\{&\mathcal{E}_h^{K-1}+\left[h\sum_{k=1}^{K-1}\Big(|f^k|_{L^2(\Omega)}+|g^k|_{H_0^1(\Omega)}\Big)\right]^2\\
&+h\sum_{k=1}^{K-1}\int_\Omega\left|f^k\frac{g^{k+1}+g^k}{2}\right|dx\Bigg\}.
\end{aligned}\tag{2.14}
$$

Finally, combining (2.13) and (2.14), we end up with the desired estimate (2.3). □

Next, we claim that the solution of system (1.10) can be expressed explicitly by means of Fourier series. Indeed, we have

Lemma 2.2. *Assume* $\varphi^K=\sum_{j=1}^{\infty}(a_j+hb_j)\Phi_j$ *and* $\varphi^{K-1}=\sum_{j=1}^{\infty}a_j\Phi_j$ *(or, equivalently,* $\varphi_0^h=\sum_{j=1}^{\infty}a_j\Phi_j$ *and* $\varphi_1^h=\sum_{j=1}^{\infty}b_j\Phi_j$*). Then the solution of system (1.10) is* *given by*

$$\begin{aligned}\varphi^k = \sum_{j=1}^{\infty} \Big\{ & e^{i\omega_j (K-k-1)} \frac{(e^{i\omega_j}-1)a_j - hb_j}{2i\sin\omega_j} \\ & + e^{-i\omega_j (K-k-1)} \frac{(1-e^{-i\omega_j})a_j + hb_j}{2i\sin\omega_j} \Big\} \Phi_j,\end{aligned} \tag{2.15}$$

where

$$\omega_j = \arccos \frac{1}{1+h^2\mu_j^2/2}. \tag{2.16}$$

Remark 2. i) From (2.15), it is easy to see that, if for some $j_0 \in \mathbb{N}$, the data φ^K and φ^{K-1} belong to $\mathrm{span}\{\Phi_j \mid j \leq j_0\}$, then the same is true for φ^k for all $1 \leq k \leq K$.

ii) From (2.15), one deduces also that, if a_j and b_j are chosen so that $(e^{-i\omega_j} - 1)a_j = hb_j$ (*resp.* $(e^{i\omega_j} - 1)a_j = hb_j$) for $j = 1, 2, \cdots$, then

$$\varphi^k = \sum_{j=1}^{\infty} a_j e^{i\omega_j (K-k-1)} \Phi_j \quad (\textit{resp.} \quad \varphi^k = \sum_{j=1}^{\infty} a_j e^{-i\omega_j (K-k-1)} \Phi_j).$$

Proof of Lemma 2.2: By Lemma 2.1, it suffices to find a solution φ^k of the form

$$\varphi^k = \sum_{j=1}^{\infty} r_j^k \Phi_j \tag{2.17}$$

such that, for $j = 1, 2, \cdots$,

$$\frac{r_j^{k+1} + r_j^{k-1} - 2r_j^k}{h^2} + \mu_j^2 \frac{r_j^{k+1} + r_j^{k-1}}{2} = 0, \qquad k = 1, 2, \cdots, K-1, \tag{2.18}$$

and

$$r_j^K = a_j + hb_j, \qquad r_j^{K-1} = a_j. \tag{2.19}$$

The characteristic polynomial of (2.18) (which is a difference equation) reads

$$p(\lambda) \triangleq \frac{\lambda^2 + 1 - 2\lambda}{h^2} + \mu_j^2 \frac{\lambda^2 + 1}{2}.$$

The roots n_j and m_j of $p(\lambda)$ are as follows

$$n_j = \frac{1 + ih\mu_j\sqrt{1 + h^2\mu_j^2/4}}{1 + h^2\mu_j^2/2}, \qquad m_j = \frac{1 - ih\mu_j\sqrt{1 + h^2\mu_j^2/4}}{1 + h^2\mu_j^2/2}. \tag{2.20}$$

Therefore, the r_j^k's satisfy

$$r_j^{k+1} - n_j r_j^k = m_j (r_j^k - n_j r_j^{k-1}).$$

By induction, the unique r_j^k satisfying (2.18)–(2.19) is given by (recalling (2.19))

$$\begin{aligned} r_j^k &= \frac{(n_j)^{K-k} - (m_j)^{K-k}}{n_j - m_j} r_j^{K-1} - \frac{(n_j)^{K-k-1} - (m_j)^{K-k-1}}{n_j - m_j} r_j^K \\ &= \frac{1}{n_j - m_j} \{ (n_j)^{K-k-1} [(n_j - 1)a_j - hb_j] \\ &\qquad - (m_j)^{K-k-1} [(m_j - 1)a_j - hb_j] \}. \end{aligned} \tag{2.21}$$

Noting the definition of ω_j in (2.16), by (2.20), it follows

$$n_j = e^{i\omega_j}, \qquad m_j = e^{-i\omega_j}.$$

Therefore, the r_j^k's given by (2.21) can be re-written as

$$r_j^k = e^{i\omega_j(K-k-1)} \frac{(e^{i\omega_j}-1)a_j - hb_j}{2i\sin\omega_j} + e^{-i\omega_j(K-k-1)} \frac{(1-e^{-i\omega_j})a_j + hb_j}{2i\sin\omega_j}. \tag{2.22}$$

Combining (2.17) and (2.22), we conclude the desired formula (2.15). □

The third one is a classical multiplier identity for the Dirichlet Laplacian:

Lemma 2.3. *Let $\varrho = (\varrho^1, \cdots, \varrho^d) \in C^1(\overline{\Omega};\ \mathbb{R}^d)$. Then, for any $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds*

$$\begin{aligned}\int_\Omega \varrho \cdot \nabla\psi \Delta\psi dx = &\ \frac{1}{2}\left[\int_\Gamma \varrho\cdot\nu \left|\frac{\partial\psi}{\partial\nu}\right|^2 d\Gamma + \int_\Omega \mathrm{div}\varrho |\nabla\psi|^2 dx\right] \\ &- \sum_{i,j=1}^d \int_\Omega \varrho^i_{x_j}\psi_{x_i}\psi_{x_j} dx.\end{aligned} \tag{2.23}$$

Identity (2.23) can be easily proved multiplying $\Delta\psi$ by $\varrho\cdot\nabla\psi$ where $\cdot$ stands for the scalar product in $\mathbb{R}^d$. We refer to [9, identity (1.25)] or to [14, Lemma 3.3] for the details.

Finally, following [9, pp. 8–9], one has

Lemma 2.4. *For any $f \in L^2(\Omega)$ and $g \in H_0^1(\Omega)$, it holds*

$$\left|\int_\Omega f\left[(x-x_0)\cdot\nabla g + \frac{d-1}{2}g\right]dx\right| \le \frac{R}{2}\int_\Omega \left(f^2 + |\nabla g|^2\right)dx, \tag{2.24}$$

where R is as in (1.7).

3. Identities via multipliers.

This section is addressed to establish two fundamental identities by means of discrete multipliers. First, we show the following one:

Lemma 3.1. *Let $\varrho = (\varrho^1, \cdots, \varrho^d) \in C^1(\overline{\Omega};\ \mathbb{R}^d)$. Then, for any $h > 0$, any $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$, any $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, and any $\{\theta^k \in H^2(\Omega)\cap H_0^1(\Omega)\}_{k=0,\cdots,K}$ satisfying (2.1), it holds*

$$\frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma \varrho\cdot\nu\left|\frac{\partial}{\partial\nu}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\right|^2 d\Gamma = U + V_1 - V_2 - W, \tag{3.1}$$

where

$$\begin{aligned}U = &\int_\Omega \varrho\cdot\left[\nabla\left(\frac{\theta^K+\theta^{K-2}}{2}\right)\frac{\theta^K-\theta^{K-1}}{h} - \nabla\left(\frac{\theta^2+\theta^0}{2}\right)\frac{\theta^1-\theta^0}{h}\right]dx \\ &+ \int_\Omega \varrho\cdot\left[\nabla\left(\frac{\theta^{K-1}-\theta^{K-2}}{2}\right)\frac{\theta^K-\theta^{K-1}}{h} + \nabla\left(\frac{\theta^2-\theta^1}{2}\right)\frac{\theta^1-\theta^0}{h}\right]dx,\end{aligned} \tag{3.2}$$

$$V_1 = h\sum_{k=1}^{K-1}\int_\Omega \mathrm{div}\varrho\frac{(\theta^{k+1}-\theta^k)(\theta^k-\theta^{k-1})}{2h^2}dx, \tag{3.3}$$

$$\begin{aligned}V_2 = \frac{h}{2}\sum_{k=1}^{K-1}\int_\Omega \Bigg[&\mathrm{div}\varrho\left|\nabla\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\right|^2 \\ &-2\sum_{i,j=1}^d \varrho^i_{x_j}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)_{x_i}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)_{x_j}\Bigg]dx,\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
W = \; & h \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) f^k dx \\
& + \frac{h}{2} \sum_{k=2}^{K-1} \int_\Omega \left(\frac{\theta^{k+1} - \theta^k}{h} + \frac{\theta^{k-1} - \theta^{k-2}}{h} \right) \left(\varrho \cdot \nabla g^k + \mathrm{div} \varrho g^k \right) dx.
\end{aligned} \tag{3.5}
$$

Proof of Lemma 3.1: Multiplying (2.1) by $\varrho \cdot \nabla(\theta^{k+1} + \theta^{k-1})/2$ (which is a discrete version of the multiplier $\varrho \cdot \nabla\theta$ for the wave equation), integrating it in Ω, summing it for $k = 1, \cdots, K-1$, it follows

$$
\begin{aligned}
& h \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \frac{\theta^{k+1} + \theta^{k-1} - 2\theta^k}{h^2} dx \\
& \quad - h \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \Delta \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) dx \\
& = h \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \left(f^k + \frac{g^{k+1} - g^k}{h} \right) dx.
\end{aligned} \tag{3.6}
$$

First, we analyze the first term in the left hand side of (3.6):

$$
\begin{aligned}
& h \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \frac{\theta^{k+1} + \theta^{k-1} - 2\theta^k}{h^2} dx \\
& = \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \left(\frac{\theta^{k+1} - \theta^k}{h} - \frac{\theta^k - \theta^{k-1}}{h} \right) dx \\
& = \int_\Omega \varrho \cdot \left[\sum_{k=2}^{K} \nabla \left(\frac{\theta^k + \theta^{k-2}}{2} \right) \frac{\theta^k - \theta^{k-1}}{h} \right. \\
& \qquad \left. - \sum_{k=1}^{K-1} \nabla \left(\frac{\theta^{k+1} + \theta^{k-1}}{2} \right) \frac{\theta^k - \theta^{k-1}}{h} \right] dx \\
& = - \sum_{k=2}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} - \theta^k + \theta^{k-1} - \theta^{k-2}}{2} \right) \frac{\theta^k - \theta^{k-1}}{h} dx \\
& \quad + \int_\Omega \varrho \cdot \left[\nabla \left(\frac{\theta^K + \theta^{K-2}}{2} \right) \frac{\theta^K - \theta^{K-1}}{h} - \nabla \left(\frac{\theta^2 + \theta^0}{2} \right) \frac{\theta^1 - \theta^0}{h} \right] dx.
\end{aligned} \tag{3.7}
$$

However,

$$
\begin{aligned}
& - \sum_{k=2}^{K-1} \int_\Omega \varrho \cdot \nabla \left(\frac{\theta^{k+1} - \theta^k + \theta^{k-1} - \theta^{k-2}}{2} \right) \frac{\theta^k - \theta^{k-1}}{h} dx \\
& = - \int_\Omega \varrho \cdot \left[\sum_{k=2}^{K-1} \nabla \left(\frac{\theta^{k+1} - \theta^k}{2} \right) \frac{\theta^k - \theta^{k-1}}{h} \right. \\
& \qquad \left. + \sum_{k=1}^{K-2} \nabla \left(\frac{\theta^k - \theta^{k-1}}{2} \right) \frac{\theta^{k+1} - \theta^k}{h} \right] dx \\
& = \int_\Omega \varrho \cdot \left[\nabla \left(\frac{\theta^{K-1} - \theta^{K-2}}{2} \right) \frac{\theta^K - \theta^{K-1}}{h} + \nabla \left(\frac{\theta^2 - \theta^1}{2} \right) \frac{\theta^1 - \theta^0}{h} \right] dx \\
& \quad - \sum_{k=1}^{K-1} \int_\Omega \varrho \cdot \nabla \left[\frac{(\theta^{k+1} - \theta^k)(\theta^k - \theta^{k-1})}{2h} \right] dx.
\end{aligned} \tag{3.8}
$$

Noting that

$$
\begin{aligned}
&-\sum_{k=1}^{K-1}\int_\Omega \varrho\cdot\nabla\left[\frac{(\theta^{k+1}-\theta^k)(\theta^k-\theta^{k-1})}{2h}\right]dx\\
&=\sum_{k=1}^{K-1}\int_\Omega \mathrm{div}\varrho\frac{(\theta^{k+1}-\theta^k)(\theta^k-\theta^{k-1})}{2h}dx,
\end{aligned}
$$

from (3.7)–(3.8), it follows

$$
h\sum_{k=1}^{K-1}\int_\Omega \varrho\cdot\nabla\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\frac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2}dx=U+V_1, \tag{3.9}
$$

where U and V_1 are defined respectively by (3.2) and (3.3).

Next, we analyze the second term in the left hand side of (3.6). Applying Lemma 2.3 (with ψ replaced by $(\theta^{k+1}+\theta^{k-1})/2$), we find

$$
\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega \varrho\cdot\nabla\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\Delta\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)dx\\
&=\frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma \varrho\cdot\nu\left|\frac{\partial}{\partial\nu}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\right|^2 d\Gamma+V_2,
\end{aligned}
\tag{3.10}
$$

where V_2 is defined by (3.4).

Further, using integration by parts and noting $g^1=g^K=0$, it follows

$$
\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega \varrho\cdot\nabla\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\frac{g^{k+1}-g^k}{h}dx\\
&=-h\sum_{k=1}^{K-1}\int_\Omega \frac{\theta^{k+1}+\theta^{k-1}}{2}\left[\varrho\cdot\nabla\left(\frac{g^{k+1}-g^k}{h}\right)+\mathrm{div}\varrho\frac{g^{k+1}-g^k}{h}\right]dx\\
&=-\sum_{k=2}^{K}\int_\Omega \frac{\theta^k+\theta^{k-2}}{2}\left(\varrho\cdot\nabla g^k+\mathrm{div}\varrho\, g^k\right)dx\\
&\quad+\sum_{k=1}^{K-1}\int_\Omega \frac{\theta^{k+1}+\theta^{k-1}}{2}\left(\varrho\cdot\nabla g^k+\mathrm{div}\varrho\, g^k\right)dx\\
&=\frac{h}{2}\sum_{k=2}^{K-1}\int_\Omega\left(\frac{\theta^{k+1}-\theta^k}{h}+\frac{\theta^{k-1}-\theta^{k-2}}{h}\right)\left(\varrho\cdot\nabla g^k+\mathrm{div}\varrho g^k\right)dx.
\end{aligned}
\tag{3.11}
$$

Finally, by (3.6), (3.9)–(3.11) and recalling the definition of W in (3.5), we conclude the desired identity (3.1). ∎

As we shall see in the next section, Lemma 3.1 is the basis to provide an important hidden regularity property of solutions of system (2.1), and via which the well-posedness of system (1.8) follows. Meanwhile, as a consequence of Lemma 3.1, we now show the following identity for the solutions of (1.10), which will play a crucial role in the proof of Theorem 6.1:

Lemma 3.2. *For any $h > 0$ and any solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.10), it holds*

$$\begin{aligned} &\frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega \left(\left| \frac{\varphi^{k+1}-\varphi^k}{h} \right|^2 + \frac{|\nabla\varphi^{k+1}|^2 + |\nabla\varphi^k|^2}{2} \right) dx + X + Y + Z \\ &= \frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma (x-x_0)\cdot\nu \left| \frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2} \right) \right|^2 d\Gamma, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} X = &\int_\Omega \left[(x-x_0)\cdot\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2} \right) + \frac{d-1}{2}\varphi^K \right] \frac{\varphi^K-\varphi^{K-1}}{h} dx \\ &-\int_\Omega \left[(x-x_0)\cdot\nabla\left(\frac{\varphi^2+\varphi^0}{2} \right) + \frac{d-1}{2}\varphi^0 \right] \frac{\varphi^1-\varphi^0}{h} dx, \end{aligned} \tag{3.13}$$

$$\begin{aligned} Y = &\frac{d}{2}\left[h^2\sum_{k=1}^{K-1}\int_\Omega \Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2} \right) \frac{\varphi^k-\varphi^{k-1}}{h} dx \right. \\ &\qquad \left. -h\int_\Omega \left| \frac{\varphi^K-\varphi^{K-1}}{h} \right|^2 dx \right] \\ &+\int_\Omega (x-x_0)\cdot \left[\nabla\left(\frac{\varphi^{K-1}-\varphi^{K-2}}{2} \right) \frac{\varphi^K-\varphi^{K-1}}{h} \right. \\ &\qquad\qquad \left. +\nabla\left(\frac{\varphi^2-\varphi^1}{2} \right) \frac{\varphi^1-\varphi^0}{h} \right] dx, \end{aligned} \tag{3.14}$$

$$\begin{aligned} Z = &\frac{(d-2)h}{8}\sum_{k=1}^{K-1}\int_\Omega \left| \nabla(\varphi^{k+1}-\varphi^{k-1}) \right|^2 dx \\ &-\frac{(d-1)h}{4}\sum_{k=0}^{K-1}\int_\Omega \left| \nabla\left(\varphi^{k+1}-\varphi^k \right) \right|^2 dx \\ &-\frac{(d-1)h}{4}\int_\Omega \left(\nabla\varphi^K\cdot\nabla\varphi^{K-1} + \nabla\varphi^1\cdot\nabla\varphi^0 \right) dx \\ &+\frac{(d-2)h}{4}\int_\Omega \left(|\nabla\varphi^{K-1}|^2 + |\nabla\varphi^1|^2 \right) dx. \end{aligned} \tag{3.15}$$

Remark 3. Identity (3.12) is a time discrete analogue of the well known identity for the wave equation (1.10) obtained by multipliers, which reads (see [9]):

$$\frac{1}{2}\int_0^T\int_\Omega \left[|\varphi_t|^2 + |\nabla\varphi|^2 \right] dxdt + \mathcal{X} = \frac{1}{2}\int_0^T\int_\Gamma (x-x_0)\cdot\nu \left| \frac{\partial\varphi}{\partial\nu} \right|^2 d\Gamma dt. \tag{3.16}$$

Here,

$$\mathcal{X} = \int_\Omega \left[(x-x_0)\cdot\nabla\varphi + \frac{d-1}{2}\varphi \right] \varphi_t dx \Big|_{t=0}^T. \tag{3.17}$$

There are clear analogies between (3.12) and (3.16). In fact the only major differences are that, in the discrete version (3.12), two extra reminder terms (Y and Z) appear, which are due to the time discretization. It is easy to see, formally, that Y and Z tend to zero as $h \to 0$. But this convergence does not hold uniformly for all solutions. Consequently, these added terms impose the need of using filtering of the high frequencies to obtain observability inequalities and also modify the observability time, as we shall see.

Proof of Lemma 3.2: We use Lemma 3.1 with $\varrho = x - x_0$, $f^k = 0$ ($k = 1, \cdots, K-1$), $g^k = 0$ ($k = 1, \cdots, K$) and $\theta^k = \varphi^k$ ($k = 0, \cdots, K$). Clearly, in this case $W = 0$ (recall (3.5) for W).

For V_1 defined in (3.3) (with θ^k replaced by φ^k), noting $\mathrm{div}\varrho = d$ and using the first equation in (1.10), one has

$$
\begin{aligned}
V_1 &= d\sum_{k=1}^{K-1}\int_\Omega \frac{(\varphi^{k+1}-\varphi^k)(\varphi^k-\varphi^{k-1})}{2h}dx\\
&= d\sum_{k=1}^{K-1}\int_\Omega \Big[(\varphi^{k+1}+\varphi^{k-1}-2\varphi^k)+(\varphi^k-\varphi^{k-1})\Big]\frac{\varphi^k-\varphi^{k-1}}{2h}dx\\
&= \frac{dh}{2}\sum_{k=1}^{K-1}\int_\Omega \left|\frac{\varphi^k-\varphi^{k-1}}{h}\right|^2 dx\\
&\quad +\frac{dh^2}{2}\sum_{k=1}^{K-1}\int_\Omega \left(\frac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2}\right)\frac{\varphi^k-\varphi^{k-1}}{h}dx\\
&= \frac{dh}{2}\left[\sum_{k=0}^{K-1}\int_\Omega \left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2 dx-\int_\Omega\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2dx\right]\\
&\quad +\frac{dh^2}{2}\sum_{k=1}^{K-1}\int_\Omega \Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\frac{\varphi^k-\varphi^{k-1}}{h}dx.
\end{aligned}
\tag{3.18}
$$

For V_2 defined in (3.4), noting $\varrho^i_{x_j} = \delta^i_j$ (the Kronecker delta) and using the elementary identity $(a+b)^2 = 2(a^2+b^2)-(a-b)^2$ for any $a, b \in \mathbb{R}$, it follows

$$
\begin{aligned}
V_2 &= \frac{(d-2)h}{2}\sum_{k=1}^{K-1}\int_\Omega \left|\nabla\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2dx\\
&= \frac{(d-2)h}{2}\sum_{k=1}^{K-1}\int_\Omega \Bigg[\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^{k-1}|^2}{2}\\
&\qquad\qquad -\left|\nabla\left(\frac{\varphi^{k+1}-\varphi^{k-1}}{2}\right)\right|^2\Bigg]dx\\
&= \frac{(d-2)h}{2}\Bigg\{\sum_{k=1}^{K-1}\int_\Omega \frac{|\nabla\varphi^{k+1}|^2}{2}dx+\sum_{k=0}^{K-2}\int_\Omega\frac{|\nabla\varphi^k|^2}{2}dx\\
&\qquad -\sum_{k=1}^{K-1}\int_\Omega\left|\nabla\left(\frac{\varphi^{k+1}-\varphi^{k-1}}{2}\right)\right|^2dx\Bigg\}\\
&= \frac{(d-2)h}{2}\Bigg\{\sum_{k=0}^{K-1}\int_\Omega \frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}dx\\
&\qquad -\int_\Omega\frac{|\nabla\varphi^{K-1}|^2+|\nabla\varphi^1|^2}{2}dx\\
&\qquad -\sum_{k=1}^{K-1}\int_\Omega\left|\nabla\left(\frac{\varphi^{k+1}-\varphi^{k-1}}{2}\right)\right|^2dx\Bigg\}.
\end{aligned}
\tag{3.19}
$$

Now, by (3.1) in Lemma 3.1, recalling the definition of U in (3.2) (with θ^k replaced by φ^k), noting $W = 0$ and (3.19), we conclude that

$$
\begin{aligned}
&\frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega\left(\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2+\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}\right)dx\\
&\quad+\int_\Omega(x-x_0)\cdot\left[\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)\frac{\varphi^K-\varphi^{K-1}}{h}\right.\\
&\qquad\qquad\left.-\nabla\left(\frac{\varphi^2+\varphi^0}{2}\right)\frac{\varphi^1-\varphi^0}{h}\right]dx\\
&\quad+\frac{(d-1)h}{2}\sum_{k=0}^{K-1}\int_\Omega\left(\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2-\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}\right)dx+Y\\
&=\frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma(x-x_0)\cdot\nu\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2d\Gamma\\
&\quad-\frac{(d-2)h}{2}\left[\int_\Omega\frac{|\nabla\varphi^{K-1}|^2+|\nabla\varphi^1|^2}{2}dx\right.\\
&\qquad\left.+\sum_{k=1}^{K-1}\int_\Omega\left|\nabla\left(\frac{\varphi^{k+1}-\varphi^{k-1}}{2}\right)\right|^2dx\right],
\end{aligned}
\tag{3.20}
$$

where Y is defined in (3.14).

On the other hand, multiplying the first equation of (1.10) by φ^k (which is a discrete version of the multiplier φ in the time-continuous setting, that leads to the identity of equipartition of energy), integrating it in Ω, summing it for $k = 1, \cdots, K-1$ and using integration by parts, we obtain:

$$
\begin{aligned}
0 &= h\sum_{k=1}^{K-1}\int_\Omega\left[\frac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2}-\Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right]\varphi^k dx\\
&=\sum_{k=1}^{K-1}\int_\Omega\left[\frac{(\varphi^{k+1}-\varphi^k)\varphi^k}{h}-\frac{(\varphi^k-\varphi^{k-1})\varphi^k}{h}\right]dx\\
&\quad-h\sum_{k=1}^{K-1}\int_\Omega\Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\varphi^k dx\\
&=\sum_{k=1}^{K-1}\int_\Omega\frac{(\varphi^{k+1}-\varphi^k)\varphi^k}{h}dx-\sum_{k=0}^{K-2}\int_\Omega\frac{(\varphi^{k+1}-\varphi^k)\varphi^{k+1}}{h}dx\\
&\quad-h\sum_{k=1}^{K-1}\int_\Omega\Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\varphi^k dx\\
&=-h\sum_{k=0}^{K-1}\int_\Omega\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2dx\\
&\quad+\int_\Omega\left[\frac{(\varphi^K-\varphi^{K-1})\varphi^K}{h}-\frac{(\varphi^1-\varphi^0)\varphi^0}{h}\right]dx\\
&\quad+h\sum_{k=1}^{K-1}\int_\Omega\nabla\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\cdot\nabla\varphi^k dx.
\end{aligned}
\tag{3.21}
$$

However

$$
\begin{aligned}
& h\sum_{k=1}^{K-1}\int_\Omega \nabla\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\cdot\nabla\varphi^k dx \\
&= h\sum_{k=1}^{K-1}\int_\Omega |\nabla\varphi^k|^2 dx \\
&\quad + h\sum_{k=1}^{K-1}\int_\Omega \left[\nabla\left(\frac{\varphi^{k+1}-\varphi^k}{2}\right)-\nabla\left(\frac{\varphi^k-\varphi^{k-1}}{2}\right)\right]\cdot\nabla\varphi^k dx \\
&= h\sum_{k=0}^{K-1}\int_\Omega \frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}dx - \frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega \left|\nabla(\varphi^{k+1}-\varphi^k)\right|^2 dx \\
&\quad -\frac{h}{2}\int_\Omega \left(\nabla\varphi^K\cdot\nabla\varphi^{K-1}+\nabla\varphi^1\cdot\nabla\varphi^0\right)dx.
\end{aligned}
\tag{3.22}
$$

Combining (3.21) and (3.22), we end up with the following equipartition of energy identity for the time semi-discrete system (1.10):

$$
\begin{aligned}
& h\sum_{k=0}^{K-1}\int_\Omega \left(\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2 - \frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}\right)dx \\
&= \int_\Omega \left[\frac{(\varphi^K-\varphi^{K-1})\varphi^K}{h} - \frac{(\varphi^1-\varphi^0)\varphi^0}{h}\right]dx \\
&\quad -\frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega \left|\nabla(\varphi^{k+1}-\varphi^k)\right|^2 dx \\
&\quad -\frac{h}{2}\int_\Omega \left(\nabla\varphi^K\cdot\nabla\varphi^{K-1}+\nabla\varphi^1\cdot\nabla\varphi^0\right)dx.
\end{aligned}
\tag{3.23}
$$

Finally, substituting (3.23) into (3.20) and recalling (3.13) and (3.15) respectively for the definition of X and Z, we conclude the desired identity (3.12). □

4. **Hidden regularity and well-posedness.**

This section is devoted to show a hidden regularity property of solutions of system (2.1) and to establish the well-posedness of system (1.8).

We begin with the following hidden regularity property of solutions of system (2.1) (recall (2.2) for the definition of $\mathcal{E}_h^k$):

Theorem 4.1. *For any time step $h>0$, any $\{f^k\in L^2(\Omega)\}_{k=1,\cdots,K-1}$, any $\{g^k\in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1=g^K=0$ in Ω, and any $\{\theta^k\in H_0^1(\Omega)\}_{k=0,\cdots,K}$ satisfying (2.1), it holds*

$$
\begin{aligned}
& h\sum_{k=1}^{K-1}\int_\Gamma \left|\frac{\partial}{\partial\nu}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\right|^2 d\Gamma \\
&\le C\left\{\min\left(\mathcal{E}_h^0,\mathcal{E}_h^{K-1}\right) + \left[h\sum_{k=1}^{K-1}\left(|f^k|_{L^2(\Omega)}+|g^k|_{H_0^1(\Omega)}\right)\right]^2 \right. \\
&\qquad \left. + h\sum_{k=1}^{K-1}\int_\Omega \left|f^k\frac{g^{k+1}+g^k}{2}\right|dx\right\}.
\end{aligned}
\tag{4.1}
$$

Remark 4. When h tends to zero, the limit of the system (2.1) is

$$\begin{cases} \theta_{tt} - \Delta\theta = f + g_t, & \text{in } (0,T)\times\Omega \\ \theta = 0, & \text{in } (0,T)\times\Gamma. \end{cases} \tag{4.2}$$

Inequality (4.1) is a time discrete analogue of the following boundary estimate of (4.2):

$$\begin{aligned} &\int_0^T \int_\Gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt \\ &\le C \left\{ \min\left(\int_\Omega \left[|\nabla\theta(0)|^2 + |\theta_t(0)|^2 \right] dx, \int_\Omega \left[|\nabla\theta(T)|^2 + |\theta_t(T)|^2 \right] dx \right) \right. \\ &\quad \left. + \left[|f|_{L^1(0,T;L^2(\Omega))} + |g|_{L^1(0,T;H_0^1(\Omega))} \right]^2 + \int_0^T \int_\Omega |fg| dx dt \right\}. \end{aligned}$$

Proof of Theorem 4.1: As in [9], we choose a vector $\varrho \in C^1(\overline{\Omega};\ \mathbb{R}^d)$ so that $\varrho = \nu$ on the boundary Γ. Then, the desired estimate (4.1) follows immediately from Lemma 3.1 and Lemma 2.1. $\square$

We now establish the well-posedness of system (1.8) by means of a discrete version of the classical transposition approach ([9]). For this purpose, for any $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$, and any $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, we consider the following adjoint problem of system (1.8):

$$\begin{cases} \dfrac{\zeta^{k+1} + \zeta^{k-1} - 2\zeta^k}{h^2} - \Delta\left(\dfrac{\zeta^{k+1}+\zeta^{k-1}}{2}\right) = f_k + \dfrac{g^{k+1} - g^k}{h}, \\ \qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1,\cdots,K-1 \\ \zeta^k = 0, \qquad\qquad\qquad\quad \text{on } \Gamma, \quad k = 0,\cdots,K \\ \zeta^K = \zeta^{K-1} = 0, \qquad\quad \text{in } \Omega. \end{cases} \tag{4.3}$$

It is easy to see that (4.3) admits a unique solution $\{\zeta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$. By Theorem 4.1, this solution has the regularity property $\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right) \in L^2(\Gamma)$, for $k = 1,\cdots,K-1$.

In order to give a reasonable definition for the solution of the non-homogenous boundary problem (1.8) in terms of the transposition method, we consider first the case when the control $\{u^k\}_{k=0,\cdots,K}$ and the initial data (y^0, y^1) are sufficiently smooth. The following result holds:

Lemma 4.2. *Assume that* $\{y^k \in H^2(\Omega)\}_{k=0,\cdots,K}$ *satisfies* (1.8). *Then*

$$\begin{aligned} &h\sum_{k=1}^{K-1}\int_\Omega f^k \frac{y^{k+1}+y^{k-1}}{2} dx \\ &\quad -\frac{h}{2}\sum_{k=2}^{K-1}\int_\Omega g^k \left(\frac{y^{k+1}-y^k}{h} + \frac{y^{k-1}-y^{k-2}}{h}\right) dx \\ &= \int_\Omega \zeta^0 \frac{y^1-y^0}{h} dx - \int_\Omega \frac{\zeta^1-\zeta^0}{h} y^0 dx \\ &\quad -h\sum_{k=1}^{K-1}\int_{\Gamma_0} \frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right) u^k d\Gamma_0. \end{aligned} \tag{4.4}$$

Proof of Lemma 4.2: Multiplying both sides of (2.1) by $(y^{k+1}+y^{k-1})/2$, integrating the resulting identity in Ω, summing it for $k=1,\cdots,K-1$, one obtains:

$$\begin{aligned}
&h\sum_{k=1}^{K-1}\Big[\int_\Omega \frac{\zeta^{k+1}+\zeta^{k-1}-2\zeta^k}{h^2}\frac{y^{k+1}+y^{k-1}}{2}dx\\
&\qquad -\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\Delta\left(\frac{y^{k+1}+y^{k-1}}{2}\right)-\int_{\Gamma_0}\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)u^k d\Gamma_0\Big]\\
&=h\sum_{k=1}^{K-1}\int_\Omega\left(f_k+\frac{g^{k+1}-g^k}{h}\right)\frac{y^{k+1}+y^{k-1}}{2}dx.
\end{aligned}\tag{4.5}$$

Recalling $\zeta^K=\zeta^{K-1}=0$ in Ω, it is clear that

$$\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega \frac{\zeta^{k+1}+\zeta^{k-1}-2\zeta^k}{h^2}\frac{y^{k+1}+y^{k-1}}{2}dx\\
&=h\sum_{k=1}^{K-1}\int_\Omega \frac{\zeta^{k+1}+\zeta^{k-1}}{2}\frac{y^{k+1}+y^{k-1}}{h^2}dx-h\sum_{k=1}^{K-1}\int_\Omega\frac{\zeta^k y^{k+1}+\zeta^k y^{k-1}}{h^2}dx\\
&=\int_\Omega\zeta^0\frac{y^1-y^0}{h}dx-\int_\Omega\frac{\zeta^1-\zeta^0}{h}y^0dx\\
&\quad +h\sum_{k=1}^{K-1}\int_\Omega\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\frac{y^{k+1}+y^{k-1}-2y^k}{h^2}dx.
\end{aligned}\tag{4.6}$$

Also, noting $g^1=g^K=0$ in Ω, it holds

$$\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega\frac{g^{k+1}-g^k}{h}\frac{y^{k+1}+y^{k-1}}{2}dx\\
&=-\frac{h}{2}\sum_{k=2}^{K-1}\int_\Omega g^k\left(\frac{y^{k+1}-y^k}{h}+\frac{y^{k-1}-y^{k-2}}{h}\right)dx.
\end{aligned}\tag{4.7}$$

Finally, from (4.5)–(4.7) and noting that $\{y^k\in H^2(\Omega)\}_{k=0,\cdots,K}$ satisfy the first equation in (1.8), the desired identity (4.4) follows. $\square$

Note that (4.4) still makes sense even if the regularity of $\{y^k\}_{k=0,\cdots,K}$ is relaxed as follows

$$\begin{cases}
y^{k+1}+y^{k-1}\in L^2(\Omega), & k=1,\cdots,K-1,\\
\dfrac{y^{k+1}-y^k}{h}+\dfrac{y^{k-1}-y^{k-2}}{h}\in H^{-1}(\Omega), & k=2,\cdots,K-1.
\end{cases}\tag{4.8}$$

This is consistent with the existence result for (1.1) (in terms of the transposition method). Indeed, under the condition $u\in L^2(\Gamma\times(0,T))$ it is well-known that the solution of (1.1) satisfies $y\in C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$. Note that formally, letting $h\to 0$, (4.8) leads to $y(t,\cdot)\in L^2(\Omega)$ and $y_t(t,\cdot)\in H^{-1}(\Omega)$. This observation motivates the definition of solution for system (1.8).

More precisely, set

$$\mathcal{H}=\left\{\{y^k\}_{k=0,\cdots,K}\;\middle|\; y^0,\cdots,y^K \text{ satisfy (4.8)}\right\}.\tag{4.9}$$

We introduce the following

Definition 4.3. We say $\{y^k\}_{k=0,\cdots,K} \in \mathcal{H}$ to be a solution of (1.8), in the sense of transposition, if $y^0 = y_0$, $y^1 = y_0 + hy_1$, and for any $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$, and $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, it holds

$$\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega f^k \frac{y^{k+1}+y^{k-1}}{2}dx - \frac{h}{2}\sum_{k=2}^{K-1}\left\langle g^k, \frac{y^{k+1}-y^k}{h} + \frac{y^{k-1}-y^{k-2}}{h}\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\
&= \left\langle \zeta^0, y_1\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \int_\Omega \frac{\zeta^1-\zeta^0}{h} y_0 dx - h\sum_{k=1}^{K-1}\int_{\Gamma_0}\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)u^k d\Gamma_0,
\end{aligned} \tag{4.10}$$

where $\{\zeta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$ is the unique solution of (4.3).

We now show the following well-posedness result for this system:

Theorem 4.4. *Assume $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $\{u^k \in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$. Then system (1.8) admits one and only one solution $\{y^k\}_{k=0,\cdots,K} \in \mathcal{H}$ in the sense of Definition 4.3. Moreover,*

i) When K is odd, $\left(y^{2\ell}, \frac{y^{2\ell+1}-y^{2\ell}}{h}\right) \in L^2(\Omega)\times H^{-1}(\Omega)$ for $\ell = 0, 1, \cdots, [\frac{K}{2}]$, and

$$\begin{aligned}
&\max_{\ell=0,1,\cdots,[\frac{K}{2}]}\left\|\left(y^{2\ell}, \frac{y^{2\ell+1}-y^{2\ell}}{h}\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \\
&\le C\left(\|(y_0,y_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} + h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\right).
\end{aligned} \tag{4.11}$$

ii) When K is even, $\left(y^{2\ell}, \frac{y^{2\ell-1}-y^{2\ell-2}}{h}\right) \in L^2(\Omega)\times H^{-1}(\Omega)$ for $\ell = 1, \cdots, \frac{K}{2}$, and

$$\begin{aligned}
&\max_{\ell=1,\cdots,\frac{K}{2}}\left\|\left(y^{2\ell}, \frac{y^{2\ell-1}-y^{2\ell-2}}{h}\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \\
&\le C\left(\|(y_0,y_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} + h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\right).
\end{aligned} \tag{4.12}$$

Furthermore, the constant $C > 0$ in the estimates (4.11) and (4.12) is independent of the time-step h.

Proof of Theorem 4.4: The proof is standard, and hence we give only a sketch. First of all, by Lemma 2.1, Theorem 4.1 and using the usual duality argument (e.g. [9]), we conclude that system (1.8) admits a solution $\{y^k\}_{k=0,\cdots,K} \in \mathcal{H}$ in the sense of Definition 4.3, which verifies

$$\begin{aligned}
&\max_{k=1,\cdots,K-1}\left\|y^{k+1}+y^{k-1}\right\|^2_{L^2(\Omega)} + \max_{k=2,\cdots,K-1}\left\|\frac{y^{k+1}-y^k}{h}+\frac{y^{k-1}-y^{k-2}}{h}\right\|^2_{H^{-1}(\Omega)} \\
&\le C\left(\|(y_0,y_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} + h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\right).
\end{aligned} \tag{4.13}$$

Inequality (4.3) implies the uniqueness of the solution of system (1.8). On the other hand, the constant C in this estimate is independent of h.

Next, we show the "regularity" properties (4.11)–(4.12) for solution $\{y^k\}_{k=0,\cdots,K}$. For this purpose, for any $\ell \in \{0, 1, \cdots, [\frac{K}{2}]\}$, we choose the test functions f^k and

g^k in (4.10) as follows:

$$f^k = \begin{cases} (-1)^{(k+3)/2} f^1, & \text{for } k = 1, 3, \cdots, 2\ell - 1 \\ 0, & \text{for } k = 2, 4, \cdots, 2\ell, 2\ell+1, 2\ell+2, \cdots, K-1, \end{cases}$$

where f^1 is arbitrary, and $g^k \equiv 0$ for all $k = 1, \cdots, K$. Now, by Lemma 2.1, Theorem 4.1 and using the usual duality argument again, similar to (4.13), one deduces that $y^{2\ell} - y^0 \in L^2(\Omega)$, and

$$\left\| y^{2\ell} - y^0 \right\|_{L^2(\Omega)}^2 \leq Ch \sum_{k=1}^{K-1} \left\| u^k \right\|_{L^2(\Gamma_0)}^2 ,$$

via which (and noting $y_0 \in L^2(\Omega)$) the boundedness of each of y^2,y^4,...$y^{2\ell}$ in $L^2(\Omega)$ follows (with a bound which is independent of the time step h). Similarly, noting $y_1 \in H^{-1}(\Omega)$, one obtains the other results in (4.11)–(4.12). $\square$

5. **Lack of controllability/observability without filtering.** This section is devoted to prove the following negative controllability/observability result:

Theorem 5.1. *For any given $h > 0$ and any nonempty open subset Γ_0 of Γ, system (1.10) is not observable, and therefore, system (1.8) is not null controllable.*

Proof of Theorem 5.1: We emphasize that, in this proof, h is fixed so that the system under consideration involves only a finite number of time-steps while it is a distributed parameter system (infinite-dimensional one) in space. This is precisely the main reason for the lack of observability results. The proof is divided into two steps.

Step 1. We first show that inequality (1.14) fails for system (1.10) when $\Gamma_0 = \Gamma$. For this, put

$$f_n = \sum_{j=1}^{n} |\mu_j|^{-d/2} \Phi_j.$$

(Recall that d is the dimension of Ω). By Weyl's formula ([7]), $\mu_k \sim C(\Omega) k^{1/d}$ as $k \to \infty$. Therefore,

$$||f_n||_{L^2(\Omega)}^2 = \sum_{j=1}^{n} |\mu_j|^{-d} \to \infty, \qquad \text{as } n \to \infty; \tag{5.1}$$

while, $\{f_n\}_{n \geq 1}$ is bounded in $H^{-s}(\Omega)$ for all $s > 0$.

It is obvious that $f_n \in H^2(\Omega) \cap H_0^1(\Omega)$ for any n. We choose the final data of (1.10) to be $(\varphi_n^K, \varphi_n^{K-1}) = (f_n, 0)$ and denote the corresponding solution by $\{\varphi_n^k\}_{K=0}^K$. Note that $\varphi_n^{K-2}, \cdots, \varphi_n^0$ are inductively determined by the following iterative elliptic systems

$$\varphi_n^{k-1} - \frac{1}{2} h^2 \Delta \varphi_n^{k-1} = 2\varphi_n^k - \varphi_n^{k+1} + \frac{1}{2} h^2 \Delta \varphi_n^{k+1}, \qquad k = K-1, \cdots 1. \tag{5.2}$$

By standard elliptic regularity theory, it is easy to see that $\varphi_n^k \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $n \in \mathbb{N}$.

On the other hand, (5.2) can be rewritten as

$$\varphi_n^{k+1} + \varphi_n^{k-1} - \frac{1}{2} h^2 \Delta \left(\varphi_n^{k+1} + \varphi_n^{k-1} \right) = 2\varphi_n^k, \qquad k = K-1, \cdots 1.$$

This, combined with the standard regularity theory for elliptic equations of second order, gives

$$\sum_{k=1}^{K-1} \left\|\varphi_n^{k+1} + \varphi_n^{k-1}\right\|_{H_0^1(\Omega)} \leq C(h) \sum_{k=1}^{K-1} \left\|\varphi_n^{k}\right\|_{H^{-1}(\Omega)} \leq C(h) \left\|f_n\right\|_{H^{-1}(\Omega)}. \tag{5.3}$$

One can also re-write (5.2) as

$$\varphi_n^{k+1} + \varphi_n^{k-1} = \frac{2}{h^2}(-\Delta)^{-1}\Big(2\varphi_n^k - \big(\varphi_n^{k+1} + \varphi_n^{k-1}\big)\Big), \qquad k = K-1, \cdots 1.$$

Therefore, using again the standard elliptic regularity theory, we conclude that for any $\tau \leq 2$, it holds

$$\begin{aligned}\sum_{k=1}^{K-1} ||\varphi_n^{k+1} + \varphi_n^{k-1}||_{H^\tau(\Omega)} &\leq C(h)||2\varphi_n^k - \big(\varphi_n^{k+1} + \varphi_n^{k-1}\big)||_{H^{\tau-2}(\Omega)} \\ &\leq C(h)(||f_n||_{H^{\tau-2}(\Omega)} + ||f_n||_{H^{-1}(\Omega)}).\end{aligned} \tag{5.4}$$

Hence, for any given $h > 0$ and $3/2 < \tau < 2$, using trace theorem, it follows from (5.4) that

$$\begin{aligned}h \sum_{k=1}^{K-1} \int_\Gamma \left| \frac{\partial}{\partial \nu} \left(\frac{\varphi_n^{k+1} + \varphi_n^{k-1}}{2} \right) \right|^2 d\Gamma \leq\ & C(h) \sum_{k=1}^{K-1} ||\varphi_n^{k+1} + \varphi_n^{k-1}||^2_{H^\tau(\Omega)} \\ &\leq C(h)||f_n||^2_{H^{\tau-2}(\Omega)}.\end{aligned} \tag{5.5}$$

The energy E_h^0 of (1.10) with data $(\varphi_n^K, \varphi_n^{K-1}) = (f_n, 0)$ reads

$$E_h^0 = E_h^{K-1} = \frac{1}{2} \int_\Omega \left(\left| \frac{f_n}{h} \right|^2 + \frac{|\nabla f_n|^2}{2} \right) dx \geq \frac{1}{2h^2} \left\|f_n\right\|^2_{L^2(\Omega)}. \tag{5.6}$$

Now, recalling that $\{f_n\}_{n\geq 1}$ is bounded in $H^{-s}(\Omega)$ for all $s > 0$, taking (5.1), (5.5) and (5.6) into account, we obtain that

$$\lim_{n\to\infty} \frac{E_h^0}{h \displaystyle\sum_{k=1}^{K-1} \int_\Gamma \left| \frac{\partial}{\partial \nu} \left(\frac{\varphi_n^{k+1} + \varphi_n^{k-1}}{2} \right) \right|^2 d\Gamma} = \infty. \tag{5.7}$$

Thus, (1.14) fails. Consequently, system (1.10) is not observable (even when $\Gamma_0 = \Gamma$).

Step 2. We now show that system (1.8) is not null controllable by means of a contradiction argument. Assume that for any $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there is a control $\{u^k \in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$ such that the solution $\{y^k\}_{k=0,\cdots,K}$ of (1.8) satisfies the null controllability property (1.9). The control is not unique, and therefore, we choose the one of minimal norm. By the closed graph theorem, we deduce that

$$\sum_{k=1}^{K-1} |u^k|_{L^2(\Gamma_0)} \leq C|(y_0, y_1)|_{L^2(\Omega)\times H^{-1}(\Omega)}. \tag{5.8}$$

Multiplying the first equation of system (1.8) by $(\varphi^{k+1} + \varphi^{k-1})/2$, integrating it in Ω, summing it for $k = 1, \cdots, K-1$ and noting Theorem 4.4, it follows

$$\left\langle \varphi^0, y_1 \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \frac{\varphi^1 - \varphi^0}{h} y_0 dx = h \sum_{k=1}^{K-1} \int_{\Gamma_0} \frac{\partial}{\partial \nu} \left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) u^k d\Gamma_0. \tag{5.9}$$

Combining (5.8) and (5.9), one deduces easily that inequality (1.14) holds. From Step 1, this is a contradiction. □

6. **Uniform observability under filtering.** In this section, we shall establish uniform observability estimates for system (1.10) (with respect to the time step h) after filtering the spurious high frequency components.

6.1. **Statement of the uniform observability result.** As mentioned in Introduction, due to the negative results stated in Theorem 5.1, we need to introduce suitable filtering spaces in which the solutions of system (1.10) evolve. Recalling the definition of Φ_j and μ_j, for any $s > 0$, define

$$\mathcal{C}_{1,s} = \{f(x) \mid f(x) = \sum_{\mu_j^2 < s} a_j \Phi_j(x), \;\; a_j \in \mathbb{C}\} \subset H_0^1(\Omega), \tag{6.1}$$

$$\mathcal{C}_{0,s} = \{g(x) \mid g(x) = \sum_{\mu_j^2 < s} b_j \Phi_j(x), \;\; b_j \in \mathbb{C}\} \subset L^2(\Omega), \tag{6.2}$$

and

$$\mathcal{C}_{-1,s} = \{z(x) \mid z(x) = \sum_{\mu_j^2 < s} c_j \Phi_j(x), \;\; c_j \in \mathbb{C}\} \subset H^{-1}(\Omega), \tag{6.3}$$

subspaces of $H_0^1(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$, respectively, with the induced topologies. It is clear that $\bigcup_{k=1}^{\infty} \mathcal{C}_{1,k}$ is dense in $H_0^1(\Omega)$, and the same can be said for $\bigcup_{k=1}^{\infty} \mathcal{C}_{0,k}$ in $L^2(\Omega)$ and $\bigcup_{k=1}^{\infty} \mathcal{C}_{-1,k}$ in $H^{-1}(\Omega)$. Denote by $\pi_{1,s}$, $\pi_{0,s}$ and $\pi_{-1,s}$ the projection operators from $H_0^1(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$ to $\mathcal{C}_{1,s}$, $\mathcal{C}_{0,s}$ and $\mathcal{C}_{-1,s}$, respectively. The space $\mathcal{C}_{-1,s}$ and the projector $\pi_{-1,s}$ will not be used in this section but we will need them later.

Our uniform observability result for system (1.10) is stated as follows:

Theorem 6.1. *Let $T > 2R$. Then there exist two constants $h_0 > 0$ and $\delta > 0$, depending only on d, T and R, such that for all $(\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$, the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.10) satisfies*

$$E_h^0 \le Ch \sum_{k=1}^{K-1} \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} \left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) \right|^2 d\Gamma_0, \tag{6.4}$$

for all $h \in (0, h_0]$.

Remark 5. In the proof we see that δ depends only on d, T and R. In particular it indicates that δ decreases as T decreases. This is natural since, as T decreases, less and less time-step iterations are involved in system (1.10) and, consequently, less Fourier components of the solutions may be observed. Further, δ tends to zero as T tends to $2R$. This is natural too since our proof of (6.4) is based on the method of multipliers which works at the continuous level for all $T > 2R$ but that, at the time-discrete level, due to the added dispersive effects, may hardly work when T is very close to $2R$, except if the filtering is strong enough.

Remark 6. In view of the hidden regularity result of Theorem 4.1, the right hand side term of (6.4) is finite.

Remark 7. In the observability result of Theorem 6.1, the filtering parameter has been taken to be of the order of h^{-2}. This is the optimal order for the filtering parameter since for higher frequencies there are solutions for which the observability constant blows-up, as Theorem 7.1 in the next section shows. However, as we shall see, the necessity of the filtering parameter δ to be small is not completely justified. In fact, our analysis of the velocity of propagation of solutions in section 7 supports that, whatever $\delta > 0$ is, observability could be expected to hold for large enough values of time T.

6.2. **A technical result.** As mentioned before, the key point in the proof of Theorem 6.1 is Lemma 3.2. We need to estimate first the term X and the error terms Y and Z in (3.12).

The following lemma provides an estimate on the term $X + Y + Z$:

Lemma 6.2. *Let K be an integer, $s > 0$ and $T > 0$. Then for any $(\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,s} \times \mathcal{C}_{0,s}$, the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.10) satisfies*

$$X + Z + Y \geq -\Big[2R + a_1 h + 3R\sqrt{s}h + T\Big(\frac{d}{2}\sqrt{s}h + a_2 sh^2\Big)\Big] E_h^0, \tag{6.5}$$

where

$$a_1 = 3d - 2 + \max\left(\frac{d-1}{2}, 2\right), \qquad a_2 = \min\left(1, (2-d)^+\right) + \frac{d-1}{2}. \tag{6.6}$$

Proof: The proof is divided in three steps, in which we estimate X, Y and Z, separately.

Note that, in view of the Fourier decomposition of solutions (see (2.15) in Remark 2), the filtering introduced in the initial data is kept for all discrete time-steps k so that

$$\int_\Omega |\nabla \varphi^k|^2 dx \leq s \int_\Omega |\varphi^k|^2 dx$$

for all $k = 0, \cdots, K$ and all solutions under consideration. This inequality will be used throughout the proof.

Step 1. First, let us consider X. We have

$$\begin{aligned}
&\left|\frac{d-1}{2}\int_\Omega \frac{\varphi^K - \varphi^{K-2}}{2}\frac{\varphi^K - \varphi^{K-1}}{h}dx\right| \\
&\leq \frac{(d-1)h}{2}\left(\int_\Omega \left|\frac{\varphi^K - \varphi^{K-2}}{2h}\right|^2 dx \int_\Omega \left|\frac{\varphi^K - \varphi^{K-1}}{h}\right|^2 dx\right)^{1/2} \\
&\leq \frac{(d-1)h}{2}\left(\int_\Omega \frac{\left|\varphi^K - \varphi^{K-1}\right|^2 + \left|\varphi^{K-1} - \varphi^{K-2}\right|^2}{2h^2}dx\right)^{1/2} \\
&\qquad \times \left(\int_\Omega \left|\frac{\varphi^K - \varphi^{K-1}}{h}\right|^2 dx\right)^{1/2} \\
&\leq (d-1)hE_h^0.
\end{aligned} \tag{6.7}$$

Further, applying Lemma 2.4 (with f and g replaced by $(\varphi^K-\varphi^{K-1})/h$ and $(\varphi^K+\varphi^{K-2})/2$), recalling (1.12) for the definition of E_h^k and using (1.13), it follows

$$\begin{aligned}
&\left|\int_\Omega \left[(x-x_0)\cdot\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)+\frac{d-1}{2}\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)\right]\frac{\varphi^K-\varphi^{K-1}}{h}dx\right|\\
&\leq \frac{R}{2}\int_\Omega\left[\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2+\left|\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)\right|^2\right]dx\\
&\leq \frac{R}{2}\int_\Omega\left(\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2+\frac{|\nabla\varphi^K|^2+|\nabla\varphi^{K-1}|^2}{2}+\frac{|\nabla\varphi^{K-2}|^2-|\nabla\varphi^{K-1}|^2}{2}\right)dx\\
&= RE_h^0+M,
\end{aligned} \tag{6.8}$$

where

$$M=\frac{Rh}{4}\int_\Omega \nabla\left(\varphi^{K-2}+\varphi^{K-1}\right)\cdot\nabla\left(\frac{\varphi^{K-2}-\varphi^{K-1}}{h}\right)dx.$$

Further, we estimate

$$\begin{aligned}
M&\leq \frac{Rh}{4}\left\{\int_\Omega\left|\nabla\left(\varphi^{K-2}+\varphi^{K-1}\right)\right|^2dx\int_\Omega\left|\nabla\left(\frac{\varphi^{K-2}-\varphi^{K-1}}{h}\right)\right|^2dx\right\}^{1/2}\\
&\leq \frac{Rh}{2}\sqrt{s}\left\{\int_\Omega\frac{\left|\nabla\varphi^{K-2}\right|^2+\left|\nabla\varphi^{K-1}\right|^2}{2}dx\right\}^{1/2}\left\{\int_\Omega\left|\frac{\varphi^{K-2}-\varphi^{K-1}}{h}\right|^2dx\right\}^{1/2}\\
&\leq \frac{h}{2}\sqrt{s}RE_h^0.
\end{aligned} \tag{6.9}$$

Combining (6.7), (6.8) and (6.9) we obtain

$$\begin{aligned}
&\left|\int_\Omega \left[(x-x_0)\cdot\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)+\frac{d-1}{2}\varphi^K\right]\frac{\varphi^K-\varphi^{K-1}}{h}dx\right|\\
&=\left|\int_\Omega \left[(x-x_0)\cdot\nabla\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)+\frac{d-1}{2}\left(\frac{\varphi^K+\varphi^{K-2}}{2}\right)\right]\frac{\varphi^K-\varphi^{K-1}}{h}dx\right.\\
&\qquad\qquad\qquad\left.+\frac{d-1}{2}\int_\Omega\frac{\varphi^K-\varphi^{K-2}}{2}\frac{\varphi^K-\varphi^{K-1}}{h}dx\right|\\
&\leq\left[R+(d-1)h+\frac{hR}{2}\sqrt{s}\right]E_h^0.
\end{aligned} \tag{6.10}$$

Similarly,

$$\begin{aligned}
&\left|-\int_\Omega\left[(x-x_0)\cdot\nabla\left(\frac{\varphi^2+\varphi^0}{2}\right)+\frac{d-1}{2}\varphi^0\right]\frac{\varphi^1-\varphi^0}{h}dx\right|\\
&\leq\left[R+(d-1)h+\frac{hR}{2}\sqrt{s}\right]E_h^0.
\end{aligned} \tag{6.11}$$

Therefore, by (6.10)–(6.11) and recalling the definition of X in (3.13), we conclude that

$$|X|\leq\left[2R+2(d-1)h+Rh\sqrt{s}\,\right]E_h^0. \tag{6.12}$$

Step 2. Next, let us consider Y. Using (1.13) and noting (2.15) in Remark 2, we obtain

$$\begin{aligned}
&\left|\int_\Omega (x-x_0)\cdot\nabla\left(\frac{\varphi^{K-1}-\varphi^{K-2}}{2}\right)\frac{\varphi^K-\varphi^{K-1}}{h}dx\right| \\
&\le \frac{Rh}{2}\left[\int_\Omega\left|\nabla\left(\frac{\varphi^{K-1}-\varphi^{K-2}}{h}\right)\right|^2dx\int_\Omega\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2dx\right]^{1/2} \\
&\le \frac{Rh\sqrt{s}}{2}\left(\int_\Omega\left|\frac{\varphi^{K-1}-\varphi^{K-2}}{h}\right|^2dx\right)^{1/2}\left(\int_\Omega\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2dx\right)^{1/2} \\
&\le Rh\sqrt{s}\,E_h^0.
\end{aligned} \tag{6.13}$$

Similarly,

$$\left|\int_\Omega (x-x_0)\cdot\nabla\left(\frac{\varphi^2-\varphi^1}{2}\right)\frac{\varphi^1-\varphi^0}{h}dx\right|\le Rh\sqrt{s}\,E_h^0. \tag{6.14}$$

Further,

$$\begin{aligned}
&\left|\frac{dh^2}{2}\sum_{k=1}^{K-1}\int_\Omega\Delta\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\frac{\varphi^k-\varphi^{k-1}}{h}dx\right| \\
&=\left|\frac{dh^2}{2}\sum_{k=1}^{K-1}\int_\Omega\nabla\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\cdot\nabla\left(\frac{\varphi^k-\varphi^{k-1}}{h}\right)dx\right| \\
&\le\frac{dh^2}{2}\sum_{k=1}^{K-1}\left[\int_\Omega\left|\nabla\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2dx\int_\Omega\left|\nabla\left(\frac{\varphi^k-\varphi^{k-1}}{h}\right)\right|^2dx\right]^{1/2} \\
&\le\frac{dh\sqrt{s}\,T}{2}E_h^0.
\end{aligned} \tag{6.15}$$

Also,

$$\left|-\frac{dh}{2}\int_\Omega\left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2dx\right|\le dhE_h^0. \tag{6.16}$$

By (6.13)–(6.16) and recalling the definition of Y in (3.14), we conclude that

$$|Y|\le h\left[d\left(\frac{\sqrt{s}\,T}{2}+1\right)+2R\sqrt{s}\right]E_h^0. \tag{6.17}$$

Step 3. Finally, we consider Z. It follows

$$\begin{aligned}
&h\sum_{k=1}^{K-1}\int_\Omega\left|\nabla(\varphi^{k+1}-\varphi^{k-1})\right|^2dx\le sh^3\sum_{k=1}^{K-1}\int_\Omega\left|\frac{\varphi^{k+1}-\varphi^{k-1}}{h}\right|^2dx \\
&\le 2sh^3\sum_{k=1}^{K-1}\int_\Omega\left(\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2+\left|\frac{\varphi^k-\varphi^{k-1}}{h}\right|^2\right)dx\le 8sh^2TE_h^0.
\end{aligned} \tag{6.18}$$

Since the first term in Z is nonnegative whenever $d\ge 2$, we get from (6.18) that

$$\frac{(d-2)h}{8}\sum_{k=1}^{K-1}\int_\Omega\left|\nabla(\varphi^{k+1}-\varphi^{k-1})\right|^2dx\ge\begin{cases}-sh^2TE_h^0, & d=1\\ 0, & d\ge 2.\end{cases} \tag{6.19}$$

Similarly,

$$h\sum_{k=0}^{K-1}\int_\Omega\left|\nabla(\varphi^{k+1}-\varphi^k)\right|^2dx\le sh^3\sum_{k=0}^{K-1}\int_\Omega\left|\frac{\varphi^{k+1}-\varphi^k}{h}\right|^2dx\le 2sh^2TE_h^0. \tag{6.20}$$

Further,

$$\begin{aligned}
&-\frac{(d-1)h}{4}\int_\Omega \nabla\varphi^K\cdot\nabla\varphi^{K-1}dx+\frac{(d-2)h}{4}\int_\Omega|\nabla\varphi^{K-1}|^2dx\\
&\geq -\frac{(d-1)h}{16}\int_\Omega|\nabla\varphi^K|^2dx+\left[\frac{(d-2)h}{4}-\frac{(d-1)h}{4}\right]\int_\Omega|\nabla\varphi^{K-1}|^2dx\\
&\geq -h\max\left(\frac{d-1}{4},1\right)E_h^0.
\end{aligned} \tag{6.21}$$

Similarly,

$$-\frac{(d-1)h}{4}\int_\Omega \nabla\varphi^1\cdot\nabla\varphi^0dx+\frac{(d-2)h}{4}\int_\Omega|\nabla\varphi^1|^2dx\geq -h\max\left(\frac{d-1}{4},1\right)E_h^0. \tag{6.22}$$

By (6.19)–(6.22), recalling the definition of Z in (3.15), we conclude that

$$Z\geq -h\left\{\left[\min(1,(2-d)^+)+\frac{d-1}{2}\right]shT+\max\left(\frac{d-1}{2},2\right)\right\}E_h^0. \tag{6.23}$$

Now, combining (6.12), (6.17) and (6.23), we arrive at the desired estimate (6.5). □

6.3. **Proof of the uniform observability result.** We are now in a position to prove the uniform observability result, i.e., Theorem 6.1.

Proof of Theorem 6.1: Combining (3.12) in Lemma 3.2 and (6.5) in Lemma 6.2, recalling the definition of Γ_0 in (1.7), we deduce that

$$\begin{aligned}
&\left\{T\left(1-\frac{d}{2}\sqrt{s}h-a_2sh^2\right)-\left[2R+a_1h+3R\sqrt{s}h\right]\right\}E_h^0\\
&\leq \frac{R}{2}h\sum_{k=1}^{K-1}\int_{\Gamma_0}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2d\Gamma_0.
\end{aligned}$$

For this inequality to yield an estimate on E_h^0 we need to choose $s=\delta h^{-2}$ with h small enough such that

$$a_2\delta+\frac{d}{2}\sqrt{\delta}<1,$$

or, more precisely,

$$0<\sqrt{\delta}<\frac{4}{\sqrt{d^2+16a_2}+d}. \tag{6.24}$$

Once this is done, for $h\in(0,h_0)$, T has to be chosen such that

$$T>\frac{2R+a_1h_0+3R\sqrt{\delta}}{1-\frac{d}{2}\sqrt{\delta}-a_2\delta}\geq 2R. \tag{6.25}$$

Hence, (6.4) holds for $h\in(0,h_0]$.

Conversely, for any $T>2R$ one can always choose h_0 and δ small enough so that (6.24) and (6.25) hold and guaranteeing the uniform observability inequality. □

7. **Optimality of the filtering parameter.** This section is addressed to analyze the optimality of the filtering mechanism introduced in Theorem 6.1.

7.1. **Optimality of the order of the filtering parameter.** We first show the following result, which indicates that the order h^{-2} of the filtering parameter that we have chosen in Theorem 6.1 is optimal.

Theorem 7.1. *Assume Γ_* is any nonempty open subset of Γ. Then, for any given $a > 2$, it follows that*

$$\lim_{h\to 0} \sup_{(\varphi_0^h,\varphi_1^h)\in \mathcal{C}_{1,h^{-a}}\times\mathcal{C}_{0,h^{-a}}} \frac{E_h^0}{h\sum_{k=1}^{K-1}\int_{\Gamma_*}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\right|^2 d\Gamma_*} = \infty. \tag{7.1}$$

Proof of Theorem 7.1: Recall that $\{\Phi_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ denotes the orthonormal basis of $L^2(\Omega)$ constituted by the eigenvectors of the Dirichlet Laplacian and $\{\mu_j^2\}_{j\geq 1}$ the corresponding eigenvalues. Since $\mu_j \to +\infty$ as $j\to\infty$, one can choose a $j_0 = j_0(h)$ so that $h^{-a/2}/2 \leq \mu_{j_0} \leq h^{-a/2}$. In view of the fact that $a > 2$, this leads to

$$\mu_{j_0} h \to \infty, \qquad \text{as } h\to 0. \tag{7.2}$$

Further, choose

$$\varphi_0^h = \frac{1}{\mu_{j_0}}\Phi_{j_0}, \qquad \varphi_1^h = \frac{e^{-i\omega_{j_0}}-1}{\mu_{j_0}h}\Phi_{j_0}, \tag{7.3}$$

where ω_{j_0} is defined by (2.16). One deduces that $(\varphi_0^h,\varphi_1^h)\in \mathcal{C}_{1,h^{-a}}\times\mathcal{C}_{0,h^{-a}}$. Noting the special choice of initial data in (7.3), by Lemma 2.2 (see also Remark 2 ii)), the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.10) is given by

$$\varphi^k = \frac{1}{\mu_{j_0}} e^{i\omega_{j_0}(K-k-1)}\Phi_{j_0}, \qquad k = 0,\cdots,K. \tag{7.4}$$

Using (2.16), it follows

$$\cos(\omega_{j_0}) = \frac{2}{2+(\mu_{j_0}h)^2}, \qquad \sin(\omega_{j_0}) = \frac{\mu_{j_0}h\sqrt{4+(\mu_{j_0}h)^2}}{2+(\mu_{j_0}h)^2}. \tag{7.5}$$

Recalling the exact form of E_h^0 in (1.12), combining (7.4) and (7.5), we compute

$$\begin{aligned} E_h^0 = E_h^{K-1} &= \frac{1}{2}\int_\Omega\left(\frac{|\nabla\varphi^K|^2+|\nabla\varphi^{K-1}|^2}{2} + \left|\frac{\varphi^K-\varphi^{K-1}}{h}\right|^2\right)dx \\ &= \frac{1}{2}\left(1+\left|\frac{e^{-i\omega_{j_0}}-1}{\mu_{j_0}h}\right|^2\right) = \frac{1}{2}\left(1+\left|\frac{\cos(\omega_{j_0})-1-i\sin(\omega_{j_0})}{\mu_{j_0}h}\right|^2\right) \\ &= \frac{4+(\mu_{j_0}h)^2}{2[2+(\mu_{j_0}h)^2]}. \end{aligned} \tag{7.6}$$

On the other hand, via (7.4) and (7.5), one has

$$\begin{aligned} &\int_{\Gamma_*}\left|\frac{\partial}{\partial\nu}\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\right|^2 d\Gamma_* \\ &\leq \left|\frac{e^{i\omega_{j_0}}+e^{-i\omega_{j_0}}}{2\mu_{j_0}}\right|^2\int_{\Gamma_*}\left|\frac{\partial\Phi_{j_0}}{\partial\nu}\right|^2 d\Gamma_* \leq \frac{\cos^2(\omega_{j_0})}{\mu_{j_0}^2}\int_\Gamma\left|\frac{\partial\Phi_{j_0}}{\partial\nu}\right|^2 d\Gamma. \end{aligned} \tag{7.7}$$

We claim that

$$\int_\Gamma\left|\frac{\partial\Phi_{j_0}}{\partial\nu}\right|^2 d\Gamma \leq C\mu_{j_0}^2. \tag{7.8}$$

Indeed, since $\Gamma \in C^2$, one can find a $\varrho_0 = (\varrho_0^1, \cdots, \varrho_0^d) \in C^1(\overline{\Omega}; \mathbb{R}^d)$ such that $\varrho_0 = \nu$ on Γ ([9]). Applying Lemma 2.3 with $\varrho = \varrho_0$ and $\psi = \Phi_{j_0}$, we get

$$\begin{aligned}&\int_\Omega \varrho_0 \cdot \nabla \Phi_{j_0} \Delta \Phi_{j_0} dx \\ &= \frac{1}{2}\left[\int_\Gamma \left|\frac{\partial \Phi_{j_0}}{\partial \nu}\right|^2 d\Gamma + \int_\Omega \mathrm{div}\varrho_0 |\nabla \Phi_{j_0}|^2 dx\right] - \sum_{i,j=1}^d \int_\Omega \partial_{x_j} \varrho_0^i \partial_{x_i} \Phi_{j_0} \partial_{x_i} \Phi_{j_0} dx.\end{aligned}$$

Recall that $\Delta \Phi_{j_0} = -\mu_{j_0}^2 \Phi_{j_0}$ in Ω. Hence, (7.8) follows from

$$\int_\Omega \varrho_0 \cdot \nabla \Phi_{j_0} \Delta \Phi_{j_0} dx = -\mu_{j_0}^2 \int_\Omega \Phi_{j_0} \varrho_0 \cdot \nabla \Phi_{j_0} dx = \frac{1}{2}\mu_{j_0}^2 \int_\Omega \mathrm{div}\varrho_0 |\Phi_{j_0}|^2 dx \le C\mu_{j_0}^2,$$

and

$$\sum_{i,j=1}^d \int_\Omega \partial_{x_j} \varrho_0^i \partial_{x_i} \Phi_{j_0} \partial_{x_i} \Phi_{j_0} dx - \frac{1}{2}\int_\Omega \mathrm{div}\varrho_0 |\nabla \Phi_{j_0}|^2 dx \le C \int_\Omega |\nabla \Phi_{j_0}|^2 dx \le C\mu_{j_0}^2.$$

Combining (7.7) and (7.8), we find

$$h \sum_{k=1}^{K-1} \int_{\Gamma_*} \left|\frac{\partial}{\partial \nu}\left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2}\right)\right|^2 d\Gamma_* \le C \cos^2(\omega_{j_0}). \tag{7.9}$$

Finally, combining (7.6) and (7.9), and noting (7.2) and (7.5), it follows

$$\begin{aligned}&\sup_{(\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,h^{-a}} \times \mathcal{C}_{0,h^{-a}}} \frac{E_h^0}{h \sum_{k=1}^{K-1} \int_{\Gamma_*} \left|\frac{\partial}{\partial \nu}\left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2}\right)\right|^2 d\Gamma_*} \\ &\ge \frac{4 + (\mu_{j_0} h)^2}{2[2 + (\mu_{j_0} h)^2]} \frac{[2 + (\mu_{j_0} h)^2]^2}{4C} = \frac{[4 + (\mu_{j_0} h)^2][2 + (\mu_{j_0} h)^2]}{8C} \to \infty \qquad \text{as } h \to 0,\end{aligned}$$

which gives (7.1). □

Some remarks are in order.

Remark 8. The argument above, based on the use of separated variables monochromatic solutions, shows that the order of filtering $\mu^2 \le Ch^{-2}$ is sharp, in the sense that the observability inequality fails to be uniform when we take into account eigenvalues μ^2 such that $\mu^2 \gg h^{-2}$. Note however that our observability results require to restrict the class of eigenvalues under consideration to $\mu^2 \le \delta h^{-2}$ with $\delta > 0$ small. The discussion above does not justify the optimality of this smallness condition on the filtering constant. Actually, as we shall show in the next section, one may expect that uniform observability and controllability properties hold within classes of filtered solutions of the form $\mu^2 \le Ch^{-2}$ with arbitrary $C > 0$ for a sufficiently large time.

Remark 9. In a first look to this problem it might seem to be surprising that the negative result in Theorem 7.1 is related to monochromatic waves. Nevertheless, the lack of uniform observability is related to the fact that the quantity in the right hand side of (7.9) is of the order of $\cos^2(\omega_{j_0})$ which tends to zero as $h \to 0$. Of course this does not happen for the continuous wave equation. Indeed, if one computes for the solution for the continuous-time wave equation (1.3) with initial data

$$\varphi_0 = \frac{1}{\mu_{j_0}}\Phi_{j_0}, \qquad \varphi_1 = \frac{e^{-i\omega_{j_0}} - 1}{\mu_{j_0} h}\Phi_{j_0},$$

the same as that in (7.3), one gets

$$\varphi = \left[\cos(\mu_{j_0}(T-t)) + \frac{e^{-i\omega_{j_0}} - 1}{\mu_{j_0} h}\sin(\mu_{j_0}(T-t))\right]\Phi_{j_0}.$$

It is easy to check that the dominant corresponding term in the continuous-time boundary observation $\int_0^T \int_{\Gamma_0} |\partial\varphi/\partial\nu|^2 \, d\Gamma_0 dt$ reads

$$\int_0^T \cos^2(\mu_{j_0}(T-t))dt.$$

Clearly, this term is bounded below (and therefore does not tend to zero) when $h \to 0$, contrarily to what happens for the corresponding discrete term $\cos^2(\omega_{j_0})$.

7.2. **A heuristic explanation.** We now give a heuristic explanation of the necessity of filtering in terms of the group velocity of propagation of the solutions of the time-discrete system (*see* [13, 17]). For doing that we consider the time-discrete wave equation (1.10) in the whole space. Applying the Fourier transform (the continuous one in space and the discrete one in time), we deduce that the symbol of the time semi-discrete system (1.10) is

$$p_h(\tau,\xi) = -\frac{4\sin^2\frac{\tau h}{2}}{h^2} + |\xi|^2\cos(\tau h), \qquad (\tau,\xi) \in \left[-\frac{\pi}{2h}, \frac{\pi}{2h}\right] \times \mathbb{R}^d.$$

It is easy to see that, for all $\tau \in \left[-\pi(2h)^{-1}, \pi(2h)^{-1}\right]$, $p_h(\tau,\xi)$ has two nontrivial roots $\xi^\pm \in \mathbb{R}^d$. The bicharacteristic rays are defined as the solutions of the following Hamiltonian system:

$$\begin{cases} \dfrac{dx(s)}{ds} = 2\xi\cos(\tau h), & \dfrac{dt(s)}{ds} = -\dfrac{2\sin(\tau h)}{h} - |\xi|^2 h\sin(\tau h), \\ \dfrac{d\xi(s)}{ds} = 0, & \dfrac{d\tau(s)}{ds} = 0. \end{cases}$$

As in the continuous case, the rays are straight lines. However, both the direction and the velocity of propagation of the rays in this time-discrete setting case are different from the time-continuous one.

Let us now illustrate the existence of bicharacteristic rays whose projection on $\mathbb{R}^d$ propagates at a very low velocity or even does not move at all. For this, we fix any $x_0 = (x_{0,1}, \cdots, x_{0,d}) \in \Omega$ and choose the initial time $t_0 = 0$. Also, we choose the initial microlocal direction $(\tau_0, \xi_0) = (\tau_0, \xi_{0,1}, \cdots, \xi_{0,d})$, as a root of P_h. Thus

$$|\xi_0|^2 = \frac{4\sin^2\frac{\tau_0 h}{2}}{h^2\cos(\tau_0 h)}, \qquad \tau_0 \in \left[-\frac{\pi}{2h}, \frac{\pi}{2h}\right].$$

Note that the above condition is satisfied for $\xi_{0,1} = 2(h)^{-1}\sin\frac{\tau_0 h}{2}\cos^{-1/2}(\tau_0 h)$ and $\xi_{0,2} = \cdots = \xi_{0,d} = 0$, for instance. In this case we get

$$\frac{dx}{dt} = \frac{dx/ds}{dt/ds} = -\frac{\cos^{3/2}(\tau_0 h)}{\cos\frac{\tau_0 h}{2}}$$

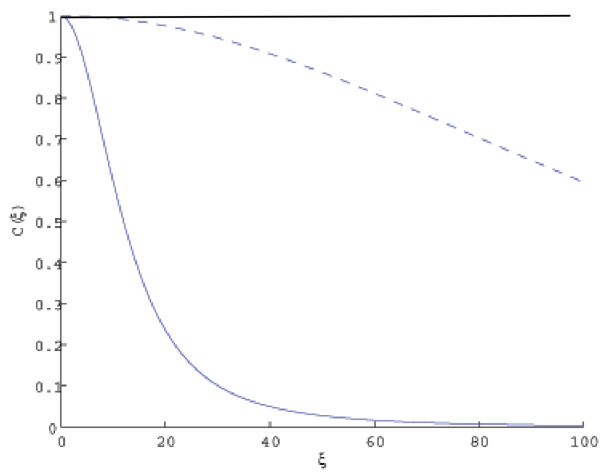

FIGURE 1. The diagram of $C(\xi)$. $h = 0.1$ (solid line) vs. $h = 0.01$ (dashed line). The thick horizontal segment corresponds to the theoretical group velocity $C(\xi) = 1$ (in the continuous case, i.e. $h = 0$).

and $x_2'(t) = \cdots = x_d'(t) = 0$. Thus, $x_j(t)$ for $j = 2, \cdots, d$ remain constant and

$$x_1(t) = x_{0,1} - t\cos^{3/2}(\tau_0 h)\cos^{-1}\frac{\tau_0 h}{2}$$

evolves with speed $-\cos^{3/2}(\tau_0 h)\cos^{-1}\frac{\tau_0 h}{2}$, which tends to 0 when $\tau_0 h \to \frac{\pi}{2}^-$, or $\tau_0 h \to -\frac{\pi}{2}^+$. This allows us to show that, as $h \to 0$, there exist rays that remain trapped on a neighborhood of x_0 for time intervals of arbitrarily large length. In order to guarantee the boundary observability these rays have to be cut-off by filtering. This can be done by restricting the Fourier spectrum of the solution to the range $|\tau| \leq \rho\pi/2h$ with $0 < \rho < 1$. This corresponds to

$$|\xi|^2 \leq \frac{4\sin^2(\rho\pi/2)}{h^2\cos(\rho\pi/2)}, \tag{7.10}$$

for the root of the symbol P_h.

This is the same scaling of the filtering operators we imposed on Theorems 6.1 and 8.1, namely, $\mu_j^2 \leq \delta/h^2$. Note however that, in (7.10), as $\rho \to 1$, the filtering parameter

$$\delta = \frac{4\sin^2(\rho\pi/2)}{\cos(\rho\pi/2)} \to \infty.$$

Thus, in principle, as mentioned above, the analysis of the velocity of propagation of bicharacteristic rays does not seem to justify the need of letting the filtering parameter δ small enough as in Theorems 6.1 and 8.1. Thus, this last restriction seems to be imposed by the rigidity of the method of multipliers rather than by the underlying wave propagation phenomena.

We can reach similar conclusions by analyzing the behavior of the so-called group velocity. Indeed, following [13], in $1-d$ the group velocity has the form

$$C(\xi)=\frac{4}{(2+h^2\xi^2)\sqrt{4+h^2\xi^2}},$$

with the graphs as in Figure 1. Obviously, it tends to zero when $h^2\xi^2$ tends to infinity. This corresponds precisely to the high frequency bicharacteristic rays constructed above for which the velocity of propagation vanishes. Based on this analysis one can show that, whatever the filtering parameter δ is, uniform observability requires the observation time to be large enough with $T(\delta)\nearrow\infty$ as $\delta\nearrow\infty$. This may be done using an explicit construction of solutions concentrated along rays (*see, for instance,* [10]). The positive counterpart of this result guaranteeing that, for any value of the filtering parameter $\delta>0$, uniform observability/controllability holds for large enough values of time, is an interesting open problem whose complete solution will require the application of microlocal analysis tools.

8. **Uniform controllability and convergence of the controls.** In this section, we present the following uniform partial controllability result for system (1.8) and the convergence result for the controls :

Theorem 8.1. *Let T, h_0 and δ be given as in Theorem 6.1, and $K>1$ be an odd integer. Then for any $h\in(0,h_0]$ and any $(y_0,y_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, there exists a control $\{u^k\in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$ such that the solution of (1.8) satisfies*

i)

$$\pi_{0,\delta h^{-2}}y^{K-1}=\pi_{-1,\delta h^{-2}}\Big(\frac{y^K-y^{K-1}}{h}\Big)=0 \quad \text{in } \Omega; \tag{8.1}$$

ii) There exists a constant $C>0$, independent of h, y^0 and y^1, such that

$$h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\le C\left\|\left(y^0,\frac{y^1-y^0}{h}\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}; \tag{8.2}$$

iii) When $h\to0$,

$$U_h\triangleq\sum_{k=1}^{K-1}u^k(x)\chi_{[kh,(k+1)h)}(t)\longrightarrow u \text{ strongly in } \quad L^2((0,T)\times\Gamma_0), \tag{8.3}$$

where u is a control of system (1.1), fulfilling (1.2);

iv) When $h\to0$,

$$\begin{aligned} y_h\triangleq\ & y^0\chi_{\{0\}}(t)+\frac1h\sum_{k=0}^{K-1}\Big[(t-kh)y^{k+1}-\Big(t-(k+1)h\Big)y^k\Big]\chi_{(kh,(k+1)h]}(t)\\ &\longrightarrow y \text{ strongly in } \quad C([0,T];L^2(\Omega))\cap H^1([0,T];H^{-1}(\Omega)), \end{aligned} \tag{8.4}$$

where y is the solution of system (1.1) with the limit control u as above.

The above theorem contains two results: the uniform partial controllability and the convergence of the controls and states as $h\to0$. The proof is standard. Indeed, the partial controllability statement follows from Theorem 6.1 and classical duality arguments ([9]); while for the convergence result, one may use the approach developed in [17]. However, for readers' convenience, we give below a sketch of the proof of Theorem 8.1.

Proof of Theorem 8.1: For any given $T > 2R$, choose a sufficiently small δ such that Theorem 6.1 guarantees the uniform observability for (1.10). Recall that for any given initial state $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ of the continuous system (1.1), the initial data of (1.8) are chosen to be $(y^0, \frac{y^1-y^0}{h}) = (y_0, y_1)$.

For any $(\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$, consider the functional

$$\begin{aligned} J_h(\varphi_0^h, \varphi_1^h) \triangleq\ & h \sum_{k=1}^{K-1} \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} \left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) \right|^2 d\Gamma_0 \\ & - \left\langle y_1, \varphi^0 \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega y_0 \frac{\varphi^1 - \varphi^0}{h} dx, \end{aligned}$$

where $\{\varphi^k\}_{k=0,\cdots,K}$ is the solution of (1.10) with data $(\varphi_0^h, \varphi_1^h)$. By Theorem 4.1, $J_h(\varphi_0^h, \varphi_1^h)$ is well-defined. Moreover J_h is convex, continuous and coercive in $\mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$, uniformly on $h > 0$. In view of Theorem 6.1, $J_h(\varphi_0^h, \varphi_1^h)$ admits one and only one minimizer $(\hat{\varphi}_0^h, \hat{\varphi}_1^h) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$.

Let $(\hat{\varphi}_0^h, \hat{\varphi}_1^h)$ be the minimizer of $J_h(\varphi_0^h, \varphi_1^h)$ in $\mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$. It is easy to check that

$$\begin{aligned} & h \sum_{k=1}^{K-1} \int_{\Gamma_0} \frac{\partial}{\partial \nu} \left(\frac{\hat{\varphi}^{k+1} + \hat{\varphi}^{k-1}}{2} \right) \frac{\partial}{\partial \nu} \left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) d\Gamma_0 \\ & = \left\langle y_1, \varphi^0 \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega y_0 \frac{\varphi^1 - \varphi^0}{h} dx, \\ & \qquad\qquad \forall\, (\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}, \end{aligned} \tag{8.5}$$

where $\{\hat{\varphi}^k\}_{k=0,\cdots,K}$ is the solution of system (1.10) with data $(\hat{\varphi}_0^h, \hat{\varphi}_1^h)$.

Multiplying the first equation of system (1.8) by $(\varphi^{k+1} + \varphi^{k-1})/2$, integrating it in Ω, summing it for $k = 1, \cdots, K-1$ and noting Theorem 4.4, it follows

$$\begin{aligned} & \left\langle \varphi^{K-1}, \frac{y^K - y^{K-1}}{h} \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \frac{\varphi^k - \varphi^{K-1}}{h} y^{K-1} dx \\ & = \left\langle \varphi^0, y_1 \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \frac{\varphi^1 - \varphi^0}{h} y_0 dx \\ & -h \sum_{k=1}^{K-1} \int_{\Gamma_0} \frac{\partial}{\partial \nu} \left(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) u^k d\Gamma_0. \end{aligned} \tag{8.6}$$

We now choose the control function $\{u^k\}_{k=1,\cdots,K-1}$ in system (1.8) as follows

$$u^k = \frac{\partial}{\partial \nu} \left(\frac{\hat{\varphi}^{k+1} + \hat{\varphi}^{k-1}}{2} \right) \Big|_{\Gamma_0}, \qquad k = 1, \cdots, K-1. \tag{8.7}$$

Then, (8.5), (8.6),(8.7) together with form of the initial data in (1.10) yield

$$\begin{aligned} & \left\langle \varphi_0^h, \frac{y^K - y^{K-1}}{h} \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_\Omega \varphi_1^h y^{K-1} dx = 0, \\ & \qquad\qquad \forall\, (\varphi_0^h, \varphi_1^h) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}. \end{aligned}$$

This gives the controllability property (8.1). The desired estimate (8.2) follows immediately from (8.7), (8.5) and Theorem 6.1.

Next, we prove the convergence of the controls. For this, recalling the exact form of U_h in (8.3) and noting its uniform boundedness with $K = 3, 5, \cdots$ (which follows from (8.2)), we see that, extracting subsequences, for some $u \in L^2((0,T)\times\Gamma_0)$ and $(\hat\varphi_0,\hat\varphi_1)\in H_0^1(\Omega)\times L^2(\Omega)$,

$$\begin{array}{rcll} U_h & \longrightarrow & u & \text{weakly in} \quad L^2((0,T)\times\Gamma_0), \\ (\hat\varphi_0^h,\hat\varphi_1^h) & \longrightarrow & (\hat\varphi_0,\hat\varphi_1) & \text{weakly in} \quad H_0^1(\Omega)\times L^2(\Omega), \end{array} \qquad \text{as } h\to 0. \tag{8.8}$$

Moreover, one can show by standard arguments, that

$$u = \left.\frac{\partial\hat\varphi}{\partial\nu}\right|_{(0,T)\times\Gamma_0}, \tag{8.9}$$

where $\hat\varphi$ is the solution of (1.3) with data $(\hat\varphi_0,\hat\varphi_1)$.

One can also use a classical Γ-convergence argument to show that the limit $(\hat\varphi_0,\hat\varphi_1)$ is the minimizer in $H_0^1(\Omega)\times L^2(\Omega)$ of the functional J corresponding to the controllability of the continuous wave equation.

Letting $K\to\infty$ in (8.5), we deduce that $\hat\varphi$ satisfies

$$\begin{aligned} \int_0^T\int_{\Gamma_0}\frac{\partial\hat\varphi}{\partial\nu}\frac{\partial\varphi}{\partial\nu}d\Gamma_0dt = \langle y_1,\varphi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega y_0\varphi_t(0)dx, \\ \forall\,(\varphi_0,\varphi_1)\in H_0^1(\Omega)\times L^2(\Omega), \end{aligned} \tag{8.10}$$

where φ is the solution of (1.3) with data (φ_0,φ_1). Similar to the above, (8.10) implies that the solution of system (1.1) with control u given by (8.9) satisfies (1.2).

On the other hand, by the weak convergence of $(\hat\varphi_0^h,\hat\varphi_1^h)$ in $H_0^1(\Omega)\times L^2(\Omega)$, recalling the definition of U_h in (8.3), noting (8.7) and (8.9), we conclude from (8.5) and (8.10) that

$$\int_0^T\int_{\Gamma_0}|U_h|^2d\Gamma_0dt \to \int_0^T\int_{\Gamma_0}|u|^2d\Gamma_0dt \qquad \text{as } h\to 0. \tag{8.11}$$

Combining (8.11) and the first convergence in (8.8), the desired strong convergence result (8.3) follows.

Once the strong convergence of the controls is known, the estimates of Theorem 4.4 allow getting a uniform bound of $\{y_h\}_{h>0}$ (defined in (8.4)) in $C([0,T];L^2(\Omega))\cap H^1([0,T];H^{-1}(\Omega))$, which yields the desired strong convergence result for the extension $\{y_h\}_{h>0}$ of the time-discrete solution $\{y^k\}_{k=0,\cdots,K}$ of (1.8) to continuous time, as indicated by (8.4). This completes the proof of Theorem 8.1. □

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