Dedicated to C. Dafermos in his 60th birthday

LARGE TIME BEHAVIOR FOR A SIMPLIFIED 1D MODEL OF FLUID-SOLID INTERACTION

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ABSTRACT

In this paper we consider a simple model in one space dimension for the interaction between a fluid and a solid represented by a point mass. The fluid is governed by the viscous Burgers equation and the solid mass, which shares the velocity of the fluid, is accelerated by the difference of pressure at both sides of it. We describe the asymptotic behavior of solutions for integrable data using energy estimates and scaling techniques. We prove that the asymptotic profile of the fluid is a self-similar solution of the Burgers equation with an appropriate total mass, and we describe the parabolic trajectory of the point mass. We also prove that, asymptotically, the difference of pressure to both sides of the point mass vanishes.

Keywords. Fluid-solid interaction, one space dimension, Burgers equations, self-similarity, large time behavior.

AMS Subject Classification. 35B40, 35K15, 35R35, 76R99 .

1 PRESENTATION OF THE MODEL AND MAIN RESULT

We consider a simplified one-dimensional model for a compressible fluid containing a solid represented as a point particle of mass $m > 0$ which floats with the fluid. The fluid density is assumed to be constant and the fluid velocity is governed by the viscous Burgers equation on both sides of the point mass location $x = h(t)$. The complete system of equations and data is

$$(1.1) \qquad \begin{cases} u_t - u_{xx} + (u^2)_x = 0, & x < h(t),\, x > h(t), \quad t > 0 \\ h'(t) = u(h(t),\, t), & t > 0 \\ m\, h''(t) = [u_x]\,(h(t), t), & t > 0 \\ u(x, 0) = u_0(x),\, x \in I\!R; h(0) = h_0,\, h'(0) = h_1. \end{cases}$$

According to the transmission conditions that hold at the point mass location $x = h(t)$, the velocity of the fluid and the solid mass coincide at this point and the mass is accelerated by the difference of pressure on its sides. We denote by $[f](x)$ the jump of the function f at the point x.

The problem of fluid-solid interaction has attracted a lot of attention from researchers in the recent past. Most of the work concerns the 2-D incompressible Navier-Stokes equations coupled with the motion of a finite number of rigid masses. The model we are considering here is a quite simplified version for several reasons: it is a scalar model both in the equations governing the motion of the point mass and the transmission conditions, and is also one-dimensional in the space variable; consequently, it does not involve an incompressibility condition. However, its very simplicity allows for a better understanding of otherwise difficult mathematical issues. In particular, we are able to perform a detailed asymptotic analysis of the interaction between the fluid and the solid as time goes to infinity, and we show that all the momentum and energy of the solid passes to the fluid, which approximates a pure Burgers-like behavior, while the solid mass is just convected along.

Reformulations. We first notice that by a simple change on units the solid may be assumed to be of unit mass, $m = 1$, and this will be accepted without loss of generality in the sequel. System (1.1) may be viewed as a free boundary problem for the velocity $u(x, t)$ and the free boundary $x = h(t)$. However, the change of variables

$$(1.2) \qquad v(x, t) = u(x + h(t),\, t)$$

allows to re-write the problem as

$$(1.3) \qquad \begin{cases} v_t - v_{xx} - h'(t)v_x + (v^2)_x = 0,\, x > 0,\, x < 0,\, t > 0 \\ v(0, t) = h'(t) \\ [v_x]\,|_{x=0} = h''(t) \\ v(x, 0) = u_0(x + h_0),\, x \in I\!R;\ h(0) = h_0,\, h'(0) = h_1. \end{cases}$$

In fact, both in model (1.1) and in (1.3) the true unknown for the solid mass is the velocity h'. Indeed, by setting $g(t) = h'(t)$, systems (1.1) and (1.3) read respectively as

$$
(1.4) \qquad \begin{cases} u_t - u_{xx} + (u^2)_x = 0, & x < h(t),\ x > h(t), \quad t > 0 \\ g(t) = u(h(t),\, t), & t > 0 \\ g'(t) = [u_x]\,(h(t), t), & t > 0 \\ u(x,0) = u_0(x),\ x \in I\!R; g(0) = h_1 \end{cases}
$$

and

$$
(1.5) \qquad \begin{cases} v_t - v_{xx} - g(t)v_x + (v^2)_x = 0,\ x > 0,\ x < 0,\ t > 0 \\ g(t) = v(0,t) \\ g'(t) = [v_x]\,|_{x=0} \\ v(x,0) = u_0(x + h_0),\ x \in I\!R;\ g(0) = h_1, \end{cases}
$$

where

$$
(1.6) \qquad h(t) = \int_0^t g(s)ds + h_0.
$$

Thus, the system under consideration turns out to be equivalent to a system coupling a parabolic equation that holds in a domain that is now independent of time (the set $x \neq 0$) with an ODE at the point $x = 0$.

Main results. Existence and uniqueness of solutions for (1.1) (or any of the equivalent versions described above) can be obtained by classical methods. Indeed, one can first analyze the linearized system to see that it generates an analytic semigroup and then check that the nonlinearity involved in the system is subcritical in the natural energy space. Global existence is due to the fact that the system possesses a good energy estimate (see Section 2). More precisely, we shall show that

Theorem 1.1 *For any $(u_0, h_0, h_1) \in H = L^2(I\!R) \times I\!R \times I\!R$ there exists a unique global solution $u \in C([0,\infty); L^2(I\!R))$, $h \in C^1([0,\infty))$ of problem* (1.1).

Notice that no sign restriction is imposed on the solutions. The proof of this result will be given in an Appendix at the end of this paper. The solution can be characterized by means of the variation of constants formula using the semigroup generated by the underlying linear evolution equation. It also satisfies an appropriate weak formulation that will be of much help when analyzing the asymptotic behavior.

As for the large-time behavior we have

Theorem 1.2 *Let $u_0 \in L^2(I\!R) \cap L^1(I\!R)$ and $h_0, h_1 \in I\!R$. Then,*

$$
(1.7) \qquad t^{(1-1/p)/2}||u(t) - \bar{u}(t)||_{L^p(I\!R)} \to 0, \qquad as \quad t \to \infty
$$

for all $1 \le p \le \infty$, *where*

$$\bar{u}(x,t) = \frac{1}{t^{1/2}} f_M(x/\sqrt{t}), \tag{1.8}$$

is the self-similar solution of Burgers' equation, satisfying

$$\begin{cases} u_t - u_{xx} + (u^2)_x = 0 \quad x \in I\!R,\, t > 0 \\ u(x,\,0) = M\delta_0, \end{cases} \tag{1.9}$$

where δ_0 *denotes the Dirac delta at the origin and the asymptotic momentum* M *is given by*

$$M = \int_{I\!R} u_0(x)dx + h_1. \tag{1.10}$$

We can also locate the asymptotic position of the point mass moving with the fluid

Theorem 1.3 *Under the above conditions, if* $M > 0$ *we have*

$$t^{-1/2}|h(t) - c\sqrt{t}| \to 0, \qquad as \quad t \to \infty \tag{1.11}$$

where $c > 0$ *is uniquely determined by the equation*

$$f_M(c) = c/2. \tag{1.12}$$

Moreover, we have a precise estimate for the particle speed

$$t^{1/2}|h'(t) - \frac{c}{2\sqrt{t}}| \to 0, \qquad as \quad t \to \infty. \tag{1.13}$$

Remarks. An analogous result holds when $M < 0$ (changing signs throughout).

It is important to observe that the self-similar profile, f_M, is known explicitly. Indeed, it can be easily computed with the aid of the Hopf-Cole transformation and the explicit expression of the Gaussian heat kernel:

$$f_M(x) = \frac{(e^M - 1)G(x)}{(e^M - 1)\int_{-\infty}^{-x} G(s)ds + 1}, \tag{1.14}$$

where $G(x) = (4\pi)^{-1/2}\exp(-|x|^2/4)$.

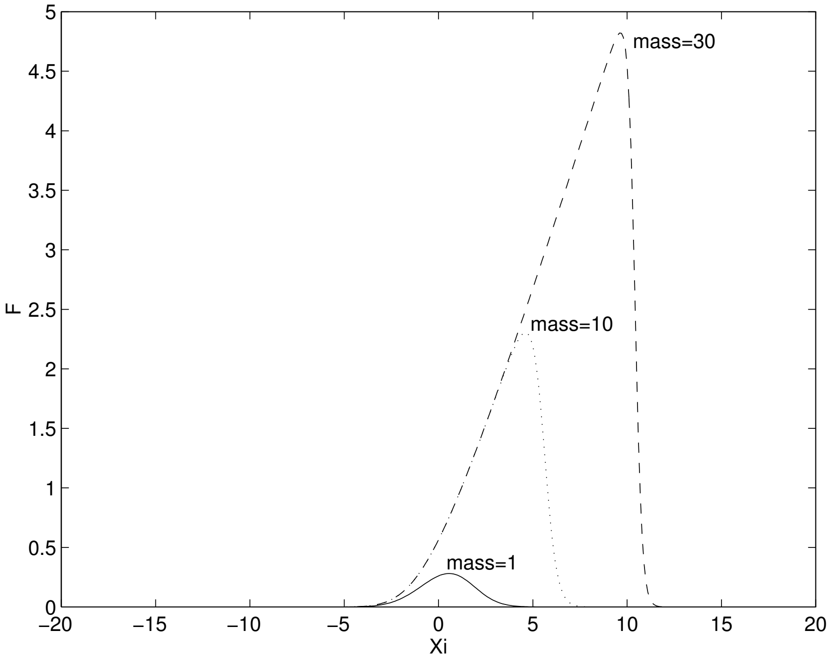

Figure 1. Self-similar solutions of Burgers equation for different masses M

Organization of the paper. We open the study in Section 2 with the derivation of momentum and energy estimates, as well as other basic properties of the solutions. Section 3 contains the time-decay estimates which are essential in order to capture the large-time behavior with the right scale. In Section 4 we introduce the scaling argument that allows to conjecture the asymptotic behavior stated in Theorem 1.2. In Section 5 we describe the weak formulation of the rescaled problem that will be used when passing to the limit. Section 6 is devoted to prove local compactness of rescaled solutions. Section 7 is devoted to prove global compactness by means of the so-called tail analysis. However, the precise convergence of solutions in $L^\infty(I\!\!R)$, as well as that of h', requires further estimates that are explained in Section 8; this allows the proof of Theorems 1.2 and 1.3 to be completed. Finally, a detailed proof of the existence and uniqueness of solutions for system (1.1) is given in an Appendix for the sake of completeness.

Related work. First of all, we would like to mention the book by J. M. Burgers [1], a standard reference in what concerns the classical Burgers equation. Besides, the following are some comments on the growing literature for the $2D$ and $3D$ versions of the fluid-solid problem. Specially worth mentioning is the paper by D. Serre [11] on the existence of solutions for the Cauchy problem in the whole space. Using a change of variables similar to (1.2), the system is written in a fixed domain (the whole space $I\!\!R^2$ minus the domain occupied by the rigid body). More recently, M.J. Esteban and B. Desjardins [4], [5], C. Conca, J. San Martin and M. Tucsnak [3], [2] and J. San Martin, V. Starovoitov and M. Tucsnak [10], have made important progress in this context. By now global existence of weak solutions is known in $I\!R^2$ or in a bounded domain even in the presence of multiple solid bodies. As far as we know, the uniqueness of weak solutions is by now an open problem. Some of the works mentioned above deal also with the $3D$ case. The works by C. Grandmont and Y. Maday [6] and M. D. Gunzburger, C.-H. Lee and A. S. Gregory [7] are also worth mentioning at this respect.

It is important to observe that when working in domains with boundary, a transformation of the form (1.2) puts the problem into a form in which the exterior boundary of the domain moves in time in a way that depends on the solution itself. This makes the problem in a bounded domain more complex than the Cauchy problem in the whole space.

On the other hand, a change of variables like (1.2) may be performed locally in time even in the presence of various masses, and this may allow to reduce the problem to a system like (1.3). But this argument, in principle, only works locally in time and fails when two (or more) different masses collide. One of the main open problems in this area is, precisely, analyzing whether two different masses collide or, in the case of a bounded domain, whether a mass may collide with the exterior boundary. In [10], where global existence of weak solutions is proved, the following alternative is obtained:

(a) If there is collision, it has to occur with zero velocity and the dynamics after the possible collision may be determined, considering the two bodies that collided as a single rigid body; or

(b) there is no collision.

Notation: All along the paper, in order to simplify the notation, we shall often simply write $\int f$ or $\int_{I\!R} f$ instead of $\int_{I\!R} f(x)dx$. Frequently the time-dependence of the functions under integration will also be ommited so that, in some cases, in the absence of ambiguity, we shall write $\int u$ instead of $\int_{I\!R} u(x,t)dx$.

2 BASIC ESTIMATES AND PROPERTIES

• Energy dissipation

Multiplying by u and integrating by parts we get:

$$\frac{1}{2}[\int_{I\!R} u^2(x,t)dx + |h'(t)|^2] + \int_0^t \int_{I\!R} |\ u_x\ |^2\ dxds = \frac{1}{2}[\int_{I\!R} u_0^2(x)\,dx + |h_1|^2] \tag{2.1}$$

for all $t > 0$. We note that a similar expression holds true if we integrate between $\tau > 0$ and t, $\tau < t \leq \infty$.

$$\frac{1}{2}[\int_{I\!R} u^2(x,t)dx + |h'(t)|^2] + \int_\tau^t \int_{I\!R} |\ u_x\ |^2\ dxds = \frac{1}{2}[\int_{I\!R} u^2(x,\tau)\,dx + |h'(\tau)|^2]. \tag{2.2}$$

We will need this observation later.

• **Conservation of momentum**

Integrating the first equation in (1.1) with respect to $x \in I\!R$ and using the jump condition we get

$$\int_{I\!R} u(x,t)\,dx + h'(t) = \int_{I\!R} u_0(x)dx + h_1, \quad \forall t > 0, \tag{2.3}$$

whenever, in addition to the assumptions of Theorem 2.1, $u_0 \in L^1(I\!R)$.

• **L^p estimates**

More generally, given a convex C^1 real function $j(u)$ we can multiply in (1.1) by $\beta(u) = j'(u)$ and integrate with respect to $x \in I\!R$ to get

$$\frac{d}{dt}\left[\int_{I\!R} j(u)dx + j(h'(t))\right] \le 0.$$

Specializing $j(u)$ to be an approximation of the function $|u|$, we deduce that the following quantity decreases in time:

$$\int_{I\!R} \mid u \mid dx + \mid h'(t) \mid, \tag{2.4}$$

whenever $u_0 \in L^1(I\!R)$.

We draw a similar conclusion when $j(u) = |u|^p$ for some $p \ge 1$. Therefore, we get L^p, $1 \le p < \infty$ estimates for the solution at all times $t > 0$ in terms of the initial data $u_0 \in L^p(I\!R)$. For the case $p = \infty$ we use the function $j(u) = (u-k)_+$ with $k \ge \|u_0\|_\infty$, $h'(0)$ to get

$$\|u(t)\|_\infty, |h'(t)| \le \max\{\|u_0\|_\infty, |h_1|\},$$

where we use the short notation $u(t)$ for $u(\cdot,t)$.

• **Positivity**

Taking $j(u)$ to be an approximation of the functions $u^+ = \max\{u, 0\}$, $u^- = (-u)^+$, we deduce that the following quantities

$$\begin{cases} \displaystyle\int u^+ dx + (h'(t))^+, \\ \displaystyle\int u^- dx + (h'(t))^- \end{cases}$$

decrease in time. Consequently, when $u_0 \ge 0$ a.e. and $h_1 \ge 0$ we deduce that

$$u(x,t) \ge 0 \quad a.e. \quad x \in I\!R, \quad \forall t > 0, \quad h'(t) \ge 0, \quad \forall t > 0. \tag{2.5}$$

Remark. As in standard parabolic theory, there is a comparison principle that works provided that both solutions share the same interface, an unlike event generally speaking. However, it applies when one of the solutions is the trivial solution since it can be viewed as a solution with an interface located wherever one wants. In this way we can prove the property of sign conservation (2.5).

3 DECAY ESTIMATES

The next step in the analysis is to establish decay rates for the solutions as time goes to infinity:

Proposition 3.1. *Assume that the initial data $(u_0, h_0, h_1) \in H = L^2(I\!R) \times I\!R \times I\!R$ are such that $u_0 \in L^1(I\!R)$. Then the solution satisfies the following decay properties:*

$$||u(t)||_{L^p(I\!R)} \leq C_p t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall t \geq 0 \tag{3.1}$$

for all $1 \leq p \leq \infty$. The constant C_p in this estimate depends on p and on the L^1-norm of u_0 and h_1. Similarly, we have

$$|h'(t)| \leq Ct^{-\frac{1}{2}}, \quad \forall t \geq 0. \tag{3.2}$$

Remark. The complete asymptotic analysis will show that these decay rates are sharp. Note that, as expected in view of the transmission condition $u = h'$ at the interface, the L^∞ norm of u decays with the same rate $(t^{-1/2})$ than h'. However, as it will be shown along the proof, this fact does not hold directly but requires a careful iterative argument.

Proof of Proposition 3.1. Estimates (3.1)-(3.2) will be obtained by means of the standard argument on scalar parabolic equations consisting on multiplying the equation by nonlinear functions of u. However, in this case, the presence of the second unknown h' requires special care.

We start with the following interpolation inequalities that are easy to prove:

$$\| u \|_p \leq \| u \|_\infty^{(p-1)/p} \| u \|_1^{1/p} \tag{3.3}$$

$$\| u \|_\infty \leq \| u \|_2^{1/2} \| u_x \|_2^{1/2}, \tag{3.4}$$

with norms taken in the variable x for fixed $t > 0$. Combining these two inequalities we get

$$\| u \|_p \leq \| u \|_\infty^{(p-1)/p} \| u \|_1^{1/p} \leq \| u \|_2^{(p-1)/2p} \| u_x \|_2^{(p-1)/2p} \| u \|_1^{1/p}. \tag{3.5}$$

When $p = 2$ we have:

$$\| u \|_2 \leq \| u_x \|_2^{1/3} \| u \|_1^{2/3}. \tag{3.6}$$

Now, multiplying by u equation (1.4) we have:

$$-\frac{d}{dt}\left[\int u^2 dx + | h' |^2\right] = 2 \int u_x^2 dx, \tag{3.7}$$

and by (3.6):

$$\int | u_x |^2 \, dx \geq ||u||_2^6 / \| u \|_1^4. \tag{3.8}$$

(Note that, at this level, the nonlinear term of the Burgers equation plays no role since, due to the continuity of u across the mass, $\int (u^2) u dx = 0$.) On the other hand, from (3.4) and (3.6) we have

$$\| u \|_\infty \leq \| u_x \|_2^{2/3} \| u \|_1^{1/3} \tag{3.9}$$

and therefore

$$\int u_x^2 dx \geq \frac{\| u \|_\infty^3}{\| u \|_1} \geq \frac{| h' |^3}{\| u \|_1}. \tag{3.10}$$

Combining (3.7), (3.8) and (3.10) we have

$$-\frac{d}{dt}\left[\frac{1}{2}\left[\int u^2 dx + | h' |^2\right]\right] \geq c\left[\| u \|_2^6 + | h' |^3\right], \tag{3.11}$$

where c depends on the L^1-norm of u, i.e., on $\|u_0\|_1 + |h_1|$. Due to the different homogeneity of the two terms in the right hand side of (3.11) we need to argue as follows:

$$\| u \|_2^6 + | h' |^3 \geq c\left[\int u^2 dx + | h' |^2\right]^3. \tag{3.12}$$

Here we are using in an essential way that $|h'|$ is known to be bounded in time, $t \geq 0$. Consequently,

$$\frac{d}{dt}\left[\int u^2 dx + | h' |^2\right] \leq -c\left[\int u^2 dx + | h' |^2\right]^3, \tag{3.13}$$

which, after integration, yields:

$$\| u \|_2 \leq c t^{-1/4} \tag{3.14}$$

$$| h' | \leq c t^{-1/4}. \tag{3.15}$$

Note that estimate (3.14) is the one we wanted for $\| u \|_2$. However the estimate (3.15) is not optimal. Indeed, we need for h' an estimate of the form $|h'| \leq C t^{-1/2}$.

As it is standard in parabolic equations, we are going to repeat this argument multiplying the equation by different powers of u and integrating by parts. In this way, when getting estimates on $\| u \|_p$ we shall keep improving our estimate on $| h' |$ that will decay as $\| u \|_p$. In particular, the estimate for $p = \infty$ will provide the desired decay rate for $| h' |$ too.

Let us now develop this argument. Multiplying by $| u |^{p-2} u$ we get

$$\frac{d}{dt}\left[\frac{1}{p}\int u^p dx + \frac{1}{p} | h' |^p\right] = -(p-1)\int | u |^{p-2} | u_x |^2 dx.$$

Following the same arguments we get, for all $1 \leq p < \infty$,

$$\begin{aligned} &\| u \|_p \leq C_p t^{-1/2(1-1/p)} \\ &| h'(t) | \leq C_p t^{-1/2(1-1/p)}. \end{aligned}$$

As in the case of the heat equation, this argument is not sufficient to get the estimate in L^∞ since we do not control the constants C_p as $p \to \infty$. Therefore, we have to argue more carefully following an iterative argument due to L. Véron, cf. [13]: when $p = 2q$ we have

$$\frac{d}{dt}\left[\| u \|_{2q}^{2q} + | h' |^{2q}\right] = -\frac{4(2q-1)}{2q}\int |(|u|^q)_x|^2\,dx. \tag{3.16}$$

From (3.8) and (3.10) we get

$$\frac{d}{dt}\left[\| u \|_{2q}^{2q} + | h' |^{2q}\right] \leq -c\frac{(2q-1)}{q}\left[\frac{\| u \|_{2q}^{6q}}{\left(\int | u |^q\,dx\right)^4} + \frac{| h' |^{3q}}{\int | u |^q\,dx}\right]. \tag{3.17}$$

Using the fact that, according to (3.16), $\left[\| u \|_p^p + | h' |^p\right]$ decreases in time along trajectories for all $1 \leq p \leq \infty$ we obtain

$$\begin{aligned} \frac{d}{dt}\left[\| u \|_{2q}^{2q} + | h' |^{2q}\right] &\leq -c\frac{(2q-1)}{q}\left[\frac{\| u \|_{2q}^{6q}}{\left[\| u_0 \|_q^q + | h_1 |^q\right]^4} + \frac{| h' |^{3q}}{\| u_0 \|_q^q + | h_1 |^q}\right] \\ &= -c\frac{(2q-1)}{q}\left[\frac{\| u \|_{2q}^{6q}}{\left[\| u_0 \|_q^q + | h_1 |^q\right]^4} + \frac{| h' |^{6q}}{| h' |^{3q}\left[\| u_0 \|_q^q + | h_1 |^q\right]}\right] \\ &= -c\frac{(2q-1)}{q\left[\| u_0 \|_q^q + | h_1 |^q\right]^4}\left[\| u \|_{2q}^{6q} + \left[\frac{\| u_0 \|_q^q + | h_1 |^q}{| h' |^q}\right]^3 | h' |^{6q}\right]. \end{aligned} \tag{3.18}$$

On the other hand,

$$\left[\frac{\| u_0 \|_q^q + | h_1 |^q}{| h' |^q}\right]^3 \geq 1.$$

Consequently,

$$\frac{d}{dt}\left[\| u \|_{2q}^{2q} + | h' |^{2q}\right] \leq -c\frac{(2q-1)}{q}\frac{\left[\| u \|_{2q}^{2q} + | h' |^{2q}\right]^3}{\left[\| u_0 \|_q^q + | h_1 |^q\right]^4}.$$

With the notation

$$E_p(t) = \| u \|_p^p + | h' |^p$$

the last inequality may be written in a simpler way as follows:

$$\frac{d}{dt}E_{2q}(t) \le -c\frac{(2q-1)}{q}\frac{(E_{2q}(t))^3}{(E_q(0))^4}, \tag{3.19}$$

and this implies that

$$E_{2q}(t) \le c\sqrt{\frac{q}{2q-1}}E_q^2(0)t^{-1/2}. \tag{3.20}$$

This is precisely the estimate one gets for the L^p-norm of solutions of the heat equation. The iteration argument by L. Véron [13] based on (3.20) (that we reproduce here for the sake of completeness) does not make use of the explicit form of the evolution equation and therefore provides the decay rates (3.1)-(3.2) we are looking for.

In view of (3.20) and taking into account that the evolution equation under consideration is time-invariant we deduce that

$$F_{2q}(t+s) \le CF_q(t)s^{-1/4q}, \tag{3.21}$$

where $F_p = E_p^{1/p}$. Given $\tau > 0$ by setting $s = \tau 2^{-n}$ and $q = 2^n$ we get

$$F_{2^{n+1}}(t+\tau 2^{-n}) \le C\tau^{-2^{-(n+2)}}2^{-n2^{-(n+2)}}F_{2^n}(t). \tag{3.22}$$

Iterating this argument we obtain

$$F_{2^{n+1}}(\tau(2^{-1}+...+2^{-n})) \le C_n(\tau)F_1(0), \tag{3.23}$$

with

$$C_n(\tau) = C\tau^{-\sum_{j=0}^n 2^{-(j+2)}}2^{-\sum_{j=0}^n (j+1)2^{-(j+2)}}. \tag{3.24}$$

On the other hand, taking into account that the functions E_p decrease with time, we have

$$F_{2^{n+1}}(\tau) \le F_{2^{n+1}}(\tau(2^{-1}+...+2^{-n})). \tag{3.25}$$

Note also that

$$C_n(\tau) \to C\tau^{-1/2}, \quad \text{as} \quad n \to \infty. \tag{3.26}$$

Letting $n \to \infty$ we deduce that

$$||u(t)||_\infty + |h'(t)| \le \liminf_{n\to\infty} F_{2^{n+1}}(\tau) \le Ct^{-1/2}[||u_0||_1 + |h_1|], \quad \forall t > 0, \tag{3.27}$$

as we wanted to prove. This completes the proof of Proposition 3.1. ∎

4 SCALING

According to the scaling properties of the Burgers equation, given (u, h) solution of (1.1) with a point mass of size $m > 0$, it is natural to introduce

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad h_\lambda(t) = \frac{1}{\lambda} h\left(\lambda^2 t\right), \tag{4.1}$$

for all $\lambda > 0$. Then, (u_λ, h_λ) is a solution of the following system:

$$\begin{cases} u_{\lambda,t} - u_{\lambda,xx} + (u_\lambda^2)_x = 0, \; x < h_\lambda, \; x > h_\lambda, \; t > 0 \\ u_\lambda\left(h_\lambda(t), t\right) = h_\lambda'(t), \; t > 0 \\ \left[u_{\lambda,x}\right]\left(h_{\lambda,t}\right) = (m/\lambda)\, h_\lambda''(t), \; t > 0 \\ u_\lambda(x,0) = u_{0,\lambda}(x), \; h_\lambda(0) = h_{0,\lambda}, \; h_\lambda'(0) = h_{1,\lambda} \end{cases} \tag{4.2}$$

with

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x), \quad h_{0,\lambda} = \frac{1}{\lambda} h_0, \qquad h_{1,\lambda}(0) = \lambda h_1. \tag{4.3}$$

We thus obtain the same system with rescaled initial data and mass m/λ. Note that we have temporarily returned to a general particle mass $m > 0$ to make clearer the effect of scaling on the solid mass. In the limit as $\lambda \to \infty$, we expect to obtain Burgers equation without a jump since, formally, as $\lambda \to \infty$, $\left[u_{\lambda,x}\right]\left(h_{\lambda,t}\right) = (m/\lambda)\, h_\lambda''(t)$ tends to zero in the distributional sense. Thus, if we denote by w and by η the limit of u_λ and h_λ (in a sense to be made precise) as $\lambda \to \infty$, respectively, one expects the following equations to be fulfilled:

$$w_t - w_{xx} + \left(w^2\right)_x = 0, \quad x \in I\!R, \; t > 0, \tag{4.4}$$

and the condition

$$w(\eta(t), \, t) = \eta'(t). \tag{4.5}$$

The latter condition determines η but is not needed to calculate w. We also expect w to be self-similar with initial datum a Dirac mass, $M\delta_0$, with $M \in I\!R$ to be determined. In fact, we expect w to be a source-type solution with δ_0 as initial data, and it turns out that w is then necessarily self-similar, of the form

$$w(x,t) = t^{-1/2} f_M\left(x/\sqrt{t}\right). \tag{4.6}$$

Note however that we can not guarantee the initial data of w to be $\left(\int_{I\!R} u_0(x)dx\right)\delta_0$ because of the possible occurence of a boundary layer at $t = 0$. As stated in Theorem 1.2, this is false in general and only holds true when $h_1 = 0$. In general we have an asymptotic initial data $M\,\delta_0$ with

$$M = \int u_0 dx + m h_1. \tag{4.7}$$

One can identify M by the following argument. By conservation of momentum, we have

$$M = \int u_0(x)\, dx + mh_1 = \int u(x,t)dx + mh'(t) = \int u_\lambda(x,1)dx + \frac{m}{\lambda}h'_\lambda(1).$$

When passing to the limit as $\lambda \to \infty$ in this quantity we get

$$M = \int w(x,1)dx. \tag{4.8}$$

This is so since the term $\frac{m}{\lambda}h'_\lambda(1)$ vanishes asymptotically because of the presence of a factor m/λ. Then, if w is a source type solution of Burgers equation with initial datum $\tilde{M}\,\delta_0$, we deduce, by conservation of momentum that, necessarily, $\tilde{M} = M$. According to this, one expects that

$$u_\lambda \to w = t^{-1/2} f_M(x/\sqrt{t}) \quad \text{in } L^p(I\!R) \text{ as } \lambda \to \infty, \tag{4.9}$$

for all $1 \le p \le \infty$ and $t > 0$, where M is as in (4.7). Conservation of momentum for Burgers is a well-known property, though usually called conservation of mass.

Moreover, in view of the self-similar structure of w and the ODE (4.5) that η satisfies, one also expects h_λ to converge to $\eta(t) = c\sqrt{t}$ for all $t > 0$ for some $c > 0$. But since the profile f_M has been identified, $c \in I\!R$ is necessarily such that

$$f_M(c) = c/2. \tag{4.10}$$

In view of the explicit form of f_M it is easy to see that equation (4.10) determines a unique value of c.

Of course, in order to be able to develop all this programme we need suitable compactness properties of the sequences (u_λ, h'_λ). The L^p decay-estimates of the previous section guarantee that, for all $t > 0$, these sequences are bounded in $L^p(I\!R) \times I\!R$ for all $1 \le p \le \infty$. But compactness needs to be proved. On the other hand, once compactness is proved, one needs to develop a careful argument in order to identify the limit w of u_λ, as well as η, limit of h_λ.

We proceed in two steps. Using the regularizing property of parabolic equations we show first local compactness. Then a careful analysis of the tails as $|x| \to \infty$ allows to show compactness in $L^p(I\!R)$. These arguments are developed in detail in the following three sections. In Section 8 we prove compactness in $L^\infty(I\!R)$, and Section 9 is devoted to analyze the behavior of the point mass.

5 WEAK FORMULATION AND SCALING

We revert in the sequel to the assumption $m = 1$. In order to pass to the limit as the parameter λ tends to ∞, instead of using the semigroup formulation of the solution, it is more convenient to use its weak formulation.

Definition. The weak formulation of the initial-value problem for system (4.2) satisfied by (u_λ, h_λ) is as follows:

$$
(5.1) \qquad \begin{cases} -\int_{I\!R}\int_0^T u_\lambda\,(\varphi_t+\varphi_{xx})\,dxdt-\int_{I\!R}\int_0^T u_\lambda^2\varphi_x\,dxdt \\ +\frac{1}{\lambda}\int_0^T h_\lambda''(t)\varphi\,(h_\lambda(t),\,t)\,dt=\int_{I\!R} u_{\lambda,0}(x)\varphi(x,0)\,dx, \\ \forall\varphi\in C_b^2\,(I\!R\times[0,T]):\varphi(x,T)\equiv 0. \end{cases}
$$

This may also be written as:

$$
(5.2) \qquad \begin{cases} -\int_{I\!R}\int_0^T u_\lambda\,(\varphi_t+\varphi_{xx})\,dxdt-\int_{I\!R}\int_0^T u_\lambda^2\varphi_x\,dxdt \\ -\frac{1}{\lambda}\int_0^T h_\lambda'(t)\,(\varphi_t+h_\lambda'\varphi_x)\,(h_\lambda(t),\,t)\,dt \\ =\int_{I\!R} u_\lambda(x,0)\varphi(x,0)dx+h_1\varphi\left(\frac{h_0}{\lambda},\,0\right), \\ \forall\varphi\in C_b^2\,(I\!R\times[0,T]):\varphi(x,T)\equiv 0. \end{cases}
$$

In order to pass to the limit, even when the test function φ is C^∞ with compact support, we need compactness of u_λ in $L^2_{loc}(I\!R\times(0,\infty))$ and the uniform boundedness of h_λ' in $L^2_{loc}(0,\infty)$.

According to the decay estimates of Section 3 we have:

$$
(5.3) \qquad \parallel u(t)\parallel_p\leq Ct^{-\frac{1}{2}(1-1/p)},\quad \forall t>0
$$

for all $1\leq p\leq\infty$, and

$$
(5.4) \qquad \mid h'(t)\mid\leq Ct^{-1/2},\quad \forall t>0.
$$

On the other hand, observe that, in view of (5.4) and the fact that $h(0)=h_0$, one also obtains

$$
(5.5) \qquad \mid h(t)\mid\leq Ct^{1/2}+|h_0|,\quad \forall t>0.
$$

It is easy to see that these bounds remain uniform for the scaled family of solutions (u_λ, h_λ). In other words, as an immediate consequence of (5.3)-(5.5), it is easy to see that

$$
(5.6) \qquad \parallel u_\lambda(t)\parallel_p\leq Ct^{-\frac{1}{2}(1-1/p)},\quad \forall t>0
$$

for all $1\leq p\leq\infty$,

$$
(5.7) \qquad \mid h_\lambda'(t)\mid\leq Ct^{-1/2},\quad \forall t>0,
$$

and

$$| h_\lambda(t) | \leq Ct^{1/2} + |h_0/\lambda|, \quad \forall t > 0, \tag{5.8}$$

for all $\lambda \geq 1$. In particular, this implies that for any finite interval $0 < \tau \leq t \leq T < \infty$,

$$u_\lambda \quad \text{is uniformly bounded in} \quad L^\infty(\tau, T; L^p(I\!R)) \tag{5.9}$$

for all $1 \leq p \leq \infty$, and

$$h_\lambda \quad \text{is uniformly bounded in} \quad W^{1,\infty}(\tau, T). \tag{5.10}$$

Of course, these estimates do not guarantee the compactness of u_λ in L^2_{loc}. But one expects that the regularizing effect of the equation will provide it. This can indeed be done in three steps: compactness holds away from the free boundary, and the uniform estimates we have allow to neglect the contribution of the interface at this level.

This allows to pass to the limit in the weak formulation (5.2). In order to recover the main result of this paper there are still two things to be done:

a) Identification of the initial data;

b) Tail analysis as $|x| \to \infty$ to show that the convergence of u_λ holds in $L^\infty(\tau, T; L^p(I\!R))$ for all $1 \leq p < \infty$.

c) Identification of the dynamics of the point mass in the limit.

6 COMPACTNESS AND LOCAL CONVERGENCE

Compactness is typically obtained using the partial differential equation, that provides for estimates on time derivatives, plus some variant of the Aubin-Lions compactness Lemma, as in J. Simon's [12]. In this problem the argument is more delicate due to the presence of the interface.

1. Regularity in the fluid. Let us first settle the situation away from $x = h(t)$. If

$$v_\lambda(x,t) = u_\lambda\left(x + h_\lambda(t),\, t\right), \tag{6.1}$$

then we have

$$v_{\lambda,t} = v_{\lambda,xx} - \left(v_\lambda^2\right)_x + h'_\lambda\, v_{\lambda,x} \tag{6.2}$$

in the domain $\{(x,t) : x \neq 0,\ t > 0\}$ (this is the rescaled version of the equation (1.3)). It satisfies the additional conditions

$$v_\lambda(0,t) = h'_\lambda(t), \qquad \left[v_{\lambda,x}\right](0) = \frac{1}{\lambda} h''_\lambda. \tag{6.3}$$

Recall that, according to Proposition 3.1,

$$u_\lambda \text{ is uniformly bounded in } L^\infty(\tau, T; L^p(I\!R)) \tag{6.4}$$

for all $1 \le p \le \infty$ and $0 < \tau < T < \infty$. On the other hand, due to the energy dissipation law u_x belongs to $L^2(0, \infty; L^2(I\!R))$ and therefore:

$$u_{\lambda,x} \text{ is uniformly bounded in } L^2\left(\tau, T; L^2(I\!R)\right) \tag{6.5}$$

for all $0 < \tau < T < \infty$. From (6.4)-(6.5) we deduce that

$$v_\lambda \text{ is uniformly bounded in } L^2\left(\tau, T; H^1(I\!R)\right) \tag{6.6}$$

and this fact, combined with the fact that v_λ is also bounded in $L^\infty(\tau, T; L^p(I\!R))$, together with the equation that v_λ satisfies to both sides of $x = 0$, yields

$$v_{\lambda,t} \text{ is uniformly bounded in } L^2\left(\tau, T; H^{-1}(I\!R^\pm)\right). \tag{6.7}$$

Note that (6.7) provides estimates for $x > 0$ and $x < 0$, but not at the interface. Note that $v_{\lambda,t}$ cannot have a singularity (Dirac delta) at $t = 0$, as is easily checked. This type of singularity will appear for $v_{\lambda,xx}$ due to the transmission condition, right-hand side of (6.3).

Now, using classical compactness results (see [12], Corol. 4, p. 85), in view of (6.4) and (6.7) the family v_λ is relatively compact in $C([\tau, T]; H^{-s}(K))$ for any compact set $K \subset (-\infty, 0) \cup (0, \infty)$ and any $s > 0$. Therefore, along a subsequence $\lambda_n \to \infty$, we have

$$v_\lambda \to \overline{v} \quad \text{strongly in } C([\tau, T]; H^{-s}(K)), \tag{6.8}$$

for all $0 < \tau < T < \infty$, K compact set of $I\!R$, and $s > 0$.

Taking into account that $v_\lambda(t)$ is uniformly bounded in $L^p(I\!R)$ for all $1 \le p \le \infty$ we also deduce that, along the same subsequence,

$$v_\lambda(t) \rightharpoonup \overline{v}(t) \quad \text{weakly in } L^p(I\!R), \tag{6.9}$$

for all $t > 0$ and $1 < p < \infty$. The convergence also holds in the weak-* topology of $L^\infty(I\!R)$.

At this point we need a technical result guaranteeing the uniform boundedness of v_λ in $W^{s,p}$-spaces for $t > 0$.

Lemma 6.1. *For all $t > 0$, the sequence $v_\lambda(t)$ is uniformly bounded in $W^{s,p}(I\!R)$ as $\lambda \to \infty$ for s and p satisfying the condition $sp < 1$.*

Assuming for the moment that this Lemma holds (we shall give a proof in Appendix B), using the fact that $W^{s,p}(K)$ is compactly embedded in $L^1(K)$ for any compact set K of $I\!R$ and the weak convergence (6.9), we deduce also local strong convergence, i.e.,

$$v_\lambda(t) \to \overline{v}(t) \quad \text{strongly in } L^p_{loc}(I\!R), \tag{6.10}$$

for all $t > 0$ and $1 \le p < \infty$. We also have

$$v_\lambda \to \overline{v} \quad \text{strongly in } L^p_{loc}(I\!R \times (0,\infty)), \tag{6.11}$$

for all $1 \le p < \infty$ and a. e. convergence.

2. The solid mass position. In view of (5.7), $h_\lambda(t)$ is uniformly bounded in $W^{1,\infty}(\tau, T)$ and in $W^{1,p}(0,T)$ for all $1 \le p < 2$. Thus, it converges to some $\overline{h}(t)$ weakly in $W^{1,p}(0,T)$ for all $1 \le p < 2$, uniformly in $[0,T]$ for all finite T, and weakly-* in $W^{1,\infty}(\tau, T)$.

3. The weak formulation. The analysis of Step 1 does not extend to a neighbourhood of the free boundary. Indeed, if we write the equation satisfied by v_λ at the interface $x = 0$ we get an extra term

$$(1/\lambda)\, h''_\lambda \otimes \delta_{x=0},$$

a distribution for which we do not have good estimates. But the previous estimates allow us to pass to the limit on the weak formulation of the equation satisfied by (v_λ, h_λ) or (u_λ, h_λ) for smooth, compactly supported test functions, see formula (5.2). It is then easy to see that $\overline{u}$, with

$$\overline{u}(x,t) = \overline{v}(x - \overline{h}(t), t),$$

is a distributional solution of

$$\overline{u}_t - \overline{u}_{xx} + (\overline{u}^2)_x = 0 \quad \text{in } I\!R \times (\tau, \infty), \tag{6.12}$$

since the only term carrying the information of the solid mass in (5.2) has a factor $1/\lambda$ and disappears in the limit. We take test functions which vanish at $t = 0$ to avoid at this moment the question of convergence at the initial time, for which we have poor estimates. On the other hand, taking into account that v_λ weakly converges to $\overline{v}$ in $L^2(\tau, T; H^1(I\!R))$, that h_λ weakly converges to $\overline{v}$ in $H^1(\tau, T)$, the continuity of the traces of v_λ at both sides of $x = 0$, and the fact that

$$v_\lambda(0,t) = h'_\lambda(t), \quad \forall t > 0, \tag{6.13}$$

in the limit we deduce that

$$\overline{v}(0,t) = \overline{h}'(t), \qquad \text{a. e.} \quad t > 0. \tag{6.14}$$

In terms of $\overline{u}$ this means,

$$\overline{u}(\overline{h}, t) = \overline{h}'(t), \qquad \text{a. e.} \quad t > 0. \tag{6.15}$$

Note that we also have

$$\overline{h}(0) = 0, \tag{6.16}$$

since $h_\lambda(0) = h_0/\lambda \to \overline{h}(0)$ as $\lambda \to \infty$.

Thus, in order to complete the proof of the Theorem 1.2 we need to identify the limit $\overline{u}$ as a particular solution of Burgers equation by identifying its initial data.

Remark 6.1. Note that by now we have not proved strong convergence of u_λ or v_λ in L^∞_{loc}. In fact, the estimates we have do not allow to get compactness in that space. Indeed, Lemma 6.1 provides uniform bounds of v_λ in $W^{s,p}(I\!R)$ for all $s < 1/p$. When p ranges from 1 to infinity and s ranges below $1/p$, by Sobolev's embedding Theorem, the uniform bound in $W^{s,p}(I\!R)$ provides compactness in all L^q_{loc} spaces for $q < \infty$ but fails to provide compactness in L^∞_{loc}. The extra estimates needed to improve the convergence will be obtained in Section 8. We have kept the whole detail of the partial convergence results of this section because they may have independent interest in other contexts.

7 INITIAL DATA IN THE LIMIT

We have been able to show that the influence of the solid point on the fluid disappears in the scaled limit in the sense that the limit $\overline{u}$ satisfies the Burgers equation. We still have to show what the limit distribution $\overline{u}$ remembers from the original situation. This happens through the initial data taken by $\overline{u}$.

When passing to the limit in the weak formulation (5.2) there is a difficulty at $t = 0$, i.e., when the test function φ does not vanish in a neighborhood of $t = 0$. Indeed, in that case, we do not have enough information to pass to the limit in the term

$$\frac{1}{\lambda}\int_0^T h'_\lambda(t)\,(\varphi_t + h'_\lambda\varphi_x)\,(h_\lambda(t),\,t)\,dt, \tag{7.1}$$

or, more precisely, in

$$\frac{1}{\lambda}\int_0^T |h'_\lambda(t)|^2\varphi_x\,(h_\lambda(t),\,t)\,dt. \tag{7.2}$$

For this reason we identify the initial datum using the technique called *tail analysis*. This issue is addressed using the ideas of the paper [9], so will be a bit sketchy.

Lemma 7.1. *Under the assumptions of Theorem 1.2, for all $\varepsilon > 0$ and $\delta > 0$ there exist $\tau > 0$ and λ_0 such that*

$$\int_{|x|\geq\delta} |u_\lambda(x,t)|dx \leq \varepsilon, \tag{7.3}$$

for all $0 \leq t \leq \tau$ and $\lambda \geq \lambda_0$. The same holds for v_λ.

Proof. Since all the interfaces h_λ have an estimate of the form

$$|h_\lambda(t)| \leq C\,t^{1/2} + \frac{h_0}{\lambda},$$

we consider the problem for u_λ in a region

$$S_1 = (\delta/2, \infty) \times (0, \tau).$$

We choose λ large and τ small enough so that the interface is excluded from that region. Then, the u_λ are solutions there of Burgers equation, and the result is a way of stating that the zero initial data are taken in the limit $\lambda \to \infty$, as $t \to 0$ and for $x \in (\delta, \infty)$. This is rather standard for Burgers equation. A similar argument applies in the symmetric region $S_2 = (-\infty, -\delta/2) \times (0, \tau)$.

We give some detail of the rigorous proof in S_1 for the reader's convenience. We multiply by an approximation of the sign function $p(u)$, $0 \le p \le 1$, and by a smooth cutoff function $0 \le \zeta \le 1$ supported in $(\delta/2, \infty)$ with $\zeta = 1$ for $x > \delta$. After integration by parts like in Section 2 and letting $p(u) \to \text{sign}(u)$, we have

$$\begin{aligned} \frac{d}{dt} \int |u| \zeta \, dx &\le \int |u^2 \zeta_x| \, dx + \int |u \zeta_{xx}| \, dx \\ &\le \frac{C}{\sqrt{t}} \int |u| \, dx + C \int |u| \, dx \, , \end{aligned}$$

where u stands for u_λ and we have used the uniform a priori estimate $|u_\lambda| \le C t^{-1/2}$. Here C denotes different constants that do not depend on λ. Using the fact that $\int |u_\lambda| \, dx$ is uniformly bounded and integrating in time gives

$$\int_{x>\delta} |u_\lambda| \, dx \le \int |u_\lambda| \zeta \, dx \le C(t + \sqrt{t}) + \int |u_{0,\lambda}| \zeta \, dx \, .$$

This proves the lemma since the last integral tends to zero as $\lambda \to \infty$. ■

As an immediate and important consequence of this Lemma we deduce that, the strong convergence of v_λ towards $\overline{v}$ for $t > 0$ fixed not only holds in $L^p_{loc}(\mathbb{R})$ but all also in $L^p(\mathbb{R})$ for all $1 \le p < \infty$.

Now we can also pass to the limit in the scaled equations even near $t = 0$ if $|x| \ge \epsilon$ and the limit of the term with initial data disappears

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}} u_\lambda(x, 0) \varphi(x, 0) dx + h_1 \varphi \left(\frac{h_0}{\lambda}, 0 \right) = 0$$

for test functions such that $\varphi(x, 0) = 0$ if $|x| \le \epsilon$. Actually, it is easily seen that $u_\lambda(x, t) \to 0$ as both $\lambda \to \infty$ and $t \to 0$, uniformly in $x \in (\epsilon, \infty)$. The same happens for $x \le -\epsilon$.

Then, necessarily, the initial data of $\overline{u}$ and $\overline{v}$ (which coincide) is a distribution concentrated at $x = 0$. Thus, it is a linear combination of the Dirac mass and its derivatives. Let us now show that it must be a measure using again the Hopf-Cole transformation: the functions $u_\lambda(t)$ are uniformly bounded in $L^1(\mathbb{R})$ for every $\lambda > 1$ and $t > 0$, cf. the L^1-estimate of section 2. The same property is inherited by the limit $\overline{u}$. Integrating,

$$w(x, t) = \int_{-\infty}^{x} \overline{u}(s, t) \, ds$$

has its total variation uniformly bounded in time. Since $W(t) = e^{w(t)}$ is a bounded solution of the heat equation, it has an initial trace U_0 which is a BV function, the same happens for w. We conclude that $\overline{u}_0 = dw_0/dx$ is a measure.

As a consequence of this argument, the initial data must be of the form $\widetilde{m}\delta_0$ necessarily. Taking into account that the solution of the Burgers equation with $\widetilde{m}\delta$ as initial data has mass $\tilde{m}$ for all $t > 0$, we deduce that $\widetilde{m} = M$.

Finally we can use the conservation of momentum (2.3) to get an asymptotic estimate, valid for any time $t > 0$:

$$\lim_{\lambda\to\infty} \int_{I\!R} u_\lambda(x,t)\, dx = M,$$

where $M = \int u_0 dx + h_1$ is the total momentum. Since we have control at infinity we can pass to the limit and get

$$\int_{I\!R} \overline{u}(x,t)\, dx = M, \quad \forall t > 0.$$

8 HIGHER ORDER ESTIMATES AND CONVERGENCE IN $L^\infty(I\!R)$

Up to now we have proved Theorem 1.2 except for the strong convergence of the solution in $L^\infty(I\!R)$. This section is devoted to complete the proof of this result. We need the following extra information on the solutions.

Proposition 8.1. *We have the following estimates:*

$$t^{3/4}\, v_x \in L^\infty(0,\infty : L^2(I\!R)), \quad t^{3/4} v_{xx} \in L^2((\tau,\infty)\times I\!R^\pm), \quad t^{3/4} h'' \in L^2(\tau,\infty). \tag{8.1}$$

These estimates are uniform bounds for the family of rescalings of a solution.

Proof. (I) Multiplying formally by v_{xx} the equation satisfied by v and integrating by parts in $x > 0$ we obtain

$$\int_0^\infty (v_{xx})^2\, dx - \int_0^\infty v_t v_{xx}\, dx + g(t)\int_0^\infty v_x v_{xx}\, dx - 2\int_0^\infty v v_x v_{xx}\, dx = 0. \tag{8.2}$$

The second term is calculated after integration by parts as

$$-\int_0^\infty v_t v_{xx}\, dx = \frac{1}{2}\frac{d}{dt}\int_0^\infty (v_x)^2\, dx + v_t(0,t) v_x(0+,t),$$

where the last expression to the right equals $g'(t)v_x(0+,t)$. The next term is

$$g(t)\int_0^\infty v_x v_{xx}\, dx = -\frac{1}{2} g(t)(v_x(0+,t))^2.$$

Finally, we have

$$2\int_0^\infty |vv_xv_{xx}|\,dx \le 2\int_0^\infty (vv_x)^2\,dx + \frac{1}{2}\int_0^\infty (v_{xx})^2\,dx.$$

Making the same calculation on the half-line $x<0$, and adding both parts we get

$$\begin{aligned}\frac{1}{2}\int_{I\!R^*}(v_{xx})^2\,dx + \frac{1}{2}\frac{d}{dt}\int (v_x)^2\,dx + (g'(t))^2 \le \\ 2\int v^2v_x^2\,dx + |g(t)g'(t)(v_x(0+,t)+v_x(0-,t))|\,,\end{aligned}$$

with integrals extending over the line, $x \in I\!R$ but for the first one where we have pointed out the domain $I\!R^* = \{x \neq 0\}$ to make sure that we do not include the contribution of (the Dirac mass of) v_{xx} at $x=0$. We now transform that last term using the elementary trace inequality

$$|v_x(0+,t)| \le \int_0^1 |v_x|\,dx + \int_0^1 |v_{xx}|\,dx,$$

and a similar one for $x<0$, so that

$$\begin{aligned}|g(t)g'(t)(v_x(0+,t)+v_x(0-,t))| &\le |g(t)g'(t)|(\textstyle\int_0^1 |v_x|\,dx + \int_0^1 |v_{xx}|\,dx) \\ &\le \tfrac{1}{2}(g'(t))^2 + \tfrac{1}{2}(g(t))^2\left(\textstyle\int |v_x|\,dx + \int |v_{xx}|\,dx\right)^2.\end{aligned}$$

Using the fact that $||v||_{L^\infty(I\!R)} + |g| = O(t^{-1/2})$ we get for t large enough

$$\int_{I\!R^*}(v_{xx})^2\,dx + \frac{d}{dt}\int (v_x)^2\,dx + (g'(t))^2 \le \frac{C}{t}\int (v_x)^2\,dx\,. \tag{8.3}$$

(II) We now concentrate on the inequality

$$\frac{d}{dt}\int (v_x)^2\,dx \le \frac{C}{t}\int (v_x)^2\,dx,$$

which we write as $Y'(t) \le CY(t)/t$ for $Y(t) = \int (v_x)^2\,dx$. We use the following result whose proof we leave to the reader

Lemma 8.1. *If a nonnegative real function $Y(t)$ satisfies $Y'(t) \le cY(t)/t$ for $t>0$ and $c \in I\!R$, then*

$$Y(t) \le \frac{C}{t}\int_{t/2}^t Y(s)\,ds, \tag{8.4}$$

where $C>0$ depends only on c.

A proof of this lemma is as follows: the hypothesis says that the function $f(t) = Y(t)t^{-c}$ is nonincreasing and nonnegative, Hence, whenever $c \neq -1$ we have

$$\int_{t/2}^t Y(s)\,ds = \int_{t/2}^t t^c f(s)\,ds \ge f(t)\int_{t/2}^t s^c\,ds = Y(t)t\frac{2^{c+1}-1}{(c+1)2^{c+1}}$$

and we get (8.4) with $C = (c+1)2^{c+1}/(2^{c+1}-1) > 0$. For $c \neq -1$ a similar calculation gives the limit value $C = 1/\log 2$. Note that the constant is sharp. ∎

Returning to our problem, the energy estimate (2.1) and the decay estimates (3.1)-(3.2) give now:

$$\int_t^\infty \int (v_x)^2 \, dxdt < 2\left[\int (v(t))^2 \, dx + |h'(t)|^2\right] \leq \frac{C}{t^{1/2}},$$

while the decay estimate (3.1) with $p = 2$ becomes:

$$\int (v_x(t))^2 \, dx \leq \frac{C_1}{t} \int_{t/2}^\infty \int (v_x)^2 \, dxdt < \frac{C}{t^{3/2}} \tag{8.5}$$

for every $t > 0$.

(III) Integrating in time (8.3) from $t > 0$ to $T > t$ after using (8.5) we easily conclude that

(8.6)

$$\int_t^T \int_{I\!R^*} (v_{xx})^2 \, dxdt + \int_{I\!R} (v(t)_x)^2 \, dx + \int_t^T (g')^2 \, dt \leq C\, t^{-1} \int_{t/4}^T \int_{I\!R} (v_x)^2 \, dxdt \leq C\, t^{-3/2},$$

where $C > 0$ is a universal constant. The slightly more general statements in (8.1) are easily obtain using the trick of integration in diadic time intervals of the form $I_n = (2^n, 2^{n+1})$ and summing in n.

(IV) The proof of the statement for the rescalings follows immediately after we make the observation that the right-hand side of estimate (8.6) depends on the solution only through the quantity $\int_t^\infty \int (v_x)^2 \, dxdt$ which is uniformly bounded for the whole family $\{v_\lambda\}_{\lambda > 1}$ in view of our previous decay estimates. Moreover, the time after which the estimates hold scales like $T_\lambda = T_1/\lambda^2$, hence they are uniform in $t \geq \tau > 0$ as $\lambda \to \infty$.

(V) We can get estimates of the quantities of the theorem also for small t by changing the form of the estimates in Step I. Indeed, we can get a trace estimate of the form

$$|v_x(0+,t)|^2 \leq \frac{1}{a}\int_0^a |v_x|^2 \, dx + \frac{a}{2}\int_0^a |v_{xx}|^2 \, dx,$$

We can replace integrals from 0 to a by integrals from 0 to ∞. We can then choose a as we please. When $a = 1/g^2$ we get

$$\begin{aligned}|g(t)g'(t)(v_x(0+,t)| &\leq \tfrac{1}{4}(g'(t))^2 + (g(t))^2 v(0+,t)^2 \leq \\ &\tfrac{1}{4}(g'(t))^2 + \tfrac{1}{2}\textstyle\int_0^\infty v_{xx}^2 \, dx + g^4(t)\int_0^\infty v_x^2 \, dx.\end{aligned}$$

Same calculation for $v(0-,t)$. We can now get the inequality

$$\int_{I\!R^*} (v_{xx})^2 \, dx + \frac{d}{dt}\int (v_x)^2 \, dx + (g'(t))^2 \leq C \int (v^2 + g(t)^4)\,(v_x)^2 \, dx. \tag{8.7}$$

and get estimates (without decay rates) much as before. ■

Application to the uniform convergence of v_λ

The extra regularity on v and h provided by Proposition 8.1 and, more precisely, the fact that these estimates are uniform for the rescaled sequences v_λ and h_λ allows to conclude the convergences claimed in Theorem 1.2, (1.7) for $p = \infty$ and in Theorem 1.3 for h'.

Indeed, $v_\lambda(t)$, according to Proposition 8.1, is bounded in $H^1(I\!R)$ for all $t > 0$. Taking into account that $v_\lambda(t)$ converges in $L^p(I\!R)$ for any finite p and $v_\lambda(t)$ is bounded in $H^1(I\!R)$, by interpolation we deduce its convergence in $L^\infty(I\!R)$.

On the other hand, $v_{\lambda,t}$ can be calculated on both subdomains $\{x > 0, t > 0\}$ and $\{x < 0, t > 0\}$ by means of the equation and we conclude that it is an L^2 function for in (x,t) for bounded t and $x \neq 0$. Now, there can be no measure supported on the line $x = 0$ for the time derivative of v by virtue of a general result that we state as a lemma

Lemma 8.1 *Let $Q = (-R,R) \times (0,T]$, $Q^+ = (0,R) \times (0,T]$, $Q^- = (-R,0) \times (0,T]$ and let v be a function in $L^1(Q)$ that has an integrable first derivative in Q^+ and Q^-, $v_t \in L^1(Q^\pm)$. Then v_t, as a distribution, is an integrable function in Q.*

Moreover, the family $v_{t,\lambda}$ is uniformly bounded in L^2_{loc}. We now use the following basic calculus result.

Lemma 8.2 *If v is an $L^1(S)$ for a bounded subset $S = I \times J \subset I\!R^2$ and $v_x \in L^\infty_t(J : L^2_x(I))$ and $v_t \in L^2(S)$, then v is Hölder continuous with exponent $1/2$ in x and $1/4$ in t, and this regularity is uniform on the data.*

As a result, the functions v_λ are uniformly Hölder continuous (locally in x,t).

9 ASYMPTOTICS FOR THE POINT MASS LOCATION

In this section we prove Theorem 1.3 about the large-time location of the solid mass. The last estimates of the last section apply to h': indeed, according Proposition 8.1, h_λ is uniformly bounded in $H^2(\tau,T)$ for all $0 < \tau < T < \infty$. Therefore, the sequence h_λ converges along a subsequence $\lambda_n \to \infty$ to a function $H(t)$ in $C^1_{loc}(0,\infty)$.

The estimates we have proved for h_λ imply that $|H(t)| \le C\,t^{1/2}$ and that $H'(t)$ is bounded for $0 < t_1 \le t \le t_2$. The convergence of h_λ allows us to define the limit the relation $v(x,t) = u(x + H(t), t)$. We now consider the interface relations

$$h'_\lambda(t) = v_\lambda(0,t) = u_\lambda(h_\lambda(t),t),$$

and pass to the limit in an interval $[t_1, t_2]$, $0 < t_1 < t_2$ to get

$$H'(t) = v(0,t) = u(H(t),t) = t^{-1/2} f(H(t)/t^{1/2}),$$

where we have used the fact that we know exactly u, the fundamental solution of Burgers equation with total integral M. We now use the change of variables $H(t) = t^{1/2} r(\tau)$, $\tau = \log(t)$ to find the convenient equation for $r = r(\tau)$

$$r'(\tau) = f(r(\tau)) - \frac{1}{2} r(\tau).$$

We know that r is a bounded solution of this equation defined for $-\infty < t < \tau$. There is only one solution of this ODE, namely the equilibrium solution $r = c$. This ends the proof of Theorem 1.3.

10 APPENDIX A: EXISTENCE AND UNIQUENESS

The existence and uniqueness of solutions of (1.1) may be proved by a careful analysis of the underlying linear evolution equation and the associated semigroup and the variation of constant formula.

In order to prove existence we work on the model (1.3) where the interface has been fixed at $x = 0$ by means of the change of variables (1.2).

We first analyze the linearized system

$$\begin{cases} v_t - v_{xx} = 0, \quad z < 0, \quad x > 0, \quad t > 0 \\ v(0,t) = g(t), \quad t > 0 \\ [v_x(0,t)] = g'(t), \quad t > 0 \\ v(x,0) = v_0(x), \quad g(0) = g_0 \end{cases} \tag{10.1}$$

with

$$g_0 = h_1, \qquad v_0(x) = u_0(x + h_0). \tag{10.2}$$

We claim that for any $v_0 \in L^2(I\!R)$ and $g_0 \in I\!R$ system (10.1) admits an unique solution

$$\begin{cases} v \in C\left([0,\infty);\, L^2(I\!R)\right) \cap L^2\left(I\!R^+;\, H^1(I\!R)\right) \\ g \in C([0,\infty)). \end{cases} \tag{10.3}$$

To see this we write system (10.1) in an abstract form as

$$V_t = AV \tag{10.4}$$

where

$$V = \begin{pmatrix} v^- \\ v^+ \\ g \end{pmatrix}. \tag{10.5}$$

Here v^- (resp. v^+) denotes the restriction of v to the interval $(-\infty, 0)$ (resp. $(0,\infty)$).

The operator A is given by

$$AV = A\begin{pmatrix} v^- \\ v^+ \\ g \end{pmatrix} = \begin{pmatrix} \partial_x^2 v^- \\ \partial_x^2 v^+ \\ \partial_x v^+(0) - \partial_x v^-(0) \end{pmatrix} \tag{10.6}$$

and it defines an unbounded operator in the Hilbert space

$$H = L^2(-\infty, 0) \times L^2(0, \infty) \times I\!R \tag{10.7}$$

with domain

$$D(A) = \left\{ V = \begin{pmatrix} v^- \\ v^+ \\ g \end{pmatrix} \in \begin{pmatrix} H^2(-\infty, 0) \\ H^2(0, \infty) \\ I\!R \end{pmatrix} : v^-(0) = v^+(0) = g \right\}. \tag{10.8}$$

We claim that A is $m-$dissipative. Indeed:

$$\begin{aligned} \langle AV, V\rangle &= \int_{-\infty}^{0} \partial_x^2 v^- v^- dx + \int_0^\infty \partial_x^2 v^+ v^+ dx + \left(\partial_x v^+(0) - \partial_x v^-(0)\right) g \\ &= -\int_{-\infty}^{0} \left|\partial_x v^-\right|^2 dx - \int_0^\infty \left|\partial_x v^+\right|^2 dx \le 0, \quad \forall V \in D(A). \end{aligned} \tag{10.9}$$

On the other hand, given

$$F = \begin{pmatrix} f^- \\ f^+ \\ p \end{pmatrix} \in H \tag{10.10}$$

system

$$-AV + V = F \tag{10.11}$$

admits an unique solution $V \in D(A)$. Indeed, system (10.11) may be rewritten as

$$\begin{cases} -\partial_x^2 v^- + v^- = f^-, \quad x < 0 \\ -\partial_x^+ v^+ + v^+ = f^+, \quad x > 0 \\ -\left[\partial_x v^+(0) - \partial_x v^-(0)\right] + g = p \end{cases} \tag{10.12}$$

and its variational formulation is: to find

$$v \in H^1(I\!R), \quad (g = v(0)) \tag{10.13}$$

such that

$$\int_{-\infty}^{\infty} v_x w_x dx + \int_{-\infty}^{\infty} vw dx + v(0)w(0) = \int_{-\infty}^{0} f^- w dx + \int_0^\infty f^+ w dx + pw(0), \quad \forall w \in H^1(I\!R). \tag{10.14}$$

It is easy to see that (10.13)-(10.14) admits an unique solution $v \in H^1(I\!R)$ such that

$$\begin{pmatrix} v^- \\ v^+ \\ g \end{pmatrix} \in D(A)$$

with v^- (resp. v^+) the restriction of v to $(-\infty, 0)$ (resp. $(0, \infty)$) and $g = v(0)$.

Moreover, A is self-adjoint.

Therefore A is the generator of an analytic semigroup of contractions in H. Moreover, according to the classical regularizing property of semigroups generated by self-adjoint operators:

$$|\langle AS(t)V_0,\, S(t)V_0\rangle| \leq \frac{1}{t}\,|V_0|_H^2\,, \quad \forall V_0 \in H \tag{10.15}$$

and

$$\int_0^\infty |< AS(t)V_0,\, S(t)V_0 >|\, dt \leq \frac{1}{2}\,|V_0|_H^2\,, \quad \forall V_0 \in H. \tag{10.16}$$

In particular, the component v of the solution of (10.1) satisfies:

$$\|v_x(t)\|^2_{L^2(I\!R)} \leq \frac{1}{t}\left[\| \; v_0 \; \|^2_{L^2(I\!R)} + |\; h_1 \;|^2\right] \tag{10.17}$$

and

$$\int_0^\infty \int_{I\!R} |v_x(x,t)|^2\, dxdt \leq \frac{1}{2}\left[\| \; v_0 \; \|^2_{L^2(I\!R)} + |h_1|^2\right] \tag{10.18}$$

for every $(v_0,\, h_0,\, h_1) \in L^2(I\!R) \times I\!R \times I\!R$.

Let us now consider the full nonlinear system (1.5):

$$\begin{cases} v_t - v_{xx} = g(t)v_x - (v^2)_x; \quad x < 0,\, x > 0,\, t > 0 \\ v(0,t) = g(t), \quad t > 0 \\ [v_x(0,t)] = g'(t), \quad t > 0 \\ v(x,0) = v_0(x), \quad x > 0,\, x < 0; \quad g(0) = g_0 = h_1. \end{cases} \tag{10.19}$$

Using the linear semigroup $S(t)$ introduced above the problem (10.19) can be rewritten as follows using the variation of constants formula.

$$V(t) = S(t)V_0 + \int_0^t S(t-s)\mathcal{N}(V(s))ds,\; t > 0 \tag{10.20}$$

where

$$V = \begin{pmatrix} v^- \\ v^+ \\ g \end{pmatrix} \tag{10.21}$$

as above, and the nonlineary $\mathcal{N}$ is also vector-valued of the form

$$\mathcal{N}(V) = \begin{pmatrix} gV_x^- - \partial_x \left[(V^-)^2\right] \\ gV_x^+ - \partial_x \left[(V^+)^2\right] \\ 0 \end{pmatrix}. \tag{10.22}$$

We shall say that

$$V \in C([0,T]; H) \tag{10.23}$$

is a *weak solution* of system (10.19) in the time interval $[0,T]$ provided

$$\int_0^T | < AV, V > | dt < \infty \tag{10.24}$$

and (10.20) holds for all $0 \leq t \leq T$.

In order to prove the global existence of an unique weak solution of (10.19) we proceed in a standard way. First we prove local existence. At the same time we establish an alternative: *global existence* or *finite time blow-up*. Finally, using an energy estimate, we show that solutions may not blow-up in finite time which leads to global existence.

• **Local existence**: Given $V_0 \in H$ we set

$$R = \| V_0 \|_H . \tag{10.25}$$

We then introduce the Banach space

$$X = \left\{ V \in C\left([0,T];\, H\right) : \langle AV,\, V\rangle \in L^1(0,T) \right\}, \tag{10.26}$$

endowed with the canonical norm

$$\| V \|_X = \| V \|_{C([0,T];\, H)} + \left[\int_0^T | < AV, V > | dt \right]^{1/2}. \tag{10.27}$$

The time T will be chosen later on (sufficiently small, in terms of the norm of the initial data) to apply the Banach fixed point Theorem.

We denote by B the ball of radious $4R$ in X, and introduce the mapping

$$[\phi(V)](t) = S(t)V_0 + \int_0^t S(t-s)\mathcal{N}(V(s))ds. \tag{10.28}$$

Our goal is to show that ϕ maps B into B and that it is a strict contraction provided T is sufficiently small. As a consequence of this, applying Banach fixed point theorem, we deduce the existence and uniqueness of a local (in time) solution of (10.20) in B.

Let us check that ϕ is in the conditions to apply Banach fixed point Theorem:

- $\phi(B) \subset B$.

Let us first check that

$$\| \phi(V) \|_{L^\infty(0,T;H)} \leq 2R. \tag{10.29}$$

We have

$$\begin{aligned} \| \phi(V) \|_{L^\infty(0,T;H)} &\leq \| S(t)V_0 \|_{L^\infty(0,T;H)} + \| \int_0^t S(t-s)\mathcal{N}(V(s))ds \|_{L^\infty(0,T;H)} \\ &\leq \| V_0 \|_H + \max_{0\leq t\leq T} \int_0^t \| S(t-s)\mathcal{N}(V(s)) \|_H \, ds \\ &\leq \| V_0 \|_H + \int_0^T \| \mathcal{N}(V(s)) \|_H \, ds \\ &\leq \| V_0 \|_H + \int_0^T \Big[| g(s) | \| v_x(s) \|_{L^2(\mathbb{R})} + 2 \| v(s) \|_{L^\infty(\mathbb{R})} \| v_x(s) \|_{L^2(\mathbb{R})} \Big] ds \\ &\leq \| V_0 \|_H + \| V \|_{L^\infty(0,T;H)} \sqrt{T} \left[\int_0^T | < AV, V > | dt \right]^{1/2} \\ &\quad + 2 \| v \|_{L^2(0,T;L^\infty(\mathbb{R}))} \| v_x \|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\leq R + \sqrt{T} 4R \, [4R]^{1/2} + R^{3/2}\sqrt{T}2^{4+1/2}. \end{aligned} \tag{10.30}$$

In the last inequality we have used the fact that

$$\| v \|_{L^2(0,T;L^\infty(\mathbb{R}))} \leq \sqrt{2}T^{1/4} \| v \|^{1/2}_{L^\infty(0,T;L^2(\mathbb{R}))} \| v_x \|^{1/2}_{L^2(0,T;L^2(\mathbb{R}))} \leq 4R\sqrt{2}T^{1/4}. \tag{10.31}$$

In view of (10.30), by choosing $T > 0$ small enough so that

$$R + \sqrt{T}(4R)^{3/2} + 2^{4+1/2}\sqrt{T}R^{3/2} \leq 2R \tag{10.32}$$

we deduce that (10.29) holds.

Note that (10.32) is equivalent to

$$4^{3/2}T^{1/2}R^{1/2} + 2^{4+1/2}T^{1/2}R^{1/2} \leq 1 \tag{10.33}$$

which holds when T is small enough with respect to R.

We now set $W = \Phi(V)$. According to (10.29), in order to see that $\Phi(B) \subset B$ it is sufficient to check that

$$\int_0^T |\langle AW, W\rangle| \, dt \leq 4R^2. \tag{10.34}$$

In view of (10.16), it is sufficient to prove that

$$\int_0^T \left|\langle A\widetilde{W}, \widetilde{W}\rangle\right| dt \leq \frac{3}{2}R^2, \tag{10.35}$$

where

$$\widetilde{W}(t) = \int_0^t S(t-s)\left[\mathcal{N}(V(s))\right] ds. \tag{10.36}$$

At this point, it is convenient to observe that $\widetilde{W}$ solves the parabolic equation

$$\begin{cases} \frac{d\widetilde{W}}{dt} - A\widetilde{W} = F = \mathcal{N}(V), \quad t > 0 \\ \widetilde{W}(0) = 0. \end{cases} \tag{10.37}$$

that we write in this abstract form to simplify the notation.

Multiplying in (10.37) by $\widetilde{W}$ and taking into account that $\mathcal{N}(V) = \partial_x[\mathcal{M}(V)]$ with

$$\mathcal{M}(V) = \begin{pmatrix} gv^- - (v^-)^2 \\ gv^+ - (v^+)^2 \\ 0 \end{pmatrix} \tag{10.38}$$

we deduce easily that

$$\int_0^T \left|\langle A\widetilde{W}, \widetilde{W}\rangle\right| dt \leq \int_0^T \| \mathcal{M}(V) \|_H^2 \, dt. \tag{10.39}$$

The last term in (10.39) is easy to estimate:

$$\begin{aligned} \int_0^T \| \mathcal{M}(\mathcal{V}) \|_H^2 \, dt &\leq \int_0^T \left\|gv - v^2\right\|_{L^2(I\!R)}^2 dt \ \leq 2\int_0^T \left[|g(t)|^2 \| v \|_{L^2(I\!R)}^2 + \| v \|_{L^4(I\!R)}^4\right] dt \\ &\leq 2T \| V \|_{L^\infty(0,T;H)}^4 + 2 \| v \|_{L^4(0,T;L^4(I\!R))}^4 \\ &\leq 2T \| V \|_{L^\infty(0,T;H)}^4 + 2\sqrt{T} \| v \|_{L^\infty(0,T;L^2(I\!R))}^3 \| v_x \|_{L^2(0,T;L^2(I\!R))} \\ &\leq 2\sqrt{T} \| V \|_{L^\infty(0,T;H)}^3 \left[\sqrt{T} \| V \|_{L^\infty(0,T;H)} + \left[\int_0^T |\langle AV, V\rangle| \, dt\right]^{1/2}\right] \\ &\leq 2\sqrt{T} \max\left(1, \sqrt{T}\right)(4R)^4. \end{aligned} \tag{10.40}$$

Here we have used the inequality

$$\begin{aligned} ||f||_{L^4(0,T;L^4(I\!R))} &\leq ||f||_{L^\infty(0,T;L^2(I\!R))}^{3/4} ||f_x||_{L^1(0,T;L^2(I\!R))}^{1/4} \\ &\leq T^{1/8} ||f||_{L^\infty(0,T;L^2(I\!R))}^{3/4} ||f_x||_{L^2(0,T;L^2(I\!R))}^{1/4}. \end{aligned} \tag{10.41}$$

In view of (10.39)-(10.40), it is easy to see that (10.35) (and, consequently, (10.34)) hold automatically by taking T small with respect to R and, more precisely, by choosing T small enough such that

$$2\sqrt{T}\max\left(1, \sqrt{T}\right)(4R)^4 \le \frac{3}{2}R^2. \tag{10.42}$$

- Φ is a strict contraction in B.

  Given V_1, $V_2 \in B$ we have now to obtain upper bounds on the norm of $\Phi(V_1)-\Phi(V_2)$.

Obviously,

$$W = \Phi(V_1) - \Phi(V_2) = \int_0^t S(t-s)\left[\mathcal{N}(V_1) - \mathcal{N}(V_2)\right] ds \tag{10.43}$$

solves

$$\begin{cases} \frac{dW}{dt} - AW = F = \mathcal{N}(V_1) - \mathcal{N}(V_2), \; t > 0 \\ W(0) = 0. \end{cases} \tag{10.44}$$

Then, as above, multiplying the equation satisfied by W in H, integrating by parts and using inequality (10.41) we deduce that

$$\begin{aligned} &||W||^2_{L^\infty(0,T;H)} + \int_0^T |\langle AW, W\rangle|\, dt \\ &\le \int_0^T \|\mathcal{M}(V_1) - \mathcal{M}(V_2)\|^2_H\, dt \le \int_0^T \left\|g_1 v_1 - v_1^2 - g_2 v_2 + v_2^2\right\|^2_{L^2(I\!R)} dt \\ &\le 2\int_0^T \left[|g_1 - g_2|^2 \|v_1\|^2_{L^2(I\!R)} + |g_2|^2 \|v_1 - v_2\|^2_{L^2(I\!R)} + \|(v_1+v_2)(v_1-v_2)\|^2_{L^2(I\!R)}\right] dt \\ &\le 2^6 R^2 T \|V_1 - V_2\|^2_{L^\infty(0,T;H)} + 2\|V_1+V_2\|^2_{L^4(I\!R\times(0,T))} \|V_1 - V_2\|^2_{L^4(I\!R\times(0,T))} \\ &\le CTR^2 ||V_1 - V_2||^2_X \end{aligned} \tag{10.45}$$

for some $C > 0$ independent of R, T and V_1 and V_2.

According to (10.45) it is clear that, by choosing T small enough with respect to R, Φ is indeed a strict contraction from B to B.

Applying Banach's fixed point theorem we deduce the existence of an unique solution V of (10.28) in B. Thus, we have local existence. In fact a standard argument allows to prove the existence of an unique solution up to a maximal time $0 < T^* \le \infty$. Moreover, the following alternative holds:

- Either $T^* < \infty$ and then the solution blows-up in time T^*, i.e.

$$\| V(t) \|_H + \left(\int_0^t | \langle AV, V \rangle | \, dt \right)^{1/2} \to \infty, \text{ as } t \nearrow T^*,$$

or,

- $T^* = \infty$ and then the solution is global in time.

Consequently, in order to conclude the proof of the global existence and uniqueness result it is sufficient to show that finite-time blow up may not occur.

But this is an immediate consequence of the energy dissipation law satisfied by the solution of (1.1) that, in terms of the variables v, g, guarantees that

$$\frac{1}{2}\left[\int_{I\!R} v^2(x,t)dx + | g(t) |^2\right] + \int_0^t \int_{I\!R} v_x^2 dxds \le \frac{1}{2}\left[\int_{I\!R} v_0^2(x)dx + | g_0 |^2\right], \quad \forall t > 0.$$

11 APPENDIX B: PROOF OF LEMMA 6.1

Let us recall that (v_λ, g_λ) satisfy

$$(11.1) \qquad \begin{cases} v_{\lambda,t} = v_{\lambda,xx} - (v_\lambda^2)_x + g_\lambda v_{\lambda,x} & \text{in} \quad I\!R \times (0,\infty) \\ v_\lambda(0,t) = g_\lambda(t) \\ [v_{\lambda,x}]\,(0,t) = \frac{1}{\lambda} g_\lambda'(t) \\ v_\lambda(x,0) = v_{0,\lambda}(x), \quad \text{in} \quad I\!R. \end{cases}$$

According to the previous estimates we have:

$$(11.2) \qquad ||v_\lambda(t)||_{L^p(I\!R)} \le Ct^{-(1-1/p)/2}, \quad \forall t > 0,$$

for all $1 \le p \le \infty$,

$$(11.3) \qquad |g_\lambda(t)| \le Ct^{-1/2}, \quad \forall t > 0,$$

$$(11.4) \qquad ||v_{0,\lambda}||_{L^1(I\!R)} \le C,$$

with $C > 0$ independent of $\lambda \ge 1$.

The statement of the lemma is equivalent to obtaining a uniform (with respect to $\lambda \ge 1$) bound of v_λ in $W^{s,p}(I\!R)$ for all $t > 0$. In fact, taking into account that v_λ is uniformly bounded in $L^p(I\!R)$, it is sufficient to get a bound in the homogeneous Sobolev space $W^{s,p}(I\!R)$, that we still denote by $W^{s,p}(I\!R)$ for simplicity. Of course, the difficulties in the proof of this result come from the interface conditions at $x = 0$.

In order to simplify the notation along the proof of this Lemma we shall denote v_λ and g_λ by v and g respectively.

We are going to prove a uniform bound for v in $W^{s,p}$ to both sides of $x = 0$. Of course this is sufficient since traces are not defined in $W^{s,p}$ for $s < 1/p$. Since the argument is similar to both sides of $x = 0$ we consider only the case $x > 0$. Thus, we rewrite the equations satisfied by v in $x > 0$ as follows:

$$\begin{cases} v_t = v_{xx} - (v^2)_x + g v_x & \text{in} \quad I\!\!R^+ \times (0,\infty) \\ v(0,t) = g(t) \\ v(x,0) = v_0(x), & \text{in} \quad I\!\!R^+. \end{cases} \tag{11.5}$$

We decompose v as follows

$$v = w + z\,, \tag{11.6}$$

where w satisfies the linear non-homogeneous boundary value problem

$$\begin{cases} w_t = w_{xx} & \text{in} \quad I\!\!R^+ \times (0,\infty) \\ w(0,t) = g(t) \\ w(x,0) = 0, & \text{in} \quad I\!\!R^+, \end{cases} \tag{11.7}$$

and z solves

$$\begin{cases} z_t = z_{xx} + g z_x + g w_x - ((z+w)^2)_x & \text{in} \quad I\!\!R^+ \times (0,\infty) \\ z(0,t) = 0 \\ z(x,0) = v_0(x), & \text{in} \quad I\!\!R^+. \end{cases} \tag{11.8}$$

Our goal is to show, in two steps, that $w(t)$ is bounded in $W^{s,p}(I\!\!R^+)$ for any $t > 0$ and then the bound of $z(t)$.

Step 1. *The linear problem (11.7).*

Given $T > 0$ we look for a bound on $w(T)$ in $W^{s,p}(I\!\!R^+)$ in terms of the estimates we have on g.

The solution of (11.7) in the time interval $(0,T)$ may be characterized by transposition. For, introduce the adjoint system:

$$\begin{cases} -\varphi_t = \varphi_{xx} & \text{in} \quad I\!\!R^+ \times (0,\infty) \\ \varphi(0,t) = 0 \\ \varphi(x,T) = \varphi_0(x), & \text{in} \quad I\!\!R^+. \end{cases} \tag{11.9}$$

We then have

$$\int_{I\!\!R^+} w(x,T)\varphi_0(x)dx = \int_0^T g(t)\varphi_x(0,t)dt. \tag{11.10}$$

Taking into account that

$$|g(t)| \le C t^{-1/2}, \tag{11.11}$$

it is then sufficient to show the existence of a constant $C > 0$ such that

$$||\varphi_x(0,t)||_{L^1(T/2,T)} \leq C||\varphi_0||_{W^{-s,p'}(I\!R^+)}. \tag{11.12}$$

Indeed, once this holds, due to the regularizing effect of the adjoint heat equation that φ satisfies in the backward sense of time, it is easy to see that

$$|\int_0^T g(t)\varphi_x(0,t)dt| \leq C||\varphi_0||_{W^{-s,p'}(I\!R^+)} \tag{11.13}$$

which, by duality, suffices to obtain the result for the linear component w of v.

This is easy to do. We introduce the odd extension of φ that, in order to simplify the notation, we still denote by φ. Then, φ satisfies

$$\begin{cases} -\varphi_t = \varphi_{xx} & \text{in} \quad I\!R \times (0,\infty) \\ \varphi(x,T) = \varphi_0(x), & \text{in} \quad I\!R^+, \end{cases} \tag{11.14}$$

where the data φ_0 at time $t = T$ is the odd extension of the data φ_0 given on $I\!R^+$.

The solution φ of (11.14) can be characterized as

$$\varphi(t) = G(T-t) * \varphi_0, \tag{11.15}$$

where G is the Gaussian heat kernel

$$G(x,t) = (4\pi t)^{-1/2}\exp(-x^2/4t). \tag{11.16}$$

Thus,

$$\varphi_x(0,t) = \int_{I\!R} G_x(-y, T-t)\varphi_0(y)dy, \tag{11.17}$$

and

$$|\varphi_x(0,t)| \leq C(T-t)^{-1}||k(y/\sqrt{T-t})||_{W^{s,p}(I\!R)}||\varphi_0(y)||_{W^{-s,p'}(I\!R)}, \tag{11.18}$$

where

$$k(y) = y\exp(-y^2/4). \tag{11.19}$$

It is easy to check that $||k(y/\sqrt{T-t})||_{W^{s,p}(I\!R)}$ is of the order of $C_{s,p}(T-t)^{(\frac{1}{p}-s)/2}$. Thus,

$$|\varphi_x(0,t)| \leq C_{s,p}(T-t)^{-1+\left(\frac{1}{p}-s\right)/2}||\varphi_0(y)||_{W^{-s,p'}(I\!R)}, \tag{11.20}$$

Therefore, (11.12) is guaranteed as soon as $s < 1/p$. By scaling, it is then easy to see that,

$$||w(t)||_{W^{s,p}(I\!R^+)} \leq C_{s,p}t^{-(1-\frac{1}{p}+s)/2}. \tag{11.21}$$

Step 2. *The nonlinear problem (11.8).*

Let us now consider equation (11.8). Its solution can be easily written as follows

$$z(t) = S(t)[v_0] + \int_0^t S(t-s)[gz + gw - ((z+w)^2)]_x(s)ds \tag{11.22}$$

where $S(\cdot)$ is the semigroup generated by the heat equation in $x > 0$ with Dirichlet boundary coditions at $x = 0$. The semigroup S may be computed explicitly. We have:

$$S(t)f(x) = (4\pi t)^{-1/2} \int_0^\infty \big[\exp\big(-|x-y|^2/4t\big) - \exp\big(-|x+y|^2/4t\big)\big] f(y)dy. \tag{11.23}$$

It is sufficient to show that $z(t)$ is bounded in $W^{s,p}(I\!R)$ for all $t > 0$. Indeed, once this is done, by scaling one then gets the decay estimate

$$||z(t)||_{W^{s,p}(I\!R^+)} \le C_{s,p} t^{-(1-\frac{1}{p}+s)/2}. \tag{11.24}$$

We take $W^{s,p}(I\!R^+)$-norms in (11.22). Applying Minkowski's and Young's inequalities and the classical L^p estimates on the heat kernel and its gradient we then have

$$\begin{aligned} ||z(t)||_{W^{s,p}(I\!R^+)} \le C||v_0||_{L^1(I\!R^+)} t^{-(1-\frac{1}{p}+s)/2} \\ +C\int_0^t (t-s)^{-(1+s)/2} ||[gz+gw-((z+w)^2)](s)||_{L^p(I\!R^+)} ds. \end{aligned} \tag{11.25}$$

On the other hand, according to the estimates we have:

$$||g(z+w)(t)||_{L^p(I\!R^+)} \le Ct^{-1/2}||(z+w)(t)||_{L^p(I\!R^+)} \le Ct^{-(2-1/p)/2},$$

$$||(z+w)^2(t)||_{L^p(I\!R)} \le C[||z(t)||_{L^{2p}(I\!R)} + ||w(t)||_{L^{2p}(I\!R)}]^2 \le Ct^{-(1-1/2p)}.$$

Note that the power of t in all these three estimates is the same. Putting all these estimates together we get,

$$||z(t)||_{W^{s,p}(I\!R)} \le C||v_0||_{L^1(I\!R)} t^{-(1-\frac{1}{p}+s)/2} + C\int_0^t (t-s)^{-(1+s)/2} s^{-(1-1/2p)} ds. \tag{11.26}$$

The integral on the right hand side of this inequality is finite for all $0 < t < \infty$ since $s < 1$. Consequently $z(t)$ is bounded in $W^{s,p}(I\!R)$ for all $t > 0$ as we wanted to prove. This completes the proof of Lemma 6.1. ■

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