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ASYMPTOTIC EXPANSION FOR THE GENERALIZED BENJAMIN-BONA-MAHONY-BURGER EQUATION

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Abstract. In this work we obtain the complete asymptotic expansion of solutions, as time goes to infinity, of both the linearized Benjamin-Bona-Mahony-Burger equation and the linear Korteweg-de Vries-Burger equation. We also compute the second term in the asymptotic expansion of the solution to the two-dimensional Benjamin-Bona-Mahony-Burger equation, with quadratic nonlinear term.

1. Introduction and main results

In this work we present some results on the asymptotic expansion, when $t \to \infty$, of the solutions to the n-dimensional Benjamin-Bona-Mahony-Burger equation:

$$\begin{cases} u_t - \Delta u_t - \Delta u + (\vec{b} \cdot \nabla u) = \nabla \cdot F(u), & x \in \mathbb{R}^n, \quad t > 0 \\ u(x,0) = u_0(x)\,. \end{cases} \tag{1.1}$$

Here, $\vec{b} \in \mathbb{R}^n$ and $F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ are a fixed vector and vector field respectively and $\nabla\cdot$ stands for the divergence operator, i.e.,

$$\nabla \cdot F(u) = \sum_{i=1}^{n} \partial_{x_i} F_i(u).$$

Equation (1.1) is an n-dimensional generalization of the well-known Benjamin-Bona-Mahony-Burger (BBMB) equation

$$u_t - u_{xxt} - u_{xx} = u u_x. \tag{1.2}$$

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The BBMB equation in one space dimension is an alternative model for the Korteweg-de Vries-Burger (KdVB) equation

$$u_t - u_{xxx} - u_{xx} = uu_x, \tag{1.3}$$

when describing the propagation of small-amplitude long unidirectional waves in nonlinear dispersive media when dissipative mechanisms are taken into account (see, for example, [1]).

Equation (1.1) is also known as the generalized Benjamin-Bona-Mahony-Burger (gBBMB) equation. The gBBMB equation may also be viewed as a perturbation of the conservation law $u_t - \nabla \cdot F(u) = 0$, adding the terms $-\Delta u_t$ and $-\Delta u$, associated with dispersive and dissipative phenomena, respectively.

The well-posedness and some asymptotic properties of solutions to this problem were studied by G. Karch in [14] and L. Zhang in [21].

The first goal of this paper is to analyze the linear equation

$$\begin{cases} u_t - \Delta u_t - \Delta u + (\vec{b} \cdot \nabla u) &= 0, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u(x,0) &= u_0(x)\,, \end{cases} \tag{1.4}$$

and to obtain the complete asymptotic expansion of solutions.

For the heat equation, this was done by J. Duoandixoetxea and E. Zuazua in [9]. Indeed, it was shown that if u is the solution of the heat equation

$$u(x,t) = (G(t) * u_0)(x),$$

where $G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel and $u_0 \in \mathbb{L}^1(\mathbb{R}^n)$ with $u_0 \in \mathbb{L}^1(1+|x|^k)$ such that $|x|^{k+1} u_0(x) \in \mathbb{L}^p(\mathbb{R}^n)$, $1 \le p \le q \le \infty$, then

$$\begin{aligned} &\Big\| G(t) * u_0 - \sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int x^\alpha u_0(x) dx \Big) D^\alpha G(t) \Big\|_q \\ &\qquad \le C t^{-(\frac{k+1}{2}) - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \big\| \, |x|^{k+1} u_0 \big\|_p, \end{aligned} \tag{1.5}$$

for all $t > 0$. This shows that, for the solutions of the heat equation, a complete asymptotic expansion may be obtained using the moments of the initial data as coefficients and the derivatives of the Gaussian heat kernel as reference profiles.

In this work we show that for the solution of the linearized equation (1.4), besides the terms

$$\sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int x^\alpha u_0(x) dx \Big) D^\alpha G(t)\ ,$$

which correspond to the heat equation, other terms due to the dispersive effects appear in its asymptotic expansion. Our result generalizes that obtained in [14], where the first and second terms of the asymptotic expansion, when $t \to \infty$, of the equation (1.4) were obtained for the case of $n = 1$.

To analyze (1.4) it is convenient to eliminate the convective term of order one in this equation. To do that we introduce the change of variables

$$v(x,t) = u(x + t\vec{b}, t). \tag{1.6}$$

Then, u solves (1.4) if and only if v satisfies

$$\begin{cases} v_t - \Delta v_t - \Delta v + \Delta(\vec{b}\cdot\nabla v) = 0\,, & \text{in } \mathbb{R}^n \times (0,\infty)\,; \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.7}$$

We shall denote by $\widetilde{S}(t)$ the linear semigroup generated by (1.7).

The following holds:

Theorem 1.1. *Let $(N, r) \in \mathbb{N}^2$, and denote*

$$D^3 = \Delta\big(\vec{b}\cdot\nabla\big)\,, \quad M_\alpha(f) = \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int f(x)x^\alpha dx\Big).$$

Then there exists a constant $C = C(N, r, n) > 0$ such that

$$\Big\|\widetilde{S}(t)v_0 - \sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\sum^{*}_{|\alpha|\le N-m-2j} M_\alpha(v_0)D^{3m}\Delta^{2j}D^\alpha G(t)\Big\|_2 \tag{1.8}$$

$$\le C\Big[t^{-([\frac{N}{2}]+1)-\frac{n}{4}}\big\|v_0\big\|_1\Big(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\Big) + t^{-\frac{N+1}{2}-\frac{n}{4}}\big\|v_0\big\|_1$$

$$+t^{-\frac{N+1}{2}-\frac{n}{4}}\sum_{m=1}^{N+1}\big\||x|^m v_0\big\|_1 + e^{-\frac{t}{8}}\big\|v_0\big\|_2 + e^{-\frac{t}{8}}t^{-\frac{n}{4}}\Big(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\Big)\Big(\sum_{j=0}^{[\frac{N}{2}]}\frac{t^{-j}}{j!}\Big)\big\|v_0\big\|_1$$

$$+ \max_{|\alpha|\le N-1}\Big\{|M_\alpha(v_0)|\Big\}t^{-\frac{1+N}{2}-\frac{n}{4}}\sum_{j=1}^{N}\big\||x|K_{2j}\big\|_1\Big],$$

for all $t > 0$ and $v_0 \in \mathbb{L}^2(\mathbb{R}^n)\cap\mathbb{L}^1(\mathbb{R}^n, 1+|x|^{N+1})$, where G is the heat kernel and K_{2j} denotes the Bessel Potential of order $2j$.

Remark 1.2. In (1.8), for a real number $s > 0$, $[s] = \max\{m \in \mathbb{N}\,; m \le s\}$ denotes its integer part and $\sum^{*}_{|\alpha|\le N-m-2j}$ means simply that the couples (m, j) such that $N - m - 2j < 0$ are not being considered in the sum.

Remark 1.3. Several comments are in order:

(i) In Theorem 1.1 we assume that $v_0 \in \mathbb{L}^2(\mathbb{R}^n)$ in addition to $v_0 \in \mathbb{L}^1(\mathbb{R}^n; 1+|x|^{N+1})$. This assumption is not needed in the context of the heat equation. However, when dealing with system (1.7) it is essential to get an exponential estimate on the $\mathbb{L}^2$-norm of the solution. Note, indeed, that in the right hand side of (1.8) there is an exponentially decaying term that is bounded in terms of the $\mathbb{L}^2$-norm of the initial data. This term does not appear when dealing with the asymptotic expansion of solutions of the heat equation.

(ii) In the left hand side of (1.8) we see two different terms in the expansion of $\widetilde{S}(t)v_0$. The first one, $\sum_{|\alpha|\leq N} M_\alpha(v_0)D^\alpha G$, appears also in the asymptotic expansion of the heat equation. The second one is due to the dispersive phenomena.

(iii) In [15] (Corollary 7.1) it is proved that, for $n = 1$, when $v_0 \in \mathbb{L}^p \cap \mathbb{L}^1(1+|x|^2)$, with $M = \int v_0(x)dx$ and $m = \int xv_0(x)dx$, the first term in the asymptotic expansion is $r_1(x,t) = MG(x,t)$ and the second one $r_2(x,t) = -m\partial_x G(x,t) - tMK * \partial_x^3 G(t)$, that is,

$$t^{(1-1/p)/2+1/2}\left\|\widetilde{S}(t)v_0 - r_1(t) - r_2(t)\right\|_p \leq Ct^{-1/2}, \quad \forall t > 1.$$

Of course, when $p = 2$, this is a particular case of Theorem 1.1. In Theorem 1.1 for $N = 1$, $n = 1$, we have replaced $tMK * \partial_x^3 G(t)$ by the first term of its asymptotic expansion, that is $tM\partial_x^3 G(t)$, to get a more explicit expression.

We now study the non-linear problem. We focus on the particular case $n = 2$ and $F(u) = u(\vec{a}\cdot\nabla u)$, for some $\vec{a} \in \mathbb{R}^2$ (which is the typical non-linearity appearing in hydrodynamics [20]). That is, we consider the problem

$$\left\{\begin{array}{rcl} u_t - \Delta u_t - \Delta u + (\vec{b}\cdot\nabla u) & = & u(\vec{a}\cdot\nabla u)\,, \\ u(x,0) & = & u_0(x)\,, \end{array}\right. \tag{1.9}$$

where $t \geq 0$ and $x \in \mathbb{R}^2$.

We compute the second term of the asymptotic expansion of the solution of (1.9) when $t \to \infty$. In [14] it was shown that, in a first approximation, the asymptotic behavior of the solution of (1.9) is given by the heat kernel:

$$t^{\frac{n}{2}(1-\frac{1}{p})}\left\|u(\cdot+tb,t) - MG(\cdot,t)\right\|_p \to 0 \quad \text{when} \quad t\to\infty\,, \tag{1.10}$$

for $1 \leq p < \frac{n}{n-1}$ with $n \geq 2$ and for $p \in [1,\infty]$ if $n = 1$ when $u_0 \in \mathbb{L}^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \cap \mathbb{L}^\infty(\mathbb{R}^n)$.

Our main result is as follows:

Theorem 1.4. *Let* $u_0 \in \mathbb{L}^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$. *Then*

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t} \left\| u(\cdot + t\vec{b}) - \widetilde{S}(t)v_0 + \frac{M^2}{8\pi} \log t \big(\vec{a} \cdot \nabla G(t)\big) \right\|_p \to 0, \tag{1.11}$$

when $t \to \infty$ *for* $1 \le p < 2$.

Of course, this result is to be complemented with Theorem 1.1 that provides a complete expansion of the linear component of $\widetilde{S}(t)v_0$ to obtain a complete description of the first and second terms of v.

This result is similar to that in [22] where the second order term Q in the asymptotic expansion, when $t \to \infty$, of the solutions to the following convection-diffusion equation was calculated for $q > 1 + 1/n$ in order to guarantee a faster decay rate of $u - MG - Q$ (it was already known that MG was the first term in the asymptotic expansion):

$$u_t - \Delta u = \vec{a} \cdot \nabla\left(|u|^{q-1}u\right), \quad x \in \mathbb{R}^n. \tag{1.12}$$

It was shown that the structure of Q and the decay rate depends on the exponent q and varies on the ranges: $1 + 1/n < q < 1 + 2/n$, $q = 1 + 2/n$ and $q > 1 + 2/n$.

Our Theorem 1.4 is similar to the case $q = 2$ and $n = 2$ in [22]. Note that when $n = 2$, $q = 2 = 1 + 2/n$, which is, somehow, a distinguished case in which the second term is still linear but is multiplied by a logarithmic function. In this respect, we recall that, when $n = 1$, G. Karch in [15] studied the equation (1.9) and obtained the second order term for the cases of $2 < q < 3$, $q = 3$, $q > 3$.

This work is organized as follows: In Section 2, we recall results obtained by G. Karch in [14] for the existence and uniqueness. In Section 3, we prove a preliminary result. In Section 4, we prove Theorem 1.1 using, essentially, the Fourier representation of solutions, Plancherel's identity and Taylor's expansion. Section 5, contains a result on the complete asymptotic expansion of the n-dimensional KdV-B linear equation where the same techniques apply. Section 6, is devoted to the proof of Theorem 1.4, based on the ideas in E. Zuazua [22], A. Carpio [7] and G. Karch [15].

The notation to be used is standard. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. For $1 \le p \le \infty$, the $\mathbb{L}^p(\mathbb{R}^n)$-norm of a Lebesgue measurable real-valued functions defined on $\mathbb{R}^n$ is denoted by $\|f\|_p$. If m is a nonnegative integer, $W^{m,p}(\mathbb{R}^n)$ will be the Sobolev space consisting of functions in $\mathbb{L}^p(\mathbb{R}^n)$ whose generalized derivatives up to order m belong also to $\mathbb{L}^p(\mathbb{R}^n)$. The case $p = 2$ deserves the special notation $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$

with the norm $\|u\|_{H^m} \equiv \big(\int_{\mathbb{R}^n} (1+|\xi|^2)^m |\widehat{u}(\xi)|^2 d\xi \big)^{1-2}$. The Fourier transform of u is given by $\mathcal{F}u(\xi) = \widehat{u}(\xi) \equiv \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx$. The heat kernel (the fundamental solution of the heat equation) is denoted by $G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. Moreover, $M = \int_{\mathbb{R}^n} u_0(x)dx$ is called the mass of the solution (one may show that $M = \int_{\mathbb{R}^n} u(x)dx$ is conserved for any $t > 0$), and $m = \int_{\mathbb{R}^n} xu_0(x)dx$ is the first moment of the initial data. For simplicity, we write $\int = \int_{\mathbb{R}^n}$. By C we will denote a generic positive constant, which does not depend on u, x and t, but that may vary from line to line.

2. Existence and uniqueness

The first results on the existence, uniqueness, and regularity of the solutions of the unidimensional problem (1.2) were obtained by T.B. Benjamin, J.L. Bona and J.J. Mahony in [2]. Later on, these results were extended by J.L. Bona and R. Smith in [5] and Ch. J. Amick, J.L. Bona, M.E. Schonbek in [1]. Using semigroup theory, G. Karch in [14] showed the existence of solutions for (1.9).

Let us first recall some properties of the Bessel Potential of order β, K_β, ($\beta > 0$), which will be useful in the study of the equation (1.9). In fact K_β is defined through its Fourier transform:

$$\widehat{K_\beta}(\xi) = \frac{1}{(1+|\xi|^2)^{\beta/2}}, \quad \beta > 0\,. \tag{2.1}$$

The Bessel Potential of order two $K_2(x)$, will be denoted by $K(x)$, i.e., $K(x) = K_2(x)$, and it is the fundamental solution of the operator $I - \Delta$ in $\mathbb{R}^n$. By (2.1) the following composition formula holds

$$K_{\beta_1+\beta_2}(x) = K_{\beta_1} * K_{\beta_2}(x)\,,\ \ \beta_1\ ,\beta_2 > 0. \tag{2.2}$$

By (2.2) and (2.1), we have for example, that

$$K_{2m}(x) = (\underbrace{K * \ldots * K}_{\text{m−times}})(x)\ ,\forall m > 0\,. \tag{2.3}$$

Also, $K_\beta\ ,\nabla K_\beta \in \mathbb{L}^1(\mathbb{R}^n)$, $\int K_\beta(x)dx = 1$ and K_β is a positive radial function, that is, $K_\beta(x) = K_\beta(s)$ with $s = |x|$. The proof of these properties can be found in [19]. If $\beta = 0$, we define

$$K_0 = \delta,\ \text{(Dirac delta), that is,} \quad (K_0 * f)(x) = f(x)\,.$$

Applying the operator $(I - \Delta)^{-1}$ to the equation (1.1), it can be written as

$$\begin{cases} u_t - K * \Delta u - K * (\vec{b}\cdot\nabla u) = K * \nabla\cdot F(u), \quad x \in \mathbb{R}^n\,, \quad t > 0. \\ u(x,0) = u_0(x). \end{cases} \tag{2.4}$$

Taking the Fourier transform in the variable x, and noting that $\widehat{K}(\xi) = 1/(1+|\xi|^2)$, the solution of the linear problem

$$u_t - K * \Delta u - K * (\vec{b} \cdot \nabla u) = 0\,,$$

can be written as the action of a linear semigroup of operators $S(t)$ on the initial data u_0

$$u(x,t) = S(t)u_0(x) = \frac{1}{(2\pi)^n}\int exp\big(t\Phi(\xi) + ix\cdot\xi\big)\widehat{u}_0(\xi)d\xi \tag{2.5}$$

with the phase function

$$\Phi(\xi) = \frac{-|\xi|^2 + i(\vec{b}\cdot\xi)}{1+|\xi|^2}\,.$$

Then the solution of problem (2.4) satisfies the following integral equation obtained from the variation of constants formula

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)K * \nabla\cdot F\big(u(\tau)\big)d\tau\,. \tag{2.6}$$

Theorem 2.1. ([14]) (Global existence). *Let $k=1$ or $k=2$. Assume that α is a number satisfying $1 \le \alpha < \infty$ if $n=2$, and $\frac{2}{n} \le \alpha \le \frac{2}{n-2}$ if $n \ge 3$. Suppose that there exists a positive constant C such that $|F'(s)| \le C(1+|s|^\alpha)$ for all $s \in \mathbb{R}^n$. If $k=1$, let us assume that $2 < p < \infty$ for $n=2$, and $max\{\frac{2}{n-2}, n\} < p < \infty$ in the case $n > 2$. If $k=2$, assume that $n/2 < p < n$ for $n \ge 2$. For $n=1$, we simply suppose that $F \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ an $1 \le p \le \infty$ for $k=1$. Then, for each initial data $u_0 \in W^{k,p}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ there exists a unique global solution u to problem* (1.1).

3. Preliminary results

We now study the asymptotic development of the solution of equation (1.7). Its solution is given by $\widetilde{S}(t)v_0(x) = S(t)v_0(x+t\vec{b})$, that is,

$$v(x,t) = \widetilde{S}(t)v_0(x) = \frac{1}{(2\pi)^n}\int \exp\big(t\widetilde{\Phi}(\xi) + ix\cdot\xi\big)\widehat{v}_0 d\xi \tag{3.1}$$

with the phase function $\widetilde{\Phi}(\xi) = \frac{-|\xi|^2 + i(\vec{b}\cdot\xi)|\xi|^2}{1+|\xi|^2}$. In this section we prove the following Theorem:

Theorem 3.1. *Let $N \in \mathbb{N}$. Then there exists a constant $C = C(N,n) > 0$ such that*

$$\big\|\widetilde{S}(t)v_0 - \sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\sum_{|\alpha|\le N-m-2j}^{*} M_\alpha(v_0)\Big(K_{2m+2j} * D^{3m}\Delta^{2j}D^\alpha G(t)\Big)\big\|_2$$

$$\le C\Big[t^{-([\frac{N}{2}]+1)-\frac{n}{4}}\Big(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\Big)\|v_0\|_1+t^{-\frac{N+1}{2}-\frac{n}{4}}\|v_0\|_1+t^{-\frac{N+1}{2}-\frac{n}{4}}\sum_{m=1}^{N+1}\||x|^m v_0\|_1$$

$$+e^{-\frac{t}{8}}\|v_0\|_2+e^{-\frac{t}{8}}t^{-\frac{n}{4}}\Big(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\Big)\Big(\sum_{j=0}^{[\frac{N}{2}]}\frac{t^{-j}}{j!}\Big)\|v_0\|_1\Big], \tag{3.2}$$

for all $v_0\in\mathbb{L}^2(\mathbb{R}^n)\cap\mathbb{L}^1(\mathbb{R}^n;1+|x|^{N+1})$.

Then, in the following section, to prove Theorem 1.1, we simplify the terms of (3.2) involving $K_{2m+2j}*D^{3m}\Delta^{2j}D^{\alpha}G(t)$ and we replace them by the first term of their asymptotic expansion, i.e., by $D^{3m}\Delta^{2j}D^{\alpha}G(t)$.

For the proof of Theorem 3.1, we will use the four following lemmas in which $\varphi\in\mathcal{C}_0^{\infty}(\mathbb{R}^n)$ denotes a cut-off function such that:

$$|\varphi(\xi)|\le 1,\ \ \forall\,\xi\in\mathbb{R}^n\ ;\quad \varphi(\xi)=1,\ \text{ if }\ |\xi|\le 1/2\ ;\quad \varphi(\xi)=0,\ \text{ if }\ |\xi|>1,$$

and

$$S_{\varphi}(x,t)=\frac{1}{(2\pi)^n}\int\exp\big(t\Phi_0(\xi)+ix\cdot\xi\big)\varphi(\xi)d\xi, \tag{3.3}$$

where $\Phi_0(\xi)=\frac{-|\xi|^2}{1+|\xi|^2}$. The following Lemma is a straightforward extension of [15] Lemma 7.1.

Lemma 3.2. *Let* $N\in\mathbb{N}$. *Then there exists a constant* $C=C(N,n)>0$ *such that*

$$\Big\|\widetilde{S}(t)v_0-\sum_{j=0}^{N}\frac{(-t)^j}{j!}K_{2j}*\big(\Delta(\vec{b}\cdot\nabla)\big)^j S_{\varphi}(t)*v_0\Big\|_2$$
$$\le Ct^{-\frac{1+N}{2}-\frac{n}{4}}\|v_0\|_1+Ce^{-\frac{t}{8}}\|v_0\|_2,$$

for all $t>0$ *and* $v_0\in\mathbb{L}^2\cap\mathbb{L}^1(\mathbb{R}^n)$.

Proof. The proof follows exactly the reasoning from Lemma 7.1 in [15].

Lemma 3.3. *Let* $m\in\mathbb{N}$, $m\ge 1$. *Then there exists a constant* $C=C(N,n)>0$ *such that*

$$\Big\|K_{2m}*\big(\Delta(\vec{b}\cdot\nabla)\big)^m S_{\varphi}(t)*v_0-\sum_{j=0}^{N}\frac{t^j}{j!}K_{2m+2j}*\big(\Delta(\vec{b}\cdot\nabla)\big)^m\Delta^{2j}G(t)*v_0\Big\|_2$$

$$\le Ct^{-(N+1)-\frac{3m}{2}-\frac{n}{4}}\|v_0\|_1+Ce^{-\frac{t}{8}}t^{-\frac{3m}{2}-\frac{n}{4}}\Bigg(\sum_{j=0}^{N}\frac{t^{-j}}{j!}\Bigg)\|v_0\|_1,$$

for all $t>0$ *and* $v_0\in\mathbb{L}^1(\mathbb{R}^n)$.

Proof. Given that $\|K_{2m}\|_1 = 1$, we have

$$\big\|K_{2m} * D^{3m} S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2m+2j} * D^{3m} \Delta^{2j} G(t) * v_0\big\|_2$$
$$\leq \big\|D^{3m} S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G(t) * v_0\big\|_2 \,.$$

We decompose G in two integrals using the cut-off function φ: $G(x,t) = G_\varphi(x,t) + G_{1-\varphi}(x,t)$. Then

$$\big\|D^{3m} S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G(t) * v_0)\big\|_2 \tag{3.4}$$
$$\leq \big\|D^{3m} S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_\varphi(t) * v_0)\big\|_2$$
$$+ \big\|\sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_{1-\varphi}(t) * v_0\big\|_2 = I_1(t) + I_2(t) \,.$$

We estimate $I_1(t)$ and $I_2(t)$ separately.

The term $I_1(t)$: We have

$$\big\|D^{3m} S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_\varphi(t) * v_0\big\|_2$$
$$\leq \big\|D^{3m} S_\varphi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_\varphi(t)\big\|_2 \, \big\|v_0\big\|_1 \,. \tag{3.5}$$

We set

$$g(x,t) = D^{3m} S_\varphi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_\varphi(t).$$

Its Fourier transform is

$$\widehat{g}(\xi,t) = \big(-i(\vec{b}\cdot\xi)|\xi|^2\big)^m \varphi(\xi)\Big(e^{-\frac{t|\xi|^2}{1+|\xi|^2}} - \sum_{j=0}^{N} \frac{t^j}{j!}\Big(\widehat{K}(\xi)|\xi|^4\Big)^j e^{-t|\xi|^2}\Big)$$
$$= \varphi(\xi)\big(-i(\vec{b}\cdot\xi)|\xi|^2\big)^m e^{-t|\xi|^2}\Big[e^{\frac{t|\xi|^4}{1+|\xi|^2}} - \sum_{j=0}^{N} \frac{t^j}{j!}\Big(\frac{|\xi|^4}{1+|\xi|^2}\Big)^j\Big]. \tag{3.6}$$

Now, using the Taylor expansion of the exponential function e^x, $x \geq 0$, we have

$$e^x - \sum_{j=0}^{N} \frac{x^j}{j!} \leq \frac{x^{N+1}}{(N+1)!} e^x.$$

Using this inequality in (3.6), we obtain

$$\begin{aligned} |\widehat{g}(\xi,t)| &\leq C\varphi(\xi)|\xi|^{3m} \frac{e^{-t|\xi|^2}}{(N+1)!} \Big(\frac{t|\xi|^4}{1+|\xi|^2} \Big)^{N+1} e^{\frac{t|\xi|^4}{1+|\xi|^2}} \qquad (3.7) \\ &= C\varphi(\xi)|\xi|^{3m} \frac{e^{-\frac{t|\xi|^2}{1+|\xi|^2}}}{(N+1)!} \Big(\frac{t|\xi|^4}{1+|\xi|^2} \Big)^{N+1}. \end{aligned}$$

From (3.7) and Plancherel's formula we see that

$$\begin{aligned} &\Big\| D^{3m} S_\varphi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m}\Delta^{2j} G_\varphi(t) \Big\|_2^2 = \int |\widehat{g}(\xi,t)|^2 d\xi \\ &\leq C_N \int |\varphi(\xi)|^2 \, |\xi|^{6m} e^{-\frac{2t|\xi|^2}{1+|\xi|^2}} \left(\frac{t|\xi|^4}{1+|\xi|^2} \right)^{2(N+1)} d\xi \qquad (3.8) \\ &= C_N t^{2(N+1)} \int_{|\xi|\leq 1} |\xi|^{6m+8(N+1)} e^{-t|\xi|^2} d\xi \leq C_N t^{-2(N+1)-3m-\frac{n}{2}}. \end{aligned}$$

In (3.8) we use that $|\xi| \leq 1$ implies $2|\xi|^2/(1+|\xi|^2) \geq |\xi|^2$. Then, returning to (3.5), by (3.8) it follows that

$$\begin{aligned} &\Big\| D^{3m} S_\varphi(t) * v_0 - D^{3m} G_\varphi(t) - \sum_{j=1}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_\varphi(t) * v_0 \Big\|_2 \\ &\qquad\qquad \leq C_N t^{-(N+1)-\frac{3m}{2}-\frac{n}{4}} \big\| v_0 \big\|_1. \qquad (3.9) \end{aligned}$$

The term $I_2(t)$: Since $\| K_{2j}\|_1 = 1$, we have

$$\Big\| \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_{1-\varphi}(t) * v_0 \Big\|_2 \leq \sum_{j=0}^{N} \frac{t^j}{j!} \big\| D^{3m}\Delta^{2j} G_{1-\varphi}(t) \big\|_2 \big\| v_0 \big\|_1, \qquad (3.10)$$

and moreover $|\xi| \geq 1/2 \Rightarrow 2t|\xi|^2 \geq (1/4)(t + t|\xi|^2)$. Then

$$\begin{aligned} \big\| D^{3m}\Delta^{2j} G_{1-\varphi}(t) \big\|_2^2 &= \int \Big| |\xi|^{4j+3m} e^{-t|\xi|^2} (1-\varphi) \Big|^2 d\xi \\ &= \int_{|\xi|\geq 1/2} |\xi|^{8j+6m} e^{-2t|\xi|^2} d\xi \end{aligned}$$

$$\leq e^{-\frac{t}{4}} \int |\xi|^{8j+6m} e^{-\frac{t|\xi|^2}{4}} d\xi \leq C e^{-\frac{t}{4}} t^{-4j-3m-\frac{n}{2}}.$$

Hence, returning to (3.10), we obtain

$$\Big\| \sum_{j=0}^{N} \frac{t^j}{j!} K_{2j} * D^{3m} \Delta^{2j} G_{1-\varphi}(t) * v_0 \Big\|_2 \leq C \sum_{j=0}^{N} \frac{t^j}{j!} e^{-\frac{t}{8}} t^{-2j-\frac{3m}{2}-\frac{n}{4}} \big\|v_0\big\|_1$$

$$\leq C e^{-\frac{t}{8}} t^{-\frac{3m}{2}-\frac{n}{4}} \Big(\sum_{j=1}^{N} \frac{t^{-j}}{j!} \Big) \big\|v_0\big\|_1. \tag{3.11}$$

Lemma 3.3 is consequence of (3.9) and (3.11). □

Lemma 3.4. *Let $(k, m, j) \in \mathbb{N}^3$. Then there exists a positive constant $C = C(k, m, j, n)$ such that*

$$\Big\| \big(\Delta(\vec{b}\cdot\nabla)\big)^m \Delta^{2j} G(t) * v_0 - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int v_0 x^\alpha dx \Big) \big(\Delta(\vec{b}\cdot\nabla)\big)^m \Delta^{2j} D^\alpha G(t) \Big\|_2$$

$$\leq C t^{-\frac{k+1+3m+4j}{2}-\frac{n}{4}} \Big\| |x|^{k+1} v_0 \Big\|_1, \tag{3.12}$$

for all $t > 0$ and $v_0 \in \mathbb{L}^1(\mathbb{R}^n, 1 + |x|^{k+1})$.

Proof. The proof consists in a direct application of the asymptotic formula from [9]

$$v_0 = \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int v_0 x^\alpha dx \Big) D^\alpha \delta + \sum_{|\alpha| = k+1} D^\alpha F_\alpha,$$

where $F_\alpha \in \mathbb{L}^1(\mathbb{R}^n)$, such that $\|F_\alpha\|_1 \leq \||x|^{1+k} v_0\|_1$. Indeed, it is sufficient to apply the convolution with $(\Delta(\vec{b} \cdot \nabla))^m \Delta^{2j} G(x, t)$ in this identity and to take L^2-norms. □

Proof of Theorem 3.1. From Lemma 3.2, it follows that

$$\Big\| \widetilde{S}(t) v_0 - \sum_{m=0}^{N} \frac{(-t)^m}{m!} K_{2m} * D^{3m} S_\varphi(t) * v_0 \Big\|_2$$

$$\leq C t^{-\frac{n}{4}-\frac{1+N}{2}} \|v_0\|_1 + C e^{-\frac{t}{8}} \|v_0\|_2. \tag{3.13}$$

Moreover, from Lemma 3.3, we have that

$$\Big\| \sum_{m=0}^{N} \frac{(-t)^m}{m!} K_{2m} * D^{3m} S_\varphi(t) * v_0 - \sum_{m=0}^{N} \frac{(-t)^m}{m!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} K_{2m+2j} * D^{3m} \Delta^{2j} G(t) * v_0 \Big\|_2$$

$$\leq Ct^{-([\frac{N}{2}]+1)-\frac{n}{4}}\left(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\right)\|v_0\|_1 + Ce^{-\frac{t}{8}}t^{-\frac{n}{4}}\left(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\right)\left(\sum_{j=0}^{[\frac{N}{2}]}\frac{t^{-j}}{j!}\right)\|v_0\|_1 . \tag{3.14}$$

Then from (3.13) and (3.14) it follows that

$$\begin{aligned}&\Big\|\widetilde{S}(t)v_0 - \sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}K_{2m+2j} * D^{3m}\Delta^{2j}G(t) * v_0\Big\|_2 \qquad (3.15)\\ &\leq Ct^{-\frac{n}{4}-\frac{1+N}{2}}\|v_0\|_1 + +Ct^{-([\frac{N}{2}]+1)-\frac{n}{4}}\left(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\right)\|v_0\|_1\\ &+ Ce^{-\frac{t}{8}}\|v_0\|_2 + Ce^{-\frac{t}{8}}t^{-\frac{n}{4}}\left(\sum_{m=0}^{N}\frac{t^{-\frac{m}{2}}}{m!}\right)\left(\sum_{j=0}^{[\frac{N}{2}]}\frac{t^{-j}}{j!}\right)\|v_0\|_1.\end{aligned}$$

On the other hand, from Lemma 3.4, we obtain

$$\begin{aligned}&\Big\|\sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}K_{2m+2j} * D^{3m}\Delta^{2j}G(t) * v_0 \qquad (3.16)\\ &-\sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\sum_{|\alpha|\leq k(m,j)}M_\alpha(v_0)K_{2m+2j} * D^{3m}\Delta^{2j}D^\alpha G(t)\Big\|_2\\ &\leq \sum_{m=0}^{N}\frac{t^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\Big\|K_{2m+2j}\ D^{3m}\Delta^{2j}G(t) * v_0\\ &-\sum_{|\alpha|\leq k(m,j)}M_\alpha(v_0)K_{2m+2j}\ D^{3m}\Delta^{2j}D^\alpha G(t)\Big\|_2\\ &\leq C\sum_{m=0}^{N}\frac{t^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\,t^{-\frac{k(m,j)+1+3m+4j}{2}-\frac{n}{4}}\Big\|\,|x|^{k(m,j)+1}v_0\Big\|_1 .\end{aligned}$$

In (3.16), $k(m,j)$ denotes a natural number and we use that $\|K_{2m+2j}\|_1 = 1$. We choose $k = N - m - 2j \geq 0$ in (3.16). Then

$$\Big\|\sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}K_{2m+2j} * D^{3m}\Delta^{2j}G(t) * v_0 \tag{3.17}$$

$$- \sum_{m=0}^{N} \frac{(-t)^m}{m!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{|\alpha|\le N-m-2j}^{*} M_\alpha(v_0) K_{2m+2j} * D^{3m}\Delta^{2j} D^\alpha G(t) \big\|_2$$

$$\le Ct^{-\frac{N+1}{2}-\frac{n}{4}} \sum_{m=0}^{N} \sum_{\substack{0\le j\le [\frac{N}{2}] \\ 0\le j\le [\frac{N-m}{2}]}} \big\| |x|^{N+1-m-2j} v_0 \big\|_1 .$$

Observe that

$$\sum_{m=0}^{N} \sum_{\substack{0\le j\le [\frac{N}{2}] \\ 0\le j\le [\frac{N-m}{2}]}} \big\| |x|^{N+1-m-2j} v_0 \big\|_1 = \sum_{m=1}^{N+1} \big\| |x|^{m} v_0 \big\|_1 ,$$

given that, for the coefficients entering in the sums one necessarily has, $1 \le N+1-m-2j \le N+1$. Finally, Theorem 3.1 is consequence of (3.15) and (3.17).

4. Complete asymptotic expansion for the gBBMB linear equation

This section is devoted to the proof of Theorem 1.1. The function entering in the $\mathbb{L}^2$-estimate of Theorem 3.1 can be written as

$$\widetilde{S}(t)v_0 - \sum_{|\alpha|\le N} M_\alpha(v_0) D^\alpha G(t) \tag{4.1}$$
$$- \sum_{m=0}^{N} \frac{(-t)^m}{m!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{|\alpha|\le N-m-2j \\ (m,j)\ne(0,0)}}^{*} M_\alpha(v_0)\, K_{2m+2j} * D^{3m}\Delta^{2j} D^\alpha G(t).$$

In this section, we simplify the terms of (4.1) that have the form

$$K_{2m+2j} * D^{3m}\Delta^{2j} D^\alpha G(t). \tag{4.2}$$

If α is an arbitrary multi-index we have

$$\int_{\mathbb{R}^n} x^\alpha K_\beta(x) dx = i^{|\alpha|} \left(D^\alpha \widehat{K_\beta} \right)(0) \ , \ \beta > 0.$$

Observing that

$$D^\alpha \widehat{K_\beta}(0) = \begin{cases} i^{|\alpha|} C_{\alpha\beta} & \text{for} \quad |\alpha| \quad \text{even} \\ 0 & \text{for} \quad |\alpha| \quad \text{odd} \ , \end{cases}$$

where $C_{\alpha\beta}$ it is a positive constant, we have

$$\int_{\mathbb{R}^n} x^\alpha K_\beta(x)dx = \begin{cases} C_{\alpha\beta}\,, & \text{for} \quad |\alpha| \quad \text{even} \\ 0 & \text{for} \quad |\alpha| \quad \text{odd}\,. \end{cases} \tag{4.3}$$

On the other hand, the solution of

$$u_t - \Delta u = 0, \quad u(x,0) = K_\beta(x), \quad \beta > 0, \tag{4.4}$$

is given by $u(x,t) = (K_\beta * G(t))(x)$. Keeping (4.3) in mind, we have $K_\beta \in \mathbb{L}^1\big(\mathbb{R}^n, (1+|x|^{r+1}\big)$, since

$$\int |x|^{r+1}|K_\beta(x)|dx \leq C\int_0^\infty s^{r+n}|K_\beta(s)|ds < \infty\,, \quad \text{where} \quad s = |x|\,.$$

As consequence of Theorem 1.5, we have the following:

Corollary 4.1. *Let $r \in \mathbb{N}$. Then there exists a constant $C = C(r,n) > 0$ such that*

$$\Big\|G(t)*K_\beta - \sum_{|\gamma|\leq r}\frac{(-1)^{|\gamma|}}{\gamma!}\Big(\int x^\gamma K_\beta(x)dx\Big)D^\gamma G(t)\Big\|_2 \leq Ct^{-\frac{r+1}{2}-\frac{n}{4}}\Big\||x|^{r+1}K_\beta\Big\|_1.$$

If instead of $G(t)$, we consider $D^{3m}\Delta^{2j}D^\alpha G(t)$ in the previous estimate we obtain the following result:

Corollary 4.2. *Let $r \in \mathbb{N}$. Then there exists a constant $C = C(r,m,j,n) > 0$ such that*

$$\begin{aligned} &\Big\|D^{3m}\Delta^{2j}D^\alpha G(t) * K_{2(m+j)} \\ &\quad - \sum_{|\gamma|\leq r}\frac{(-1)^{|\gamma|}}{\gamma!}\Big(\int x^\gamma K_{2(m+j)}(x)dx\Big)D^\gamma D^{3m}\Delta^{2j}D^\alpha G(t)\Big\|_2 \\ &\leq Ct^{-\frac{r+1+3m+4j+|\alpha|}{2}-\frac{n}{4}}\Big\||x|^{r+1}K_{2(m+j)}\Big\|_1, \quad (m,j)\in\mathbb{N}^2 \;\; /m+j\neq 0. \end{aligned}$$

And, as an immediate consequence of Corollary 4.2 we have:

Corollary 4.3. *Let $r \in \mathbb{N}$. Then there exists a constant $C = C(r,m,j,n) > 0$ such that*

$$\begin{aligned} &\Big\|\sum_{m=0}^{N}\frac{(-t)^m}{m!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}\sum_{|\alpha|\leq N-m-2j}^{*} M_\alpha(v_0)\Big(K_{2(m+j)} * D^{3m}\Delta^{2j}D^\alpha G(t) \\ &- \sum_{|\gamma|\leq r} M_\gamma\big(K_{2(m+j)}\big)D^\gamma D^{3m}\Delta^{2j}D^\alpha G(t)\Big)\Big\|_2 \end{aligned}$$

$$\leq C \sum_{m=0}^{N} \frac{t^m}{m!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{|\alpha| \leq N-m-2j}^{*} \big|M_\alpha(v_0)\big|\, t^{-\frac{r+1+3m+4j+|\alpha|}{2}-\frac{n}{4}} \big\||x|^{r+1} K_{2(m+j)}\big\|_1$$

$$\leq C \max_{|\alpha|\leq N-1} \Big\{\big|M_\alpha(v_0)\big|\Big\} t^{-\frac{r+1+N}{2}-\frac{n}{4}} \sum_{j=1}^{N} \big\||x|^{r+1} K_{2j}\big\|_1 .$$

Then the proof of Theorem 1.1 follows directly from Corollary 4.3, equation (4.1) and Theorem 3.1:

$$\Big\|S(t)v_0 - \sum_{|\alpha|\leq N} M_\alpha(v_0) D^\alpha G(t) \tag{4.5}$$

$$- \sum_{m=0}^{N} \frac{(-t)^m}{m!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{|\alpha|\leq N-m-2j \\ (m,j)\neq(0,0)}}^{*} M_\alpha(v_0) \sum_{|\gamma|\leq r} M_\gamma\big(K_{2(m+j)}\big) D^\gamma D^{3m} \Delta^{2j} D^\alpha G(t)\Big\|_2$$

$$\leq Ct^{-([\frac{N}{2}]+1)-\frac{n}{4}} \big\|v_0\big\|_1 \Big(\sum_{m=0}^{N} \frac{t^{-\frac{m}{2}}}{m!}\Big) + Ct^{-\frac{N+1}{2}-\frac{n}{4}} \big\|v_0\big\|_1 + Ct^{-\frac{N+1}{2}-\frac{n}{4}} \sum_{m=1}^{N+1} \big\||x|^m v_0\big\|_1$$

$$+ Ce^{-\frac{t}{8}} \big\|v_0\big\|_2 + Ce^{-\frac{t}{8}} t^{-\frac{n}{4}} \Bigg(\sum_{m=0}^{N} \frac{t^{-\frac{m}{2}}}{m!}\Bigg)\Bigg(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!}\Bigg) \big\|v_0\big\|_1$$

$$+ \ C \max_{\big|\alpha\big|\leq N-1} \Big\{|M_\alpha(v_0)|\Big\}\ t^{-\frac{r+1+N}{2}-\frac{n}{4}} \sum_{j=1}^{N} \big\||x|^{r+1} K_{2j}\big\|_1 .$$

Notice that in (4.5) the best choice of r is $r = 0$ because, in this case, all the terms in the right member of (4.5) have the same decay rate. When $r \geq 1$, the last term on the right-hand side of (4.5) will have a better decay and therefore it will be "negligible" compared with the others. □

Remark 4.4. We have, for example, the following particular results depending on the number of terms we take in the asymptotic development:
(a) If $N = 0$ in Theorem 1.1, with $M = \int v_0(x)dx$, the first term in the asymptotic expansion is $r_1(x,t) = MG(t)$. Then for $v_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n, 1+|x|)$ we get

$$t^{\frac{n}{4}} \big\|\widetilde{S}(t)v_0 - MG(t)\big\|_2 \leq Ct^{-\frac{1}{2}}, \quad \text{for} \quad t \geq 1.$$

(b) If $N = 1$ in Theorem 1.1, with $m = \int x v_0(x)dx$, the first term in the asymptotic expansion is $r_1(x,t) = MG(t)$ and the second one is $r_2(x,t) = -\ m.\nabla G(t) - tM\Delta(\vec{b} \cdot \nabla G(t)$. Then for $v_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n, 1+|x|^2)$ we have

$$t^{\frac{n}{4}+\frac{1}{2}}\big\|\widetilde{S}(t)v_0 - \sum_{j=1}^{2} r_j(t)\big\|_2 \le Ct^{-\frac{1}{2}}, \quad \text{for} \quad t \ge 1 .$$

(c) If $N = 2$ in Theorem 1.1, we get that the first term is $r_1(x,t) = MG(t)$, the second one being $r_2(x,t) = -m \cdot \nabla G(t) - tM\Delta\big(\vec{b} \cdot \nabla G(t)\big)$. The third one is

$$r_3(x,t) = \sum_{|\alpha|=2} \frac{1}{\alpha!}\Big(\int x^\alpha v_0 dx\Big) D^\alpha G(t) + t\Delta\big(\vec{b}\cdot\nabla\big)m \cdot \nabla G(t) + tM\Delta^2 G(t)$$
$$+ \frac{t^2}{2} M\big(\Delta(\vec{b}\cdot\nabla)\big)^2 G(t).$$

Then for $v_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n, 1+|x|^3)$, we have

$$t^{\frac{n}{4}+1}\big\|\widetilde{S}(t)v_0 - \sum_{j=1}^{3} r_j(t)\big\|_2 \le Ct^{-\frac{1}{2}}, \quad \text{for} \quad t \ge 1 .$$

On the other hand, in view of the density of $\mathbb{L}^1(\mathbb{R}^n, 1+|x|)$ in $\mathbb{L}^1(\mathbb{R}^n)$, in [23] it is proven that if $v_0 \in \mathbb{L}^1(\mathbb{R}^n)$, then

$$t^{\frac{n}{4}}\big\|G(t) * v_0 - MG(t)\big\|_2 \to 0, \quad \text{when} \quad t \to \infty . \tag{4.6}$$

Hence, as a consequence of the Remark 4.4 (a), we have the result in Proposition 2.1 of [14] in the case $p = 2$. Indeed:

Theorem 4.5. *Let $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^1(\mathbb{R}^n)$. Then the solution $v(x,t) = \widetilde{S}(t)v_0(x)$ of* (1.4) *satisfies*

$$t^{\frac{n}{4}}\big\|\widetilde{S}(t)v_0 - MG(t)\big\|_2 \to 0, \quad \textit{when} \quad t \to \infty. \tag{4.7}$$

Proof. The proof is similar to that of Theorem 3.1. Indeed, if $N = 0$, from Lemmas 3.2 and 3.3 (with $m = 0$), we have

$$t^{\frac{n}{4}}\big\|S_\varphi(t) * v_0 - G(t) * v_0\big\|_2 \to 0, \quad \text{when} \quad t \to \infty , \tag{4.8}$$

$$t^{\frac{n}{4}}\big\|\widetilde{S}(t)v_0 - S_\varphi(t) * v_0\big\|_2 \to 0, \quad \text{when} \quad t \to \infty . \tag{4.9}$$

Hence, (4.8) and (4.9) imply that

$$t^{\frac{n}{4}}\big\|\widetilde{S}(t)v_0 - G(t) * v_0\big\|_2 \to 0, \quad \text{when} \quad t \to \infty . \tag{4.10}$$

Then from (4.10) and (4.6) it follows that $t^{\frac{n}{4}}\big\|S(t)v_0 - MG(t)\big\|_2 \to 0$, when $t \to \infty$.

5. Complete asymptotic expansion of the n-dimensional KdVB linear equation

Let us consider now the n-dimensional version of the linearized Korteweg-de Vries-Burger equation

$$\begin{cases} u_t + \Delta(\vec{a}\cdot\nabla u) - \Delta u = 0, & x\in\mathbb{R}^n, t>0 \\ u(x,0) = u_0(x)\,. \end{cases} \tag{5.1}$$

Its solution takes the form $u(x,t) = T(\cdot,t) * u_0(x)$, with

$$T(x,t) = \frac{1}{(2\pi)^n}\int e^{t\psi(\xi)+ix\cdot\xi}d\xi.$$

The phase function is now $\psi(\xi) = -|\xi|^2 + i|\xi|^2(\vec{a}\cdot\xi)$, which is a bit simpler than the phase function $\widetilde{\Phi}$ of the linear gBBMB equation. Proceeding as in the proof of Theorem 3.1 of Section 2 the following result can be proved:

Theorem 5.1. *For any $N\in\mathbb{N}$, there exists a constant $C = C(N) > 0$ such that*

$$\Big\|T(t)*u_0 - \sum_{k=0}^{N}\frac{(-t)^k}{k!}\big(\Delta(\vec{a}\cdot\nabla)\big)^k\sum_{|\alpha|\le N-k}M_\alpha(u_0)D^\alpha G(t)\Big\|_2 \le C\,t^{-\frac{n}{4}-\frac{N+1}{2}}\big\|u_0\big\|_1 + C\,t^{-\frac{n}{4}-\frac{N+1}{2}}\sum_{k=1}^{N+1}\big\|\,|x|^k u_0\big\|_1\,,$$

for all $t>0$ and $u_0\in\mathbb{L}^1(\mathbb{R}^n, 1+|x|^{N+1})$.

The proof of this result is very close to the one of Theorem 1.1 on the gBBMB equation. Therefore, we omit the details.

Remark 5.2. For instance, if $N = 2$, in Theorem 5.1 we have that the first term is $r_1(x,t) = MG(t)$. The second term is $r_2(x,t) = -\,m\nabla G(t) - tM\Delta\big(\vec{a}\cdot\nabla G(t)\big)$, and the third one

$$r_3(x,t)=\sum_{|\alpha|=2}\frac{1}{\alpha!}\Big(\int x^\alpha v_0 dx\Big)D^\alpha G(t) + t\Delta(\vec{a}\cdot\nabla)m\cdot\nabla G(t) + \frac{t^2}{2}M(\Delta(\vec{a}\cdot\nabla))^2G(t).$$

Then for $u_0\in\mathbb{L}^1(\mathbb{R}, 1+|x|^3)$, we have

$$t^{\frac14+1}\|T(t)u_0 - \sum_{j=1}^{3}r_j(t)\|_2 \le Ct^{-\frac12}\,, \quad \text{for} \quad t\ge 1\,.$$

Remark 5.3. In agreement with Theorems 1.1 and 5.1, the successive terms appearing in the asymptotic expansion of the solutions of (1.7) and (5.1) have the form

$$\sum_{m=0}^{N} \frac{(-t)^m}{m!} \left(\Delta(\vec{b}\cdot\nabla)\right)^m \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!}\, \Delta^{2j} \sum_{|\alpha|\le N-m-2j}^{*} M_\alpha(v_0) D^\alpha G(t) \tag{5.2}$$

and

$$\sum_{k=0}^{N} \frac{(-t)^k}{k!} \left(\Delta(\vec{a}\cdot\nabla)\right)^k \sum_{|\alpha|\le N-k} M_\alpha(u_0) D^\alpha G(t)\,, \tag{5.3}$$

respectively. We see that, the term due to dispersive effects of $\Delta(\vec{a}\cdot\nabla)u$ in (5.1) and $\Delta(\vec{a}\cdot\nabla)v$ and Δv_t in (1.7), appear in the asymptotic expansions starting at the second term.

6. Quadratic nonlinear term in two space dimensions: First and second term of the asymptotic expansion

In this section we calculate the first and second order terms in the asymptotic expansion of the solutions of (1.9) in $\mathbb{R}^2$. We begin by a change of variables defining $v(x,t) = u(x + t\vec{b}, t)$. Then $v = v(x,t)$ satisfies the new equation

$$\begin{cases} v_t - \Delta v_t - \Delta v + \Delta(\vec{b}\cdot\nabla v) = v(\vec{a}\cdot\nabla v), & \text{in } \mathbb{R}^2\times(0,\infty) \\ v(x,0) = v_0(x), & \text{in } \mathbb{R}^2. \end{cases} \tag{6.1}$$

The solution of (1.9) may also be characterized by means of the integral equation

$$v(t) = \widetilde{S}(t)v_0 + \frac{1}{2}\int_0^t \widetilde{S}(t-\tau)\vec{a}\cdot\nabla K * v^2(\tau)d\tau\,. \tag{6.2}$$

Recall that K is the Bessel Potential of order two such that $\widehat{K}(\xi) = \frac{1}{1+|\xi|^2}$. It is also convenient to recall a result by L. Zhang in [21], obtained applying the Fourier-Splitting method introduced by M. Schonbek in the case of the parabolic conservation laws [16] and the Navier-Stokes equation [17] [18], on the equation:

$$\begin{cases} u_t - \Delta u_t - \Delta u + \vec{b}\cdot\nabla u = \varphi(u)\cdot\nabla u, & \text{in } \mathbb{R}^2\times(0,\infty) \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^2\,, \end{cases} \tag{6.3}$$

with $\varphi \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^2)$ satisfying the following conditions:

$$\begin{aligned} &|\varphi(u)| \le c|u| \quad \text{for} \quad |u| \quad \text{small}, \\ &|\varphi(u)| \le c|u|^p \quad \text{for} \quad |u| \quad \text{large}, \quad p\ge 1\,(\text{integer}). \end{aligned} \tag{6.4}$$

The following was proved:

Theorem 6.1. ([21]) *Let* $u_0 \in \mathbb{L}^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$ *and* $\varphi \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^2)$ *satisfying* (6.4). *Then the solution of* (6.3) *satisfies the following decay properties:*

$$\big\|u(t)\big\|_2 \le C(1+t)^{-1/2}; \quad \big\|\nabla u(t)\big\|_2 \le C(1+t)^{-1}; \quad \big\|u(t)\big\|_\infty \le C(1+t)^{-1}, \tag{6.5}$$

where $C > 0$ *depends on the* $\mathbb{L}^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$*-norm of* u_0 *and on the nonlinearity* φ.

Let us notice that the function involved in (6.3) can be written as $\varphi(s) = \vec{a}s$, so that the conditions (6.4) are verified for $p = 1$. Thus, we are dealing with a quadratic nonlinearity.

We now prove some preliminary results which will be useful in the proof of Theorem 1.4.

Theorem 6.2. *Let* $v = v(x,t)$ *be the solution of* (6.1) *corresponding to the initial data* $v_0 \in \mathbb{L}^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$. *Then*

$$t^{(1-\frac{1}{p})}\big\|v(t) - MG(\cdot,t)\big\|_p \to 0 \quad \textit{when} \quad t \to \infty\,, \tag{6.6}$$

for $1 \le p < \infty$, *with* $M = \int u_0(x)dx$.

Proof. From (6.5) we have $\|v(t)\|_\infty \le Ct^{-1}$ and, as $\|G(t)\|_\infty \le Ct^{-1}$, then

$$\big\|v(t) - MG(t)\big\|_\infty \le Ct^{-1}\,, \forall\, t > 0\,. \tag{6.7}$$

Moreover, by (1.10), with $p = 1$, we have

$$\big\|v(t) - MG(t)\big\|_1 \to 0\,, \quad \text{when} \quad t \to \infty\,. \tag{6.8}$$

On the other hand, if $1 \le p < \infty$ by Gagliardo-Nirenberg inequality we deduce that

$$\big\|v - MG\big\|_p \le \big\|v - MG\big\|_\infty^{\frac{p-1}{p}} \big\|v - MG\big\|_1^{\frac{1}{p}}\,.$$

Hence, from (6.7) it follows that

$$\big\|v(t) - MG(t)\big\|_p \le Ct^{-\frac{p-1}{p}} \big\|v(t) - MG(t)\big\|_1^{\frac{1}{p}}\,.$$

Therefore, from (6.8), we obtain

$$t^{1-\frac{1}{p}}\big\|v(t) - MG(t)\big\|_p \le C\big\|v(t) - MG(t)\big\|_1^{\frac{1}{p}} \to 0\,, \quad \text{when} \quad t \to \infty\,.$$

This finishes the proof of Theorem 6.2. □

Remark 6.3. Theorem 6.2 for $1 \le p < 2$ is a direct application of [14, Th.2.2 and Prop.2.1].

The following Lemma is used for the proof of Lemma 6.5.

Lemma 6.4. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be such that $\varphi(\xi) = 0$, for $|\xi| \geq 1$. Let moreover*

$$J_\alpha(x,t) = \int_{\mathbb{R}^2} \xi^\alpha \varphi(\xi) e^{ix\cdot\xi} \Big(e^{-t\widetilde{\Phi}(\xi)} \widehat{K}(\xi) - e^{-t|\xi|^2} \Big) d\xi.$$

Then there exists a positive constant C independent of t such that

$$\big\| J_\alpha(t) \big\|_p \leq C(1+t)^{-(1-\frac{1}{p})-(\frac{|\alpha|}{2}+\frac{1}{2})}, \quad \forall\, t>0\,, \quad \forall\, 1 \leq p \leq \infty. \tag{6.9}$$

Proof. The complete proof is based on the methods from Karch [14]. We omit the details. Roughly speaking, one obtains an estimate for $\big\|J_\alpha(t)\big\|_2$, $\big\|J_\alpha(t)\big\|_1$ and $\big\|J_\alpha(t)\big\|_\infty$ using the interpolation inequalities

$$\big\|w\big\|_\infty \leq C\big\|w\big\|_2^{1-\frac{n}{2N}} \sum_{|k|=N} \big\|\partial_x^k w\big\|_2^{\frac{n}{2N}} \text{ and } \big\|\widehat{w}\big\|_1 \leq C\big\|w\big\|_2^{1-\frac{n}{2N}} \sum_{|k|=N} \big\|\partial_x^k w\big\|_2^{\frac{n}{2N}}.$$

The remaining estimates are obtained applying the classical inequality

$$\big\|J_\alpha(t)\big\|_p \leq \big\|J_\alpha(t)\big\|_\infty^{1-1/p} \big\|J_\alpha(t)\big\|_1^{1/p}. \qquad \square$$

Lemma 6.5. *Let $G(x,t)$ be the heat kernel. Then*

$$\big\|\widetilde{S}(t)K - G(t)\big\|_p \leq Ct^{-(1-\frac{1}{p})-\frac{1}{2}}\,, if \quad 1 \leq p < \infty\,; \tag{6.10}$$

$$\big\|\widetilde{S}(t)\vec{a}\cdot\nabla K - \vec{a}\cdot\nabla G(t)\big\|_p \leq Ct^{-(1-\frac{1}{p})-1}\,, if \quad 1 \leq p < 2\,. \tag{6.11}$$

The inequality (6.10) for $p = 2$ is a particular case of Theorem 1.1 with $v_0 = K$.

Proof. We first prove (6.11). We have

$$\widetilde{S}(t)\vec{a}\cdot\nabla K - \vec{a}\cdot\nabla G(t) = \int_{\mathbb{R}^2} \varphi(\xi)\big(\xi\cdot\vec{a}\big)\Big(e^{-t\widetilde{\Phi}(\xi)}\widehat{K}(\xi) - e^{-t|\xi|^2}\Big)e^{ix\cdot\xi}d\xi$$
$$+ \int_{\mathbb{R}^2} \big(1-\varphi(\xi)\big)\big(\xi\cdot\vec{a}\big)\Big(e^{-t\widetilde{\Phi}(\xi)}\widehat{K}(\xi) - e^{-t|\xi|^2}\Big)e^{ix\cdot\xi}d\xi = F_1(x,t) + F_2(x,t). \tag{6.12}$$

Using (6.9) with $|\alpha| = 1$, we obtain

$$\big\|F_1(t)\big\|_p \leq Ct^{-1-(1-\frac{1}{p})}, \quad \forall \quad 1 \leq p \leq \infty\,.$$

Now, the term

$$F_2(t) = \int_{\mathbb{R}^2} \big(1-\varphi(\xi)\big)\big(\xi\cdot\vec{a}\big)\Big(e^{-t\widetilde{\Phi}(\xi)}\widehat{K}(\xi) - e^{-t|\xi|^2}\Big)e^{ix\cdot\xi}d\xi$$

is a sum of two terms whose $\mathbb{L}^p$-norms decay exponentially (away from a neighborhood of $x=0$) for $1 \le p < 2$. We refer to [14] for more details on the analysis of the term $\int_{\mathbb{R}^2} \big(1-\varphi(\xi)\big)\big(\xi \cdot \vec{a}\big) e^{-t\widetilde{\Phi}(\xi)+ix\cdot\xi}\widehat{K}(\xi)d\xi$.

We now prove (6.10). We have

$$\widetilde{S}(t)K(x) - G(x,t) = \int_{\mathbb{R}^2} \varphi(\xi)\Big(e^{-t\widetilde{\Phi}(\xi)}\widehat{K}(\xi) - e^{-t|\xi|^2}\Big)e^{ix\cdot\xi}d\xi$$
$$+ \int_{\mathbb{R}^2} \big(1-\varphi(\xi)\big)\Big(e^{-t\widetilde{\Phi}(\xi)}\widehat{K}(\xi) - e^{-t|\xi|^2}\Big)e^{ix\cdot\xi}d\xi = Y_1(x,t) + Y_2(x,t)\,.$$

Using (6.9) with $|\alpha| = 0$, we obtain

$$\big\|Y_1(t)\big\|_p \le Ct^{-\frac{1}{2}-(1-\frac{1}{p})}, \quad \forall \quad 1 \le p \le \infty\,.$$

Now, observe that Y_2 is the sum of two terms which are exponentially decaying in $\mathbb{L}^p$ (away from a neighborhood of $x=0$) for $1 \le p < \infty$. We refer to [14] for more details on the analysis of $\int_{\mathbb{R}^2} \big(1-\varphi(\xi)\big)e^{-t\widetilde{\Phi}(\xi)+ix\cdot\xi}\widehat{K}d\xi$. $\square$

Lemma 6.6. *Let $v(x,t)$ be the solution to* (6.1). *Then there exists a constant C such that*

$$\big\|\widetilde{S}(t-\tau)\vec{a}\cdot\nabla K * v^2(\tau)\big\|_p \le C \begin{cases} (t-\tau)^{-(1-\frac{1}{p})-\frac{1}{2}}(1+\tau)^{-1}\,, 1 \le p < 2; \\ (t-\tau)^{-(1-\frac{1}{p})}(1+\tau)^{-3/2}\,, 1 \le p < \infty, \end{cases} \tag{6.13}$$

and

$$\big\|\,\widetilde{S}(t-\tau)\vec{a}\cdot\nabla K - \vec{a}\cdot\nabla G(t-\tau)\,) * v^2(\tau)\big\|_p$$
$$\le C \begin{cases} (t-\tau)^{-(1-\frac{1}{p})-1}(1+\tau)^{-1}\,, 1 \le p < 2\,; \\ (t-\tau)^{-(1-\frac{1}{p})-\frac{1}{2}}(1+\tau)^{-3/2}\,, 1 \le p < \infty\,, \end{cases} \tag{6.14}$$

for all $t > 0$, $\tau \in (0,t)$.

Proof. For the proof of (6.13) recall that, by Corollary 4.2 from [14] with $n=2$, we have

$$\big\|\widetilde{S}(t)K\big\|_p \le C_p(1+t)^{-(1-\frac{1}{p})} \quad \text{for} \quad 1 \le p < \infty\,, \tag{6.15}$$

and

$$\big\|\widetilde{S}(t)\nabla K\big\|_p \le C_p(1+t)^{-(1-\frac{1}{p})-\frac{1}{2}} \quad \text{for} \quad 1 \le p < 2\,. \tag{6.16}$$

Moreover, using estimates (6.5) it follows that

$$\big\|v(t)\big\|_2 \le C(1+t)^{-1/2} \quad \text{and} \quad \big\|\nabla v(t)\big\|_2 \le C(1+t)^{-1}\,.$$

These two properties and (6.10) with (6.11) provide the proof of (6.14). $\square$

Proof of Theorem 1.4. The proof follows an analogous reasoning as in [15] (Lemma 6.1). Therefore we shall be brief in details. By (6.2), it is sufficient to prove that

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\Big\| \int_0^t \widetilde{S}(t-\tau)\vec{a}\cdot\nabla K * v^2(\tau)d\tau - \frac{M^2}{8\pi}\log t\big(\vec{a}\cdot\nabla G(t)\big)\Big\|_p \to 0\,, \quad (6.17)$$

when $t\to\infty$ with $1\leq p<2$. To show (6.17), we note that

$$\begin{aligned}
&\Big\| \int_0^t \widetilde{S}(t-\tau)\vec{a}\cdot\nabla K * v^2(\tau)d\tau - \frac{M^2}{8\pi}\log t\big(\vec{a}\cdot\nabla G(t)\big)\Big\|_p \qquad (6.18)\\
&\quad\leq \int_0^t \big\|\widetilde{S}(t-\tau)\vec{a}\cdot\nabla K * v^2(\tau) - \vec{a}\cdot\nabla G(t-\tau) * v^2(\tau)\big\|_p d\tau\\
&\qquad + \Big\| \int_0^t \vec{a}\cdot\nabla G(t-\tau) * v^2(\tau)d\tau - \vec{a}\cdot\nabla G(t)\int_0^t\int_{\mathbb{R}^2} v^2(y,\tau)dyd\tau\Big\|_p\\
&\qquad + \Big\|\vec{a}\cdot\nabla G(t)\int_0^t\int_{\mathbb{R}^2} v^2(y,\tau)dyd\tau - \frac{M^2}{8\pi}\log t\big(\vec{a}\cdot\nabla G(t)\big)\Big\|_p\\
&\qquad = I_1(t)+I_2(t)+I_3(t).
\end{aligned}$$

Hence, we will prove that each term $t^{(1-\frac{1}{p})+\frac{1}{2}}I_i(t)/\log t$ tends to zero as $t\to\infty$. Concerning I_1, we claim that

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\Big\| \int_0^t \Big\{\widetilde{S}(t-\tau)\vec{a}\cdot\nabla K - \vec{a}\cdot\nabla G(t-\tau)\Big\} * v^2(\tau)d\tau\Big\|_p \to 0\,, \quad (6.19)$$

when $t\to\infty$. This is a consequence of (6.11) and (6.14).

Concerning I_3, we have

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\Big\|\vec{a}\cdot\nabla G(t)\int_0^t\int_{\mathbb{R}^2} v^2(y,\tau)dyd\tau - \frac{M^2}{8\pi}\log t\big(\vec{a}\cdot\nabla G(t)\big)\Big\|_p \to 0\,, \quad (6.20)$$

when $t\to\infty$. Indeed, (6.20) is a consequence of the following Lemma:

Lemma 6.7. *Let* $v_0\in\mathbb{L}^1(\mathbb{R}^2)\cap\mathbb{L}^\infty(\mathbb{R}^2)\cap H^3(\mathbb{R}^2)$ *and* v *satisfy the equation* (6.1). *Then*

$$\lim_{t\to\infty}\frac{1}{\log t}\int_0^t\int_{\mathbb{R}^2} v^2(y,\tau)dyd\tau = \frac{M^2}{8\pi}\,.$$

Proof. It is similar to Lemma 6.1 from [15]. □

Concerning I_2, we claim that

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\Big\| \int_0^t \vec{a}\cdot\nabla G(\cdot-y,t-\tau)v^2(y,\tau)dyd\tau - \vec{a}\cdot\nabla G(\cdot,t)\int_0^t\int_{\mathbb{R}^2} v^2dyd\tau\Big\|_p \to 0, \quad (6.21)$$

when $t \to \infty$. To show (6.21), we proceed as in Carpio [7] and Karch [15]. We fix $0 < \delta < 1$ and we decompose the area of integration $(0,t) \times \mathbb{R}^2$ into three parts as follows

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \vec{a} \cdot \nabla G(\cdot - y, t-\tau) v^2(y,\tau) dy d\tau - \int_0^t \int_{\mathbb{R}^2} \vec{a} \cdot \nabla G(\cdot, t) v^2(y,\tau) dy d\tau \\
&= \int_0^{\delta t} \int_{|y| \leq \delta\sqrt{t}} \Big\{ \vec{a} \cdot \nabla G(\cdot - y, t-\tau) - \vec{a} \cdot \nabla G(\cdot, t) \Big\} v^2(y,\tau) dy d\tau \\
&\quad + \int_{\delta t}^t \int_{\mathbb{R}^2} \Big\{ \vec{a} \cdot \nabla G(\cdot - y, t-\tau) - \vec{a} \cdot \nabla G(\cdot, t) \Big\} v^2(y,\tau) dy d\tau \\
&\quad\quad + \int_0^{\delta t} \int_{|y| \geq \delta\sqrt{t}} \Big\{ \vec{a} \cdot \nabla G(\cdot - y, t-\tau) - \vec{a} \cdot \nabla G(\cdot, t) \Big\} v^2(y,\tau) dy d\tau \\
&\quad\quad = J_\delta^1 + J_\delta^2 + J_\delta^3. \qquad (6.22)
\end{aligned}$$

We now estimate the three terms $t^{(1-\frac{1}{p})+\frac{1}{2}} \|J_\delta^i\|_p / \log t$ separately.

The term $\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t} \|J_\delta^1\|_p$: Using the self-similar structure of $G(x,t)$, we observe that

$$\begin{aligned}
&G(x,t) = t^{-1} G(xt^{-1/2}, 1) \Rightarrow \nabla G(x,t) = t^{-1-1/2} \nabla G(xt^{-1/2}, 1), \quad \text{and} \\
&G(x-y, t-\tau) = t^{-1} G\Big(\frac{x-y}{t^{1/2}}, 1 - \frac{\tau}{t}\Big) \Rightarrow \\
&\nabla G(x-y, t-\tau) = t^{-1-1/2} \nabla G\Big(\frac{x-y}{t^{1/2}}, 1 - \frac{\tau}{t}\Big).
\end{aligned}$$

Hence, making the change of variables $xt^{-1/2} = \overline{x}$, we have

$$\begin{aligned}
&\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t} \|J_\delta^1\|_p \qquad (6.23) \\
&\leq \frac{1}{\log t} \Big\| \int_0^{\delta t} \int_{|y| \leq \delta\sqrt{t}} \{ \vec{a} \cdot \nabla G\big(\cdot - \frac{y}{t^{1/2}}, 1 - \frac{t}{\tau}\big) - \vec{a} \cdot \nabla G(\cdot, 1) \} v^2(y,\tau) dy d\tau \Big\|_p.
\end{aligned}$$

Applying Minkowski inequality in (6.23), we have

$$\begin{aligned}
&\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t} \|J_\delta^1\|_p \qquad (6.24) \\
&\leq \frac{1}{\log t} \int_0^{\delta t} \int_{|y| \leq \delta\sqrt{t}} \big\| \vec{a} \cdot \nabla G\big(\cdot - \frac{y}{t^{1/2}}, 1 - \frac{t}{\tau}\big) - \vec{a} \cdot \nabla G(\cdot, 1) \big\|_p \big| v^2(y,\tau) \big| dy d\tau.
\end{aligned}$$

Thanks to the continuity of the translation operator in $\mathbb{L}^p(\mathbb{R}^2)$ and the continuity of $G(x,t)$ with respect to t, given ε, we can choose $0 < \delta < 1$ such that

$$\sup_{\substack{|y|\le\delta\sqrt{t}\\ \tau\le\delta\tau}} \left\|\vec{a}\cdot\nabla G\Big(\,\cdot - \frac{y}{t^{1/2}}, 1-\frac{t}{\tau}\Big) - \vec{a}\cdot\nabla G(\cdot,1)\right\|_p \le \varepsilon\,. \tag{6.25}$$

Therefore,

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\|J_\delta^1\|_p \le \frac{\varepsilon}{\log t}\int_0^{\delta t}\|v(\tau)\|_2^2 d\tau \le \frac{C\varepsilon}{\log t}\int_0^{\delta t}(1+\tau)^{-1}d\tau \le C\varepsilon, \tag{6.26}$$

where C is a constant independent of t.

Estimate of $\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\|J_\delta^2\|_p$**:**

$$\begin{aligned}\left\|J_\delta^2\right\|_p &\le \int_{\delta t}^t \left\|\vec{a}\cdot\nabla G(t-\tau) * v^2(\tau)\right\|_p d\tau + \left\|\vec{a}\cdot\nabla G(t)\right\|_p \int_{\delta t}^t \left\|v(\tau)\right\|_2^2 d\tau \\ &\le C\int_{\delta t}^t (t-\tau)^{-(1-\frac{1}{p})-\frac{1}{2}}\tau^{-1}d\tau + Ct^{-(1-\frac{1}{p})-\frac{1}{2}}\int_{\delta t}^t \tau^{-1}d\tau = Ct^{-(1-\frac{1}{p})-\frac{1}{2}}.\end{aligned} \tag{6.27}$$

Therefore, (6.27) implies that

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\left\|J_\delta^2\right\|_p \to 0\,,\quad \text{with}\quad t\to\infty\,, \tag{6.28}$$

for $1\le p<2$.

Estimate of $\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\left\|J_\delta^3\right\|_p$**:** Now, we prove that

$$\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\left\|J_\delta^3\right\|_p \to 0\,,\quad \text{with}\quad t\to\infty \tag{6.29}$$

for $1\le p<2$. Indeed

$$\begin{aligned}\frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\|J_\delta^3\|_p &\le \frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\int_0^{\delta t}\int_{|y|\ge\delta\sqrt{t}}\|\vec{a}\cdot\nabla G(\cdot-y,t-\tau)\|_p|v(y,\tau)|^2dyd\tau \\ &+ \frac{t^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\int_0^{\delta t}\int_{|y|\ge\delta\sqrt{t}}\left\|\vec{a}\cdot\nabla G(\cdot,t)\right\|_p \left|v(y,\tau)\right|^2dyd\tau \qquad (6.30) \\ &\le \frac{Ct^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\int_0^{\delta t}\int_{|y|\ge\delta\sqrt{t}}(t-\tau)^{-(1-\frac{1}{p})-\frac{1}{2}}\left|v(y,\tau)\right|^2dyd\tau \\ &+ \frac{Ct^{(1-\frac{1}{p})+\frac{1}{2}}}{\log t}\int_0^{\delta t}\int_{|y|\ge\delta\sqrt{t}}t^{-(1-\frac{1}{p})-\frac{1}{2}}\left|v(y,\tau)\right|^2dyd\tau\end{aligned}$$

$$\leq \frac{C(1-\delta)^{-(1-\frac{1}{p})-\frac{1}{2}}}{\log t}\int_0^{\delta t}\int_{|y|\geq\delta\sqrt{t}}\left|v(y,\tau)\right|^2 dyd\tau$$

$$+\frac{C}{\log t}\int_0^{\delta t}\int_{|y|\geq\delta\sqrt{t}}\left|v(y,\tau)\right|^2 dyd\tau.$$

Hence, (6.29) follows from the following Lemma:

Lemma 6.8. *Let* $v_0 \in \mathbb{L}^1 \cap H^3(\mathbb{R}^2)$. *If* v *is the solution of* (6.1), *then*

$$\lim_{t\to\infty}\frac{1}{\log t}\int_0^{\delta t}\int_{|y|\geq\delta\sqrt{t}}\left|v(y,\tau)\right|^2 dyd\tau = 0\,.$$

Proof. Using Theorem 6.2, it is similar to inequality (6.6) in [15]. □

Finally, (6.19), (6.20) and (6.21) imply (6.17).

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