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OPTIMAL SENSOR LOCATION FOR WAVE AND SCHRÖDINGER EQUATIONS

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Abstract. This paper summarizes the research we have carried out recently on the problem of the optimal location of sensors and actuators for wave equations, which has been the object of the talk of the third author at the *Hyp2012 Conference* held in Padova (Italy). We also address the same issues for the Schrödinger equations and present some possible perspectives of future research.

We consider the multi-dimensional wave or Schrödinger equations in a bounded domain Ω, with usual boundary conditions (Dirichlet, Neumann or Robin). We investigate the problem of optimal sensor location, in other words, the problem of designing what is the best possible subdomain of a prescribed measure on which one can observe the solutions. We present two mathematical problems modeling this question. The first one, in which the initial data under consideration are fixed, leads to optimal sets whose complexity depends on the regularity of the initial data. In the second one, the optimal set is searched so as to be uniform with respect to all initial data, and leads to a criterium of spectral nature, the answer being intimately related to the concentration properties of the eigenfunctions of the Laplacian. Under quantum ergodicity assumptions on the domain Ω we compute the optimal value of this problem, and show that this optimal value can be interpreted as the best possible observability constant of a corresponding time-asymptotic or randomized observability inequality. Although optimal sets do exist in some specific situations, we show that the existence of an optimal set cannot be expected in general. Finally, we study a spectral approximation of that problem and construct a maximizing sequence of sets.

1. Introduction.

1.1. Preliminaries: the problem of optimal observation. Let $T > 0$, $n \in \mathbb{N}^*$, and $\Omega \subset \mathbb{R}^n$ be a bounded open connected subset. In this article we consider both the homogeneous wave equation

$$\partial_{tt} y = \triangle y, \tag{1}$$

and the Schrödinger equation

$$i\partial_t y = \triangle y, \tag{2}$$

for almost all $(t, x) \in (0, T) \times \Omega$, with Dirichlet boundary conditions for the sake of simplicity (other conditions are considered at the end of the article).

For any measurable subset ω of Ω of positive Lebesgue measure, we consider in both cases the observable variable

$$z(t, x) = \chi_\omega(x) y(t, x), \tag{3}$$

where χ_ω denotes the characteristic function of ω.

In this article we investigate the question of knowing whether there exists a best possible subset ω in order to observe the equation (1) or (2). To make the problem more precise, throughout the article we fix a real number $L \in (0, 1)$, and from now on we restrict our search to all measurable subsets $\omega \subset \Omega$ which are of Lebesgue measure $|\omega| = L|\Omega|$. This determines the volume fraction of sensors that one would like to place in the domain Ω, in the best possible way.

Let us next model and define what the wording "best possible way" can mean.

1.2. Mathematical modeling of two optimal design problems. In this context there are several possible ways of defining a concept of domain optimization. Certainly, the first problem that can be raised is the following.

First problem: best observation domain for fixed initial data.

- **Wave equation** (1)**:** *given fixed initial data* $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$*, we investigate the problem of maximizing the functional*

$$G_T(\chi_\omega) = \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt, \tag{4}$$

over all possible measurable subsets ω *of* Ω *of Lebesgue measure* $|\omega| = L|\Omega|$*, where* $y \in C^0(0,T; L^2(\Omega,\mathbb{C})) \cap C^1(0,T; H^{-1}(\Omega,\mathbb{C}))$ *is the unique solution of* (1) *such that* $y(0,\cdot) = y^0(\cdot)$ *and* $\partial_t y(0,\cdot) = y^1(\cdot)$*.*
- **Schrödinger equation** (2)**:** *given* $y^0 \in L^2(\Omega,\mathbb{C})$*, we investigate the problem of maximizing the functional* G_T *defined by* (4) *over all possible measurable subsets* ω *of* Ω *of Lebesgue measure* $|\omega| = L|\Omega|$*, where* $y \in C^0(0,T; L^2(\Omega,\mathbb{C}))$ *is the unique solution of* (2) *such that* $y(0,\cdot) = y^0(\cdot)$*.*

This problem appears as a mathematical benchmark, and is the first problem that one can raise in order to give a sense to the notion of best observation. However, this problem is not well suited in view of practical applications since it depends on the initial conditions. In applications, obviously, the location of sensors should be independent on the initial data. This problem is however interesting from an analytical point of view. As we will see, solving this problem is easy and optimal sets are level sets of a given function, that depends on the solution under consideration in a very sensitive way.

Let us now come to the definition of a uniform optimal design problem, independent on the initial data. In view of defining such a problem, relevant for practical issues, let us first recall the notion of observability inequality.

The system (1)-(3) is said to be observable on ω in time T if and only if there exists $C_T^{(W)}(\chi_\omega) > 0$ such that

$$C_T^{(W)}(\chi_\omega) \|(y^0, y^1)\|^2_{L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})} \leq \int_0^T \int_\omega |y(t,x)|^2 \, dxdt, \tag{5}$$

for all $(y^0, y^1) \in L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})$. This is the so-called *observability inequality.* It is well known that within the class of $\mathcal{C}^\infty$ domains Ω, this observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [3]), according to which every ray of Geometric Optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ intersects ω within time T.

Similarly, system (2)-(3) is said to be observable on ω in time T if and only if there exists $C_T^{(S)}(\chi_\omega) > 0$ such that

$$C_T^{(S)}(\chi_\omega) \|y^0\|^2_{L^2(\Omega,\mathbb{C})} \leq \int_0^T \int_\omega |y(t,x)|^2 \, dxdt, \tag{6}$$

for every $y^0 \in L^2(\Omega,\mathbb{C})$. If there exists T^* such that the pair (ω, T^*) satisfies the *Geometric Control Condition* then the observability inequality (35) holds for every $T > 0$ (see [18]). In some sense the Schrödinger equation can be viewed as a wave equation with an infinite speed of propagation.

In the sequel, the quantities $C_T^{(W)}(\chi_\omega)$ and $C_T^{(S)}(\chi_\omega)$ denote the largest possible nonnegative constants for which the inequalities (34) and (35) hold, that is,

$$C_T^{(W)}(\chi_\omega) = \inf_{\|(y^0,y^1)\|_{L^2\times H^{-1}}=1} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt, \tag{7}$$

and

$$C_T^{(S)}(\chi_\omega) = \inf_{\|y^0\|_{L^2}=1} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt. \tag{8}$$

They are called the *observability constants.*

These remarks being done, in view of defining a uniform optimal design problem for the observability of wave or Schrödinger equations, it is natural to raise the problem of maximizing the above observability constants over all possible subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$. However, this problem appears to be:

1. Very difficult to handle: indeed when considering spectral expansions of the solutions, difficulties arise due to crossed terms, as in the interesting open problem of determining the best constants in Ingham's inequalities (see [13, 14], see also [23] for such considerations in the one-dimensional case);
2. Finally, not so relevant. Indeed the above inequalities are *deterministic*, and hence, in some sense, the observability constants are pessimistic, since they give an account for the worst possible observability scenario. In practice one is led to handle a large number of solutions but not all of them, and the deterministic observability constant will rarely be reached. We are then going to define a randomized version of the observability constant, which appears to be more relevant.

These two points appeal further comments.

Let us first present the Fourier expansion of solutions of the spectral basis of the Laplacian. Let $(\phi_j)_{j\in\mathbb{N}^*}$ be a Hilbertian basis of $L^2(\Omega)$ consisting of eigenfunctions[1] of the Dirichlet Laplacian operator on Ω, associated with the negative eigenvalues $(-\lambda_j^2)_{j\in\mathbb{N}^*}$. Then, for given initial data $(y^0, y^1) \in L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})$, the corresponding solution of (1) is

$$y(t,x) = \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}\right) \phi_j(x), \tag{9}$$

where the sequences $(a_j)_{j\in\mathbb{N}^*}$ and $(b_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$ and are defined by

$$\begin{aligned} a_j &= \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dx - \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dx\right), \\ b_j &= \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dx + \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dx\right). \end{aligned} \tag{10}$$

for every $j \in \mathbb{N}^*$. Moreover,

$$\|(y^0,y^1)\|^2_{L^2\times H^{-1}} = 2\sum_{j=1}^{+\infty}(|a_j|^2 + |b_j|^2). \tag{11}$$

[1] Note that this Hilbertian basis is not necessarily unique in case of multiple eigenvalues. What follows depends a priori on the specific choice of the basis of eigenfunctions which is done at this step of our analysis.

With such a spectral expansion, note that

$$G_T(\chi_\omega) = \sum_{j,k=1}^{+\infty} \alpha_{jk} \int_\omega \phi_i(x)\phi_j(x)\,dx, \tag{12}$$

where the coefficients α_{jk}, $(j,k) \in (\mathbb{N}^*)^2$ (which can be easily computed) depend only on the initial data (y^0, y^1) and the observation time T. It can be noted that, since ω is a proper subset of Ω, there holds in general $\int_\omega \phi_i(x)\phi_j(x)\,dx \neq 0$. Because of these crossed terms, the observability constant $C_T^{(W)}(\chi_\omega)$ defined by (7) can be interpreted as the infimum of eigenvalues of an infinite dimensional nonnegative symmetric matrix (called *Gramian*), which is far from diagonal due to nonzero nondiagonal terms.

The observability constant $C_T^{(W)}(\chi_\omega)$ could be easily expressed if the Gramian were to be a diagonal matrix. This is actually one of the nice consequences of the randomization procedure mentioned in the second point. Let us explain briefly this procedure (full details are provided in [26]). Following [6], we randomize some initial data determined by their Fourier coefficients (10), by defining $a_j^\nu = \beta_{1,j}^\nu a_j$ and $b_j^\nu = \beta_{2,j}^\nu b_j$, where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of independent Bernoulli random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, satisfying

$$\mathbb{P}(\beta_{1,j}^\nu = \pm 1) = \mathbb{P}(\beta_{2,j}^\nu = \pm 1) = \frac{1}{2} \qquad \text{and} \qquad \mathbb{E}(\beta_{1,j}^\nu \beta_{2,k}^\nu) = 0$$

for all j and k in $\mathbb{N}^*$ and every $\nu \in X$. Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$. Let y_ν denote the corresponding solution,

$$y_\nu(t,x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t}\right) \phi_j(x).$$

Then, instead of considering the deterministic observability inequality (34), we consider the randomized one

$$C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{L^2\times H^{-1}}^2 \leq \mathbb{E}\left(\int_0^T \int_\omega |y_\nu(t,x)^2|\,dx\,dt\right), \tag{13}$$

for all $y^0(\cdot) \in L^2(\Omega,\mathbb{C})$ and $y^1(\cdot) \in H^{-1}(\Omega,\mathbb{C})$. Here, the constant $C_{T,\mathrm{rand}}^{(W)}(\chi_\omega)$, called the *randomized observability constant* for the wave equation, is a new constant which is a priori different from its deterministic counterpart $C_T^{(W)}(\chi_\omega)$. A similar consideration is done for the Schrödinger equation, with a randomized observability constant $C_{T,\mathrm{rand}}^{(S)}(\chi_\omega)$.

It is proved in [26] that, for every measurable subset ω of Ω, there holds

$$2\,C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) = C_{T,\mathrm{rand}}^{(S)}(\chi_\omega) = T \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx = TJ(\chi_\omega). \tag{14}$$

In other words, the randomization procedure sketched permits to kill the crossed terms and hence, up to considering random initial data and an averaged version of the observability inequality, to provide a concept of randomized Gramian, which is a diagonal infinite dimensional matrix.

As mentioned above, the randomized observability inequality (13) appears to be more relevant than its classical deterministic version (34) in view of applications. The first problem in which the initial data are given and fixed is not very relevant.

But in practice one does not need to consider all possible solutions either. The above randomization procedure, provides a reasonable mathematical modeling of this practical optimal design problem.

It follows from all above considerations that a way to define a relevant uniform optimal design problem is the following.

Second problem: uniform optimal design problem. *We investigate the problem of maximizing the functional*

$$J(\chi_\omega) = \inf_{j\in I\!N^*} \int_\omega \phi_j(x)^2\, dx, \tag{15}$$

over all possible subsets ω *of* Ω *of Lebesgue measure* $|\omega| = L|\Omega|$.

This problem consists of maximizing an eigenfunction energy concentration criterion. As we will see, solving this problem leads to highly interesting mathematical considerations related to quantum ergodicity properties of the domain Ω.

Remark 1. It is proved in [26] that, if the domain Ω is such that every eigenvalue of A is simple, then, similarly to (14), there holds

$$2\, C_\infty^{(W)}(\chi_\omega) = C_\infty^{(S)}(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx = J(\chi_\omega), \tag{16}$$

for every measurable subset ω of Ω, where $C_\infty^{(W)}(\chi_\omega)$ and $C_\infty^{(S)}(\chi_\omega)$ are *time asymptotic observability constants*, defined respectively as the largest possible nonnegative constant for which the time asymptotic observability inequality

$$C_\infty^{(W)}(\chi_\omega)\|(y^0,y^1)\|^2_{L^2\times H^{-1}} \leq \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega |y(t,x)^2|\, dx\, dt, \tag{17}$$

holds for all $(y^0,y^1)\in L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})$, for the wave equation, and

$$C_\infty^{(S)}(\chi_\omega)\|y^0\|^2_{L^2} \leq \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega |y(t,x)^2|\, dx\, dt, \tag{18}$$

holds for every $y^0(\cdot)\in L^2(\Omega,\mathbb{C})$, for the Schrödinger equation.

1.3. **Some bibliographical comments.** The problem of optimal measurement locations for state estimation in linear partial differential equations has been widely considered in engineering problems (see e.g. [9, 15, 16, 21, 28, 29] and the many references therein), the aim being to optimize the number, the place and the type of sensors or actuators in order to improve the estimation or more generally some performance index. Fields of applications are very numerous and concern for example active structural acoustics, piezzoelectric issues, vibration control in mechanical structures, damage detection processes, chemical reactions, just to name a few of them. A usual approach popular in the engineering community consists of recasting the optimal sensor location problem for distributed systems as an optimal control problem with an infinite dimensional Riccati equation, having a statistical model interpretation, and then of computing approximations with optimization techniques. However, on the one part, their techniques rely on an exhaustive search over a predefined set of possible candidates and are faced with combinatorial difficulties due to the selection problem and thus with the usual flaws of combinatorial optimization methods. On the other part, in all these references approximations are used to determine the optimal sensor or actuator location. The optimal performance and the corresponding sensor or actuator location of the approximating sequence are then expected to converge to the exact optimal performance and location. Among the possible approximation processes, the closest one to our present study consists of considering Fourier expansion representations and using modal approximation schemes.

However, in these references there is no systematic mathematical study of the optimal design problem. The search of optimal domains relies on finite-dimensional approximations and no convergence analysis is led. However, in the present article we show that modal approximation procedures may fail and Γ-convergence properties may not hold when passing to the limit from a finite number of eigenfunction components to all of them.

Although the optimal design problems under consideration in this article have been widely studied in the engineering community, in particular because of their great importance in practical problems, there exist only few mathematical results. An important difficulty arising when focusing on an optimal shape problem is the generic non-existence of classical solutions, as explained and surveyed in [2], thus leading to consider relaxation procedures. In [4] the authors investigate the problem modeled in [27] of finding the best possible distributions of two materials (with different elastic Young modulus and different density) in a rod in order to minimize the vibration energy in the structure. For this optimal design problem in wave propagation, the authors of [4] prove existence results and provide relaxation and optimality conditions. The authors of [1] also propose a relaxation formulation of eigenfrequency optimization problems applied to optimal design. In [7] the authors discuss several possible criteria for optimizing the damping of abstract wave equations in Hilbert spaces, and derive optimality conditions for a certain criterion related to a Lyapunov equation. In [11, 12], the authors consider the problem of determining the best possible shape and position of the damping subdomain of given measure for a 1D wave equation. In [20, 22] the authors investigate numerically the optimal location of the support of the control for the 1-D wave equation. Their numerical methods are then mostly based on gradient techniques or level set methods combined with shape and topological derivatives (we refer the reader e.g. to [5] for a survey on variational methods in shape optimization problems). In [23] we investigated the second problem presented previously in the one-dimensional case, and in [24] we studied the related dual problem of finding the optimal location of the support of the control for the one-dimensional wave equation. In [25] we solved in a complete way the first problem (optimal observation domain for the problem with fixed initial data), and in [26] we solved the second problem (uniform with respect to initial data), emphasizing close connections with the quantum chaos theory, as explained further.

2. **Statement of the main results.**

2.1. **First problem: best observation domain for fixed initial data.**

Consider fixed initial data $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ (resp., $y^0 \in L^2(\Omega, \mathbb{C})$) for the wave equation (1) (resp., for the Schrödinger equation (2)), and let y be their corresponding solution. We define the integrable function

$$\varphi(x) = \int_0^T |y(t,x)|^2 dt, \tag{19}$$

for every $x \in \Omega$. Note that $G_T(\chi_\omega) = \int_\omega \varphi(x)\, dx$ for every measurable subset $\omega \subset \Omega$.

Theorem 2.1. [25] *There exists at least one measurable subset ω of Ω, solution of the first problem, characterized as follows. There exists a real number λ such that every optimal set ω is contained in the level set $\{\varphi \geq \lambda\}$, where the function φ defined by (19) is integrable on Ω.*

Moreover, if there exists $R > 0$ such that

$$\sum_{j=0}^{+\infty} \frac{R^j}{j!} \left(\|A^j y^0\|_{L^2}^2 + \|A^{j-1} y^1\|_{L^2}^2 \right)^{1/2} < +\infty, \tag{20}$$

in the case of the wave equation, and

$$\sum_{j=0}^{+\infty} \frac{R^j}{j!} \|A^j y^0\|_{L^2} < +\infty, \tag{21}$$

in the case of the Schrödinger equation, where $A = \sqrt{-\triangle}$ (square root of the Dirichlet-Laplacian), then the first problem has a unique[2] *solution χ_ω, where ω is a measurable subset of Ω of measure $L|\Omega|$, satisfying moreover the following properties:*

- *there exists $\eta > 0$ such that $d(\omega, \partial\Omega) > \eta$, where d denotes the Euclidean distance on $\mathbb{R}^n$;*
- *ω is semi-analytic*[3]*, and has a finite number of connected components;*
- *if Ω is symmetric with respect to an hyperplane and $y^0 \circ \sigma = y^0$ and $y^1 \circ \sigma = y^1$, where σ denotes the symmetry operator with respect to this hyperplane, then ω enjoys the same symmetry property.*

Remark 2. The optimal set is not necessarily unique, whenever the function φ is constant on some subset of Ω of positive measure. We refer to [23, 25] for explicit examples.

Theorem 2.1 states that, if the initial data belong to some analyticity spaces, then the (unique) optimal set ω is the union of a finite number of connected components. Using a careful harmonic analysis construction, it is proved in [25] that there exist C^∞ initial data for which the optimal set ω may have a fractal structure and, more precisely, may be of Cantor type. More precisely, one has the following result.

Theorem 2.2. [25] *Let $\Omega = (0, 2\pi)$ and let $T > 0$ be an integer multiple of 4π. There exist C^∞ initial data (y^0, y^1) defined on Ω for which the first problem has a unique solution ω; moreover ω has a fractal structure and in particular it has an infinite number of connected components.*

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[2] Similarly to the definition of elements of L^p-spaces, the subset ω is unique within the class of all measurable subsets of Ω quotiented by the set of all measurable subsets of Ω of zero measure.

[3] A subset ω of a real analytic finite dimensional manifold M is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions, that is, for every $x \in \omega$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} (with $1 \leq i \leq p$ and $1 \leq j \leq q$) such that

$$\omega \cap U = \bigcup_{i=1}^{p} \{ y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0, \; j = 1, \dots, q \}.$$

We recall that such semi-analytic (and more generally, subanalytic) subsets enjoy nice properties, for instance they are stratifiable in the sense of Whitney.

2.2. **Uniform optimal design.** In this section, we focus on the second problem, defined as

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega), \tag{22}$$

with

$$J(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx,$$

and

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } |\omega| = L|\Omega|\}. \tag{23}$$

2.2.1. *Convexification.* To ensure compactness properties, we consider the convex closure of $\mathcal{U}_L$ for the weak star topology of L^∞,

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega, [0,1]) \mid \int_\Omega a(x)\, dx = L|\Omega| \right\}. \tag{24}$$

The convexified version of the second problem (22) is

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a), \tag{25}$$

where

$$J(a) = \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\, dx. \tag{26}$$

By upper semi-continuity of J for the weak star topology of L^∞, it is clear that the problem (25) has at least one solution. For instance in dimension one there is an infinite number of solutions, characterized through their Fourier coefficients (see [23]). Note that taking $a(\cdot) = L$ yields $\sup_{a\in\overline{\mathcal{U}}_L} J(a) \geq L$, and note that a priori, $\sup_{\chi_\omega\in\mathcal{U}_L} J(\chi_\omega) \leq \sup_{a\in\overline{\mathcal{U}}_L} J(a)$. The question of knowing if this inequality is an equality or not (gap or no-gap) is not obvious, and cannot be treated using standard Γ-convergence arguments due to the lack of lower semi-continuity of J.

2.2.2. *Main results.* We make the following assumptions on the Hilbertian basis $(\phi_j^2)_{j\in\mathbb{N}^*}$ of eigenfunctions under consideration.

> ***Weak Quantum Ergodicity on the base (WQE) property.*** *There exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2\, dx$ converging vaguely to the uniform measure $\frac{1}{|\Omega|}\, dx$.*

> ***Uniform L^∞-boundedness property.*** *There exists $A > 0$ such that*
> $$\|\phi_j\|_{L^\infty(\Omega)} \leq A, \tag{27}$$
> *for every $j \in \mathbb{N}^*$.*

These assumptions above imply what we call the L^∞*-Weak Quantum Ergodicity on the base (L^∞-WQE) property*, that is, there exists a subsequence of $(\phi_j^2)_{j\in\mathbb{N}^*}$ converging to $\frac{1}{|\Omega|}$ for the weak star topology of $L^\infty(\Omega)$. This property obviously implies that

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a) = \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\, dx = L, \tag{28}$$

and moreover the supremum is reached with the constant function $a = L$ on Ω.

Theorem 2.3. [26] *If the WQE and uniform L^∞-boundedness properties hold, then*

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L, \tag{29}$$

for every $L \in (0,1)$. In other words, under these assumptions there is no gap between the original problem (22) and the convexified one.

As a consequence, the maximal value of the randomized observability constants $2\, C^{(W)}_{T,\text{rand}}(\chi_\omega) = C^{(S)}_{T,\text{rand}}(\chi_\omega)$ over the set $\mathcal{U}_L$ is equal to TL. Moreover if the spectrum of A is simple then the maximal value of the time asymptotic observability constants $2C^{(W)}_\infty(\chi_\omega) = C^{(S)}_\infty(\chi_\omega)$ over the set $\mathcal{U}_L$ is equal to L.

We now define the set $\mathcal{U}_L^b = \{\chi_\omega \in \mathcal{U}_L \mid |\partial\omega| = 0\}$, and we make the following assumptions.

> ***Quantum Unique Ergodicity on the base (QUE) property.*** *The whole sequence of probability measures $\mu_j = \phi_j^2 \, dx$ converges vaguely to the uniform measure $\frac{1}{|\Omega|} \, dx$.*

> ***Uniform L^p-boundedness property.*** *There exist $p \in (1,+\infty]$ and $A > 0$ such that*
>
> $$\|\phi_j\|_{L^{2p}(\Omega)} \leq A, \tag{30}$$
>
> *for every $j \in \mathbb{N}^*$.*

Theorem 2.4. [26] *If $\partial\Omega$ is Lipschitz and if the QUE and uniform L^p-boundedness properties hold, then*

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L, \tag{31}$$

for every $L \in (0,1)$.

Actually the statement of Theorem 2.4 holds true as well whenever the set $\mathcal{U}_L^b$ is replaced by the set of all measurable subsets ω of Ω, of measure $|\omega| = L|\Omega|$, that are moreover open either with Lipschitz boundary or bounded perimeter.

The ergodicity assumptions made above are sufficient but are not sharp. For instance it is proved in [26] that, if Ω is the unit disk of the Euclidean two-dimensional space, then, for every $p \in (1,+\infty]$ and for any basis of eigenfunctions, the uniform L^p-boundedness property is not satisfied, and QUE does not hold as well; however (29) and (31) hold true. And this, in spite of the phenomenon of whispering galleries, which gives an account for the existence of certain semi-classical measures (weak limits of the probability measures $\phi_j^2 \, dx$) such as the Dirac measure along the boundary.

Remark 3. The assumptions made in the above theorems obviously hold in dimension one (Dirichlet-Laplacian on a bounded interval). In higher dimensions they are related to deep questions arising in mathematical physics (indeed, in quantum mechanics $\mu_j = \phi_j^2 \, dx$ is the probability of being in the state ϕ_j), related to Shnirelman's Theorem. This celebrated result asserts that, if the domain Ω is a convex ergodic billiard with piecewise smooth boundary, then there exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2 \, dx$ of density one converging vaguely to the uniform measure $\frac{1}{|\Omega|} dx$ (see [10, 31]). This property is referred to as *Quantum Ergodicity on the base* (in short, QE on the base). Actually the result is stronger and holds in the full phase space, for pseudo-differential operators (see [30] for a recent survey). Of course, QUE implies QE which in turn implies WQE.

Note that Shnirelman Theorem lets open the possibility of having an exceptional subsequence of μ_j converging vaguely to some measure different from the uniform one, for instance, to a measure carried by closed geodesics (concentration phenomenon known as *scar*, see e.g. [8]). The QUE assumption made above postulates that this scarring phenomenon does not occur. Up to now there is no example of a domain in dimension more than one in which QUE has been proved to hold, and this is a deep open question in this thematics. We refer the reader to [26] for a more detailed discussion on such quantum ergodicity issues in relation with shape optimization problems.

Remark 4. In general we do not expect the supremum in (29) or (31) to be reached. This is an open question. But it is reached in several very particular situations. This is the case for instance in dimension one for a very specific value of L: when $\Omega = [0,\pi]$, then the supremum of J over $\mathcal{U}_L$ (which is equal to L) is reached if and only if $L = 1/2$; in that case, it is reached for all measurable subsets $\omega \subset [0,\pi]$ of measure $\pi/2$ such that ω and its symmetric image $\omega' = \pi - \omega$ are disjoint and complementary in $[0,\pi]$ (see [26]).

3. **Spectral approximation of the uniform optimal design problem.** Given the functional J defined by (15), in view of designing a spectral approximation it is natural to consider the truncated functional defined by

$$J_N(\chi_\omega) = \min_{1\leq j\leq N} \int_\omega \phi_j(x)^2\, dx, \tag{32}$$

for every $N \in \mathbb{N}^*$ and every measurable subset ω of Ω. The spectral approximation of the second problem (uniform optimal design problem) is then

$$\sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega). \tag{33}$$

Accordingly, J_N is extended to $\overline{\mathcal{U}}_L$ by $J_N(a) = \min_{1\leq j\leq N} \int_\Omega a(x)\phi_j(x)^2\, dx$ for every $a \in \overline{\mathcal{U}}_L$.

Theorem 3.1. [26]

1. *For every measurable subset ω of Ω, the sequence $(J_N(\chi_\omega))_{N\in\mathbb{N}^*}$ is non-increasing and converges to $J(\chi_\omega)$.*
2. *There holds*
$$\lim_{N\to+\infty} \max_{a\in\overline{\mathcal{U}}_L} J_N(a) = \max_{a\in\overline{\mathcal{U}}_L} J(a).$$
Moreover, if $(a^N)_{n\in\mathbb{N}^*}$ is a sequence of maximizers of J_N in $\overline{\mathcal{U}}_L$, then up to a subsequence, it converges to a maximizer of J in $\overline{\mathcal{U}}_L$ for the weak star topology of L^∞.
3. *For every $N \in \mathbb{N}^*$, the problem (33) has a unique solution χ_{ω^N}, where $\omega^N \in \mathcal{U}_L$. Moreover, ω^N is semi-analytic (see Footnote 3) and thus has a finite number of connected components.*

Remark 5. It is proved in [12, 23] that, in the one-dimensional case, the optimal set ω_N maximizing J_N is the union of N intervals concentrating around equidistant points and that ω_N is actually the worst possible subset for the problem of maximizing J_{N+1}. This is the so-called *spillover phenomenon* which is a serious drawback from the practical point of view since it makes it impossible the implementation of a spectral approximation procedure.

The next numerical simulations, based on the above spectral approximation, confirm this pathological behavior. Consider $\Omega = [0,\pi]^2$. The normalized eigenfunctions of the Dirichlet-Laplacian are $\phi_{j,k}(x_1,x_2) = \frac{2}{\pi}\sin(jx_1)\sin(kx_2)$, for every $(x_1,x_2)\in[0,\pi]^2$. Let $N\in\mathbb{N}^*$. We use an interior point line search filter method to solve the optimization problem $\sup_{\chi_\omega\in\mathcal{U}_L} J_N(\chi_\omega)$, with

$$J_N(\chi_\omega) = \min_{1\leq j,k\leq N} \int_0^\pi \int_0^\pi \chi_\omega(x_1,x_2)\phi_{j,k}(x_1,x_2)^2\,dx_1\,dx_2.$$

Some results are provided on Figure 1 in the Dirichlet case. They show very clearly that the number of connected components of the optimal set increases as N grows. We have thus constructed a maximizing sequence of sets for the second problem (uniform optimal design problem) which is evidently far from converging in any reasonable sense.

4. **Further comments and perspectives.**

4.1. **Generalization to other boundary conditions.** Up to now we have restricted ourselves to Dirichlet boundary conditions. Actually, as shown in [26], our analysis can be developed in the more general framework where Ω is an open bounded connected subset of M, and (M,g) is a smooth n-dimensional Riemannian manifold, with $n\geq 1$. In that case, the Dirichlet-Laplacian is replaced with the Laplace-Beltrami operator $\triangle_g$ on M for the metric g. The boundary of Ω can be empty: in this case, Ω is a compact connected n-dimensional Riemannian manifold. If $\partial\Omega\neq\emptyset$ then we consider boundary conditions $By=0$ on $(0,T)\times\partial\Omega$, where B can be either:

- the usual Dirichlet trace operator, $By = y_{|\partial\Omega}$,
- or Neumann, $By = \frac{\partial y}{\partial n}_{|\partial\Omega}$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$,
- or mixed Dirichlet-Neumann, $By = \chi_{\Gamma_0} y_{|\partial\Omega} + \chi_{\Gamma_1}\frac{\partial y}{\partial n}_{|\partial\Omega}$, where $\partial\Omega = \Gamma_0\cup\Gamma_1$ with $\Gamma_0\cap\Gamma_1=\emptyset$, and χ_{Γ_i} is the characteristic function of Γ_i, $i=0,1$,
- or Robin, $By = \frac{\partial y}{\partial n}_{|\partial\Omega} + \beta y_{|\partial\Omega}$, where β is a nonnegative bounded measurable function defined on $\partial\Omega$, such that $\int_{\partial\Omega}\beta > 0$.

The Lebesgue measure dx must be replaced with the canonical measure dV_g induced by the canonical Riemannian volume V_g on M.

Also, to encompass all possible boundary conditions settled above, we replace the observability inequalities (34) and (35) with

$$C_T^{(W)}(\chi_\omega)\|(y^0,y^1)\|^2_{D(A^{1/2})\times X} \leq \int_0^T\int_\omega |\partial_t y(t,x)|^2\,dV_g\,dt, \tag{34}$$

for all $(y^0,y^1)\in D(A^{1/2})\times X$, and

$$C_T^{(S)}(\chi_\omega)\|y^0\|^2_{D(A)} \leq \int_0^T\int_\omega |\partial_t y(t,x)|^2\,dV_g\,dt, \tag{35}$$

for every $y^0\in D(A)$. Here, the following notations are used: $A=-\triangle_g$ is the Laplace operator defined on $D(A)=\{y\in X \mid Ay\in X \text{ and } By=0\}$ with one of

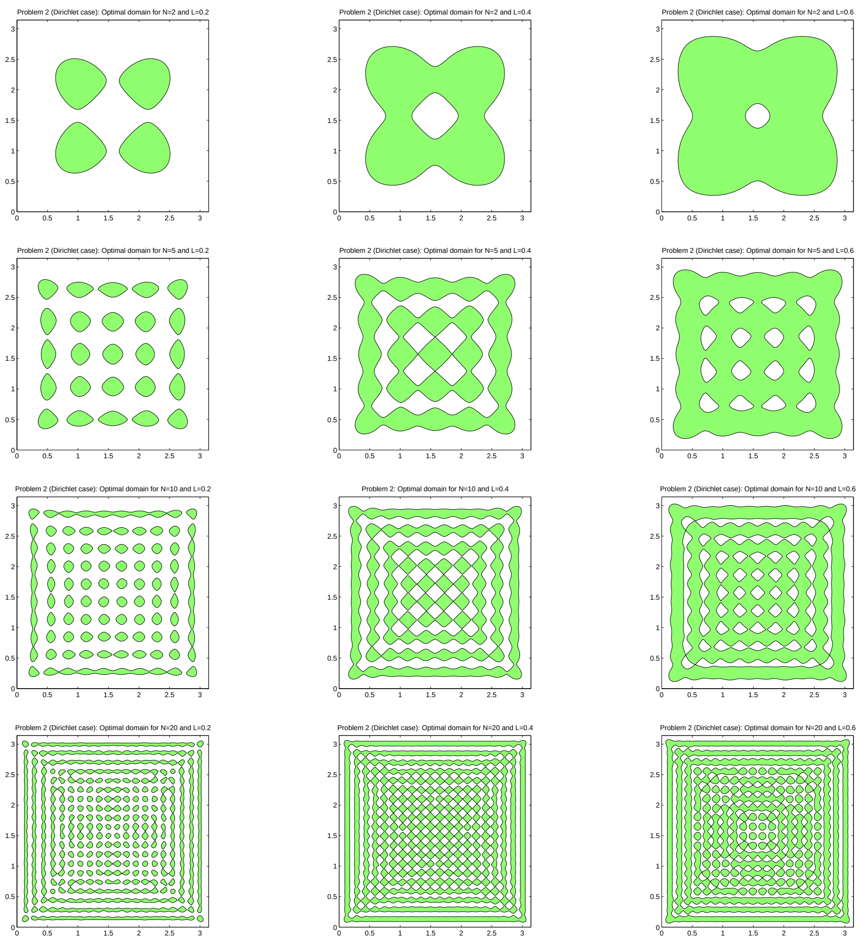

FIGURE 1. On this figure, $\Omega = [0,\pi]^2$. Line 1, from left to right: optimal domain (in green) in the Dirichlet case for $N = 2$ (4 eigenmodes) and $L \in \{0.2, 0.4, 0.6\}$. Line 2, from left to right: optimal domain (in green) for $N = 5$ (25 eigenmodes) and $L \in \{0.2, 0.4, 0.6\}$. Line 3, from left to right: optimal domain (in green) for $N = 10$ (100 eigenmodes) and $L \in \{0.2, 0.4, 0.6\}$. Line 4, from left to right: optimal domain (in green) for $N = 20$ (400 eigenmodes) and $L \in \{0.2, 0.4, 0.6\}$.

the above boundary conditions whenever $\partial\Omega \neq \emptyset$, and X is the space $L^2(\Omega,\mathbb{C})$ in the case of Dirichlet, mixed or Robin boundary conditions, and otherwise

$$X = L^2_0(\Omega,\mathbb{C}) = \{y \in L^2(\Omega,\mathbb{C}) \mid \int_\Omega y(x)\, dV_g = 0\}.$$

Defined in this space, the operator A is then selfadjoint and positive definite. In the case of Dirichlet boundary conditions, one has $D(A) = H^2(\Omega,\mathbb{C}) \cap H^1_0(\Omega,\mathbb{C})$

and $D(A^{1/2}) = H^1_0(\Omega,\mathbb{C})$. For Neumann boundary conditions, one has

$$D(A) = \{y \in H^2(\Omega,\mathbb{C}) \mid \frac{\partial y}{\partial n}_{|\partial\Omega} = 0 \text{ and } \int_\Omega y(x)\, dV_g = 0\}$$

and

$$D(A^{1/2}) = \{y \in H^1(\Omega,\mathbb{C}) \mid \int_\Omega y(x)\, dV_g = 0\}.$$

In the mixed Dirichlet-Neumann case (with $\Gamma_0 \neq \emptyset$), one has

$$D(A) = \{y \in H^2(\Omega,\mathbb{C}) \mid y_{|\Gamma_0} = \frac{\partial y}{\partial n}_{|\Gamma_1} = 0\},$$

and

$$D(A^{1/2}) = H^1_{\Gamma_0}(\Omega,\mathbb{C}) = \{y \in H^1(\Omega,\mathbb{C}) \mid y_{|\Gamma_0} = 0\}$$

(see e.g. [17]).

4.2. **An intrinsic variant of the uniform optimal design problem.** As said before, the second problem (15) depends a priori on the orthonormal Hilbertian basis $(\phi_j)_{j\in\mathbb{N}^*}$ of $L^2(\Omega)$ which has been fixed at the beginning of the analysis, at least whenever the spectrum of A is not simple. If the eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$ of A are multiple, then the choice of the basis $(\phi_j)_{j\in\mathbb{N}^*}$ is an issue. One possible way to get rid of this dependence is to consider the infimum of the criteria J defined by (15) over all possible choices of orthonormal bases of eigenfunctions. This leads to the following intrinsic variant of the second problem. We adopt the framework and the notations of the previous section.

> **Intrinsic uniform optimal design problem.** *We investigate the problem of maximizing the functional*
>
> $$J_{int}(\chi_\omega) = \inf_{\phi\in\mathcal{E}} \int_\omega \phi(x)^2\, dV_g, \tag{36}$$
>
> *over all possible subsets ω of Ω of measure $V_g(\omega) = LV_g(\Omega)$, where $\mathcal{E}$ denotes the set of all normalized eigenfunctions of A.*

Note that $C_T^{(W)}(\chi_\omega) \leq \frac{T}{2} J_{\text{int}}(\chi_\omega) \leq C_{T,\text{rand}}^{(W)}(\chi_\omega)$ and $C_T^{(S)}(\chi_\omega) \leq T J_{\text{int}}(\chi_\omega) \leq C_{T,\text{rand}}^{(S)}(\chi_\omega)$. As before, the functional J_{int} is extended to $\overline{\mathcal{U}}_L$ by setting $J_{\text{int}}(a) = \inf_{\phi\in\mathcal{E}} \int_\Omega a(x)\phi(x)^2\, dV_g$ for every $a \in \overline{\mathcal{U}}_L$. The following results are the intrinsic counterpart of Theorems 2.3 and 2.4.

Theorem 4.1. [26] *Assume that the uniform measure $\frac{1}{V_g(\Omega)}\, dV_g$ is a closure point of the family of probability measures $\mu_\phi = \phi^2\, dV_g$, $\phi \in \mathcal{E}$, for the vague topology, and that the whole family of eigenfunctions in $\mathcal{E}$ is uniformly bounded in $L^\infty(\Omega)$. Then*

$$\sup_{\chi_\omega\in\mathcal{U}_L} \inf_{\phi\in\mathcal{E}} \int_\omega \phi(x)^2\, dV_g = \sup_{a\in\overline{\mathcal{U}}_L} \inf_{\phi\in\mathcal{E}} \int_\Omega a(x)\phi(x)^2\, dV_g = L, \tag{37}$$

for every $L \in (0,1)$. In other words, there is no gap between the intrinsic uniform optimal design problem and its convexified version.

Theorem 4.2. [26] *Assume that the uniform measure $\frac{1}{V_g(\Omega)}\, dV_g$ is the unique closure point of the family of probability measures $\mu_\phi = \phi^2\, dV_g$, $\phi \in \mathcal{E}$, for the vague* *topology, and that the whole family of eigenfunctions in $\mathcal{E}$ is uniformly bounded in $L^{2p}(\Omega)$, for some $p \in (1, +\infty]$. Then*

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{\phi \in \mathcal{E}} \int_\omega \phi(x)^2 \, dV_g = L, \tag{38}$$

for every $L \in (0, 1)$.

Remark 6. We are able to provide examples where there is a gap between the intrinsic second problem (36) and its convexified version. This occurs in any of the two following examples (see [26]):

- $\Omega = S^2$, the unit sphere in $\mathbb{R}^3$, endowed with the usual flat metric;
- Ω is the unit half-sphere in $\mathbb{R}^3$, endowed with the usual flat metric, and Dirichlet conditions are imposed on the great circle (boundary of Ω).

In both cases, if L is close enough to 1 then $\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) < L$, and hence there is a gap between the problem (36) and its convexified version.

4.3. **Optimal location of internal controllers.** By duality, our previous results provide an answer to the question of determining the shape and location of the control domain for wave or Schrödinger equations that minimizes the L^2 norm of the controllers realizing null controllability. For simplicity we restrict ourselves to the internally controlled wave equation on Ω with Dirichlet boundary conditions,

$$\begin{array}{ll} \partial_{tt} y(t,x) - \triangle_g y(t,x) = h_\omega(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ y(0,x) = y^0(x), \ \partial_t y(0,x) = y^1(x), & x \in \Omega, \end{array} \tag{39}$$

where h_ω is a control supported in $[0,T] \times \omega$ and ω is a measurable subset of Ω. Note that the Cauchy problem (39) is well posed for all initial data $(y^0, y^1) \in H_0^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$ and every $h_\omega \in L^2((0,T) \times \Omega, \mathbb{C})$, and its solution y belongs to $C^0(0,T; H_0^1(\Omega, \mathbb{C})) \cap C^1(0,T; L^2(\Omega, \mathbb{C})) \cap C^2(0,T; H^{-1}(\Omega, \mathbb{C}))$. The exact null controllability problem settled in these spaces consists of finding a control h_ω steering the control system (39) to

$$y(T, \cdot) = \partial_t y(T, \cdot) = 0. \tag{40}$$

It is well known that, for every subset ω of Ω of positive measure, the exact null controllability problem is by duality equivalent to the fact that the observability inequality

$$C \|(\phi^0, \phi^1)\|^2_{L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})} \leq \int_0^T \int_\omega |\phi(t,x)|^2 \, dV_g \, dt, \tag{41}$$

holds, for all $(\phi^0, \phi^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, for a positive constant C (only depending on T and ω), where ϕ is the (unique) solution of the adjoint system

$$\begin{array}{ll} \partial_{tt} \phi(t,x) - \triangle_g \phi(t,x) = 0, & (t,x) \in (0,T) \times \Omega, \\ \phi(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ \phi(0,x) = \phi^0(x), \ \partial_t \phi(0,x) = \phi^1(x), & x \in \Omega. \end{array} \tag{42}$$

The Hilbert Uniqueness Method (HUM, see [19]) provides a way to design the unique control solving the control problem (39)-(40) and having moreover a minimal $L^2((0,T) \times \Omega, \mathbb{C})$ norm. This control is referred to as the HUM control and is characterized as follows. Define the HUM functional J_ω by

$$J_\omega(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_\omega \phi(t,x)^2 \, dV_g \, dt - \langle \phi^1, y^0 \rangle_{H^{-1}, H_0^1} + \langle \phi^0, y^1 \rangle_{L^2}. \tag{43}$$

The notation $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ stands for the duality bracket between $H^{-1}(\Omega,\mathbb{C})$ and $H_0^1(\Omega,\mathbb{C})$, and the notation $\langle\cdot,\cdot\rangle_{L^2}$ stands for the usual scalar product of $L^2(\Omega,\mathbb{C})$. If (41) holds then the functional J_ω has a unique minimizer (still denoted (ϕ^0,ϕ^1)) in the space $L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})$, for all $(y^0,y^1)\in H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})$. The HUM control h_ω steering (y^0,y^1) to $(0,0)$ in time T is then given by

$$h_\omega(t,x)=\chi_\omega(x)\phi(t,x), \tag{44}$$

for almost all $(t,x)\in(0,T)\times\Omega$, where ϕ is the solution of (42) with initial data (ϕ^0,ϕ^1) minimizing J_ω.

The HUM operator Γ_ω is defined by

$$\begin{array}{rrcl}\Gamma_\omega: & H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C}) & \longrightarrow & L^2((0,T)\times\Omega,\mathbb{C})\\ & (y^0,y^1) & \longmapsto & h_\omega\end{array}$$

Optimal design control problem. We investigate the problem of minimizing the norm of the operator Γ_ω,

$$\|\Gamma_\omega\|=\sup_{\|(y^0,y^1)\|_{H_0^1)\times L^2}=1}\|h_\omega\|_{L^2((0,T)\times\Omega,\mathbb{C})} \tag{45}$$

over the set $\mathcal{U}_L$.

Here, we formulate the optimal design control problem in terms of minimizing the operator norm of Γ_ω in order to discard the dependence with respect to the initial data (y^0,y^1) and improve the robustness of the cost function.

By a duality argument, it is proved in [26] that, for every measurable subset ω of Ω, if $C_T^{(W)}(\chi_\omega)>0$ then

$$\|\Gamma_\omega\|=\frac{1}{C_T^{(W)}(\chi_\omega)},$$

and if $C_T^{(W)}(\chi_\omega)=0$, then $\|\Gamma_\omega\|=+\infty$. It follows that, for the optimal design control problem,

$$\inf_{\chi_\omega\in\mathcal{U}_L}\|\Gamma_\omega\|=\left(\sup_{\chi_\omega\in\mathcal{U}_L}C_T^{(W)}(\chi_\omega)\right)^{-1},$$

and therefore the problem is equivalent to the problem of maximizing the observability constant. Then, all considerations before can be applied as well to the optimal design control problem.

4.4. **Conclusions and perspectives.** We have provided a mathematical rigorous modeling of the problem of optimizing the shape and placement of sensors over a domain in which one considers the wave or the Schrödinger equation, with Dirichlet, Neumann, mixed or Robin boundary conditions whenever the boundary is nonempty.

First, when a specific choice of the initial data is given and therefore we deal with a particular solution, we have shown that the problem always admits at least one solution that can be regular or of fractal type depending on the regularity of the initial data.

In view of practical applications, we have defined a uniform optimal design problem, which does not depend on the initial data. Through spectral decompositions, we have motivated a second problem which consists of maximizing a spectral functional that can be viewed as a measure of eigenfunction concentration. Roughly speaking, the subset ω has to be chosen so to maximize the minimal trace of the squares of all eigenfunctions. This spectral criterion can be obtained and interpreted in two ways: on the one hand, it corresponds to a time asymptotic observability constant as the observation time interval tends to infinity, and on the other hand, to a randomized version of the deterministic observability inequality. We have also considered the convexified formulation of the problem. Under appropriate quantum ergodicity assumptions on Ω, we have a no-gap result between the initial problem and its convexified version, and we have computed the optimal value.

We have then provided spectral approximations, permitting to construct a maximizing sequence, and presented some numerical simulations that show the increasing complexity of the optimal sets.

Overall, our results highlight precise connections between optimal observability issues and quantum ergodic properties of the domain under consideration.

Our results open new directions for future research. We mention hereafter some of them.

1. As mentioned before, we expect that the second problem (uniform optimal design problem) not to have any optimal solution in general, except in very particular (degenerate) situations. In other words, in general, an optimal set probably does not exist. Besides, when implementing spectral approximations of the second problem the spillover phenomenon has been underlined and the increasing complexity has been put in evidence on numerical simulations. This indicates the lack of suitability of this spectral approximation procedure of common use in engineering applications. Further investigation is needed to formulate variants of these problems not presenting these instabilities. We mention here two possibilities:
   (a) In [26] we propose a slight modification of the observability inequality under consideration, which consists, e.g. in the Dirichlet case, of replacing the H^1_0 norm by the full H^1 one. Surprisingly enough, we show that the situation is then very different and that, if L is not too small then under QUE type assumptions there exists an optimal set. Consequently, the reinforcement of the observed norm by a compact term contributes to the existence of optimal sets. This can be even achieved by every value of the volume fraction L by means of a suitable modification of the observed norm (essentially by adding to the H^1_0-norm the L^2 one multiplied by a sufficiently large positive constant). Note however that, when reinforcing the observed norm, the corresponding observability constant decreases. It would then be natural to look for a compromise between ensuring the existence of optimal sets but at the price of deteriorating the observability constant.
   (b) A second idea is to define a variant of the criterion (15), by using Cesaro means. This idea is close to the filtering procedures used in [32] in the context of the numerical approximation of controls. The use of Cesaro means should also permit to weaken ergodicity assumptions (see [10]).

   In any case, an interesting direction for research is to model and define other kinds of spectral criteria permitting to avoid the spillover phenomenon to recover the existence of an optimal set.
2. In this work we considered wave and Schrödinger equations. In an ongoing work, we are studying the case of the heat equation. As it could be expected, the conclusion is then very different since optimal sets then exist much more easily, due to the intrinsic strong damping of the heat equation.

3. A crucial from the point of view of applications but fully open question is that of the numerical approximation of the optimal sets or densities. Two approaches are then to be considered, the continuous and the discrete one. In this setting a natural question is as follows: do the numerical optimal designs corresponding to discrete dynamics obtained by numerical approximation of the wave equation converge to the continuous optimal design as the mesh size tends to 0? According to the results of [32], one can expect the answer to be negative because of the effect of high-frequency spurious numerical solutions.

   If this were the case the numerical optimal design problem should be reformulated by means of suitable high-frequency filtering techniques.
4. Similar issues can be formulated in the context of homogenization. For instance, we could consider the optimal design problem above on a perforated domain Ω_ε, a rapidly oscillating manifold M_ϵ or for elliptic operators with rapidly oscillating coefficients. The question would then be to know whether, as ϵ tends to zero, the optimal designs do converge in some suitable sense to the optimal design of the limit homogenization problem. Once again one expects the result not to be true in general, due to the distortion that the high-frequency solutions may introduce in the highly heterogeneous medium, with respect to the limit homogeneous one. These issues have been the object of intensive research in the context of controllability problems (see [33]), but, as far as we know, have not been treated so far in the frame of the optimal design problems discussed in this paper.

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REFERENCES

[1] G. Allaire, S. Aubry, F. Jouve, *Eigenfrequency optimization in optimal design*, Comput. Methods Appl. Mech. Engrg. **190** (2001), 3565–3579.

[2] G. Allaire, A. Henrot, *On some recent advances in shape optimization*, C. R. Acad. Sci. Paris, t. 329, Série II b (2001), 383–396.

[3] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., **30** (1992), no. 5, 1024–1065.

[4] J.C. Bellido, A. Donoso, *An optimal design problem in wave propagation*, J. Optim. Theory Appl. **134** (2007), 339–352.

[5] D. Bucur, G. Buttazzo, *Variational methods in shape optimization problems*, Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

[6] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. **173** (2008), no. 3, 449–475.

[7] F. Fahroo, K. Ito, *Optimum damping design for an abstract wave equation*, New directions in control and automation, I (Limassol, 1995), Kybernetika (Prague) 32 (1996), no. 6, 557–574.

[8] F. Faure, S. Nonnenmacher, S. De Bièvre, *Scarred eigenstates for quantum cat maps of minimal periods*, Comm. Math. Phys. **239** (2003), no. 3, 449–492.

[9] M.I. Frecker, *Recent advances in optimization of smart structures and actuators*, Journal of Intelligent Material Systems and Structures **14** (2003), 207–216.

[10] P. Gérard, E. Leichtnam, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J. **71** (1993), 559–607.

[11] P. Hébrard, A. Henrot, *Optimal shape and position of the actuators for the stabilization of a string*, Syst. Cont. Letters **48** (2003), 199–209.

[12] P. Hébrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. **44** 2005, 349–366.

[13] A.E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Zeitschrift **41** (1936), 367–379.

[14] S. Jaffard, M. Tucsnak, E. Zuazua, *On a theorem of Ingham*, J. Fourier Anal. Appl. **3** (1997), 577–582.

[15] C.S. Kubrusly, H. Malebranche, *Sensors and controllers location in distributed systems - a survey*, Automatica **21** (1985), 117–128.

[16] S. Kumar, J.H. Seinfeld, *Optimal location of measurements for distributed parameter estimation*, IEEE Trans. Autom. Contr. **23**(1978), 690–698.

[17] I. Lasiecka, R. Triggiani, *Exact controllability of the wave equation with Neumann boundary control*, Appl. Math. Optim. **19** (1989), 243–290.

[18] G. Lebeau, *Contrôle de l'equation de Schrödinger*, J. Math. Pures Appl., **71** (1992), 267–291.

[19] J.-L. Lions, *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Rev. **30** (1988), 1–68.

[20] A. Münch, *Optimal location of the support of the control for the 1-D wave equation: numerical investigations*, Comput. Optim. Appl. **42** (2009), 443–470.

[21] S.L. Padula, R.K. Kincaid, *Optimization strategies for sensor and actuator placement*, Report NASA TM-1999-209126, 1999.

[22] F. Periago, *Optimal shape and position of the support for the internal exact control of a string*, Syst. Cont. Letters **58** (2009), no. 2, 136–140.

[23] Y. Privat, E. Trélat, E. Zuazua, *Optimal observation of the one-dimensional wave equation*, J. Fourier Anal. Appl. **19** (2013), no. 3, 514–544.

[24] Y. Privat, E. Trélat, E. Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire (2013).

[25] Y. Privat, E. Trélat, E. Zuazua, *Complexity and regularity of maximal energy domains for the wave equation with fixed initial data*, Preprint Hal (2013), 17 pages. Submitted.

[26] Y. Privat, E. Trélat and E. Zuazua, *Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains*, Preprint Hal (2013), 63 pages. Submitted.

[27] O. Sigmund, J.S. Jensen, *Systematic design of phononic band-gap materials and structures by topology optimization*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **361** (2003), no. 1806, 1001–1019.

[28] D. Ucinski, M. Patan, *Sensor network design fo the estimation of spatially distributed processes*, Int. J. Appl. Math. Comput. Sci. **20** (2010), no. 3, 459–481.

[29] M. van de Wal, B. Jager, *A review of methods for input/output selection*, Automatica **37** (2001), no. 4, 487–510.

[30] S. Zelditch, *Local and global analysis of eigenfunctions on Riemannian manifolds*, Preprint (2012).

[31] S. Zelditch, M. Zworski, *Ergodicity of eigenfunctions for ergodic billiards*, Comm. Math. Phys. **175** (1996), no. 3, 673–682.

[32] E. Zuazua, Propagation, observation, control and numerical approximation of waves approximated by finite difference method, SIAM Review, **47** (2005), no. 2, 197–243.

[33] E. Zuazua Controllability and Observability of Partial Differential Equations: Some results and open problems in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, 2006, 527–621.

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