

ASYMPTOTIC EXPANSION FOR DAMPED WAVE EQUATIONS WITH PERIODIC COEFFICIENTS

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We consider a linear dissipative wave equation in \mathbf{R}^N with periodic coefficients. By means of Bloch wave decomposition, we obtain an expansion of solutions as $t \rightarrow \infty$ and conclude that, in a first approximation, the solutions behave as the homogenized heat kernel.

1. Introduction

1.1. Setting of the problem

This paper is concerned with the analysis of the asymptotic behavior, as $t \rightarrow \infty$, of the solutions of

$$\begin{cases} \rho(x)u_{tt} - \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) + a_0 \rho(x)u_t = 0 & \text{in } \mathbf{R}^N \times (0, \infty) \\ u(x, 0) = \varphi^0(x) \\ u_t(x, 0) = \varphi^1(x) \end{cases} \quad (1.1)$$

where the coefficients satisfy

$$\begin{cases} a_{k\ell} \in L^\infty_\#(Y) \text{ where } Y =]0, 2\pi[^N, \text{ i.e., each } a_{k\ell} \text{ is a} \\ Y\text{-periodic bounded measurable function defined on } \mathbf{R}^N, \\ \exists \alpha > 0 \text{ such that } a_{k\ell}(x)\eta_k\eta_\ell \geq \alpha|\eta|^2 \quad \forall \eta \in \mathbf{R}^N, \text{ a.e. } x \in \mathbf{R}^N, \\ a_{k\ell} = a_{\ell k} \quad \forall k, \ell = 1, \dots, N, \text{ and} \\ a_0 \text{ is a positive constant,} \end{cases} \quad (1.2)$$

and

$$\begin{cases} \rho \in L^\infty_\#(Y), \text{ i.e., } \rho \text{ is } Y\text{-periodic, and} \\ \exists \rho_0, \rho_1 \in \mathbf{R}_+, \text{ such that } 0 < \rho_0 \leq \rho(x) \leq \rho_1, \text{ a.e. } x \in Y. \end{cases} \quad (1.3)$$

The equation (1.1) has a dissipative nature. Indeed, the energy associated to (1.1), given by

$$E(t) = \frac{1}{2} \int_{\mathbf{R}^N} \left[\rho(x)|u_t|^2 + a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \frac{\partial u}{\partial x_k} \right] dx,$$

is decreasing

$$\frac{\partial E}{\partial t} = -a_0 \int_{\mathbf{R}^N} \rho(x)|u_t|^2 dx.$$

Moreover, solutions of (1.1) satisfy the conservation law

$$\frac{\partial}{\partial t} \int_{\mathbf{R}^N} (u_t + a_0 u) \rho(x) dx = \int_{\mathbf{R}^N} \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) dx = 0.$$

That is, the mass of $u_t + a_0 u$ with respect to the weight $\rho(\cdot)$ is conserved along time:

$$m_\rho(u_t + a_0 u) = m_\rho(\varphi^1 + a_0 \varphi^0) = \int_{\mathbf{R}^N} (\varphi^1 + a_0 \varphi^0) \rho(x) dx. \quad (1.4)$$

Using Bloch waves decomposition together with a choice of a convenient Lyapunov function we obtain the asymptotic expansion of the solutions of (1.1). In particular we conclude that solutions behave as the homogenized heat kernel as $t \rightarrow \infty$. In fact, equation (1.1) can be viewed as a heat equation “perturbed” by the second order term $\rho(x)u_{tt}$, that introduces oscillations that, according to our analysis, are not strong enough to change the behavior of solutions as $t \rightarrow \infty$ in a first approximation.

To be more precise, equation (1.1) can be viewed as a perturbation of the parabolic equation

$$\begin{cases} \rho(x)a_0 u_t - \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) = 0 & \text{in } \mathbf{R}^N \times (0, \infty) \\ u(x, 0) = \varphi^0(x). \end{cases} \quad (1.5)$$

The asymptotic behavior of solutions of (1.5) is well known when $\rho = 1$, $a_0 = 1$. The first term in the asymptotic expansion was obtained in ⁶. It was shown that

$$t^{\frac{N}{2}(1-\frac{1}{p})} \|u(t) - m(\varphi)G_h(\cdot, t)\|_p \rightarrow 0, \text{ as } t \rightarrow \infty, 1 \leq p \leq \infty,$$

where

$$m(\varphi) = \int_{\mathbf{R}^N} \varphi dx$$

and G_h is the fundamental solution of the homogenized system

$$\begin{cases} u_t - q_{k\ell} \frac{\partial^2 u}{\partial x_\ell \partial x_k} = 0 & \text{in } \mathbf{R}^N \times (0, \infty) \\ u(x, 0) = \delta_0(x). \end{cases}$$

Here and in the sequel we denote by δ_0 the Dirac delta at the origin and by $\{q_{jk}\}_{j,k=1}^N$ the homogenized coefficients associated to the periodic matrix with coefficients (1.2). The constant $m(\varphi)$ is the mass of the solution of (1.5) which is conserved along time, i.e.,

$$\frac{\partial}{\partial t}[m(u(\cdot, t))] = 0, \quad \forall t \in (0, \infty).$$

Later on, in J.H. Ortega and E. Zuazua ⁷, the asymptotic expansion as $t \rightarrow \infty$ of the solutions of (1.5) with $L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ initial data was obtained by means of the Bloch wave decomposition (see ³ and ⁴ for an introduction to Bloch waves). Their first result was established in the $L^2(\mathbf{R}^N)$ -setting and then, thanks to the parabolic regularizing effect, a converge result in $L^\infty(\mathbf{R}^N)$ was derived. The results in ⁷ are, to some extent, an extension of those of ⁵ on the constant coefficient heat equation. The analysis in ⁷ can be easily adapted to the general parabolic equation (1.5) with variable, periodic density ρ .

This work is devoted to adapt the analysis in ⁷ to the case of the dissipative wave equation (1.1) under consideration.

Our main result will only be given for the solution u of (1.1) in the $L^2(\mathbf{R}^N)$ -setting with $L^2 \cap L^1(\mathbf{R}^N) \times H^{-1} \cap L^1(\mathbf{R}^N)$ initial data but, as it will become clear during the proofs, a similar analysis allows to obtain the asymptotic expansion of (u, u_t) in $H^s(\mathbf{R}^N) \times H^{s-1}(\mathbf{R}^N)$ with initial data in $H^s \cap L^1(\mathbf{R}^N) \times H^{s-1} \cap L^1(\mathbf{R}^N)$. An asymptotic expansion in $L^\infty(\mathbf{R}^N)$ can also be given, but here, in the absence of the parabolic regularizing effect used in ⁷, we should consider sufficiently smooth initial data so that the asymptotic expansion holds in H^s with $s > 0$ large enough so that $H^s \hookrightarrow C^0$.

1.2. Main results

The well-posedness of the equation (1.1) under the conditions (1.2) and (1.3) can be easily obtained writing (1.1) as an abstract evolution equation in the space of finite energy $H = H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$, with the inner product

$$((u, v), (\tilde{u}, \tilde{v}))_H = \int_{\mathbf{R}^N} u \tilde{u} dx + \int_{\mathbf{R}^N} a_{k\ell}(x) \frac{\partial u}{\partial x_k} \frac{\partial \tilde{u}}{\partial x_\ell} dx + \int_{\mathbf{R}^N} v \tilde{v} \rho(x) dx,$$

with $\{a_{k\ell}\}$ as in (1.2) and ρ as in (1.3), whenever $(u, v), (\tilde{u}, \tilde{v}) \in H$. Under these conditions the operator associated to (1.1) is maximal and dissipative on H . Then, Lummer-Phillip's theorem guarantees that the operator associated to (1.1) is the infinitesimal generator of a continuous semigroup. Thus, we deduce that for any initial data $(\varphi^0, \varphi^1) \in L^2(\mathbf{R}^N) \times H^{-1}(\mathbf{R}^N)$ the equation (1.1) has a unique weak solution $u = u(x, t)$ such that

$$u \in C^0(\mathbf{R}^+, L^2(\mathbf{R}^N)) \cap C^1(\mathbf{R}^+, H^{-1}(\mathbf{R}^N)).$$

Let us now state the main result.

Theorem 1 Assume that $\varphi^0 \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ and $\varphi^1 \in L^1(\mathbf{R}^N) \cap H^{-1}(\mathbf{R}^N)$ with $|x|^{k+1}\varphi^0(x), |x|^{k+1}\varphi^1(x) \in L^1(\mathbf{R}^N)$ for some $k \geq 0$. Let $u = u(x, t)$ be the solution of (1.1). Then, there exist periodic functions $c_\alpha \in L^\infty_\#(Y)$, with $|\alpha| \leq k$, and constants $c_{\beta, n}$, with $n \leq \frac{k}{2}$ and $4 \leq |\beta| \leq 2k$, depending on the initial data, the coefficients $\{a_{k\ell}\}$ and ρ , such that the solution u satisfies

$$\|u(\cdot, t) - \sum_{|\alpha| \leq k} c_\alpha(\cdot) [G_\alpha^*(\cdot, t) + \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha, n)} \sum_{|\beta|=4n+2m} c_{\beta, n} G_{\alpha+\beta}^*(\cdot, t)]\| \leq c_k t^{-\frac{2k+2+N}{4}} \quad (1.6)$$

as $t \rightarrow \infty$. Here $p(\alpha) = \left\lfloor \frac{k-|\alpha|}{2} \right\rfloor$, $a(\alpha, n) = p(\alpha) - n$,

$$G_\alpha^*(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \xi^\alpha e^{-\frac{q_{k\ell}}{a_0 \bar{\rho}} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi, \quad (1.7)$$

and $\bar{\rho}$ is the averaged density:

$$\bar{\rho} = \frac{1}{(2\pi)^N} \int_Y \rho(x) dx. \quad (1.8)$$

We observe that $G_\alpha^* = (-i)^{|\alpha|} (\partial^\alpha G^* / \partial x_\alpha)$ where $G^* = G^*(x, t)$ is the fundamental solution of the underlying parabolic homogenized system

$$\begin{cases} \bar{\rho} a_0 G_t^* - q_{k\ell} \frac{\partial^2 G^*}{\partial x_\ell \partial x_k} = 0 & \text{in } \mathbf{R}^N \times (0, \infty) \\ G^*(x, 0) = \delta_0(x). \end{cases} \quad (1.9)$$

The convergence result in (1.6) indicates that, roughly, the solution u of the equation (1.1) may be approximated at any order by a linear combination of the derivatives of the fundamental solution of the heat equation, modulated by the periodic functions $c_\alpha(\cdot)$. The role of these coefficients is to adapt the gaussian asymptotic profiles to the periodicity of the medium where the solution u evolves.

We consider $v(x, t) = c_0(x) G^*(x, t)$, the first term of the asymptotic expansion of the solution of (1.1). As we shall see below (see Section 5), $c_0(\cdot)$ turns out to be a constant and more precisely

$$c_0 = \frac{1}{a_0 \bar{\rho}} m_\rho(u_t + a_0 u). \quad (1.10)$$

It is important to observe that the mass of $v(x, t)$ with respect to the weight $\rho(\cdot)$, as $t \rightarrow \infty$, is the same as the mass of $u_t + a_0 u$ associated to (1.1) which is constant in time as seen in (1.4). In fact, we have

$$\begin{aligned} m_\rho(v_t) &= -\frac{c_0}{(2\pi)^N} \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \frac{q_{k\ell}}{a_0 \bar{\rho}} \xi_k \xi_\ell e^{-\frac{q_{k\ell}}{a_0 \bar{\rho}} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi \rho(x) dx \quad (x = y\sqrt{t}, \xi = \frac{\eta}{\sqrt{t}}) \\ &= -\frac{c_0}{t} \int_{\mathbf{R}^N} \rho(y\sqrt{t}) \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \frac{q_{k\ell}}{a_0 \bar{\rho}} \eta_k \eta_\ell e^{-\frac{q_{k\ell}}{a_0 \bar{\rho}} \eta_k \eta_\ell} e^{iy \cdot \eta} d\eta dy, \end{aligned}$$

and

$$m_\rho(v) = c_0 \int_{\mathbb{R}^N} \rho(y\sqrt{t}) \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-\frac{q_{k\ell}}{a_0\bar{\rho}} \eta_k \eta_\ell} e^{iy \cdot \eta} d\eta dy.$$

Since $\rho(y\sqrt{t}) \rightarrow \bar{\rho}$ weakly-* in $L^\infty(\mathbb{R}^N)$ as $t \rightarrow \infty$, then

$$m_\rho(v_t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

and, thanks to

$$\frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-\frac{q_{k\ell}}{a_0\bar{\rho}} \eta_k \eta_\ell} e^{iy \cdot \eta} d\eta dy = 1$$

we have

$$m_\rho(v) \rightarrow \bar{\rho} c_0, \quad \text{as } t \rightarrow \infty.$$

Thus, we obtain as $t \rightarrow \infty$,

$$m_\rho(v_t + a_0 v) \rightarrow a_0 \bar{\rho} c_0 = m_\rho(u_t + a_0 u).$$

Therefore, as expected, the total mass of the solution is captured by the first term in the asymptotic expansion.

The rest of the paper is organized as follows. First, in Section 2 and 3, we study the simpler problem in which $\rho \equiv 1$:

$$\begin{cases} u_{tt} - \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) + a_0 u_t = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = \varphi^0(x) \\ u_t(x, 0) = \varphi^1(x). \end{cases} \quad (1.11)$$

In Section 2 we recall some basic results on Bloch wave decomposition. In Section 3 we prove some basic lemmas and obtain the asymptotic behavior of the solutions of (1.11). In Section 4 we prove the main result, Theorem 1, in the general case. Finally, in Section 5, we analyze the periodic functions c_α and the constants $c_{\beta,n}$ entering in the asymptotic expansion.

2. Bloch wave decomposition

All along this section we assume that $\rho \equiv 1$. The general case will be discussed in Section 4.

In this section we recall some basic results on Bloch wave decompositions. We refer to ³ and to ⁴ for the notations and the proofs.

Let us consider the following spectral problem parametrized by $\xi \in \mathbb{R}^N$: To find $\lambda = \lambda(\xi) \in \mathbb{R}$ and $\psi = \psi(x; \xi)$ (no identically zero) such that

$$\begin{cases} A\psi(\cdot; \xi) = \lambda(\xi)\psi(\cdot; \xi) & \text{in } \mathbb{R}^N, \\ \psi(\cdot; \xi) \text{ is } (\xi, Y)\text{-periodic, i.e.,} \\ \psi(y + 2\pi m; \xi) = e^{2\pi i m \cdot \xi} \psi(y) & \forall m \in \mathbb{Z}^N, y \in \mathbb{R}^N, \end{cases} \quad (2.12)$$

where A is the elliptic operator in divergence form

$$A \stackrel{\text{def}}{=} - \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial}{\partial x_\ell} \right), \quad y \in \mathbb{R}^N. \quad (2.13)$$

We can write $\psi(x; \xi) = e^{ix \cdot \xi} \phi(x, \xi)$, ϕ being Y -periodic in the variable x . It is clear from (2.12) that the (ξ, Y) -periodicity is unaltered if we replace ξ by $(\xi + m)$ with $m \in \mathbb{Z}^N$. Therefore, ξ can be confined to the dual cell $\xi \in Y' = [-\frac{1}{2}, \frac{1}{2}]^N$. Under these conditions, it is known (see ⁴) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \leq \lambda_1(\xi) \leq \dots \leq \lambda_m(\xi) \leq \dots \rightarrow \infty, \\ \lambda_m(\xi) \text{ is a Lipschitz function of } \xi \in Y', \forall m \geq 1. \end{cases} \quad (2.14)$$

Besides, the corresponding eigenfunctions denoted by $\psi_m(\cdot; \xi)$ and $\phi_m(\cdot; \xi)$, form orthonormal basis in the subspaces of $L^2_{loc}(\mathbb{R}^N)$ which are (ξ, Y) -periodic and Y -periodic, respectively. Moreover, as a consequence of the min-max principle, it follows that (see ⁴)

$$\lambda_2(\xi) \geq \lambda_2^{(N)} > 0, \quad \forall \xi \in Y', \quad (2.15)$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary conditions on ∂Y .

The Bloch waves introduced above enable us to describe the spectral resolution of the unbounded self-adjoint operator A in $L^2(\mathbb{R}^N)$, in the orthogonal basis of Bloch waves

$$\{\psi_m(x; \xi) = e^{ix \cdot \xi} \phi_m(x; \xi) : m \geq 1, \xi \in Y'\}.$$

Thus, we have

Proposition 1 *Let $g \in L^2(\mathbb{R}^N)$. The m^{th} Bloch coefficient of g is defined as follows:*

$$\widehat{g}_m(\xi) = \int_{\mathbb{R}^N} g(x) e^{-ix \cdot \xi} \overline{\phi_m(x; \xi)} dx \quad \forall m \geq 1, \xi \in Y'.$$

Then the following inverse formula holds:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \widehat{g}_m(\xi) e^{ix \cdot \xi} \phi_m(x; \xi) d\xi.$$

Further, we have Parseval's identity:

$$\int_{\mathbb{R}^N} |g(x)|^2 dx = \int_{Y'} \sum_{m=1}^{\infty} |\widehat{g}_m(\xi)|^2 d\xi.$$

Finally, for all g in the domain of A , we have

$$Ag(x) = \int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi) \widehat{g}_m(\xi) e^{ix \cdot \xi} \phi_m(x; \xi) d\xi,$$

and, consequently, the equivalence of norms in $H^1(\mathbb{R}^N)$ and in $H^{-1}(\mathbb{R}^N)$:

$$\begin{aligned}\|g\|_{H^1(\mathbb{R}^N)}^2 &= \int_{Y'} \sum_{m=1}^{\infty} (1 + \lambda_m(\xi)) |\widehat{g}_m(\xi)|^2 d\xi, \\ \|g\|_{H^{-1}(\mathbb{R}^N)}^2 &= \int_{Y'} \sum_{m=1}^{\infty} \frac{|\widehat{g}_m(\xi)|^2}{1 + \lambda_m(\xi)} d\xi.\end{aligned}$$

Using Proposition 1, the solution of (1.11) can be written as follows.

Lemma 1 *Let $u(x, t)$ be the solution of (1.11). Then*

$$u(x, t) = \int_{Y'} \sum_{m=1}^{\infty} \left(\beta_m^1(\xi) e^{-\alpha_m^1(\xi)t} + \beta_m^2(\xi) e^{-\alpha_m^2(\xi)t} \right) e^{ix \cdot \xi} \phi_m(x, \xi) d\xi, \quad (2.16)$$

where

$$\alpha_m^1(\xi) = \frac{1}{2}(a_0 - \sqrt{a_0^2 - 4\lambda_m(\xi)}), \quad (2.17)$$

$$\alpha_m^2(\xi) = \frac{1}{2}(a_0 + \sqrt{a_0^2 - 4\lambda_m(\xi)}), \quad (2.18)$$

and

$$\beta_m^1(\xi) = \frac{\alpha_m^2(\xi)}{\sqrt{a_0^2 - 4\lambda_m(\xi)}} \widehat{\varphi}_m^0(\xi) + \frac{1}{\sqrt{a_0^2 - 4\lambda_m(\xi)}} \widehat{\varphi}_m^1(\xi), \quad (2.19)$$

$$\beta_m^2(\xi) = -\frac{\alpha_m^1(\xi)}{\sqrt{a_0^2 - 4\lambda_m(\xi)}} \widehat{\varphi}_m^0(\xi) - \frac{1}{\sqrt{a_0^2 - 4\lambda_m(\xi)}} \widehat{\varphi}_m^1(\xi), \quad (2.20)$$

where $\widehat{\varphi}_m^0(\xi)$ and $\widehat{\varphi}_m^1(\xi)$ are the Bloch coefficients of the initial data φ^0 and φ^1 .

Proof. Since $u(x, t) \in L^2(\mathbb{R}^N)$ for all $t > 0$, we have that

$$u(x, t) = \int_{Y'} \sum_{m=1}^{+\infty} \widehat{u}_m(\xi, t) e^{ix \cdot \xi} \phi_m(x, \xi) d\xi, \quad (2.21)$$

where $\widehat{u}_m(\xi, t)$ is defined by Proposition 1 and satisfies for any $m \geq 1$ the following differential equation

$$\begin{cases} \partial_t^2 \widehat{u}_m + a_0 \partial_t \widehat{u}_m + \lambda_m(\xi) \widehat{u}_m = 0 & \text{in } Y' \times (0, +\infty) \\ \widehat{u}_m(\cdot, 0) = \widehat{\varphi}_m^0, \quad \frac{\partial \widehat{u}_m}{\partial t}(\cdot, 0) = \widehat{\varphi}_m^1 & \text{in } Y'. \end{cases} \quad (2.22)$$

Here, ∂_t denotes the derivative with respect to t . Solving the differential equation (2.22) we find

$$\widehat{u}_m(\xi, t) = \beta_m^1(\xi) e^{-\alpha_m^1(\xi)t} + \beta_m^2(\xi) e^{-\alpha_m^2(\xi)t}, \quad (2.23)$$

where $\{\alpha_m^i(\xi), i = 1, 2\}$ are defined by (2.17) and (2.18), and they are the two roots of the characteristic equation

$$y^2 + a_0 y + \lambda_m(\xi) = 0.$$

The constants $\{\beta_m^i(\xi), i = 1, 2\}$ as in (2.19)-(2.20) are obtained in order to meet the initial data in (1.11)□.

We also have the following result on the dependence of λ_1 and ϕ_1 with respect to the parameter ξ (see ⁴ and ²).

Proposition 2 *Assume that the coefficients $a_{k\ell}$ satisfy (1.2). Then there exists $\delta_1 > 0$ such that the first eigenvalue λ_1 is an analytic function on $B_{\delta_1} = \{\xi : |\xi| < \delta_1\}$ and satisfies*

$$c_1|\xi|^2 \leq \lambda_1(\xi) \leq c_2|\xi|^2, \quad \forall \xi \in Y', \quad (2.24)$$

and

$$\begin{aligned} \lambda_1(0) = \partial_k \lambda_1(0) &= 0, & k = 1, \dots, N, \\ \partial_{k\ell}^2 \lambda_1(0) &= 2q_{k\ell}, & k, \ell = 1, \dots, N, \\ \partial^\alpha \lambda_1(0) &= 0 & \forall \alpha \text{ such that } |\alpha| \text{ is odd.} \end{aligned} \quad (2.25)$$

Futhermore, there is a choice of the first eigenfunction $\phi_1(x, \xi)$ satisfying

$$\begin{cases} \xi \rightarrow \phi_1(x, \xi) \in L^\infty \cap H_{\#}^1(Y) \text{ is analytic on } B_{\delta_1} \\ \phi_1(x, 0) = (2\pi)^{-\frac{N}{2}}. \end{cases}$$

The coefficients $q_{k\ell}$ are those of the homogenized matrix associated with the family $(a_{k\ell}^\varepsilon)$, where $a_{k\ell}^\varepsilon(x) = a_{k\ell}(x/\varepsilon)$ as $\varepsilon \rightarrow 0$. Since $\alpha_1^1(\xi)$ and $\beta_1^1(\xi)$ is defined by (2.19), respectively, we have:

Proposition 3 *Assume the same hypotheses as in Proposition 2. Then there exists $\delta > 0$, with $\delta \leq \delta_1$, such that $\alpha_1^1(\xi)$ and $\beta_1^1(\xi)$ are analytic functions on B_δ . Futhermore, $\alpha_1^1(\xi)$ satisfies*

$$c_3|\xi|^2 \leq \alpha_1^1(\xi) \leq c_4|\xi|^2, \quad \forall \xi \in B_\delta, \quad (2.26)$$

and

$$\begin{aligned} \alpha_1^1(0) = \partial_k \alpha_1^1(0) &= 0 & k = 1, \dots, N, \\ \partial_{k\ell}^2 \alpha_1^1(0) &= 2\frac{q_{k\ell}}{a_0}, & k, \ell = 1, \dots, N, \\ \partial^\beta \alpha_1^1(0) &= 0 & \forall \beta \text{ such that } |\beta| \text{ is odd.} \end{aligned} \quad (2.27)$$

3. Asymptotic expansion when $\rho \equiv 1$

3.1. Bloch component of u with exponential decay

We start this section proving that, in (2.16), the terms corresponding to the eigenvalues $\lambda_m(\xi)$, $m \geq 2$, decay exponentially as $t \rightarrow \infty$. Further, we also prove that the term corresponding to $\lambda_1(\xi)$ goes to zero exponentially, as $t \rightarrow \infty$, whenever $\xi \in U = \{\xi \in Y' : |\xi| > \delta\}$, with $\delta > 0$.

Lemma 2 *Let $\widehat{u}_m = \widehat{u}_m(\xi, t)$, $m \geq 1$, be the Bloch coefficients associated to the solution $u = u(x, t)$ of (1.11) given in (2.16). Then, there exists positive constants α and β , such that*

$$\int_{Y'} \sum_{m=2}^{\infty} |\widehat{u}_m(\xi, t)|^2 d\xi \leq \alpha e^{-\beta t} (\|\varphi^0\|^2 + \|\varphi^1\|_{H^{-1}}^2). \quad (3.28)$$

Proof. We consider the Lyapunov function

$$L_m(\xi, t) = E_m(\xi, t) + \varepsilon F_m(\xi, t),$$

where

$$\begin{aligned} E_m(\xi, t) &= \frac{1}{2} \left(|\partial_t \widehat{u}_m(\xi, t)|^2 + \lambda_m(\xi) |\widehat{u}_m(\xi, t)|^2 \right), \\ F_m(\xi, t) &= \partial_t \widehat{u}_m(\xi, t) \overline{\widehat{u}_m(\xi, t)} + \frac{a_0}{2} |\widehat{u}_m(\xi, t)|^2, \end{aligned}$$

and ε is a suitable constant to be chosen later. Here, $\bar{\cdot}$ denotes the complex conjugate. It follows from (2.15) that for $m \geq 2$,

$$\begin{aligned} |L_m(\xi, t) - E_m(\xi, t)| &\leq \varepsilon \left[\frac{1}{2\lambda_2^{(N)}} |\partial_t \widehat{u}_m(\xi, t)|^2 + \left(\frac{\lambda_m(\xi)}{2} + \frac{a_0 \lambda_m(\xi)}{2\lambda_2^{(N)}} \right) |\widehat{u}_m(\xi, t)|^2 \right] \\ &\leq \varepsilon c_0 E_m(\xi, t), \quad \text{with } c_0 = \max \left(\frac{1}{\lambda_2^{(N)}}, \frac{a_0}{\lambda_2^{(N)}} + 1 \right). \end{aligned}$$

Consequently, we have for $\varepsilon < 1/c_0$ that

$$(1 - \varepsilon c_0) E_m(\xi, t) \leq L_m(\xi, t) \leq (1 + \varepsilon c_0) E_m(\xi, t). \quad (3.29)$$

Now, we claim that

$$\partial_t L_m(\xi, t) \leq -c L_m(\xi, t) \quad (3.30)$$

holds for some positive constant independent of ξ whenever $m \geq 2$. Clearly, from (3.29) and (3.30) we conclude the proof of (3.28) for all $m \geq 2$.

In order to prove the claim, we proceed as follows: Multiplying the equation (2.22) by $\overline{\partial_t \widehat{u}_m(\xi, t)}$, we obtain

$$\partial_t E_m(\xi, t) = -a_0 |\partial_t \widehat{u}_m(\xi, t)|^2.$$

Next, we multiply the equation (2.22) by $\widehat{u}_m(\xi, t)$ to obtain

$$\partial_t \left(\partial_t \widehat{u}_m(\xi, t) \widehat{u}_m(\xi, t) + \frac{a_0}{2} |\widehat{u}_m(\xi, t)|^2 \right) = -\lambda_m(\xi) |\widehat{u}_m(\xi, t)|^2 + |\partial_t \widehat{u}_m(\xi, t)|^2$$

Adding the identities above and choosing ε small we deduce that

$$\begin{aligned} \partial_t L_m(\xi, t) &= -a_0 |\partial_t \widehat{u}_m(\xi, t)|^2 - \varepsilon \lambda_m(\xi) |\widehat{u}_m(\xi, t)|^2 + \varepsilon |\partial_t \widehat{u}_m(\xi, t)|^2 \\ &\leq -c_1(\varepsilon) E_m(\xi, t) \quad \text{with } 0 < c_1(\varepsilon) = 2 \min(\varepsilon, a_0 - \varepsilon). \end{aligned} \quad (3.31)$$

Now, using (3.29) with ε sufficiently small and satisfying (3.31), in particular $\varepsilon < a_0$, there exists $c_2 = c_2(\varepsilon) > 0$ such that

$$\partial_t L_m(\xi, t) \leq -c_2 L_m(\xi, t), \quad \text{where } c_2 = \frac{c_1}{1 - \varepsilon(c_0)^{-1}} > 0.$$

Therefore,

$$L_m(\xi, t) \leq L_m(\xi, 0) e^{-c_2 t},$$

and using again (3.29) we obtain

$$E_m(\xi, t) \leq c E_m(\xi, 0) e^{-c_2 t}, \quad (3.32)$$

where $c = 1 + \varepsilon(c_0)^{-1}$. Recalling the definition of $E_m(\xi, t)$ we have from (3.32) and (2.14) that

$$\lambda_m(\xi) |\widehat{u}_m(\xi, t)|^2 \leq c e^{-c_2 t} \{ |\partial_t \widehat{u}_m(\xi, 0)|^2 + \lambda_m(\xi) |\widehat{u}_m(\xi, 0)|^2 \}$$

and

$$|\widehat{u}_m(\xi, t)|^2 \leq c e^{-c_2 t} \left\{ \frac{1}{\lambda_2^{(N)}} |\partial_t \widehat{u}_m(\xi, 0)|^2 + |\widehat{u}_m(\xi, 0)|^2 \right\}.$$

Adding the inequalities above it follows that

$$(1 + \lambda_m(\xi)) |\widehat{u}_m(\xi, t)|^2 \leq c e^{-c_2 t} \left(1 + \frac{1}{\lambda_2^{(N)}} \right) \{ |\partial_t \widehat{u}_m(\xi, 0)|^2 + (1 + \lambda_m(\xi)) |\widehat{u}_m(\xi, 0)|^2 \},$$

and consequently

$$|\widehat{u}_m(\xi, t)|^2 \leq c e^{-c_2 t} \left(1 + \frac{1}{\lambda_2^{(N)}} \right) \left\{ |\widehat{u}_m(\xi, 0)|^2 + \frac{|\partial_t \widehat{u}_m(\xi, 0)|^2}{1 + \lambda_m(\xi)} \right\}.$$

Therefore,

$$\int_{Y'} \sum_{m=2}^{\infty} |\widehat{u}_m(\xi, t)|^2 d\xi \leq c_3 e^{-c_2 t} \int_{Y'} \sum_{m=2}^{\infty} \left\{ |\widehat{\varphi}_m^0(\xi)|^2 + \frac{|\widehat{\varphi}_m^1(\xi)|^2}{1 + \lambda_m(\xi)} \right\} d\xi,$$

where $\widehat{\varphi}_m^0(\xi)$ and $\widehat{\varphi}_m^1(\xi)$ are the Bloch coefficients of the initial data φ^0 and φ^1 , respectively. This completes the proof of (3.28) \square .

Remark 1 We choose $\delta > 0$ in Proposition 3 such that $\delta \leq \delta_1$ (δ_1 being the radius of the ball where λ_1 and ϕ_1 are analytic as in Proposition 2) and satisfying that

$$a_0^2 - 4\lambda_1(\xi) \geq a_0^2 - 4c_2\delta^2 = c_\delta > 0, \quad \forall \xi \in B_\delta. \quad (3.33)$$

Obviously this can be done since $\lambda_1(0) = \nabla_\xi \lambda(0) = 0$.

Lemma 3 Let $\hat{u}_1 = \hat{u}_1(\xi, t)$ be the first Bloch coefficient of the solution u of (1.11) given in (2.16). Then, there exist positive constants α and β , such that

$$\int_{Y' - B_\delta} |\hat{u}_1(\xi, t)|^2 d\xi \leq \alpha e^{-\beta t} (\|\varphi^0\|^2 + \|\varphi^1\|_{H^{-1}}^2), \quad (3.34)$$

with δ satisfying (3.33). On the other hand, the second component $\beta_1^2(\xi)e^{-\alpha_1^2(\xi)t}$ of $\hat{u}_1(\xi, t)$ as in (2.23) satisfies

$$\int_{B_\delta} |\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} d\xi \leq C e^{-a_0 t} (\|\varphi^0\|^2 + \|\varphi^1\|_{H^{-1}}^2). \quad (3.35)$$

Proof. In order to prove (3.34) we argue as in Lemma 2. We consider the Lyapunov function $L_1(\xi, t)$ and, instead of (2.15), we use Proposition 2 and Remark 1 to obtain (3.29) for $\xi \in Y' - B_\delta$ and $m = 1$. We observe that (2.24) gives us

$$\frac{1}{\lambda_1(\xi)} < \frac{1}{c_1\delta^2}, \quad \forall \xi \in Y' - B_\delta,$$

and (3.34) is obtained following the same steps of Lemma 2.

To prove (3.35) it is enough to observe that, thanks to (3.33), and since, according to (2.18) $\alpha_1^2(\xi) \geq a_0/2$ for all $\xi \in B_\delta$, we have

$$\begin{aligned} (1 + \lambda_1(\xi))|\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} &\leq (1 + c_2|\xi|^2)|\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} \\ &\leq (1 + c_2\delta^2) \frac{4 + (a_0 - \sqrt{c_\delta})^2}{2c_\delta} e^{-a_0 t} (|\hat{\varphi}_0^1(\xi)|^2 + |\hat{\varphi}_1^1(\xi)|^2) \\ &\leq (1 + c_2\delta^2) \frac{4 + (a_0 - \sqrt{c_\delta})^2}{2c_\delta} e^{-a_0 t} ((1 + \lambda_1(\xi))|\hat{\varphi}_0^1|^2 + |\hat{\varphi}_1^1|^2). \end{aligned}$$

Consequently,

$$|\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} \leq C(\delta) e^{-a_0 t} \left(|\hat{\varphi}_1^0(\xi)|^2 + \frac{|\hat{\varphi}_1^1(\xi)|^2}{1 + \lambda_1(\xi)} \right),$$

and the result follows \square .

3.2. Bloch component of u with polynomial decay

Thanks to Lemmas 2 and 3, to conclude the proof of the Theorem 1 it is sufficient to analyse

$$I(x, t) = \int_{\tilde{B}_\delta} \beta_1^1(\xi) e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi} \phi_1(x; \xi) d\xi, \quad (3.36)$$

since the other components of u have an exponentially decay in $L^2(\mathbf{R}^N)$. To do that we make use of classical asymptotic lemmas (see Lemma 5 and Lemma 6 below) and assume that the initial data $\varphi^0 \in L^1(\mathbf{R}^N)$ and $\varphi^1 \in L^1(\mathbf{R}^N) \cap H^{-1}(\mathbf{R}^N)$ are such that $|x|^{k+1}\varphi^0(x)$, $|x|^{k+1}\varphi^1(x) \in L^1(\mathbf{R}^N)$. Under these conditions the first Bloch coefficients $\widehat{\varphi}_1^0(\xi)$ and $\widehat{\varphi}_1^1(\xi)$ of the initial data belong to $C^{k+1}(B_\delta)$ (see Lemma 4), what is crucial in the proof of the asymptotic expansion (see the definitions of $\beta_1^1(\xi)$ and $\alpha_1^1(\xi)$ in Lemma 1).

Lemma 4 *Let $\varphi \in L^1(\mathbf{R}^N)$ be a function such that $|x|^k \varphi \in L^1(\mathbf{R}^N)$. Then, its first Bloch coefficient $\widehat{\varphi}_1(\xi)$, belongs to $C^k(B_\delta)$, where B_δ is a neighborhood of $\xi = 0$ where the first Bloch wave $\phi_1(x; \xi)$ is analytic.*

Proof. Since

$$\widehat{\varphi}_1(\xi) = \int_{\mathbf{R}^N} \varphi(x) e^{-i \cdot \xi} \overline{\phi_1(x; \xi)} dx,$$

for all $\alpha \in (\mathbf{N} \cup \{0\})^N$ with $|\alpha| \leq k$, we have

$$\frac{\partial \widehat{\varphi}_1}{\partial \xi_\alpha}(\xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbf{R}^N} \varphi(x) (-i)^{|\beta|} x^\beta e^{-i \cdot \xi} \frac{\partial \overline{\phi_1}}{\partial \xi_{\alpha-\beta}}(x; \xi) dx,$$

where $\beta \leq \alpha$ if and only if $\beta_j \leq \alpha_j$ for all $j = 1, \dots, N$, and

$$\binom{\alpha}{\beta} = \prod_{k=1}^N \binom{\alpha_j}{\beta_j}.$$

Moreover, Proposition 2 gives us that the function $\xi \rightarrow \phi_1(x; \xi)$ is analytic with values in $L^\infty_\#(Y)$, what guarantees that

$$\begin{aligned} \left| \frac{\partial \widehat{\varphi}_1}{\partial \xi_\alpha}(\xi) \right| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} c_\beta \int_{\mathbf{R}^N} |\varphi(x) x^\beta| dx, \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} c_\beta \int_{\mathbf{R}^N} (1 + |x|^k) |\varphi(x)| dx, \end{aligned}$$

for $0 \leq |\alpha| \leq k$, where c_β is a positive constant. Thus, using again Proposition 2, we have that the map $\xi \rightarrow e^{-i\xi \cdot x} \partial_\xi^\alpha \phi_1(x; \xi)$ is continuous, and the result follows \square .

Now, we are going to present some basic asymptotic results. The following definition will be useful to simplify the notation.

Definition 1 Given $f, g \in C(\mathbf{R}; \mathbf{R})$, we say that f and g are of the same order as $t \rightarrow \infty$ and we denote it by $f \sim g$ when

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 1.$$

The following basic lemmas on asymptotic analysis are needed (see ¹, p. 263):

Lemma 5 Let $f : [0, b] \rightarrow \mathbf{R}$ be a continuous function such that it has the uniform asymptotic series expansion

$$f(x) = x^\alpha \sum_{n=1}^{\infty} a_n x^{\beta_n}, \quad x \in [0, b],$$

with $\alpha > 1$ and $\beta_n > 0$. Then

$$\int_0^b e^{-tx} f(x) dx \sim \sum_{n=1}^{\infty} a_n \frac{\Gamma(\alpha + \beta_n + 1)}{t^{(\alpha + \beta_n + 1)}}, \quad \text{as } t \rightarrow \infty.$$

When $f(x) = x^\alpha$ we have that

$$\int_0^b e^{-tx} x^\alpha dx \sim \frac{\Gamma(\alpha + 1)}{t^{(\alpha + 1)}}, \quad \text{as } t \rightarrow \infty,$$

where

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx, \quad z > 0.$$

As a consequence of Lemma 5, the following holds in ⁷:

Lemma 6 Let $c > 0$. Then

$$\int_{b_s} e^{-c|\xi|^2 t} |\xi|^k d\xi \sim c_k t^{-\frac{k+N}{2}}, \quad \text{as } t \rightarrow \infty,$$

for all $k \in \mathbf{N}$, where c_k is a positive constant that may be computed explicitly. On the other hand, if $q = (q_{ij})$ is a symmetric positive matrix, we also have

$$\int_{Y'} e^{-\frac{q_{ij}}{a_0} \xi_i \xi_j t} \xi^\beta d\xi \sim c_{|\beta|} t^{-\frac{(|\beta|+N)}{2}}, \quad \text{as } t \rightarrow \infty,$$

for all multi-index $\beta \in (\mathbf{N} \cup \{0\})^N$ and for a suitable constant $c_{|\beta|}$ that may be computed as well.

In the sequel we prove three lemmas which are related to the asymptotic behavior of the terms $\beta_1^1(\xi)$, $\phi_1(x; \xi)$ and $e^{-\alpha_1^1(\xi)t}$, respectively, that appear in (3.36).

We recall that the idea we have in mind is to prove that the solutions of (1.1) may be approximated by a linear combination of the derivatives of the fundamental solution of the homogenized heat equation and, as we said in the beginning of this section, in view of Lemma 5 and 6, our analysis may be restricted to consider (3.36). Thus, our first step in this direction is to prove that (3.36) can be replaced by

$$J(x, t) = \int_{B_\delta} \sum_{|\alpha| \leq k} d_\alpha \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi} \phi_1(x; \xi) d\xi, \quad (x, t) \in \mathbf{R}^N \times \mathbf{R}^+, \quad (3.37)$$

where

$$d_\alpha = \frac{1}{\alpha!} \partial^\alpha \beta_1^1(0), \quad (3.38)$$

which are obtained by means of the Taylor expansion of $\beta_1^1(\xi)$ in $\xi = 0$.

Lemma 7 *Let $\varphi^0 \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ and $\varphi^1 \in L^1(\mathbf{R}^N) \cap H^{-1}(\mathbf{R}^N)$ be such that $|x|^{k+1}\varphi^0(x), |x|^{k+1}\varphi^1(x) \in L^1(\mathbf{R}^N)$. Consider $I(x, t)$ as in (3.36) and $J(x, t)$ as in (3.37). Then, there exist $c_k > 0$ such that*

$$\|I(\cdot, t) - J(\cdot, t)\| \leq c_k t^{-\frac{2k+2+N}{4}} \text{ as } t \rightarrow \infty.$$

Proof. Since $\varphi^0 \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ and $\varphi^1 \in L^1(\mathbf{R}^N) \cap H^{-1}(\mathbf{R}^N)$ with $|x|^{k+1}\varphi^0, |x|^{k+1}\varphi^1 \in L^1(\mathbf{R}^N)$ then, from Lemma 4, we have that $\widehat{\varphi}_1^0, \widehat{\varphi}_1^1 \in C^{k+1}(B_\delta)$, and, according to (2.19), $\beta_1^1 \in C^{k+1}(B_\delta)$ as well. Thus, thanks to (3.38), from the Taylor expansion we have that for all $\xi \in B_\delta$,

$$|\beta_1^1(\xi) - \sum_{|\alpha| \leq k} d_\alpha \xi^\alpha| \leq C_k |\xi|^{k+1}, \quad C_k > 0. \quad (3.39)$$

These constants can be computed explicitly in terms of $\partial^\alpha \lambda_1(0)$ and $\partial^\alpha \phi_1(0)$, and the mass of the initial datum (see Section 5). Indeed, from Parseval's identity, we have

$$\|I(\cdot, t) - J(\cdot, t)\|^2 = \int_{B_\delta} |\beta_1^1(\xi) - \sum_{|\alpha| \leq k} d_\alpha \xi^\alpha|^2 e^{-2\alpha_1^1(\xi)t} d\xi.$$

Then, thanks to estimate (2.26) in Proposition 3 and Lemma 6, we obtain

$$\begin{aligned} \|I(\cdot, t) - J(\cdot, t)\|^2 &= \int_{B_\delta} |\beta_1^1(\xi) - \sum_{|\alpha| \leq k} d_\alpha \xi^\alpha|^2 e^{-2c_3|\xi|^2 t} d\xi \\ &\leq c_k \int_{B_\delta} |\xi|^{2(k+1)} e^{-2c_3|\xi|^2 t} d\xi \sim c_k t^{-\frac{2k+2+N}{2}}, \text{ as } t \rightarrow \infty. \end{aligned}$$

This concludes the proof of Lemma 7 \square .

In a second step, we compute the Taylor expansion of $\phi_1(x, \xi)$ around $\xi = 0$, and prove that all the terms entering in the definition (3.37) of J and that we denote by J_α with $\alpha \in (\mathbf{N} \cup 0)^N$,

$$J_\alpha(x, t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi} \phi_1(x; \xi) d\xi, \quad (x, t) \in \mathbf{R}^N \times \mathbf{R}^+, \quad (3.40)$$

may be approximated in L^2 -setting by a linear combination of the form

$$\frac{1}{(2\pi)^N} \sum_{|\gamma| \leq k - |\alpha|} \tilde{d}_\gamma(x) \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi},$$

where \tilde{d}_γ are periodic functions defined by

$$\tilde{d}_\alpha(\cdot) = \frac{1}{\alpha!} \partial_\xi^\alpha \phi_1(\cdot; 0) \in L^\infty_\#(\mathbf{R}^N). \quad (3.41)$$

This way be done obtaining the same rate of decay as in Lemma 7.

Now we present a result from ² that will be needed when stating and proving in Lemma 9 the facts metioned above.

Lemma 8 *Let us introduce*

$$G(x) = \int_{Y'} g(\xi) e^{ix \cdot \xi} \omega(x; \xi) d\xi, \quad x \in \mathbf{R}^N, \quad (3.42)$$

where $g \in L^2(Y')$ and $\omega \in L^\infty(Y'; L^2_\#(Y))$. Then we have

$$\|G\|^2 = \int_{Y'} |g(\xi)|^2 \|\omega(\cdot; \xi)\|_{L^2(Y)}^2 d\xi.$$

Proof. To check this result we expand $\omega(x; \xi)$ as a function of x in the orthonormal basis $\{\phi_m(x; \xi)\}_{m=1}^\infty$ where $\xi \in Y'$ is a parameter:

$$\omega(x; \xi) = \sum_{m=1}^\infty a_m(\xi) \phi_m(x; \xi).$$

Introducing this expression in (3.42), we get

$$G(x) = \int_{Y'} g(\xi) \sum_{m=1}^\infty a_m(\xi) e^{ix \cdot \xi} \phi_m(x; \xi) d\xi.$$

Applying the Parseval's identity of Proposition 1, it follows that

$$\|G\|^2 = \int_{Y'} |g(\xi)|^2 \sum_{m=1}^\infty |a_m(\xi)|^2 d\xi.$$

This completes the proof of the lemma if we use the Parseval's identity in $L^2(Y)$:

$$\|\omega(\cdot; \xi)\|_{L^2(Y)}^2 = \sum_{m=1}^{\infty} |a_m(\xi)|^2 \quad \forall \xi \in Y' \square.$$

Lemma 9 *We consider $J_\alpha(x, t)$ defined in (3.40), with $|\alpha| \leq k$, and*

$$I_\alpha(x, t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi} d\xi, \quad (x, t) \in \mathbf{R}^N \times \mathbf{R}^+, \quad (3.43)$$

Then there exists periodic functions $\tilde{d}_\gamma = \tilde{d}_\gamma(x)$ defined in (3.41), such that

$$\|J_\alpha(\cdot, t) - \sum_{|\gamma| \leq k - |\alpha|} \tilde{d}_\gamma(\cdot) I_{\gamma + \alpha}(\cdot, t)\| \leq c_{k, |\alpha|} t^{-\frac{2k+2+N}{4}}, \quad \text{as } t \rightarrow \infty.$$

Proof. We set

$$R_k(x; \xi) = \phi_1(x; \xi) - \sum_{|\alpha| \leq k} \tilde{d}_\alpha(x) \xi^\alpha,$$

where $\tilde{d}_\alpha(\cdot)$ is defined in (3.41). Since ϕ_1 is an analytic function with respect to ξ in B_δ and values in $L^2(Y)$ we have, for all $\xi \in B_\delta$,

$$\|R_k(\cdot; \xi)\|_{L^2(Y)} \leq C_k |\xi|^{k+1}, \quad C_k > 0.$$

Thus, $R_k \in L^\infty(Y'; L^2_\#(Y))$. Then, for $\alpha \in (\mathbf{N} \cup \{0\})^N$ with $|\alpha| \leq k$, we have

$$J_\alpha(x, t) - \sum_{|\gamma| \leq k - |\alpha|} \tilde{d}_\gamma(x) I_{\gamma + \alpha}(x, t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix \cdot \xi} R_k(x; \xi) d\xi,$$

and since $R_k(y; \xi)$ is Y -periodic in the variable y , it follows from Lemma 8 that

$$\begin{aligned} \|J_\alpha(\cdot, t) - \sum_{|\gamma|=0}^{k-|\alpha|} \tilde{d}_\gamma(\cdot) I_{\gamma + \alpha}(\cdot, t)\|^2 &\leq \int_{B_\delta} \frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}} e^{-2\alpha_1^1(\xi)t} \|R_{k-|\alpha|}(\cdot; \xi)\|_{L^2(Y)}^2 d\xi \\ &\leq C_{k-|\alpha|} \int_{B_\delta} |\xi|^{2k+2} e^{-2\alpha_1^1(\xi)t} d\xi, \end{aligned}$$

and we conclude as in Lemma 7, using the Proposition 2 and Lemma 6 \square .

The proofs of Lemmas 7 and 9, as well as that of Lemma 10 that we present below, provide a systematic way of computing the coefficients that appear in the statement of Theorem 1. We note that the functions $c_\alpha(\cdot)$ are related to the derivatives of the first Bloch eigenfunction $\phi_1(x; \xi)$ with respect to ξ in $\xi = 0$. We shall describe how to compute them explicitly in Section 5.

Now, we are going to study the asymptotic behavior of the integral $I_\alpha(x, t)$ defined in Lemma 9. We observe that, according to Proposition 3,

$$\begin{aligned}\alpha_1^1(0) &= \partial_k \alpha_1^1(0) = 0, & k &= 1, \dots, N, \\ \partial_{k\ell}^2 \alpha_1^1(0) &= \frac{2q_{k\ell}}{a_0}, & \ell, k &= 1, \dots, N,\end{aligned}$$

where the coefficients $q_{k\ell}$ are those of the homogenized matrix associated with the family $(a_{k\ell}^\varepsilon)$, where $a_{k\ell}^\varepsilon(x) = a_{k\ell}(x/\varepsilon)$, as $\varepsilon \rightarrow 0$. Then, for all $\xi \in B_\delta$, we have

$$e^{-\alpha_1^1(\xi)t} \sim e^{-\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t}, \quad \text{as } t \rightarrow \infty.$$

This fact provides a first idea of the behavior of $I_\alpha(x, t)$, defined in (3.43), as $t \rightarrow \infty$. Furthermore, in Lemma 10 it turns evident that solutions of (1.1) behave as a linear combination of functions $G_\alpha^*(x, t)$ introduced in (1.7), which are the derivatives $(-i)^{|\alpha|} \partial_x^\alpha$ of the fundamental solution G^* of the homogenized heat equation.

Lemma 10 *We consider the function $I_\alpha(x, t)$ defined in (3.43) with $|\alpha| \leq k$. Then, there exist constants $c_{\beta, n}$, with $4 \leq |\beta| \leq 4p(\alpha)$, such that*

$$\left\| I_\alpha(\cdot, t) - I_\alpha^*(\cdot, t) - \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha, n)} \sum_{|\beta|=4n+2m} c_{\beta, n} I_{\alpha+\beta}^*(\cdot, t) \right\| \leq c_{k, |\alpha|} t^{-\frac{2k+2+N}{4}},$$

as $t \rightarrow \infty$, with $p(\alpha) = \left[\frac{k-|\alpha|}{2} \right]$, $a(\alpha, n) = p(\alpha) - n$, and where for $\alpha \in (\mathbf{N} \cup 0)^N$

$$I_\alpha^*(x, t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi, \quad (x, t) \in \mathbf{R}^N \times \mathbf{R}^+. \quad (3.44)$$

Here and in the sequel $[\cdot]$ denotes the integer part.

Remark 2 *In view of (1.7) with $\bar{p} = 1$ and (3.44) the analogy between I_α^* and G_α^* is clear. I_α^* is in fact obtained by integrating the same quantity as in G_α^* but this time in B_δ instead of \mathbf{R}^N . In particular, $I_\alpha^*(x, t)$ has the same polynomial decay as $G_\alpha^*(x, t)$, because*

$$\|G_\alpha^*(\cdot, t) - I_\alpha^*(\cdot, t)\| \leq ce^{-c(\delta)t}, \quad \text{with } c(\delta) > 0. \quad (3.45)$$

Indeed, by Parseval's identity and the coercitivity of coefficients $\{q_{k\ell}\}$ we have

$$\begin{aligned}\|G_\alpha^*(\cdot, t) - I_\alpha^*(\cdot, t)\|^2 &= \frac{1}{(2\pi)^{2N}} \int_{\mathbf{R}^N - B_\delta} |\xi|^{2|\alpha|} e^{-2\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} d\xi \\ &\leq \frac{1}{(2\pi)^{2N}} \int_{\mathbf{R}^N - B_\delta} |\xi|^{2|\alpha|} d\xi e^{-2\frac{c}{a_0} |\xi|^2 t} \leq ce^{-c(\delta)t}.\end{aligned}$$

Proof of Lemma 10. We consider the map $\xi \in B_\delta \rightarrow \nu(\xi)$ given by

$$\nu(\xi) = \alpha_1^1(\xi) - \frac{1}{2} \partial_{k\ell}^2 \alpha_1^1(0) \xi_k \xi_\ell = \alpha_1^1(\xi) - \frac{q_{k\ell}}{a_0} \xi_k \xi_\ell. \quad (3.46)$$

Thanks to Proposition 3, the map $\nu = \nu(\xi)$ is analytic in B_δ . Moreover, it follows from (2.27) that for $\xi \in B_\delta$

$$|\nu(\xi)| = \left| \alpha_1^1(\xi) - \frac{q_{k\ell}}{a_0} \xi_k \xi_\ell \right| \leq c |\xi|^4. \quad (3.47)$$

The function $(e^{-\nu(\xi)t} - 1)$ is also analytic, and by (3.47) we have

$$\left| e^{-\nu(\xi)t} - \sum_{n=0}^p \frac{t^n}{n!} (-\nu(\xi))^n \right| \leq C_p (\nu(\xi)t)^{p+1} \leq c_p |\xi|^{4p+4} t^{p+1}. \quad (3.48)$$

Thus, defining for $p \geq 1$

$$\nu_{\alpha,p}(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha \left[\sum_{n=0}^p \frac{t^n}{n!} (-\nu(\xi))^n \right] e^{-\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi, \quad (x,t) \in \mathbf{R}^N \times \mathbf{R}^+,$$

and replacing (3.46) in $I_\alpha(x,t)$, we get

$$I_\alpha(x,t) - \nu_{\alpha,p}(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} \left[e^{-\nu(\xi)t} - \sum_{n=0}^p \frac{t^n}{n!} (-\nu(\xi))^n \right] e^{ix \cdot \xi} d\xi.$$

Then, from Parseval's identity and (3.48) it follows that

$$\begin{aligned} \|I_\alpha(\cdot, t) - \nu_{\alpha,p}(\cdot, t)\| &= \int_{B_\delta} \frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}} \left| e^{-\nu(\xi)t} - \sum_{n=0}^p \frac{t^n}{n!} (-\nu(\xi))^n \right|^2 e^{-2\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} d\xi \\ &\leq c t^{2p+2} \int_{B_\delta} |\xi|^{2|\alpha|+8p+8} e^{-2\frac{q_{k\ell}}{a_0} \xi_k \xi_\ell t} d\xi \\ &\sim \tilde{c} t^{2p+2} t^{-\frac{2|\alpha|+8p+8+N}{2}} = \tilde{c} t^{-\frac{2|\alpha|+4p+4+N}{2}} \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (3.49)$$

Now we choose p such that the above decay rate is of the order of $(2k+2+N)/2$, i.e., p satisfies $(2|\alpha|+4p+4+N) \geq 2k+N+2$, or, equivalently,

$$2p \geq k - |\alpha| - 1. \quad (3.50)$$

Thus, we have

$$\|I_\alpha(\cdot, t) - \nu_{\alpha,p}(\cdot, t)\| \leq \tilde{c} t^{-\frac{2k+n+2}{4}}, \quad \text{as } t \rightarrow \infty. \quad (3.51)$$

To conclude the proof we are going to study the asymptotic behavior of the integral $\nu_{\alpha,p}(x,t)$ defined above. But before doing it, we note that if we consider the Taylor expansion of $\nu(\xi)$ around $\xi = 0$, we obtain

$$\nu_{\alpha,p}(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha \left\{ 1 + \sum_{n=1}^p \frac{t^n}{n!} \left(\sum_{|\beta|=0}^{\infty} \frac{1}{\beta!} \partial_\xi^\beta \nu(0) \xi^\beta \right)^n \right\} e^{-\frac{qk\ell}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi.$$

Indeed, since $\partial^\beta \nu(0) = 0$ for $|\beta| < 4$ and $|\beta|$ odd (see (3.46) and Proposition 3) we have

$$\nu(\xi) = \sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial_\xi^\beta \alpha_1^1(0),$$

and, consequently

$$(-\nu(\xi))^n = (-1)^n \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial_\xi^\beta \alpha_1^1(0) \right)^n = \sum_{m=0}^{\infty} \sum_{|\beta|=4n+2m} c_{\beta,n} \xi^\beta, \quad (3.52)$$

for suitable constants $c_{\beta,n}$ that will be computed in Section 5. This fact suggests the following approximation

$$\nu_{\alpha,p}(x,t) \sim \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha \left\{ 1 + \sum_{n=1}^p \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n} \xi^\beta \right\} e^{-\frac{qk\ell}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi,$$

where $a(n)$ is an index to be chosen for any $n = 1, \dots, p$. Thus, let us define

$$\begin{aligned} \omega_{\alpha,p}(x,t) &= \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha \left\{ 1 + \sum_{n=1}^p \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n} \xi^\beta \right\} e^{-\frac{qk\ell}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi \\ &= I_\alpha^*(x,t) + \sum_{n=1}^p \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n} I_{\beta+\alpha}^*(x,t), \end{aligned}$$

with $a(n)$ to be chosen later, and consider the difference

$$\begin{aligned} \nu_{\alpha,p}(x,t) - \omega_{\alpha,p}(x,t) &= \\ &= \frac{1}{(2\pi)^N} \sum_{n=1}^p \frac{t^n}{n!} \int_{B_\delta} \xi^\alpha \left[(-\nu(\xi))^n - \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n} \right] e^{-\frac{qk\ell}{a_0} \xi_k \xi_\ell t} e^{ix \cdot \xi} d\xi. \end{aligned}$$

Now, computing in (3.52) the Taylor expansion of order $(4n + 2a(n))$ of $(-\nu(\xi))^n$, we obtain the existence of a positive constant $C_{a(n)} > 0$ satisfying, for all $\xi \in B_\delta$,

$$\left| (-\nu(\xi))^n - \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n} \right| \leq C_{a(n)} |\xi|^{4n+2a(n)+2}.$$

Hence,

$$\begin{aligned}
& \|\nu_{\alpha,p}(\cdot, t) - \omega_{\alpha,p}(\cdot, t)\|^2 \leq \\
& \leq \sum_{n=1}^p \frac{t^{2n}}{(n!)^2} \int_{B_\delta} \frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}} |(-\nu(\xi))^n - \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n}|^2 e^{-2\frac{qk\ell}{a_0} \xi_k \xi_\ell t} d\xi \\
& \leq \frac{1}{(2\pi)^{2N}} \sum_{n=1}^p \frac{t^{2n}}{(n!)^2} C_\nu^2 \int_{B_\delta} |\xi|^{2|\alpha|} |\xi|^{2(4n+2a(n)+2)} e^{-2\frac{qk\ell}{a_0} \xi_k \xi_\ell t} d\xi \\
& \leq c \sum_{n=1}^p t^{-\frac{N+4n+4+4a(n)+2|\alpha|}{2}}, \text{ as } t \rightarrow \infty.
\end{aligned}$$

Finally, in order to obtain the same decay rate as in Lemma 9 we take into account (3.49), and choose $a(n)$ satisfying $(N + 4n + 4 + 4a(n) + 2|\alpha|) \geq 2|\alpha| + 4p + 4 + N$, i.e.,

$$a(n) \geq p - n.$$

Recalling that p satisfies (3.50), we choose

$$p(\alpha) = \left\lfloor \frac{k - |\alpha|}{2} \right\rfloor \quad \text{and} \quad a(\alpha, n) = \left\lfloor \frac{k - |\alpha|}{2} \right\rfloor - n.$$

This gives us that

$$\left\| \nu_{\alpha,p}(\cdot, t) - I_\alpha^*(\cdot, t) - \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(n,\alpha)} \sum_{|\beta|=4n+2m} c_{\beta,n} I_{\beta+\alpha}^*(\cdot, t) \right\| \leq ct^{-\frac{2k+n+2}{4}},$$

as $t \rightarrow \infty$, and, since (3.51) is satisfied, we conclude the proof of Lemma 10 \square .

In the sequel, we are going to prove the main result of this section, i.e., we obtain the complete asymptotic expansion of the solution of (1.11) when $\rho \equiv 1$.

Proof of Theorem 1 with $\rho \equiv 1$.

Firstly, for $(x, t) \in \mathbf{R}^N \times \mathbf{R}^+$, we denote by

$$H(x, t) = \sum_{|\alpha| \leq k} c_\alpha(x) [G_\alpha^*(x, t) + \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha,n)} \sum_{|\beta|=4n+2m} c_{\beta,n} G_{\alpha+\beta}^*(x, t)],$$

the asymptotic expansion presented in Theorem 1, where $G_\alpha^*(x, t)$ was defined in (1.7).

From Lemma 2 and Lemma 3 it follows that

$$\begin{aligned}
\|u(\cdot, t) - H(\cdot, t)\| & \leq \|u(\cdot, t) - I(\cdot, t)\| + \|I(\cdot, t) - H(\cdot, t)\| \\
& \leq ce^{-at} + \|I(\cdot, t) - H(\cdot, t)\|,
\end{aligned}$$

where $I(x, t)$ is defined in (3.36). Then, to conclude the proof it is enough to prove that

$$\|I(\cdot, t) - H(\cdot, t)\| \leq c_k t^{-\frac{2k+2+N}{4}}, \text{ as } t \rightarrow \infty.$$

In fact, from Lemma 7 we have

$$\begin{aligned} \|I(\cdot, t) - H(\cdot, t)\| &\leq \|I(\cdot, t) - J(\cdot, t)\| + \|J(\cdot, t) - H(\cdot, t)\| \\ &\leq c_k t^{-\frac{2k+2+N}{4}} + \|J(\cdot, t) - H(\cdot, t)\|, \text{ as } t \rightarrow \infty, \end{aligned} \quad (3.53)$$

and recalling that

$$J(x, t) = \sum_{|\alpha| \leq k} (2\pi)^N d_\alpha J_\alpha(x, t),$$

where both, $J(x, t)$ and $J_\alpha(x, t)$, were defined in (3.37) and (3.40), respectively, we obtain from Lemma 9,

$$\begin{aligned} \|J(\cdot, t) - H(\cdot, t)\| &= \left\| \sum_{|\alpha| \leq k} (2\pi)^N d_\alpha \{J_\alpha(\cdot, t) - \sum_{|\gamma| \leq k-|\alpha|} \tilde{d}_\gamma(\cdot) I_{\gamma+\alpha}(\cdot, t)\} \right\| \\ &\leq c'_k t^{-\frac{2k+2+N}{4}}, \text{ as } t \rightarrow \infty. \end{aligned} \quad (3.54)$$

Thus, if we define the periodic functions as

$$c_\alpha(x) = (2\pi)^N \sum_{\gamma \leq \alpha} \tilde{d}_\gamma(x) d_{\alpha-\gamma}, \quad (3.55)$$

thanks to (3.53) and (3.54), it follows that

$$\|I(\cdot, t) - H(\cdot, t)\| \leq c' t^{-\frac{2k+2+N}{4}} + \left\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) I_\alpha(\cdot, t) - H(\cdot, t) \right\|, \text{ as } t \rightarrow \infty.$$

Now, we are going to prove that the following holds:

$$\left\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) I_\alpha(\cdot, t) - H(\cdot, t) \right\| \leq c_k t^{-\frac{2k+2+N}{4}}, \text{ as } t \rightarrow \infty.$$

Using Lemma 10, we obtain, as $t \rightarrow \infty$,

$$\left\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) [I_\alpha(\cdot, t) - I_\alpha^*(\cdot, t) - \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha, n)} \sum_{|\beta|=4n+2m} c_{\beta, n} I_{\alpha+\beta}^*(\cdot, t)] \right\| \leq C_k t^{-\frac{2k+2+N}{4}},$$

where C_k depends on the L^∞ -norm of the functions $c_\alpha(\cdot)$ and the function $I_\alpha^*(x, t)$ defined in (3.44). Finally, Remark 8 gives us that

$$\left\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) (I_\alpha^*(\cdot, t) + \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha, n)} \sum_{|\beta|=4n+2m} c_{\beta, n} I_{\alpha+\beta}^*(\cdot, t)) - H(\cdot, t) \right\| \leq c e^{-c(\delta)t},$$

as $t \rightarrow \infty$, and, returning to (3.53) we obtain

$$\left\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) I_\alpha(\cdot, t) - H(\cdot, t) \right\| \leq C_k t^{-\frac{2k+2+N}{4}} + ce^{-c(\delta)t}, \quad \text{as } t \rightarrow \infty,$$

what concludes the proof \square .

4. Proof of the general case

Theorem 1 is proved following the same steps of Section 2 and 3 for the case $\rho \equiv 1$. However, the Bloch wave decomposition used for the equation (1.11) in Section 2 can not be applied for the problem (1.1), due to variable density ρ . Consequently, we need to introduce a different spectral problem.

Given $\xi \in Y'$ we consider the spectral problem of finding numbers $\lambda = \lambda(\xi) \in \mathbf{R}$ and functions $\psi = \psi(x; \xi)$ (no identically zero) such that

$$\begin{cases} A\psi(\cdot; \xi) = \lambda(\xi)\psi(\cdot; \xi)\rho(\cdot) & \text{in } \mathbf{R}^N, \\ \psi(\cdot; \xi) \text{ is } (\xi, Y)\text{-periodic, i.e.,} \\ \psi(y + 2\pi m; \xi) = e^{2\pi i m \cdot \xi} \psi(y) & \forall m \in \mathbf{Z}^N, y \in \mathbf{R}^N, \end{cases} \quad (4.56)$$

where A is the elliptic operator in divergence form defined in (2.13) and ρ satisfies (1.3). If we consider $\psi(x; \xi) = e^{ix \cdot \xi} \phi(x; \xi)$, the variational formulation obtained for (4.56) for any $\varphi \in H_{\#}^1(Y)$ is given by

$$\langle A(\xi)\phi, \varphi \rangle = \int_Y a_{k\ell}(x) \left(\frac{\partial \phi}{\partial x_k} + i\xi_k \phi \right) \overline{\left(\frac{\partial \varphi}{\partial x_\ell} + i\xi_\ell \varphi \right)} dx = \lambda(\xi) \int_Y \phi \bar{\varphi} \rho(x) dx.$$

Since the operator associated with (4.56) is uniformly elliptic and self-adjoint, defined in a bounded domain, it is known (see ³ and ⁴) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \leq \lambda_1(\xi) \leq \dots \leq \lambda_m(\xi) \leq \dots \rightarrow \infty, \\ \lambda_m(\xi) \text{ is a Lipschitz function of } \xi \in Y', \forall m \geq 1. \end{cases} \quad (4.57)$$

Besides, the corresponding eigenfunctions denoted by $\psi_m(\cdot; \xi) = e^{i\xi \cdot x} \phi_m(\cdot; \xi)$, where the functions $\{\phi_m(x; \xi)\}$ form orthonormal basis in the space of periodic functions in $L_{loc}^2(\mathbf{R}^N; \rho(x)dx)$, i.e.,

$$\int_Y \phi_m \bar{\phi}_n \rho(x) dx = \delta_{mn} \quad (\text{Kronecker's delta}).$$

The eigenfunctions $\psi_m(\cdot, \xi)$ and $\phi_m(\cdot, \xi)$ are (ξ, Y) -periodic and Y -periodic, respectively. Moreover, as a consequence of the min-max principle (see ⁴) we have

$$\lambda_2(\xi) \geq \frac{\lambda_2^{(N)}}{\rho_1} > 0, \quad \forall \xi \in Y', \quad (4.58)$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary condition on ∂Y for $\rho \equiv 1$ and ρ_1 is defined in (1.3).

Now, with the orthonormal basis of Bloch waves $\{e^{ix \cdot \xi} \phi_m(x; \xi) : m \geq 1, \xi \in Y'\}$, we have a similar Bloch wave decomposition as in Proposition 1:

Proposition 4 *Let $g \in L^2(\mathbb{R}^N)$. The m^{th} Bloch coefficient of g is defined as follows:*

$$\widehat{g}_m(\xi) = \int_{\mathbb{R}^N} g(x) e^{-ix \cdot \xi} \bar{\phi}_m(x; \xi) \rho(x) dx \quad \forall m \geq 1, \xi \in Y'.$$

Then the following inverse formula holds:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \widehat{g}_m(\xi) e^{ix \cdot \xi} \phi_m(x; \xi) d\xi.$$

Further, we have Parseval's identity:

$$\|g\|_{L^2(\rho)}^2 = \int_{\mathbb{R}^N} |g(x)|^2 \rho(x) dx = \int_{Y'} \sum_{m=1}^{\infty} |\widehat{g}_m(\xi)|^2 d\xi.$$

Finally, for all g in the domain of A , we have

$$Ag(x) = \rho(x) \int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi) \widehat{g}_m(\xi) e^{ix \cdot \xi} \phi_m(x; \xi) d\xi.$$

Using Proposition 4, the equation (1.1) can be written as follows:

$$\int_{Y'} \sum_{m=1}^{\infty} (\partial_t^2 \widehat{u}_m(\xi, t) + \lambda_m(\xi) \widehat{u}_m(\xi, t) + a_0 \partial_t \widehat{u}_m(\xi, t)) e^{ix \cdot \xi} \phi_m(x; \xi) \rho(x) d\xi = 0.$$

Since $\{e^{ix \cdot \xi} \phi_m(x; \xi) : m \geq 1, \xi \in Y'\}$ form an orthonormal basis, this is equivalent to the family of the differential equations

$$\partial_t^2 \widehat{u}_m(\xi, t) + \lambda_m(\xi) \widehat{u}_m(\xi, t) + a_0 \partial_t \widehat{u}_m(\xi, t) = 0, \quad \forall m \geq 1, \xi \in Y'.$$

Once these differential equations are solved, (1.1) is solved as in (2.16) and Lemma 1 holds. The developments of Section 3 apply with minor changes and Theorem 1 holds.

In order to understand the type of changes that the variable density $\rho(\cdot)$ causes in the fundamental solution, we are going to study the Taylor expansion of the first Bloch eigenvalue and eigenvector. For a more complete analysis the reader is referred to ². We observe that

$$\lambda_1(0) = 0 \quad \text{and} \quad \phi(x; 0) = (2\pi)^{-\frac{N}{2}} \rho^{-\frac{1}{2}},$$

where $\bar{\rho}$ is defined in (1.8). We consider the equation

$$A(\xi)\phi_1(\cdot; \xi) = \lambda_1(\xi)\rho(\cdot)\phi_1(\cdot; \xi), \quad (4.59)$$

where

$$A(\xi) = - \left(\frac{\partial}{\partial x_k} + i\xi_k \right) \left[a_{k\ell}(x) \left(\frac{\partial}{\partial x_\ell} + i\xi_\ell \right) \right].$$

If we differentiate the equation (4.59) with respect to ξ_k with $k = 1, \dots, N$ and if we take scalar product with $\phi_1(x; \xi)$ in $\xi = 0$, we get

$$\partial_k \lambda_1(0) = 0.$$

Furthermore, we observe that

$$A \partial_k \phi_1(\cdot; 0) = i(2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}} \frac{\partial a_{k\ell}}{\partial x_\ell},$$

then

$$\partial_k \phi_1(x; 0) = i(2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}} \chi^k(x),$$

where χ^k is the classical test function in homogenization theory, solution of the cell problem

$$\begin{cases} A\chi^k = \frac{\partial a_{k\ell}}{\partial y_\ell} & \text{in } Y, \\ \chi^k \in H_{\#}^1(Y), \quad \frac{1}{|Y|} \int_Y \chi^k dy = 0. \end{cases} \quad (4.60)$$

This is the same test function as in the case $\rho \equiv 1$. If we differentiate again the eigenvalue equation, we have that

$$\partial_{k\ell}^2 \lambda_1(0) = \frac{1}{\bar{\rho}} \frac{1}{(2\pi)^N} \int_Y (2a_{k\ell} + a_{km} \frac{\partial \chi^\ell}{\partial x_m} + a_{m\ell} \frac{\partial \chi^k}{\partial x_m}) dx = \frac{2q_{k\ell}}{\bar{\rho}},$$

with $q_{k\ell}$ the homogenized coefficients as in previous section (see in ⁴).

Since $\alpha_1^1(\xi)$ is defined in (2.17) and thanks to the analysis above for the eigenvalue λ_1 , we obtain

$$\begin{aligned} \alpha_1^1(0) &= \partial_k \alpha_1^1(0) = 0 & k &= 1, \dots, N, \\ \partial_{k\ell}^2 \alpha_1^1(0) &= \frac{2q_{k\ell}}{\bar{\rho} a_0} & k, \ell &= 1, \dots, N. \end{aligned}$$

Then, for all $\xi \in B_\delta$ we have

$$e^{-\alpha_1^1(\xi)t} \sim e^{-\frac{q_{k\ell}}{\bar{\rho} a_0} \xi_k \xi_\ell t}.$$

5. Analysis of the periodic functions and constants entering in the asymptotic expansion

To finish this work we describe the periodic functions $c_\alpha(\cdot)$ and constants $c_{\beta,n}$, where $\alpha, \beta \in (\mathbf{N} \cup \{0\})^N$ and $n \geq 1$, that appear in the statement of Theorem 1.

Computation of $c_\alpha(\cdot)$. According to (3.38), (3.41) and (3.55),

$$c_\alpha(x) = \sum_{\beta \leq \alpha} \frac{(2\pi)^N}{(\alpha - \beta)! \beta!} \partial^{\alpha - \beta} \phi_1(x; 0) \partial^\beta \beta_1^1(0), \quad (5.61)$$

where, recalling the definition of $\beta_1^1(\xi)$ given in (2.19), we have

$$\begin{aligned} \partial^\beta \beta_1^1(0) &= \partial^\beta \widehat{\varphi}_1^0(0) + \sum_{0 \neq \gamma \leq \beta} \frac{a_0}{2} \partial^{\beta - \gamma} \widehat{\varphi}_1^0(0) \partial^\gamma \left((a_0^2 - 4\lambda_1(\xi))^{-\frac{1}{2}} \right) (0) \\ &\quad + \sum_{\gamma \leq \beta} \partial^{\beta - \gamma} \widehat{\varphi}_1^1(0) \partial^\gamma \left((a_0^2 - 4\lambda_1(\xi))^{-\frac{1}{2}} \right) (0) \end{aligned}$$

and for $j = 0, 1$ (the first Bloch coefficients of the initial data)

$$\partial^\gamma \widehat{\varphi}_1^j(0) = \int_{\mathbf{R}^N} \varphi^j(x) \sum_{\alpha \leq \gamma} [(-i)^{|\gamma - \alpha|} x^{\gamma - \alpha} \partial^\alpha \phi_1(x; 0)] dx.$$

We observe that the higher order derivatives of λ_1 and ϕ_1 in $\xi = 0$ may be computed as in the previous section.

First, note that

$$c_0(x) = (2\pi)^N \phi_1(x; 0) \beta_1^1(0) = (2\pi)^N \phi_1(x, 0) \left(\widehat{\varphi}_1^0(0) + \frac{\widehat{\varphi}_1^1(0)}{a_0} \right),$$

and since $\phi_1(x, 0) = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}}$, it follows that c_0 is constant. Futhermore, according to Proposition 4, we have for $i = 0, 1$

$$\widehat{\varphi}_1^i(0) = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}} \int_{\mathbf{R}^N} \varphi^i(x) \rho(x) dx = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}} m_\rho(\varphi^i).$$

Thus,

$$c_0 := c_0(x) = \frac{1}{\bar{\rho}} m_\rho(\varphi^0 + \frac{1}{a_0} \varphi^1),$$

and, since (1.4) is satisfied, c_0 is defined as in (1.10).

For $|\alpha| = 1$, we consider $\alpha = e_k$, any of the canonical vectors. The corresponding periodic function is

$$c_k(x) = \frac{-i}{\bar{\rho}} \left(\chi^k(x) m_\rho(\varphi^0 + \frac{\varphi^1}{a_0}) + m_\rho((\chi^k + x_k)(\varphi^0 + \frac{\varphi^1}{a_0})) \right),$$

with χ^k the periodic test function, solution of (4.60). Observe that two different terms appear in c_k . First the total mass of the solution multiplied by the periodic

function $\chi^k(\cdot)$. Second, a constant term in which the periodic function $\chi^k(\cdot)$ enters as well, but this time as a weight when computing the corresponding moment of the initial data. In both cases we see how the periodicity of the medium affects the value of c_k that varies substantially with respect to the case of constant coefficients. This fact was already pointed out in ⁷ when studying the heat equation with periodic coefficients.

The remaining values of the functions $c_\alpha(\cdot)$ with multiindexes α with $|\alpha| \geq 2$ may be computed by taking successive derivatives in (4.59).

Computation of $c_{\beta,n}$. We recall that the constants $c_{\beta,n}$ were defined in (3.52) and satisfy

$$\sum_{m=1}^{\infty} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n} = (-1)^n \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial^\beta \alpha_1^1(0) \right)^n,$$

where $\alpha_1^1(\xi)$ is given in (2.17). Moreover, we have for $|\beta|$ even and $|\beta| \geq 4$ that

$$\partial^\beta \alpha_1^1(0) = \sum_{m=1}^{|\beta|} \sum_{\substack{|\beta'|=m \\ \beta', \beta_1, \dots, \beta_N \in \mathbf{N}^N \\ \beta' + \beta_1 + \dots + \beta_N = \beta}} \frac{2^{m-1} f(m)}{a_0^{2m-1}} \partial^{\beta_1} [(\partial_1 \lambda_1)^{\beta_1'}](0) \dots \partial^{\beta_m} [(\partial_N \lambda_1)^{\beta_N'}](0),$$

where, for $m \in \mathbf{N}$,

$$f(m) = \begin{cases} 1 & m = 1, \\ \frac{(2m-3)!}{2^{m-2}(m-2)!} & m \geq 2. \end{cases}$$

Observe that the constants $\partial^\beta \alpha_1^1(0)$ depend on the derivatives of λ_1 at $\xi = 0$, computed in Section 4. Thus, we may write the constants $c_{\beta,n}$ as

$$c_{\beta,n} = \sum_{\substack{s_1, \dots, s_n \in \mathbf{N} \\ s_1 + \dots + s_n = \frac{|\beta| - 4n}{2}}} \sum_{\substack{|\beta_i|=4+s_i \\ \beta_1, \dots, \beta_n \in \mathbf{N}^N \\ \beta_1 + \dots + \beta_n = \beta}} (-1)^n \frac{1}{\beta_1!} \dots \frac{1}{\beta_n!} \partial^{\beta_1} \alpha_1^1(0) \dots \partial^{\beta_n} \alpha_1^1(0).$$

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