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Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential

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Abstract. In this paper, we analyze a unique continuation problem for the linearized Benjamin-Bona-Mahony equation with space-dependent potential in a bounded interval with Dirichlet boundary conditions. The underlying Cauchy problem is a characteristic one. We prove two unique continuation results by means of spectral analysis and the (generalized) eigenvector expansion of the solution, instead of the usual Holmgren-type method or Carleman-type estimates. It is found that the unique continuation property depends very strongly on the nature of the potential and, in particular, on its zero set, and not only on its boundedness or integrability properties.

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1. Introduction and main results

Let us consider the following linearized Benjamin-Bona-Mahony equation

$$\begin{cases} u_t - u_{txx} = [\alpha(x)u]_x + \beta(x)u & \text{in } (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0 & t \in (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $T > 0$ is a given time, $\alpha(\cdot) \in L^\infty(0,1)$ and $\beta(\cdot) \in L^2(0,1)$ are given potentials.

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As we shall see below (*see* Lemma 2.1 in Section 2), system (1.1) is well posed in $L^2(0, 1)$ and $H_0^1(0, 1)$. In particular, for any $u_0 \in L^2(0, 1)$, (1.1) admits a unique solution $u \in C([0, T]; L^2(0, 1))$.

Let $F \subset (0, 1)$ be an open, non-empty subset. We are interested in the property of unique continuation for (1.1), that is, whether

$$u(t, x) = 0 \quad \text{in } (0, T) \times F \tag{1.2}$$

implies that u vanishes identically.

This problem is motivated by questions related to the controllability and the stabilization of Benjamin-Bona-Mahony like equations (*see* Section 7). At this respect it is also worth mentioning that if the unique continuation property above were known to hold also for potentials depending both in space and time, one could deduce decay properties for solutions of the true nonlinear Benjamin-Bona-Mahony equation, with damping terms localized in space in F (we refer to [15] for a similar analysis in the context of the KdV equation). But this is an open problem that we do not address here and the methods developed in this paper do not seem to suffice to handle it.

There exist a very extensive literature on unique continuation problems. Moreover, the equation under consideration is $1 - D$ in space and therefore, one could expect the problem above to fit in some of the existing results. But this does not seem to be the case. At this respect, it is important to note that both $x = Const.$ and $t = Const.$ are characteristic lines for the first equation in (1.1). Hence the main difficulty in our problem consists in the fact that the Cauchy problem involved in the unique continuation one is characteristic. Therefore we can not apply Holmgren uniqueness theorem even in the simplest case in which $\alpha(\cdot)$ and $\beta(\cdot)$ are analytic functions. Moreover, it is well-known that solutions of characteristic Cauchy problems may be non-unique (*see* for example [11]). Thus, one can not exclude *a priori* the above uniqueness property to fail. On the other hand, when coefficients fail to be analytic, the main tool to prove unique continuation properties is the so-called Carleman-type estimates ([11], [12], [20], for instance). In Carleman-type estimates the lower order terms (with bounded coefficients or even with unbounded coefficients under suitable integrability conditions) of the equation can be controlled in some weighted norms by the principal part of the operator. However, as we shall see, for our problem, the unique continuation property depends very strongly on the nature of the potentials $\alpha(\cdot)$ and $\beta(\cdot)$ and, in particular, on its zero set, and not only on its boundedness or integrability properties. Therefore, in the present situation, one can not expect to apply Carleman-type estimates. Note also that most of the literature on Carleman inequalities is related to non-characteristic Cauchy problems, which is not the case here. There exist also a few works concerning the uniqueness of characteristic Cauchy problems ([1] and the references therein). For instance, the main theorem (Theorem 2) in [1] is a uniqueness result for solutions of differential equations (with smooth coefficients) satisfying certain decay conditions at infinity and it is based on Carleman-type estimates. Therefore, as we remarked above, it seems that it can not be applied to our problem. Finally, we would like to mention a recent work [19]. By means of Carleman estimate, [19] proves a nice unique continuation result for (1.1) across the characteristic under conditions (1.2) and $u(0,x) \equiv 0$ in $(0,1)$ (but without the null boundary conditions in (1.1), also the coefficients of (1.1) may be both time- and space-dependent). The condition $u(0,x) \equiv 0$ in $(0,1)$ plays a key role in the proof of the main result of [19]. We remark that the unique continuation problem considered in [19] is different from ours.

Now, we give a simple negative result showing that the unique continuation property necessarily depends on the zero sets of the potentials $\alpha(\cdot)$ and $\beta(\cdot)$.

Example 1.1. Let $\alpha(\cdot) = \beta(\cdot) \equiv 0$. Then any time-independent function $u = u(x) \in C_0^\infty\big((0,1) \setminus \overline{F}\big)$ satisfies (1.1) and (1.2). Therefore, the unique continuation property does not hold for this simple case unless $(0,1) \setminus \overline{F} = \emptyset$.

Remark 1.1. Example 1.1 shows that the unique continuation results on Benjamin-Bona-Mahony equations in [5] (Theorems 3.1–3.4) are not correct without further assumptions.

More generally, we have the following necessary condition for unique continuation of (1.1)–(1.2).

Theorem 1.1. *Let $F \subset (0,1)$ be an open, non-empty subset, $\alpha(\cdot) \in L^\infty(0,1)$ and $\beta(\cdot) \in L^2(0,1)$. Suppose that the unique continuation property above holds. Then $\alpha(\cdot)$ and $\beta(\cdot)$ can not vanish simultaneously in any open, non-empty subset of $(0,1) \setminus \overline{F}$.*

The proof is immediate: If α and $\beta(\cdot)$ vanish simultaneously in an open, non-empty subset U of $(0,1) \setminus \overline{F}$, then any time-independent function $u = u(x) \in C_0^\infty(U)$ solves (1.1) and satisfies (1.2). Thus u is not necessarily identically equal to zero.

On the other hand, it was essentially proved (although not stated explicitly) in [14] for the case $\alpha(x) \equiv -1$ and $\beta(x) \equiv 0$ that the weak solution $u \in C([0,T]; H_0^1(0,1))$ for (1.1) vanishes identically if

$$u_x(t,0) = 0, \qquad \forall\, t \in (0,T). \tag{1.3}$$

Of course, this provides a positive answer to our problem in the particular case $\alpha(x) \equiv -1$ and $\beta(x) \equiv 0$.

The main idea of the proof in [14] was to use the explicit series expansion of the solution in terms of the eigenvectors of the generator of the underlying semigroup and its time analyticity. In a first step, the analyticity in time allows to show that, when (1.3) holds, then $u_x(t,0)$ vanishes for all time. The series development of the solution on the basis of the eigenvectors of the generator of the underlying semigroup allows one to reduce the problem to the unique continuation of the eigenvectors, which can be solved by ODE methods.

The approach in [14] does not apply directly in our case. Even though we assume that the potential $\beta(x) \equiv 0$, the generator of the semigroup associated with (1.1) is not skew-adjoint when $\alpha(x)$ is not constant. Thus we can not apply Fourier series method directly to reduce the problem to the analysis of the eigenvectors and we shall require important further developments in order to justify the use of eigenvector expansions for the solutions of (1.1).

In view of Theorem 1.1, we need to impose conditions on potentials α and β, and more precisely in their zero sets in order for the unique continuation property to be true. And therefore, it is natural to analyze under what conditions on $\alpha(\cdot)$ and β, the unique continuation for (1.1) and (1.2) holds.

For this purpose, we need to introduce some notations. For any interval (c, d), we denote by

$$W(c, d)$$

the set of all weight functions on (c, d), i.e. the set of all measurable, bounded functions which are positive almost everywhere in (c, d). For any $\alpha(\cdot) \in W(c, d)$, we denote by $L^2(c, d; \alpha)$ the Hilbert space of the completion of $C_0^\infty(c, d)$ with respect to the norm

$$||f||_{2,\alpha} \stackrel{\triangle}{=} \int_c^d \alpha(x)|f(x)|^2 dx, \qquad \forall\, f \in C_0^\infty(c, d). \tag{1.4}$$

It is easy to see that $L^2(c, d) \subset L^2(c, d; \alpha)$ topologically and algebraically.

We have the following unique continuation results.

Theorem 1.2. *Let* $\beta(\cdot) = 0$ *and either* $\alpha(\cdot) \in W(0, 1)$ *or* $-\alpha(\cdot) \in W(0, 1)$. *Let* $0 < a < b < 1$, $T > 0$ *and* $u_0 \in L^2(0, 1)$. *Suppose that the weak solution* $u \in C([0, T]; L^2(0, 1))$ *of (1.1) satisfies (1.2) with* $F = (0, a) \cup (b, 1)$. *Then*

$$u \equiv 0 \qquad in\ \mathbb{R} \times (0, 1). \tag{1.5}$$

Note that in Theorem 1.2 it is assumed that u vanishes in a neighborhood of both extremes $x = 0,\ 1$ of the interval $(0, 1)$ and that $\beta(\cdot) \equiv 0$. If we impose more regularity conditions on α, we have the following better result, which allows, in particular, a non-zero potential β.

Theorem 1.3. *Let* $\beta(\cdot) \in L^\infty(0, 1)$ *and* $\alpha(\cdot) \in W^{2,\infty}(0, 1)$ *with* $\min_{x\in[0,1]} |\alpha(x)| > 0$. *Let* $0 \le a < b \le 1$, $T > 0$ *and* $u_0 \in L^2(0, 1)$. *Suppose that the weak solution* $u \in C([0, T]; L^2(0, 1))$ *of (1.1) satisfies (1.2) with* $F = (a, b)$. *Then*

$$u \equiv 0 \qquad in\ \mathbb{R} \times (0, 1). \tag{1.6}$$

The proofs of Theorems 1.2 and 1.3 will be given in Sections 3 and 6 respectively.

Remark 1.2. Theorems 1.1–1.3 show that the unique continuation property for system (1.1) under condition (1.2) depends very strongly in the nature of the coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ of the lower order terms of the first equation in (1.1). As far as we know, such a phenomenon was not observed and analyzed in the existing literature.

Remark 1.3. For the proof of Theorem 1.2, it would be useful to have an eigenvector expansion of the solution to (1.1). However, due to the $x-$dependence of $\alpha(x)$, the eigenvectors may not be computed explicitly. In fact, we do not know at present whether the (generalized) eigenvectors of the generator of the underlying semigroup form a Riesz basis of $L^2(0, 1)$ without further regularity conditions on $\alpha(\cdot)$. In order to overcome this difficulty, we need to introduce a special Hilbert space $\mathcal{H}$ where the equation evolves by means of a semigroup generated by a compact, skew-adjoint operator and therefore, where Fourier series may be used. We also show that solutions of (1.1) satisfying (1.2) with $F = (0, a) \cup (b, 1)$ belong to this space $\mathcal{H}$. This allows us to use well known spectral theorems to obtain the eigenvector expansion of solutions of (1.1) satisfying (1.2) and to reduce the problem to the unique continuation of the eigenvectors which may be easily solved by ODE techniques. This is sufficient to complete the proof of Theorem 1.2.

Remark 1.4. In order to prove Theorem 1.3, we will show that the generalized eigenvectors of the generator of the underlying semigroup form a Riesz basis of $H_0^1(0, 1)$. At first, by means of a series of transformations, we can show that the "high frequency" eigenvectors are quadratically close to a subsequence of some known Riesz basis in $H_0^1(0, 1)$. In order to conclude the proof of Theorem 1.3, the completeness of the generalized eigenvectors is also required. But the existing results, for instance those in [7] (that have been successfully applied in several other problems, see, for example, [3], [4], [6], [2], [16], [18] and so on), do not seem to apply in our case. Section 5 is devoted to overcome this difficulty by means of a new abstract result which is strongly inspired in the works by Guo ([8]) and Guo & Yu ([9]).

Remark 1.5. In Theorems 1.2 and 1.3 we impose some technical conditions on $\alpha(\cdot)$, $\beta(\cdot)$ and F, especially we require $\alpha(\cdot)$ to be of constant sign. We remark that, in the proof of Theorem 1.2, we use in an essential way the facts that u vanishes in a neighborhood of both extremes $x = 0$ and $x = 1$ (*see* Lemma 3.1) and $\beta(\cdot) = 0$ (*see* Lemma 3.2), and we need the constant sign condition on $\alpha(\cdot)$ to check that $||\cdot||_{2,\alpha}$ defined by (1.4) (or $||\cdot||_{2,-\alpha}$ in the case that $\alpha(\cdot)$ is negative) is a norm in $C_0^\infty(0, 1)$. On the other hand, in the proof of Theorem 1.3, we need the constant sign condition on $\alpha(\cdot)$, too (*see* Lemma 2.3), and we use the fact that $\alpha(\cdot) \in W^{2,\infty}(0, 1)$ when we analyze the asymptotic behavior of the eigenvalues and eigenvectors of the generator of the underlying semigroup (*see* (4.21)). It is reasonable to expect the (necessary) conditions on $\alpha(\cdot)$, $\beta(\cdot)$ and F in Theorem

1.1 to be also sufficient for the unique continuation property to hold but this is by now an open problem.

2. Some preliminaries

In order to prove Theorems 1.2–1.3, we need some preliminaries. First of all, we denote by A the operator in $L^2(0,1)$:

$$\begin{cases} D(A) \stackrel{\triangle}{=} H_0^1(0,1) \cap H^2(0,1), \\ Au = u_{xx}, \quad \forall\, u \in D(A). \end{cases} \tag{2.1}$$

The well-posedness of system (1.1) is easy to get.

Lemma 2.1. *Let $\alpha(\cdot) \in L^\infty(0,1)$ and $\beta(\cdot) \in L^2(0,1)$ (resp. $\alpha(\cdot) \in L^2(0,1)$ and $\beta(\cdot) \in L^1(0,1)$). For any $u_0 \in L^2(0,1)$ (resp. $u_0 \in H_0^1(0,1)$), there exists a unique solution of (1.1) in the class $u \in C([0,T], L^2(0,1))$ (resp. $u \in C([0,T], H_0^1(0,1))$).*

Proof. We consider only the case $u_0 \in L^2(0,1)$ (the case $u_0 \in H_0^1(0,1)$ can be treated similarly). Existence and uniqueness may be proved easily by standard methods on the basis of the following energy estimate. Denote by

$$E(t) \stackrel{\triangle}{=} \frac{1}{2}[|u|^2_{L^2(0,1)} + |u|^2_{H^{-1}(0,1)}] \tag{2.2}$$

the energy of solutions of (1.1). It suffices to prove that for any given $T > 0$

$$E(t) \le CE(0), \qquad \forall\, t \in [0,T] \tag{2.3}$$

for some constant $C > 0$.

For this purpose, we multiply both sides of the first equation in (1.1) by $(-A)^{-1}u$ and integrate it on $(0,1)$. Using integration by parts and Sobolev embedding theorem, we get easily

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^1 [(\alpha u)_x + \beta u][(-A)^{-1}u]dx \\ &= \int_0^1 \{(-A)^{-1/2}[(\alpha u)_x + \beta u]\}[(-A)^{-1/2}u]dx \\ &\le |(-A)^{-1/2}[(\alpha u)_x + \beta u]|_{L^2(0,1)}|(-A)^{-1/2}u|_{L^2(0,1)} \\ &\le C|u|_{L^2(0,1)}|(-A)^{-1/2}u|_{L^2(0,1)} \le CE(t). \end{aligned} \tag{2.4}$$

Now, (2.3) follows from (2.4) and Gronwall's inequality immediately. □

In order to prove Theorem 1.2, let us re-write (1.1) equivalently as

$$\begin{cases} u_t = \mathcal{A}u, & t > 0, \\ u(0) = u_0, \end{cases} \tag{2.5}$$

where $\mathcal{A} : L^2(0,1;\alpha) \to L^2(0,1;\alpha)$ is a bounded linear operator given by (recall that $\beta(\cdot) = 0$ and for simplicity we consider only $\alpha(\cdot) \in W(0,1)$)

$$\mathcal{A}f = (I - A)^{-1}\partial_x(\alpha f), \qquad \forall\, f \in L^2(0,1;\alpha). \tag{2.6}$$

We need the following explicit expression for $\mathcal{A}$.

Lemma 2.2. *Let $\alpha(\cdot) \in W(0,1)$. Then for any $f \in L^2(0,1;\alpha)$, it holds*

$$\begin{aligned} \mathcal{A}f(x) \;=& \frac{e^x - e^{-x}}{2(e - e^{-1})}\int_0^1 \Big(e^{1-s} + e^{s-1}\Big)\alpha(s)f(s)ds \\ &- \frac{1}{2}\int_0^x \Big(e^{x-s} + e^{s-x}\Big)\alpha(s)f(s)ds, \qquad \forall\, x \in (0,1). \end{aligned} \tag{2.7}$$

Proof. It is easy to check that for any $f \in C_0^\infty(0,1)$ it holds

$$\begin{aligned} (I - A)^{-1}f(x) \;=& \frac{e^x - e^{-x}}{2(e - e^{-1})}\int_0^1 \Big(e^{1-s} - e^{s-1}\Big)f(s)ds \\ &- \frac{1}{2}\int_0^x \Big(e^{x-s} - e^{s-x}\Big)f(s)ds. \end{aligned} \tag{2.8}$$

Therefore, for $f \in C_0^\infty(0,1)$, integrating by parts we have

$$\begin{aligned} \Big((I - A)^{-1}\partial_x f\Big)(x) \;=& \frac{e^x - e^{-x}}{2(e - e^{-1})}\int_0^1 \Big(e^{1-s} + e^{s-1}\Big)f(s)ds \\ &- \frac{1}{2}\int_0^x \Big(e^{x-s} + e^{s-x}\Big)f(s)ds. \end{aligned} \tag{2.9}$$

Hence by the density of $C_0^\infty(0,1)$ in $L^2(0,1;\alpha)$, we get the desired result immediately. □

As before, in order to prove Theorem 1.3, we re-write (1.1) equivalently as

$$\begin{cases} u_t = \widetilde{\mathcal{A}}u, & t > 0, \\ u(0) = u_0, \end{cases} \tag{2.10}$$

where $\widetilde{\mathcal{A}} : H_0^1(0,1) \to H_0^1(0,1)$ is a bounded linear operator given by

$$\widetilde{\mathcal{A}}v = (I - A)^{-1}\Big(\partial_x(\alpha v) + \beta v\Big), \qquad \forall\, v \in H_0^1(0,1). \tag{2.11}$$

Remark 2.1. Recall that we assume $u_0 \in L^2(0,1)$ in Theorem 1.3. However, from the proof of Theorem 1.3 (*see* Section 6), one sees that it suffices to consider the case $u_0 \in H_0^1(0,1)$. Therefore, we will consider the operator $\widetilde{\mathcal{A}}$ in $H_0^1(0,1)$ in what follows.

We need the following simple result.

Lemma 2.3. *Let $\alpha(\cdot) \in H^1(0,1)$ and $\beta(\cdot) \in L^2(0,1)$. Then the operator $\widetilde{\mathcal{A}}$ defined by (2.11) is compact in $H^1(0,1)$, its adjoint operator $\widetilde{\mathcal{A}}^*$ is given by*

$$\widetilde{\mathcal{A}}^* w = (I-A)^{-1}\Big[-\alpha w_x + \beta w\Big], \qquad \forall\, w \in H_0^1(0,1). \tag{2.12}$$

Furthermore, if $\min_{x\in[0,1]} |\alpha(x)| > 0$, then $0 \notin \sigma_p(\widetilde{\mathcal{A}}^*)$, the set of eigenvalues of $\widetilde{\mathcal{A}}^*$.

Proof. The compactness of $\widetilde{\mathcal{A}}$ is obvious. Let us derive (2.12). For any $v, w \in H_0^1(0,1)$, we have

$$\begin{aligned}
(\widetilde{\mathcal{A}}v, w)_{H_0^1(0,1)} &= (\widetilde{\mathcal{A}}v, w)_{L^2(0,1)} + \Big((\widetilde{\mathcal{A}}v)_x, w_x\Big)_{L^2(0,1)} \\
&= (\widetilde{\mathcal{A}}v, w)_{L^2(0,1)} - \Big((\widetilde{\mathcal{A}}v)_{xx}, w\Big)_{L^2(0,1)} \\
&= \Big((\alpha v)_x + \beta v, w\Big)_{L^2(0,1)} = (v, -\alpha w_x + \beta w)_{L^2(0,1)} \\
&= \Big(v, (I-A)^{-1}\Big[-\alpha w_x + \beta w\Big]\Big)_{H_0^1(0,1)}.
\end{aligned} \tag{2.13}$$

Thus, we get (2.12).

Now, let us show that $0 \notin \sigma_p(\widetilde{\mathcal{A}}^*)$. If $0 \in \sigma_p(\widetilde{\mathcal{A}}^*)$, then we can find a $\eta \in H_0^1(0,1)$, $\eta \neq 0$, such that $\widetilde{\mathcal{A}}^*\eta = 0$. Thus, by (2.12), we see that

$$\begin{cases} -\alpha\eta_x + \beta\eta = 0, & 0 < x < 1, \\ \eta(0) = \eta(1) = 0. \end{cases} \tag{2.14}$$

By $\min_{x\in[0,1]} |\alpha(x)| > 0$, from (2.14) we conclude that $\eta = 0$. Therefore we arrive at a contradiction. This completes the proof of Lemma 2.3. □

Also, we have the following result.

Lemma 2.4. *Let $\alpha(\cdot) \in L^2(0,1)$ and $\beta(\cdot) \in L^1(0,1)$. Assume that $u_0 \in H_0^1(0,1)$. Then the solution u of (2.10) (or equivalently (1.1)) has the regularity*

$$u \in C^\omega(\mathbb{R}; H_0^1(0,1)), \tag{2.15}$$

where $C^\omega(\mathbb{R}; H_0^1(0,1))$ stands for the class of analytic functions defined in $\mathbb{R}$ with values in $H_0^1(0,1)$.

Proof. It is easy to check that under the assumptions in Lemma 2.4, the operator $\widetilde{\mathcal{A}}$ defined by (2.11) is a bounded linear operator in $H_0^1(0,1)$. Thus, by means of the semigroup theory (*see* for example [10]), one gets Lemma 2.4 immediately. □

Note that our goal is to reduce the unique continuation of (1.1)–(1.2) in Theorems 1.2 and 1.3 to the same problem for the eigenvectors of $\mathcal{A}$ or $\widetilde{\mathcal{A}}$. Thus, we need the following simple result.

Lemma 2.5. *Let* $\xi(\cdot) \in L^\infty(0,1)$ *and* $\eta(\cdot) \in L^2(0,1)$. *Suppose* $u \in L^2(0,1)$ *satisfies*

$$\begin{cases} u - u_{xx} = [\xi(x)u]_x + \eta(x)u \text{ in } (0,1), \\ u(0) = u_x(0) = u(1) = 0. \end{cases} \tag{2.16}$$

Then $u \equiv 0$ *in* $(0,1)$.

Proof. We set

$$v = v(x) \overset{\triangle}{=} \int_0^x u(s)ds. \tag{2.17}$$

By (2.16), it is easy to see that v satisfies

$$\begin{cases} v_{xx} = -\xi(x)v_x + \int_0^x \eta(s)v_x(s)ds + v \quad \text{ in } (0,1), \\ v(0) = v_x(0) = 0. \end{cases} \tag{2.18}$$

By the first equation in (2.18), it is easy to see that

$$\begin{aligned} \frac{d}{dx}(v_x^2 + v^2) &= 2v_x(v_{xx} + v) = 2v_x[\xi(x)v_x + \int_0^x \eta(s)v_x(s)ds + 2v] \\ &\le C\Big[v_x^2 + v^2 + |v_x|\Big(\int_0^x \eta^2(s)ds\Big)^{1/2}\Big(\int_0^x v_x^2(s)ds\Big)^{1/2}\Big] \\ &\le C\Big(v_x^2 + v^2 + \int_0^x v_x^2(s)ds\Big). \end{aligned} \tag{2.19}$$

Integrating (2.19) from 0 to x, and noting the second equation in (2.18), we get

$$v_x^2(x)+v^2(x) \le C\int_0^x \Big[v_x^2(t)+v^2(t)+\int_0^t v_x^2(s)ds\Big]dt \le C\int_0^x [v_x^2(s)+v^2(s)]ds. \tag{2.20}$$

Therefore by means of Gronwall's inequality, we have $v \equiv 0$ in $(0,1)$, which implies the desired result immediately. □

3. Proof of Theorem 1.2

As we mentioned in Remark 1.3, in order to prove Theorem 1.2, we need to introduce a special Hilbert space $\mathcal{H}$ where the equation evolves by means of a semigroup generated by a compact, skew-adjoint operator. For this purpose, we need some properties of the solutions of (1.1) and (1.2) with $F = (0, a) \cup (b, 1)$. Fix any $\alpha(\cdot) \in W(0, 1)$, and set

$$\begin{array}{l} \beta_0(x) = e^{1-x}, \qquad \gamma_0(x) = e^{x-1}, \\ \beta_n(x) = \int_x^1 \beta_{n-1}(s)(e^{x-s} + e^{s-x})\alpha(s)ds, \\ \gamma_n(x) = \int_x^1 \gamma_{n-1}(s)(e^{x-s} + e^{s-x})\alpha(s)ds, \end{array} \tag{3.1}$$

where $n = 1, 2, \cdots$. We have the following result.

Lemma 3.1. *Suppose that assumptions in Theorem 1.2 hold. Then*

$$\int_0^1 \beta_n(s)\alpha(s)u(t, s)ds = \int_0^1 \gamma_n(s)\alpha(s)u(t, s)ds = 0, \qquad \forall\, t \in (0, T), \tag{3.2}$$

where $n = 0, 1, 2, \cdots$.

Proof. We divide the proof into two steps.

Step 1. First of all, let us prove that (3.2) holds for $n = 0$. Recalling that (1.1) is equivalent to (2.5), by Lemma 2.2, we get

$$\begin{array}{rl} u_t(t, x) = & \frac{e^x - e^{-x}}{2(e-e^{-1})} \int_0^1 \left(e^{1-s} + e^{s-1}\right)\alpha(s)u(t, s)ds \\ & - \frac{1}{2} \int_0^x \left(e^{x-s} + e^{s-x}\right)\alpha(s)u(t, s)ds, \quad \forall\, (t, x) \in (0, T) \times (0, 1). \end{array} \tag{3.3}$$

By (3.3), it is easy to see that $u_t \in C([0, T]; H_0^1(0, 1))$. However, by (1.2) (recall $F = (0, a) \cup (b, 1)$), we have

$$u_t(t, x) = 0, \qquad (t, x) \in (0, T) \times (0, a) \tag{3.4}$$

and

$$\int_0^x \left(e^{x-s} + e^{s-x}\right)\alpha(s)u(t, s)ds = 0, \qquad (t, x) \in (0, T) \times (0, a) \tag{3.5}$$

Combining (3.3)–(3.5), it is easy to see that

$$\int_0^1 \left(e^{1-s} + e^{s-1}\right)\alpha(s)u(t, s)ds = 0, \qquad \forall\, t \in [0, T]. \tag{3.6}$$

Therefore, we arrive at

$$u_t(t, x) = -\frac{1}{2} \int_0^x \left(e^{x-s} + e^{s-x}\right)\alpha(s)u(t, s)ds, \qquad \forall\, (t, x) \in (0, T) \times (0, 1). \tag{3.7}$$

By (3.7), we get

$$u_{tx}(t,x) = -\alpha(x)u(t,x) - \frac{1}{2}\int_0^x \Big(e^{x-s} - e^{s-x}\Big)\alpha(s)u(t,s)ds \tag{3.8}$$

for $(t,x) \in (0,T) \times (0,1)$ almost everywhere. However, using (1.2) (recall $F = (0,a) \cup (b,1)$) again, we see that

$$u_{tx}(t,x) = 0 \qquad \forall\, (t,x) \in (0,T) \times (b,1). \tag{3.9}$$

Combining (3.8)–(3.9), we have

$$\int_0^1 \Big(e^{1-s} - e^{s-1}\Big)\alpha(s)u(t,s)ds = 0, \qquad \forall\, t \in [0,T]. \tag{3.10}$$

By (3.6) and (3.10), we see that

$$\int_0^1 e^{1-s}\alpha(s)u(t,s)ds = \int_0^1 e^{s-1}\alpha(s)u(t,s)ds = 0, \qquad \forall\, t \in [0,T]. \tag{3.11}$$

Now, (3.2) for $n = 0$ follows from (3.11) and (3.1) immediately.

Step 2. Let us prove (3.2) in the general case. Note that, by Step 1, we have

$$\int_0^1 \beta_0(s)\alpha(s)u(t,s)ds = 0, \qquad \forall\, t \in (0,T). \tag{3.12}$$

Differentiating (3.12) with respect to t and noting (3.7), we get

$$\int_0^1 \beta_0(s)\alpha(s)\left(\int_0^s \Big(e^{x-s} + e^{s-x}\Big)\alpha(x)u(t,x)dx\right)ds = 0, \qquad \forall\, t \in (0,T). \tag{3.13}$$

Exchanging the order of integration in (3.13), we arrive at

$$\int_0^1 \left(\int_x^1 \beta_0(s)\Big(e^{x-s} + e^{s-x}\Big)\alpha(s)ds\right)\alpha(x)u(t,x)dx = 0, \qquad \forall\, t \in (0,T). \tag{3.14}$$

Now, by (3.14) and (3.1), we get

$$\int_0^1 \beta_1(x)\alpha(x)u(t,x)dx = 0, \qquad \forall\, t \in (0,T). \tag{3.15}$$

Similarly, one gets

$$\int_0^1 \gamma_1(x)\alpha(x)u(t,x)dx = 0, \qquad \forall\, t \in (0,T). \tag{3.16}$$

Therefore we have proved (3.2) for $n = 1$. Repeating the above procedure, one gets (3.2) for $n = 2, 3, \cdots$. □

The Hilbert space $\mathcal{H}$ we need is as follows

$$\mathcal{H} \stackrel{\triangle}{=} \{f \in L^2(0,1;\alpha);\ \int_0^1 \beta_n(s)\alpha(s)f(s)ds = \int_0^1 \gamma_n(s)\alpha(s)f(s)ds = 0 \text{ for } n = 0, 1, 2\cdots\}, \tag{3.17}$$

where β_n and γ_n are as in (3.1).

It is easy to see that $\mathcal{H}$ is a closed subspace in $L^2(0,1;\alpha)$. Therefore $\mathcal{H}$ is an Hilbert space with the topology inherited from $L^2(0,1;\alpha)$, i.e. the norm in $\mathcal{H}$ is that in $L^2(0,1;\alpha)$.

Remark 3.1. Note that Lemma 3.1 reduces the unique continuation problem to the case where the initial datum u_0 belongs to $\mathcal{H}$. Thus, if we were able to show that $\mathcal{H} = \{0\}$, then Theorem 1.2 would hold immediately.

When $\alpha(x) \equiv \alpha_0 \neq 0$, where α_0 is a constant, we can show directly that $\mathcal{H} = \{0\}$. In fact, for any $f(\cdot) \in \mathcal{H}$, taking $n = 0$ in (3.17) and noting (3.1), we get

$$\int_0^1 e^{1-s} f(s)ds = \int_0^1 e^{s-1} f(s)ds = 0. \tag{3.18}$$

On the other hand, by (3.1), it is easy to check that

$$\beta_1(x) = -\alpha_0\Big(xe^{1-x} + \frac{e^{x-1} - 3e^{1-x}}{2}\Big). \tag{3.19}$$

Therefore, by taking $n = 1$ in (3.17) and noting (3.18)–(3.19), we get

$$\int_0^1 se^{1-s} f(s)ds = 0. \tag{3.20}$$

Similarly

$$\int_0^1 se^{s-1} f(s)ds = 0. \tag{3.21}$$

Repeating this procedure, one gets

$$\int_0^1 s^n e^{1-s} f(s)ds = \int_0^1 s^n e^{s-1} f(s)ds = 0, \tag{3.22}$$

where $n = 2, 3, \cdots$. Now, by (3.18), (3.20)–(3.22), and noting that $\{1, s, s^2, \cdots, s^n, \cdots\}$ form a basis of $L^2(0,1)$, it is easy to conclude that $f(\cdot) \equiv 0$.

In the general case, it is not clear whether $\mathcal{H} = \{0\}$ for any $\alpha(\cdot) \in W(0,1)$. The problem may be reformulated as follows:

Problem (P): Does the set $\{\beta_0(\cdot), \cdots, \beta_n(\cdot), \cdots, \gamma_0(\cdot), \cdots, \gamma_n(\cdot), \cdots\}$ form a basis of $L^2[0,1]$?

Problem (P) is of independent interest.

In what follows we will avoid to prove $\mathcal{H} = \{0\}$ although we conjecture that this does hold. Instead we assume that $\dim \mathcal{H} > 0$. Analyzing the structure of operator $\mathcal{A}$ (defined by (2.6)) in $\mathcal{H}$ we shall see that if an element of $\mathcal{H}$ is such that (1.2) (with $F = (0, a) \cup (b, 1)$) holds, necessarily it is identically zero and this is sufficient to prove Theorem 1.2.

The following lemma will play a fundamental role in the sequel.

Lemma 3.2. *Let $\alpha(\cdot) \in W(0, 1)$ and $\mathcal{A}$ and $\mathcal{H}$ be defined by (2.6) and (3.17) respectively. Then $\mathcal{A}$ is a compact, skew-adjoint operator in $\mathcal{H}$.*

Proof. We divide the proof into three steps.

Step 1. Let us prove that

$$\mathcal{A}f \in \mathcal{H}, \qquad \forall\, f \in \mathcal{H}. \tag{3.23}$$

By Lemma 2.2 when $f \in \mathcal{H}$, we have

$$\mathcal{A}f(x) = -\frac{1}{2}\int_0^x \Big(e^{x-s} + e^{s-x}\Big)\alpha(s) f(s)ds, \qquad \forall\, x \in (0, 1). \tag{3.24}$$

Therefore, for any $n = 0, 1, 2, \cdots$, by (3.24), (3.1) and the definition of $\mathcal{H}$, we get

$$\begin{aligned}
&\int_0^1 \beta_n(s)\alpha(s)\mathcal{A}f(s)ds\\
&= -\frac{1}{2}\int_0^1 \beta_n(s)\alpha(s)\left(\int_0^s \Big(e^{x-s} + e^{s-x}\Big)\alpha(x) f(x)dx\right) ds\\
&= -\frac{1}{2}\int_0^1 \left(\int_x^1 \beta_n(s)\Big(e^{x-s} + e^{s-x}\Big)\alpha(s)ds\right)\alpha(x) f(x)dx\\
&= -\frac{1}{2}\int_0^1 \beta_{n+1}(x)\alpha(x) f(x)dx = 0.
\end{aligned} \tag{3.25}$$

Similarly

$$\int_0^1 \gamma_n(s)\alpha(s)\mathcal{A}f(s)ds = 0. \tag{3.26}$$

Combining (3.25)–(3.26), we see that $\mathcal{A}f \in \mathcal{H}$.

Step 2. Let us show that $\mathcal{A}$ is a compact operator in $\mathcal{H}$. For this purpose, assume G is a bounded subset in $\mathcal{H}$. By (3.24), it is easy to see that

$$\mathcal{A}G \overset{\triangle}{=} \{\mathcal{A}f;\ f \in G\} \tag{3.27}$$

is a bounded subset in $H_0^1(0, 1)$. From this and noting that $L^2(0, 1) \subset L^2(0, 1; \alpha)$ topologically, one concludes that $\mathcal{A}G$ is a compact set in $L^2(0, 1; \alpha)$, which gives the compactness of $\mathcal{A}$ in $\mathcal{H}$ immediately.

Step 3. Let us show that $\mathcal{A}$ is a skew-adjoint operator in $\mathcal{H}$. For this purpose, take any two elements $f, g \in \mathcal{H}$. By the definition of $\mathcal{H}$, we have

$$\int_0^1 e^{-s}\alpha(s)g(s)ds = \int_0^1 e^{s}\alpha(s)g(s)ds = 0, \tag{3.28}$$

which implies

$$\int_0^x e^{\pm s}(s)\alpha(s)g(s)ds = -\int_x^1 e^{\pm s}\alpha(s)g(s)ds, \qquad \forall\, x \in (0, 1). \tag{3.29}$$

Thus, by (3.24) and (3.29), we have

$$\begin{aligned}
(\mathcal{A}f, g)_{\mathcal{H}} &= -\frac{1}{2}\int_0^1 \alpha(x)\left(\int_0^x \left(e^{x-s} + e^{s-x}\right)\alpha(s)f(s)ds\right) g(x)dx \\
&= -\frac{1}{2}\Big[\int_0^1 e^{-s}\alpha(s)f(s)\Big(\int_s^1 e^{x}\alpha(x)g(x)dx\Big)ds \\
&\quad + \int_0^1 e^{s}\alpha(s)f(s)\Big(\int_s^1 e^{-x}\alpha(x)g(x)dx\Big)ds\Big] \\
&= \frac{1}{2}\Big[\int_0^1 e^{-s}\alpha(s)f(s)\Big(\int_0^s e^{x}\alpha(x)g(x)dx\Big)ds \\
&\quad + \int_0^1 e^{s}\alpha(s)f(s)\Big(\int_0^s e^{-x}\alpha(x)g(x)dx\Big)ds\Big] \\
&= \frac{1}{2}\int_0^1 \alpha(x)f(x)\left(\int_0^x \left(e^{x-s} + e^{s-x}\right)\alpha(s)g(s)ds\right) dx \\
&= -(f, \mathcal{A}g)_{\mathcal{H}}, \qquad \forall\, f, g \in \mathcal{H},
\end{aligned} \tag{3.30}$$

which gives the desired result immediately. □

Now, similar to Lemma 2.4, one has the following result.

Lemma 3.3. *Assume that $u_0 \in \mathcal{H}$. The solution u of (2.5) (or equivalently (1.1)) has the regularity*

$$u \in C^{\omega}(\mathbb{R}; \mathcal{H}), \tag{3.31}$$

where $C^{\omega}(\mathbb{R}; \mathcal{H})$ stands for the class of analytic functions defined in $\mathbb{R}$ with values in $\mathcal{H}$.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let u be a solution of (1.1) satisfying (1.2) with $F = (0, a) \cup (b, 1)$. By Lemma 3.1, we get

$$\int_0^1 \beta_n(s)\alpha(s)u(t, s)ds = \int_0^1 \gamma_n(s)\alpha(s)u(t, s)ds = 0, \qquad \forall\, t \in (0, T), \tag{3.32}$$

where $n = 0, 1, 2, \cdots$. On the other hand, by Lemma 2.1, the weak solution of (1.1) satisfies $u \in C([0, T]; L^2(0, 1))$. Taking $t = 0$ in (3.32), we see that

$$\int_0^1 \beta_n(s)\alpha(s)u_0(s)ds = \int_0^1 \gamma_n(s)\alpha(s)u_0(s)ds = 0, \qquad \forall\, t \in (0, T), \quad (3.33)$$

where $n = 0, 1, 2, \cdots$. Therefore, by the definition of $\mathcal{H}$ (see (3.17)) and noting that $L^2(0, 1) \subset L^2(0, 1; \alpha)$, we conclude that

$$u_0 \in \mathcal{H}. \quad (3.34)$$

Now, by Lemma 3.2, we know that $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a compact, skew-adjoint operator. Therefore, $i\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a compact, self-adjoint operator. Applying the classical theorem on the spectral decomposition of compact self-adjoint operators (cf., for example, Theorem 4.2 of Chapter VI in [17]) to $i\mathcal{A}$ in $\mathcal{H}$, one can find a sequence of eigenvalues $\{\lambda_n\}$ of $\mathcal{A}$, $\lambda_n \in i\mathbb{R}$ with λ_n tending to zero as $n \to \infty$. For each λ_n ($n = 1, 2, \cdots$), one can choose the corresponding eigenvectors $e_n^1, \cdots, e_n^{\mu_n}$ such that

$$e_1^1, \cdots, e_1^{\mu_1}, \cdots, e_n^1, \cdots, e_n^{\mu_n}, \cdots \quad (3.35)$$

form an orthonormal basis of $\mathcal{H}$. Recall $\alpha = \alpha(\cdot) \in W(0, 1)$. Therefore it is easy to see that

$$\lambda_n \neq 0, \qquad n = 1, 2, \cdots. \quad (3.36)$$

Indeed, if $\lambda = 0$ were an eigenvalue, there would exist $e \in H_0^1(0, 1)$, $e \neq 0$, such that $(I - A)^{-1}(\alpha e)_x \equiv 0$. Thus $(\alpha e)_x \equiv 0$, which implies $\alpha e = Const$. Since $e \in H_0^1(0, 1)$ and $e \neq 0$, this implies that either $Const = 0$ or α is unbounded. The second possibility has to be excluded since $\alpha \in W(0, 1)$. Thus $\alpha e \equiv 0$ in $(0, 1)$. But then, since $\alpha \in W(0, 1)$, we have $e = 0$ a.e. in $(0, 1)$, which is in contradiction with the fact that $e \neq 0$.

By (3.34), we can assume that the initial datum u_0 of (1.1) is decomposed as

$$u_0 = \sum_{n=1}^{\infty}\sum_{j=1}^{\mu_n} a_n^j e_n^j \qquad \text{in } \mathcal{H}, \quad (3.37)$$

where a_n^j are constants. Then it follows that the corresponding solution u of (1.1) can be expressed as

$$u(t, x) = \sum_{n=1}^{\infty}\Big(\sum_{j=1}^{\mu_n} a_n^j e_n^j(x)\Big)e^{\lambda_n t} \qquad (t, x) \in \mathbb{R} \times (0, 1). \quad (3.38)$$

However, by Lemma 3.3, we see that $u \in C^{\omega}(\mathbb{R}; \mathcal{H})$. Therefore, by (1.2) (recall $F = (0, a) \cup (b, 1)$), we obtain that

$$u(t, x) = 0 \qquad \text{in} \quad \mathbb{R} \times (0, a), \quad (3.39)$$

i.e.

$$\sum_{m=1}^{\infty}\Big(\sum_{j=1}^{\mu_m} a_n^j e_m^j(x)\Big)e^{\lambda_m t}=0, \qquad (t,x)\in\mathbb{R}\times(0,a). \tag{3.40}$$

Note that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{irt}e^{-ist}dt=\begin{cases}1 & \text{if } r=s\\ 0 & \text{if } r\neq s\end{cases}, \qquad \forall\, r,s\in\mathbb{R}. \tag{3.41}$$

Hence, by (3.40)–(3.41), for each $n\in\mathbb{N}$, we have

$$\sum_{j=1}^{\mu_n} a_n^j e_n^j(x)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{-\lambda_n t}\sum_{m=1}^{\infty}\Big(\sum_{j=1}^{\mu_m} a_n^j e_m^j(x)\Big)e^{\lambda_m t}dt=0,\ \forall\, x\in(0,a). \tag{3.42}$$

Note that $\lambda_n\neq 0$. Taking into account that

$$e(x)=\sum_{j=1}^{\mu_n} a_n^j e_n^j(x)$$

satisfies $(I-A)^{-1}(\alpha e)_x=\lambda_n e$ together with $e(0)=e(1)=0$, by (3.42) and Lemma 2.5, we conclude that

$$\sum_{j=1}^{\mu_n} a_n^j e_n^j(x)=0, \qquad \forall\, x\in(0,1). \tag{3.43}$$

Therefore

$$a_n^j=0, \qquad j=1,\cdots,\mu_n,\ n=1,2,\cdots. \tag{3.44}$$

Combining (3.37) and (3.44), we see that $u_0=0$. Consequently $u\equiv 0$. Thus the proof of Theorem 1.2 is completed. □

4. Asymptotic analysis of the eigenvalues and eigenvectors of operator $\widetilde{\mathcal{A}}$

As we mentioned in the introduction, the method of proof of Theorem 1.2 we have developed in the previous section does not apply when $\beta(\cdot)\neq 0$ or $F\neq(0,a)\cup(b,1)$. In order to prove Theorem 1.3 we need to justify developing the solutions $u(\cdot)$ of (1.1) in the series of the generalized eigenvectors of the operator $\widetilde{\mathcal{A}}$ defined by (2.11). For this purpose, the first step is to get a sharp asymptotic description of the "high frequency" eigenvalues and eigenvectors of $\widetilde{\mathcal{A}}$. This is the object of this section.

Without loss of generality, we suppose

$$\min_{x\in[0,1]}\alpha(x)>0. \tag{4.1}$$

The case $\min_{x\in[0,1]}\big(-\alpha(x)\big) > 0$ can be considered similarly. Also, we recall that $\alpha(\cdot) \in W^{2,\infty}(0,1)$ and $\beta(\cdot) \in L^\infty(0,1)$.

Denote

$$a_0 \stackrel{\triangle}{=} \int_0^1 \alpha(s)ds. \tag{4.2}$$

We need the following variable transformation

$$\tilde{x} \stackrel{\triangle}{=} \frac{1}{a_0}\int_0^x \alpha(s)ds, \qquad x \in [0,1]. \tag{4.3}$$

Then by (4.1), we see that $\tilde{x} \in [0,1]$ and $\tilde{x} \to x$ is a one-to-one and onto map. Denote

$$\begin{cases} \tilde{u}(t,\tilde{x}) = u(t,x),\\ \tilde{\alpha}(\tilde{x}) = \alpha(x),\\ \tilde{\beta}(\tilde{x}) = \beta(x),\\ \tilde{u}(t,\tilde{x}) = u(t,x). \end{cases} \tag{4.4}$$

Then, by (1.1), we see that $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \frac{\tilde{\alpha}^2}{a_0^2}\tilde{u}_{t\tilde{x}\tilde{x}} - \frac{\tilde{\alpha}\tilde{\alpha}_{\tilde{x}}}{a_0^2}\tilde{u}_{t\tilde{x}} = \frac{\tilde{\alpha}^2}{a_0}\tilde{u}_{\tilde{x}} + \Big(\frac{\tilde{\alpha}\tilde{\alpha}_{\tilde{x}}}{a_0} + \tilde{\beta}\Big)\tilde{u} & \text{in } (0,T)\times(0,1),\\ \tilde{u}(t,0) = \tilde{u}(t,1) = 0 & t \in (0,T),\\ \tilde{u}(0,\tilde{x}) = \tilde{u}_0(\tilde{x}) & \text{in } (0,1). \end{cases} \tag{4.5}$$

In what follows, we denote $\tilde{x}$, $\tilde{u}$, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{u}_0$ simply by x, u, α, β and u_0 respectively. Thus, (4.5) reads

$$\begin{cases} \frac{a_0^2}{\alpha^2}u_t - u_{txx} - \frac{\alpha_x}{\alpha}u_{tx} = a_0 u_x + \frac{a_0\alpha_x\alpha + a_0^2\beta}{\alpha^2}u & \text{in } (0,T)\times(0,1),\\ u(t,0) = u(t,1) = 0 & t \in (0,T),\\ u(0,x) = u_0(x) & \text{in } (0,1). \end{cases} \tag{4.6}$$

By means of the above variable transformation, we can define a linear operator $\mathcal{F}:\ H_0^1(0,1) \to H_0^1(0,1)$ by

$$(\mathcal{F}f)(x) = f\left(\frac{1}{a_0}\int_0^x \alpha(s)ds\right)(= f(\tilde{x}(x))), \qquad \forall\ f \in H_0^1(0,1). \tag{4.7}$$

Then $\mathcal{F}^{-1}:\ H_0^1(0,1) \to H_0^1(0,1)$ is given by

$$(\mathcal{F}^{-1}f)(x) = f(\tilde{x}^{-1}(x)), \qquad \forall\ f \in H_0^1(0,1). \tag{4.8}$$

It is easy to check the following simple result, which will play an important role in the sequel.

Lemma 4.1. *Let $\alpha(\cdot) \in L^\infty(0,1)$ with $\min_{x\in[0,1]}|\alpha(x)| > 0$. Then both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bounded linear operators from $H_0^1(0,1)$ to $H_0^1(0,1)$. Furthermore, if $\alpha(\cdot) \in W^{1,\infty}(0,1)$, then both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bounded linear operators from $H^2(0,1)$ to $H^2(0,1)$.*

We claim that for any $\phi \in L^2(0,1)$, the following equation

$$\begin{cases} \frac{a_0^2}{\alpha^2}\psi - \psi_{xx} - \frac{\alpha_x}{\alpha}\psi_x = \phi & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0 \end{cases} \tag{4.9}$$

admits one and only one solution $\psi \in H_0^1(0,1) \cap H^2(0,1)$, and that the mapping $\phi \in L^2(0,1) \to \psi \in H_0^1(0,1) \cap H^2(0,1)$ is continuous. To see this, we note that solving (4.9) is equivalent to solving

$$\begin{cases} \mathcal{F}^{-1}\psi - (\mathcal{F}^{-1}\psi)_{xx} = \mathcal{F}^{-1}(\frac{\alpha^2}{a_0^2}\phi) & \text{in } (0,1), \\ \mathcal{F}^{-1}\psi(0) = \mathcal{F}^{-1}\psi(1) = 0, \end{cases} \tag{4.10}$$

which has, indeed, a unique solution $\mathcal{F}^{-1}\psi \in H_0^1(0,1) \cap H^2(0,1)$. Therefore, the operator $\left(\frac{a_0^2}{\alpha^2} - \partial_{xx} - \frac{\alpha_x}{\alpha}\partial_x\right)^{-1}$ is well-defined in $L^2(0,1)$ and it has the continuity property claimed above.

Now, we define an operator $\mathcal{B}: H_0^1(0,1) \to H_0^1(0,1)$ by

$$\mathcal{B}f = \left(\frac{a_0^2}{\alpha^2} - \partial_{xx} - \frac{\alpha_x}{\alpha}\partial_x\right)^{-1}\left(a_0 f_x + \frac{a_0\alpha\alpha_x + a_0^2\beta}{\alpha^2}f\right), \qquad \forall f \in H_0^1(0,1). \tag{4.11}$$

Then (4.6) can be re-written equivalently as

$$\begin{cases} u_t = \mathcal{B}u, & t > 0, \\ u(0) = u_0. \end{cases} \tag{4.12}$$

Let μ be an eigenvalue of $\mathcal{B}$ and $y \in H_0^1(0,1)$ be the corresponding eigenvector. It is easy to show that

$$\mu \neq 0. \tag{4.13}$$

Put

$$\lambda = \frac{1}{\mu}. \tag{4.14}$$

Then, we see that λ and y satisfy

$$\begin{cases} y'' = -\left(\frac{\alpha'}{\alpha} + a_0\lambda\right)y' + \left(\frac{a_0^2}{\alpha^2} - \frac{a_0\alpha\alpha' + a_0^2\beta}{\alpha^2}\lambda\right)y, & 0 < x < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{4.15}$$

where we denote $' = \partial/\partial x$.

We need the following transformation:

$$y = e^{\phi}z, \tag{4.16}$$

where $\phi = \phi(x)$ will be given below. It is easy to check that z satisfies

$$\begin{cases} z'' = -\Big(\frac{\alpha'}{\alpha} + a_0\lambda + 2\phi'\Big)z' + \Big[\frac{a_0^2}{\alpha^2} - \frac{a_0\alpha\alpha' + a_0^2\beta}{\alpha^2}\lambda \\ \qquad -\Big(\frac{\alpha'}{\alpha} + a_0\lambda\Big)\phi' - \phi'' - (\phi')^2\Big]z, \qquad 0 < x < 1, \\ z(0) = z(1) = 0. \end{cases} \tag{4.17}$$

Let us take

$$\phi = \phi(x) \stackrel{\triangle}{=} -\frac{1}{2}(\ln\alpha + a_0\lambda x). \tag{4.18}$$

Then we get

$$\begin{cases} z'' = \Big(\frac{a_0^2}{4}\lambda^2 - g\lambda + h\Big)z, \qquad 0 < x < 1, \\ z(0) = z(1) = 0, \end{cases} \tag{4.19}$$

where

$$g \stackrel{\triangle}{=} \frac{a_0\alpha\alpha' + 2a_0^2\beta}{2\alpha^2} \tag{4.20}$$

and

$$h \stackrel{\triangle}{=} \frac{4a_0^2 + 2\alpha\alpha'' - (\alpha')^2}{4\alpha^2}. \tag{4.21}$$

It is obvious that $z \in H_0^1(0,1)$. Conversely, if $\lambda \in \mathbb{C}$, $0 \neq z \in H_0^1(0,1)$ satisfies (4.19), then it is easy to check that λ and y given by (4.16) satisfy (4.15).

We need the following rough estimate on the eigenvalues of (4.19).

Theorem 4.1. *Let* $\beta(\cdot) \in L^\infty(0,1)$ *and* $\alpha(\cdot) \in W^{2,\infty}(0,1)$ *with* $\min_{x\in[0,1]}|\alpha(x)| > 0$. *Then the set of eigenvalues of (4.19) is symmetric about the real axis and is contained in the set* $\Lambda_1 \cup \Lambda_2$, *where*

$$\begin{aligned} &\Lambda_1 \stackrel{\triangle}{=} \{\lambda \in \mathbb{R};\ |\lambda| < \delta_1\}, \ \textit{and} \\ &\Lambda_2 \stackrel{\triangle}{=} \{\lambda \in \mathbb{C};\ \operatorname{Im}\lambda \neq 0, |\operatorname{Re}\lambda| \leq \delta_2\} \end{aligned} \tag{4.22}$$

with

$$\delta_1 \stackrel{\triangle}{=} \frac{2\Big(|g|_{L^\infty(0,1)} + \sqrt{|g|^2_{L^\infty(0,1)} + a_0^2|h|_{L^\infty(0,1)}}\Big)}{a_0^2} \quad \textit{and} \quad \delta_2 \stackrel{\triangle}{=} \frac{2|g|_{L^\infty(0,1)}}{a_0^2}. \tag{4.23}$$

Proof. Multiplying both sides of the first equation in (4.19) by $\bar{z}$, integrating it on $(0,1)$, using integration by parts, we get

$$\frac{a_0^2}{4}\int_0^1 |z|^2 dx\lambda^2 - \int_0^1 g|z|^2 dx\lambda + \int_0^1 h|z|^2 dx + \int_0^1 |z'|^2 dx = 0. \tag{4.24}$$

Case 1. $\operatorname{Im}\lambda = 0$. In this case, we have

$$\frac{a_0^2}{4}\int_0^1 |z|^2 dx\lambda^2 - \int_0^1 g|z|^2 dx\lambda + \int_0^1 h|z|^2 dx$$
$$\geq \left(\frac{a_0^2}{4}\lambda^2 - |g|_{L^\infty(0,1)}|\lambda| - |h|_{L^\infty(0,1)}\right)\int_0^1 |z|^2 dx. \tag{4.25}$$

Note that if $|\lambda| \geq \delta_1$, we have

$$\frac{a_0^2}{4}\lambda^2 - |g|_{L^\infty(0,1)}|\lambda| - |h|_{L^\infty(0,1)} \geq 0. \tag{4.26}$$

Combining (4.24)–(4.26), we arrive at

$$z \equiv 0, \tag{4.27}$$

which contradicts the fact that z is an eigenvector. Therefore, we conclude that $|\lambda| < \delta_1$.

Case 2. $\operatorname{Im}\lambda \neq 0$. In this case, from (4.24), we see that

$$\operatorname{Re}\lambda = \frac{2\int_0^1 g|z|^2 dx}{a_0^2\int_0^1 |z|^2 dx}. \tag{4.28}$$

Thus it is obvious that $|\operatorname{Re}\lambda| \leq \delta_2$. □

Let us solve the (nonlinear) eigenvalue problem (4.19). For this purpose, we use the so-called "shooting method". Fix $\lambda \in \mathbb{C}$, we consider the following Cauchy problem:

$$\begin{cases} w'' = \left(\frac{a_0^2}{4}\lambda^2 - g\lambda + h\right)w, & x > 0, \\ w(0,\lambda) = 0, \quad w'(0,\lambda) = 1. \end{cases} \tag{4.29}$$

Clearly the zeros of $\lambda \to w(1,\lambda)$ are the eigenvalues of problem (4.19). In addition, it can be checked that the algebraic multiplicity of an eigenvalue is the order to which $w(1,\lambda)$ vanishes.

The "main part" of (4.29) (when λ is large) is the following equation:

$$\begin{cases} v'' = \frac{a_0^2\lambda^2}{4}v, & x > 0, \\ v(0,\lambda) = 0, \quad v'(0,\lambda) = 1. \end{cases} \tag{4.30}$$

It is easy to check that the unique solution of (4.30) reads

$$v(x,\lambda) = \frac{2}{a_0\lambda}\sinh\frac{a_0\lambda x}{2}. \tag{4.31}$$

Therefore, using the variation of constants formula, similar to Theorem 3.1 in [2], we have the following asymptotic estimates.

Theorem 4.2. *Let $\beta(\cdot) \in L^\infty(0, 1)$ and $\alpha(\cdot) \in W^{2,\infty}(0, 1)$ with $\min_{x\in[0,1]} |\alpha(x)| > 0$. Then there is a constant $C_1 = C_1(\alpha, \beta) > 0$ such that for the solution of (4.29), the following estimates*

$$\left| w(x, \lambda) - \frac{2}{a_0\lambda} \sinh\left(\frac{a_0\lambda}{2}x - \frac{1}{a_0}\int_0^x gds\right)\right| \le \frac{C_1}{|\lambda|^2} \tag{4.32}$$

and

$$\left| w'(x, \lambda) - \cosh\left(\frac{a_0\lambda}{2}x - \frac{1}{a_0}\int_0^x gds\right)\right| \le \frac{C_1}{|\lambda|}. \tag{4.33}$$

hold uniformly for $x \in [0, 1]$, $|\lambda| \ge 1$ and $|\mathrm{Re}\,\lambda| < \delta_1$, where δ_1 is the constant in (4.23).

Let us denote

$$b_0 \triangleq -\frac{1}{a_0}\int_0^1 g(s)ds. \tag{4.34}$$

We choose

$$N_0 \triangleq \left[\frac{a_0^2 C_1}{\pi}\right] + 1, \tag{4.35}$$

where C_1 is the constant in Theorem 4.2. Put

$$\Gamma_n \triangleq \{\lambda \in \mathbb{C};\ |\frac{a_0}{2}\lambda + b_0 \mp in\pi| = 2C_1/(n\pi)\}, \qquad |n| > N_0. \tag{4.36}$$

We need the following estimate, which is a simple consequence of Lemma 5.2 in [3].

Lemma 4.2. *For any $\lambda \in \Gamma_n$ with $|n| \ge N_0$, it holds*

$$\left|\frac{2}{a_0}\sinh(\frac{a_0}{2}\lambda + b_0)\right| > \frac{C_1}{|\lambda|}, \tag{4.37}$$

where C_1 is the constant in Theorem 4.2.

Now, we can estimate the location of the eigenvalues of (4.19).

Theorem 4.3. *Let $\beta(\cdot) \in L^\infty(0, 1)$ and $\alpha(\cdot) \in W^{2,\infty}(0, 1)$ with $\min_{x\in[0,1]} |\alpha(x)| > 0$. Then there is an integer N_1, depending only on α and β, such that (4.19) has one simple eigenvalue in Γ_n for each $|n| > N_1$.*

Proof. By the definition of Γ_n in (4.36), it is easy to see that one can find $N_1 \ge N_0$ such that

$$\Gamma_n \subset \{\lambda \in \mathbb{C};\ |\mathrm{Re}\,\lambda| < \delta_1, |\lambda| \ge 1\}, \qquad \forall\ |n| > N_1. \tag{4.38}$$

Thus, by Theorem 4.2 and Lemma 4.2, we see that the following estimate

$$\left| w(1, \lambda) - \frac{2}{a_0\lambda}\sinh(\frac{a_0}{2}\lambda + b_0)\right| \le \frac{C_1}{|\lambda|^2} \le \left|\frac{2}{a_0\lambda}\sinh(\frac{a_0}{2}\lambda + b_0)\right| \tag{4.39}$$

holds for any $|n| > N_1$. Hence, by (4.39) and Rouchés Theorem, we conclude that $w(1, \lambda)$ possesses the same number of zeros in Γ_n. □

Let us return to the asymptotic estimate on the eigenvalues and the corresponding eigenvectors of $\mathcal{B}$.

Theorem 4.4. *Let $\beta(\cdot) \in L^\infty(0,1)$ and $\alpha(\cdot) \in W^{2,\infty}(0,1)$ with $\min_{x\in[0,1]} |\alpha(x)| > 0$. Then for any $|n| > N_1$, where N_1 is the integer given by Theorem 4.3, the eigenvalue μ_n of $\mathcal{B}$ is algebraically simple and has the following asymptotic expansion:*

$$\mu_n = \frac{a_0}{2(-b_0 + in\pi)} + O(|n|^{-3}); \tag{4.40}$$

the corresponding eigenvector y_n of $\mathcal{B}$ satisfies

$$y_n(x) = \frac{e^{b_0 x - in\pi x}}{in\pi\sqrt{\alpha}} \sinh\Big(in\pi x - b_0 x - \frac{1}{a_0}\int_0^x g ds\Big) + O(n^{-2}) \tag{4.41}$$

and

$$y_n'(x) = \frac{e^{2b_0 x + \frac{1}{a_0}\int_0^x g ds}}{\sqrt{\alpha}} e^{-2in\pi x} + O(|n|^{-1}). \tag{4.42}$$

Proof. By Theorem 4.3 and (4.36), we see that the eigenvalue of (4.19) has the following asymptotic expansion:

$$\lambda_{\pm n} = \frac{2}{a_0}(-b_0 \pm in\pi) + O(|n|^{-1}). \tag{4.43}$$

Thus, by Theorem 4.2, we conclude that the corresponding eigenvectors of (4.19) satisfy

$$\begin{aligned} z(x, \lambda_{\pm n}) &= \frac{2}{a_0\lambda_{\pm n}} \sinh\Big(\frac{a_0\lambda_{\pm n}}{2} x - \frac{1}{a_0}\int_0^x g ds\Big) + O(|\lambda_{\pm n}|^{-2}) \\ &= \frac{1}{-b_0 \pm in\pi} \sinh\Big(\pm in\pi x - b_0 x - \frac{1}{a_0}\int_0^x g ds\Big) + O(n^{-2}) \\ &= \frac{1}{\pm in\pi} \sinh\Big(\pm in\pi x - b_0 x - \frac{1}{a_0}\int_0^x g ds\Big) + O(n^{-2}) \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} z'(x, \lambda_{\pm n}) &= \cosh\left(\frac{a_0\lambda_{\pm n}}{2} x - \frac{1}{a_0}\int_0^x g ds\right) + O(|\lambda_{\pm n}|^{-1}) \\ &= \cosh\left(\pm in\pi x - b_0 x - \frac{1}{a_0}\int_0^x g ds\right) + O(|n|^{-1}). \end{aligned} \tag{4.45}$$

Denote

$$y_{\pm n}(x) = e^{\phi(x,\lambda_{\pm n})} z(x, \lambda_{\pm n}), \tag{4.46}$$

where ϕ is the function given by (4.18). Then, it is easy to see that every $\lambda_{\pm n}^{-1}$ is a eigenvalue of $\mathcal{B}$, and $y_{\pm n} (\neq 0)$ is a corresponding eigenvector. Also, by (4.43)–(4.46) and (4.18), we get the asymptotic expansions (4.41)–(4.42) immediately. □

Now, let us denote

$$U_n(x) = \frac{1}{\sqrt{1+n^2\pi^2}} e^{-i\text{sgn}\,(n)\sqrt{1+n^2\pi^2}x} \sin(n\pi x), \qquad n = \pm 1,\ \pm 2, \cdots. \tag{4.47}$$

Then by Proposition 2.3 in [14], we know that the family $\{U_n\}_{|n|=1}^{\infty}$ forms an orthonormal basis of $H_0^1(0,1)$. Thus $\{\text{sgn}\,(n)U_n\}_{|n|=1}^{\infty}$ forms an orthonormal basis of $H_0^1(0,1)$, too.

Further, we denote

$$u_n(x) = \frac{e^{2b_0x+\frac{1}{a_0}\int_0^x g ds}}{\sqrt{\alpha}} \frac{\text{sgn}\,(n)}{\sqrt{1+n^2\pi^2}} e^{-i\text{sgn}\,(n)\sqrt{1+n^2\pi^2}x} \times \sin(n\pi x),\ n = \pm 1,\ \pm 2, \cdots. \tag{4.48}$$

Recall that a Riesz basis of a Hilbert space H is obtained from an orthonormal basis by means of a bounded invertible operator transformation in H. Therefore, noting that u_n is obtained from $\text{sgn}\,(n)U_n$ by simple multiplication by a C^1 function

$$\frac{e^{2b_0x+\frac{1}{a_0}\int_0^x g ds}}{\sqrt{\alpha}} (> 0 \text{ for any } x \in [0,1])$$

which is independent of n, we see that

Lemma 4.3. *The family $\{u_n\}_{|n|=1}^{\infty}$ forms a Riesz basis of $H_0^1(0,1)$.*

We have the following crucial result.

Theorem 4.5. *Let $\beta(\cdot) \in L^\infty(0,1)$ and $\alpha(\cdot) \in W^{2,\infty}(0,1)$ with $\min_{x\in[0,1]} |\alpha(x)| > 0$. Then the eigenvectors y_n ($|n| > N_1$) of $\mathcal{B}$, where N_1 is the integer given by Theorem 4.3, satisfy*

$$\sum_{|n|=N_1+1}^{\infty} |y_n - u_n|^2_{H_0^1(0,1)} < \infty. \tag{4.49}$$

Proof. By (4.41)–(4.42) and (4.48), we get

$$\begin{aligned}
&\sum_{|n|=N_1+1}^{\infty} |y_n - u_n|^2_{H_0^1(0,1)} \\
&= \sum_{|n|=N_1+1}^{\infty} \Big(|y_n - u_n|^2_{L^2(0,1)} + |y_n' - u_n'|^2_{L^2(0,1)} \Big) \\
&\le C\Big(\sum_{|n|=N_1+1}^{\infty} \frac{1}{n^2} + \sum_{|n|=N_1+1}^{\infty} \int_0^1 \Big| e^{-2in\pi x} + ie^{-i\text{sgn}\,(n)\sqrt{1+n^2\pi^2}x} \sin(n\pi x) \\
&\qquad - \text{sgn}\,(n) \frac{n\pi}{\sqrt{1+n^2\pi^2}} e^{-i\text{sgn}\,(n)\sqrt{1+n^2\pi^2}x} \cos(n\pi x) \Big|^2 dx \Big).
\end{aligned} \tag{4.50}$$

However

$$\sum_{|n|=N_1+1}^{\infty}\int_0^1 \left|e^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}-e^{-in\pi x}\right|^2 dx<\infty \tag{4.51}$$

and

$$\sum_{|n|=N_1+1}^{\infty}\left|1-\mathrm{sgn}\,(n)\frac{n\pi}{\sqrt{1+n^2\pi^2}}\right|^2<\infty. \tag{4.52}$$

Thus, by

$$e^{-2in\pi x}=-ie^{-in\pi x}\sin(n\pi x)+e^{-in\pi x}\cos(n\pi x),$$

and noting (4.51) and (4.52), we see that

$$\begin{aligned}
&\sum_{|n|=N_1+1}^{\infty}\int_0^1 \Big|e^{-2in\pi x}+ie^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}\sin(n\pi x)\\
&\qquad -\mathrm{sgn}\,(n)\frac{n\pi}{\sqrt{1+n^2\pi^2}}e^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}\cos(n\pi x)\Big|^2 dx\\
&=\sum_{|n|=N_1+1}^{\infty}\int_0^1 \Big|i\left(e^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}-e^{-in\pi x}\right)\sin(n\pi x)\\
&\qquad +e^{-in\pi x}\left(1-\mathrm{sgn}\,(n)\frac{n\pi}{\sqrt{1+n^2\pi^2}}\right)\cos(n\pi x)\\
&\qquad +\mathrm{sgn}\,(n)\frac{n\pi}{\sqrt{1+n^2\pi^2}}\left(e^{-in\pi x}-e^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}\right)\cos(n\pi x)\Big|^2 dx\\
&\le C\sum_{|n|=N_1+1}^{\infty}\Big[\int_0^1 \left|e^{-i\mathrm{sgn}\,(n)\sqrt{1+n^2\pi^2}x}-e^{-in\pi x}\right|^2 dx\\
&\qquad +\left|1-\mathrm{sgn}\,(n)\frac{n\pi}{\sqrt{1+n^2\pi^2}}\right|^2\Big]<\infty.
\end{aligned} \tag{4.53}$$

Thus, combining (4.50) and (4.53), we obtain (4.49). □

Now, it is easy to deduce the following result, which is the main result of this section.

Theorem 4.6. *Let* $\beta(\cdot)\in L^\infty(0,1)$ *and* $\alpha(\cdot)\in W^{2,\infty}(0,1)$ *with* $\min_{x\in[0,1]}|\alpha(x)|>0$. *Then for any* $|n|>N_1$, *where* N_1 *is as in Theorem 4.3,* μ_n *is a simple eigenvalue of* $\widetilde{\mathcal{A}}$ *with eigenvector* $\mathcal{F}^{-1}y_n$ *which satisfies*

$$\sum_{|n|=N_1+1}^{\infty}|\mathcal{F}^{-1}y_n-\mathcal{F}^{-1}u_n|^2_{H_0^1(0,1)}<\infty. \tag{4.54}$$

Note that, by Lemma 4.3, we see that $\{\mathcal{F}^{-1}u_n\}_{|n|=1}^{\infty}$ forms a Riesz basis of $H_0^1(0,1)$, too. Thus Theorem 4.6 tells us that the "high frequency" eigenvectors of $\widetilde{\mathcal{A}}$ are quadratically close to a subsequence of the Riesz basis $\{\mathcal{F}^{-1}u_n\}_{|n|=1}^{\infty}$ in $H_0^1(0,1)$. We will show in the next section that we can choose a sequence of generalized eigenvectors of $\widetilde{\mathcal{A}}$ to form a Riesz basis of $H_0^1(0,1)$.

5. Riesz basis property of the generalized eigenvectors of compact operators

This section is devoted to derive a sufficient condition for checking the Riesz basis property of the generalized eigenvectors of compact operators, which is of independent interest.

Throughout this section, H is a complex Hilbert space, G is a linear operator in H. We denote by $\rho(G)$, $\sigma(G)$ and $\sigma_p(G)$ the resolvent set of G, the spectrum of G and the point spectrum of G (i.e. the set of eigenvalues of G) respectively.

First of all, let us recall that a non-zero vector $\eta \in H$ is called a generalized eigenvector of G, corresponding to some $\lambda \in \sigma_p(G)$, if $(\lambda I - G)^m \eta = 0$ for some positive integer m.

Next, we recall that a sequence of vectors $\{h_j\}_{j=1}^{\infty}$ in H is said to be $\omega-$linearly independent if

$$\sum_{j=1}^{\infty} c_j h_j = 0, \qquad c_j \in \mathbb{C} \text{ for } j \in \mathbb{N}$$

is not possible for

$$0 < \sum_{j=1}^{\infty} |c_j h_j|^2 < \infty.$$

The problem of verifying whether a set of generalized eigenvectors of a linear operator forms a Riesz basis is important both from a theoretical and a practical point of view. In this direction, the main tool is the following Bari's theorem (*see* for example [7]).

Theorem 5.1. *Any $\omega-$linearly independent sequence $\{g_n\}_{n=1}^{\infty}$ of H which is quadratically close to some Riesz basis $\{f_n\}_{n=1}^{\infty}$, i.e.*

$$\sum_{n=1}^{\infty} |g_n - f_n|^2 < \infty, \tag{5.1}$$

is itself a Riesz basis.

In some applications (as in Section 4 above), by means of asymptotic analysis techniques, it is relatively easy to find a sequence of generalized eigenvectors

$\{g_n\}_{n=N+1}^{\infty}$ (N large enough) of G, which is quadratically close to a subsequence of some known Riesz basis $\{f_n\}_{n=1}^{\infty}$, i.e.

$$\sum_{n=N+1}^{\infty} |g_n - f_n|^2 < \infty. \tag{5.2}$$

The most common approach to construct a Riesz basis of H via the generalized eigenvectors of G is then to analyze carefully the number of the remaining independent generalized eigenvectors and to show that $\overline{sp(G)}$, the root space of G, i.e. the closed subspace spanned by the generalized eigenvectors of G, is complete in H.

Very recently, Guo ([8]) and Guo & Yu ([9]) proved an interesting result, which says that the last step described above is in fact not necessary for many problems. Their result reads as follows:

Theorem 5.2. *Let G be a densely defined linear operator with compact resolvent in H. Let $\{f_n\}_{n=1}^{\infty}$ be a Riesz basis of H. Suppose a sequence of generalized eigenvectors $\{g_n\}_{n=N+1}^{\infty}$ of G satisfies (5.2) for some $N \in \mathbb{N}$. Then one can find an integer $M \geq N$ and some generalized eigenvectors $\{g_{n0}\}_{n=1}^{M}$ of G such that*

$$\{g_{n0}\}_{n=1}^{M} \cup \{g_n\}_{n=M+1}^{\infty} \tag{5.3}$$

forms a Riesz basis of H.

Note, however, that one can not apply Theorem 5.2 to our problem. The reason is that our operator $\widetilde{\mathcal{A}}$ (defined by (2.11)) is itself a compact operator (and therefore a bounded operator). Therefore its resolvent can not be compact because the underlying Hilbert space $H_0^1(0, 1)$ is infinite-dimensional. One could also try to apply Theorem 5.2 to $\widehat{\mathcal{A}} \triangleq (\lambda_0 - \widetilde{\mathcal{A}})^{-1}$ for some $\lambda_0 \in \rho(\widetilde{\mathcal{A}})$. But this is not possible since $\widehat{\mathcal{A}}$ is always a bounded linear operator and hence the resolvent of $\widehat{\mathcal{A}}$ can not be compact for the same reason as above. Therefore, we need to derive a new sufficient condition for the Riesz basis property of the generalized eigenvectors of compact operators to hold. Our result is the following.

Theorem 5.3. *Let* $\dim H = \infty$. *Let G be a compact operator in H and $0 \notin \sigma_p(G^*)$. Let $\{f_n\}_{n=1}^{\infty}$ be a Riesz basis of H. Suppose a sequence of generalized eigenvectors $\{g_n\}_{n=N+1}^{\infty}$ of G satisfies (5.2) for some $N \in \mathbb{N}$. Then there exist an integer $M \geq N$ and some generalized eigenvectors $\{g_{n0}\}_{n=1}^{M}$ of G such that the sequence (5.3) forms a Riesz basis of H.*

Remark 5.1. It is easy to see that Theorem 5.2 is a consequence of Theorem 5.3. In fact, if $\dim H < \infty$, the result in Theorem 5.2 is (trivially) correct. If $\dim H = \infty$ and G be a densely defined linear operator with compact resolvent in H, then $\widetilde{G} = (\lambda_0 I - G)^{-1}$ with $\lambda_0 \in \rho(G)$ is compact and it is easy to check that $0 \notin \sigma_p(\widetilde{G}^*)$. Thus noting that G and $\widetilde{G}$ have the same generalized eigenvectors, our assertion follows immediately by applying Theorem 5.3 to $\widetilde{G}$.

Remark 5.2. From the proof of Theorem 5.3, it is easy to see that we can choose $M = N$ in Theorem 5.3 provided that $\{g_n\}_{n=N+1}^{\infty}$ is $\omega-$linearly independent in H.

Combining Theorem 5.3, Theorem 4.6 and Lemma 2.3, we obtain the following conclusion immediately.

Theorem 5.4. *Let $\beta(\cdot) \in L^{\infty}(0, 1)$ and $\alpha(\cdot) \in W^{2,\infty}(0, 1)$ with $\min_{x\in[0,1]} |\alpha(x)| > 0$. Then there exist finitely many generalized eigenvectors $u_1, u_2, \cdots, u_M$ of $\widetilde{\mathcal{A}}$ corresponding respectively to eigenvalues $\mu_1, \mu_2, \cdots, \mu_M$ for some integer $M \geq N_1$, where N_1 is as in Theorem 4.3, such that*

$$\{u_k\}_{k=1}^{M} \cup \{\mathcal{F}^{-1} y_n\}_{|n|=M+1}^{\infty} \tag{5.4}$$

forms a Riesz basis of $H_0^1(0, 1)$, where $\mathcal{F}^{-1} y_n$ is an eigenvector of $\widetilde{\mathcal{A}}$ corresponding to a simple eigenvalue μ_n for each $|n| > M$.

In order to prove Theorem 5.3, we need the following three known lemmata.

Lemma 5.1. ([8]–[9]) *Let $\{f_n\}_{n=1}^{\infty}$ be a Riesz basis of H. Suppose that for some $N \in \mathbb{N}$, $\{g_n\}_{n=N+1}^{\infty}$ is a sequence in H satisfying (5.2). Then there is a $M \geq N$ such that*

$$\{f_n\}_{n=1}^{M} \cup \{g_n\}_{n=M+1}^{\infty} \tag{5.5}$$

forms a Riesz basis of H.

Lemma 5.2. ([16]) *Let $\{f_n\}_{n=1}^{\infty}$ be a Riesz basis in H. Let $\{g_n\}_{n=N+1}^{\infty}$ with some $N \in \mathbb{N}$ be a $\omega-$linearly independent sequence in H satisfying (5.2). Then $\{g_n\}_{n=N+1}^{\infty}$ forms a Riesz basis in the subspace spanned by itself in H.*

Lemma 5.3. ([7], pp. 17) *Let G be a compact operator in H for which $\overline{sp(G)} \neq H$, and let $\mathcal{P}$ be the orthogonal projector from H to $\overline{sp(G)}^{\perp}$. Then $\mathcal{P}G\mathcal{P}$ is a Volterra operator in H, i.e. a compact operator in H with no nonzero eigenvalues.*

Proof of Theorem 5.3. The proof is similar to [9]. However, for the sake of completeness and also for the reader's convenience, we give the details here. The proof is divided into several steps.

Step 1. By Lemma 5.1, we can find a $M \geq N$ such that $\{f_n\}_{n=1}^{M} \cup \{g_n\}_{n=M+1}^{\infty}$ form a Riesz basis of H. Thus $\{g_n\}_{n=M+1}^{\infty}$ is $\omega-$linearly independent.

Let $\{g_\tau\}$ be an arbitrary set such that

$$\{g_\tau\} \cup \{g_n\}_{n=M+1}^{\infty} \tag{5.6}$$

is a "maximal" $\omega-$linearly independent subset of generalized eigenvectors of G, i.e. $\{g_\tau\} \cup \{g_n\}_{n=M+1}^{\infty}$ is a $\omega-$linearly independent subset of generalized eigenvectors of G, and for any other generalized eigenvector g of G, the set

$$\{g\} \cup \{g_\tau\} \cup \{g_n\}_{n=M+1}^{\infty} \tag{5.7}$$

is not $\omega-$linearly independent anymore. In this way, we see that $\{g_\tau\} \cup \{g_n\}_{n=M+1}^{\infty}$ spans the root subspace $\overline{sp(G)}$ of G.

Step 2. We claim that the number of $\{g_\tau\}$ is not greater than M. In fact, if we assume that

$$\{g_\tau\} \supset \{g_1, g_2, \cdots, g_{M+1}\}, \tag{5.8}$$

then noting that $\{f_n\}_{n=1}^{M} \cup \{g_n\}_{n=M+1}^{\infty}$ is a Riesz basis of H, we see that

$$g_j = \sum_{n=1}^{M} a_n^j f_n + \sum_{n=M+1}^{\infty} a_n^j g_n \tag{5.9}$$

for some sequence $\{a_n^j\}_{n=1}^{\infty}$ in $\mathbb{C}$ $(j = 1, 2, \cdots, M+1)$. Denote

$$\gamma_j = (a_1^j, a_2^j, \cdots, a_M^j), \qquad j = 1, 2, \cdots, M+1. \tag{5.10}$$

Then $\gamma_1, \gamma_2, \cdots, \gamma_{M+1}$ must be linearly dependent, i.e. there exist $b_j \in \mathbb{C}$ $(j = 1, 2, \cdots, M+1)$, $\sum_{j=1}^{M+1} |b_j| > 0$, such that

$$\sum_{j=1}^{M+1} b_j \gamma_j = 0. \tag{5.11}$$

Thus, by (5.9) and (5.11), we see that

$$\sum_{j=1}^{M+1} b_j g_j = \sum_{n=M+1}^{\infty} \Big(\sum_{j=1}^{M+1} b_j a_n^j \Big) g_n, \tag{5.12}$$

which contradicts the $\omega-$linearly independence of $\{g_\tau\} \cup \{g_n\}_{n=M+1}^{\infty}$.

Step 3. Now, by Step 2, we denote $\{g_\tau\} = \{g_{n0}\}_{n=1}^{L}$ with $L \le M$. Then, by Lemma 5.2, we conclude that $\{g_{n0}\}_{n=1}^{L} \cup \{g_n\}_{n=M+1}^{\infty}$ forms a Riesz basis of $\overline{sp(G)}$.

We claim $\overline{sp(G)} = H$. Otherwise we have the following orthogonal decomposition

$$H = H_1 \bigoplus \overline{sp(G)}, \tag{5.13}$$

where $H_1 \triangleq \overline{sp(G)}^{\perp} \neq \{0\}$. Noting that

$$H_1 \subset \Big(\text{span}\, \{g_{M+1}, g_{M+2}, \cdots\} \Big)^{\perp} \tag{5.14}$$

and by the fact that $\{f_n\}_{n=1}^M \cup \{g_n\}_{n=M+1}^\infty$ forms a Riesz basis of H, it is easy to see that

$$\dim H_1 < \infty. \tag{5.15}$$

We denote by $\mathcal{P}$ the orthogonal projector from H to H_1. Then, by Lemma 5.3, we see that $\mathcal{P}G\mathcal{P}$ is a Volterra operator, and so is its adjoint operator $\mathcal{P}G^*\mathcal{P}$. Thus $\mathcal{P}G^*\mathcal{P}$ is a compact operator and has no nonzero eigenvalues.

On the other hand, since $\overline{sp(G)}$ is an invariant subspace of G, H_1 is an invariant subspace of G^*. Note that

$$\mathcal{P}G^*\mathcal{P}H_1 = G^*H_1. \tag{5.16}$$

Thus $G^*\big|_{H_1}$, the restriction of G^* to H_1, has no nonzero eigenvalues. Consequently, by (5.15), we conclude that

$$0 \in \sigma_p(G^*\big|_{H_1}) \subset \sigma_p(G^*), \tag{5.17}$$

which contradicts our assumption.

Step 4. We claim that $L = M$. In fact, if $L < M$, then noting that (by Step 3) $\{g_{n0}\}_{n=1}^L \cup \{g_n\}_{n=M+1}^\infty$ form a Riesz basis of H, we see that

$$f_j = \sum_{n=1}^{L} c_n^j g_{n0} + \sum_{n=M+1}^{\infty} c_n^j g_n \tag{5.18}$$

for some sequence $\{c_n^j\}_{n=1}^L \cup \{c_n^j\}_{n=M+1}^\infty$ in $\mathbb{C}$ $(j = 1, 2, \cdots, M)$. Then arguing as in Step 2, we conclude that

$$\sum_{j=1}^{M} d_j f_j = \sum_{n=M+1}^{\infty} \Big(\sum_{j=1}^{M} d_j c_n^j\Big) g_n \tag{5.19}$$

for some $d_j \in \mathbb{C}$ $(j = 1, 2, \cdots, M)$ with $\sum_{j=1}^M |d_j| > 0$. It is easy to see that (5.19) contradicts the $\omega-$linearly independence of $\{f_n\}_{n=1}^M \cup \{g_n\}_{n=M+1}^\infty$ (recall Lemma 5.1). This completes the proof of Theorem 5.3. □

6. Proof of Theorem 1.3

We divide the proof into several steps.

Step 1. Recall equation (1.1) and that $u_0 \in L^2(0, 1)$. We claim that without loss of generality, we may assume that $u_0 \in H_0^1(0, 1)$. To see this, let us denote (recall (2.1) for A)

$$v \stackrel{\triangle}{=} u_t, \qquad v_0 \stackrel{\triangle}{=} u_t(0) = (I - A)^{-1}\Big((\alpha u_0)_x + \beta u_0\Big), \tag{6.1}$$

where $u(\cdot)$ is the solution of (1.1). Then v satisfies (recall (2.11) for $\widetilde{\mathcal{A}}$)

$$\begin{cases} v_t = \widetilde{\mathcal{A}} v, & t > 0, \\ v(0) = v_0 \end{cases} \tag{6.2}$$

and

$$v_0 \in H_0^1(0,1). \tag{6.3}$$

We note that it suffices to prove

$$v_0 = 0. \tag{6.4}$$

In fact, if (6.4) holds, then it follows from (6.1) and (1.1) that

$$\begin{cases} (\alpha u_0)_x + \beta u_0 = 0, & x \in (0,1), \\ u_0(0) = u_0(1) = 0. \end{cases} \tag{6.5}$$

Thus, by $\min_{x\in[0,1]} |\alpha(x)| > 0$, from (6.5), we get $u_0 = 0$. Therefore by the well-posedness of (1.1), we see that $u(t) \equiv 0$.

In the sequel, we assume that $u_0 \in H_0^1(0,1)$.

Step 2. By Theorem 5.4, we know that $\{u_n\}_{n=1}^M \cup \{\mathcal{F}^{-1} y_n\}_{|n|=M+1}^\infty$ forms a Riesz basis of $H_0^1(0,1)$. Recall that u_j ($j = 1, \cdots, M$) are generalized eigenvectors of $\widetilde{\mathcal{A}}$, and $\mathcal{F}^{-1} y_n$ ($|n| \geq M+1$) are eigenvectors of $\widetilde{\mathcal{A}}$ with simple eigenvalues μ_n. We also claim that, without loss of generality, we may assume that $\{u_n\}_{n=1}^M$ can be re-arranged as follows

$$u_{1,0}, \cdots, u_{1,m_1-1}, \cdots, u_{n_0,0}, \cdots, u_{n_0,m_{n_0}-1} \tag{6.6}$$

for some positive integers n_0 and m_k ($k = 1, \cdots, n_0$), where $\{u_{k,j}\}_{j=0}^{m_k-1}$ is the associated Jordan chain of the corresponding generalized eigenvectors of $\widetilde{\mathcal{A}}$ with respect to μ_k and $u_{k,0}$, i.e.

$$\begin{aligned} &\widetilde{\mathcal{A}} u_{k,0} = \mu_k u_{k,0}, \\ &\widetilde{\mathcal{A}} u_{k,j} = \mu_k u_{k,j} + u_{k,j-1}, \qquad j = 1, \cdots m_k - 1. \end{aligned} \tag{6.7}$$

In fact, if $\{u_n\}_{n=1}^M$ is a proper subset of $\{u_{1,0}, \cdots, u_{1,m_1-1}, \cdots, u_{n_0,0}, \cdots, u_{n_0,m_{n_0}-1}\}$, say $u_{k^0,j^0} \notin \{u_n\}_{n=1}^M$ for some $k^0 \in \{1, \cdots, n_0\}$ and $j^0 \in \{0, \cdots, m_{k^0} - 1\}$, then we simply take $a_{k^0,j^0} = 0$ in (6.9) and the following proof works in the same way. Therefore, for simplicity, we may assume that

$$\{u_{1,0}, \cdots, u_{1,m_1-1}, \cdots, u_{n_0,0}, \cdots, u_{n_0,m_{n_0}-1}\} \cup \{\mathcal{F}^{-1} y_n\}_{|n|=M+1}^\infty \tag{6.8}$$

forms a Riesz basis of $H_0^1(0,1)$.

Now, by $u_0 \in H_0^1(0,1)$, we may suppose

$$u_0 = \sum_{k=1}^{n_0} \sum_{j=0}^{m_k-1} a_{k,j} u_{k,j} + \sum_{|n|=M+1}^{\infty} a_n \mathcal{F}^{-1} y_n \tag{6.9}$$

for some complex numbers $a_{k,j}$ and a_n. Therefore, the solution of (6.2) can be expressed as follows

$$u(t,x) = \sum_{k=1}^{n_0} e^{\mu_k t} \sum_{j=0}^{m_k-1} a_{k,j} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} u_{k,s}(x) + \sum_{|n|=M+1}^{\infty} e^{\mu_n t} a_n \mathcal{F}^{-1} y_n(x), \tag{6.10}$$

where $(t,x) \in \mathbb{R} \times (0,1)$.

Step 3. We need to make some modification on (6.10). If $\mu_k = \mu_\ell$ for some $k \neq \ell$ with $k, \ell \in \{1, \cdots, n_0\}$, then we rewrite the sum

$$\begin{aligned} & e^{\mu_k t} \sum_{j=0}^{m_k-1} a_{k,j} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} u_{k,s} + e^{\mu_\ell t} \sum_{j=0}^{m_\ell-1} a_{\ell,j} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} u_{\ell,s} \\ & = e^{\mu_k t} \sum_{j=0}^{\max(m_k,m_\ell)-1} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} (a_{k,j} u_{k,s} + a_{\ell,j} u_{\ell,s}), \end{aligned} \tag{6.11}$$

where we assume $a_{k,j} = 0$ if $j > m_k$ (or $a_{\ell,j} = 0$ if $j > m_\ell$).

Note that μ_n is an simple eigenvalue of $\widetilde{\mathcal{A}}$ when $|n| \geq M+1$, thus it is easy to see that $\mu_k \neq \mu_\ell$ for any $k \neq \ell$ with $|\ell| \geq M+1$. Thus, we may rewrite (6.10) as follows

$$u(t,x) = \sum_{k=1}^{k_0} e^{\tilde{\mu}_k t} \sum_{j=0}^{\tilde{m}_k-1} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_{k,j}^{\ell} \tilde{u}_{k,s}^{\ell}(x) + \sum_{k=k_0+1}^{\infty} e^{\tilde{\mu}_k t} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_k^{\ell} \tilde{u}_k^{\ell}(x) \tag{6.12}$$

for some integer $k_0 \geq 0$, where we have renumbered the eigenvalues $\{\mu_n\}_{n=1}^{n_0} \cup \{\mu_n\}_{|n|=M+1}^{\infty}$ of $\widetilde{\mathcal{A}}$ as $\{\tilde{\mu}_k\}_{k=1}^{\infty}$ such that

$$\tilde{\mu}_k \neq \tilde{\mu}_\ell \qquad \text{for any } k \neq \ell; \tag{6.13}$$

also we have renumbered the corresponding eigenvectors and/or generalized eigenvectors (6.8) of $\widetilde{\mathcal{A}}$ as

$$\{\tilde{u}_{k,0}^1, \cdots, \tilde{u}_{k,0}^{\hat{m}_k}, \cdots, \tilde{u}_{k,\tilde{m}_k-1}^1, \cdots, \tilde{u}_{k,\tilde{m}_k-1}^{\hat{m}_k}\}_{k=1}^{k_0} \cup \{\tilde{u}_k^1, \cdots, \tilde{u}_k^{\hat{m}_k}\}_{k=k_0+1}^{\infty} \tag{6.14}$$

with

$$\tilde{m}_k \in [2, \infty), \qquad k = 1, \cdots, k_0, \tag{6.15}$$

$$\hat{m}_k \in [1, \infty), \qquad k = 1, 2, \cdots \tag{6.16}$$

and

$$\hat{m}_k = 1 \qquad \text{(at least) for } k \text{ large enough.} \tag{6.17}$$

On the other hand, by Lemma 2.4, we know that $u \in C^\omega(\mathbb{R}; H_0^1(0,1))$. Therefore, by (1.2) (recall $F = (a,b)$), we obtain that

$$u(t,x) = 0 \qquad \text{in } \mathbb{R} \times (a,b). \tag{6.18}$$

Thus

$$\sum_{k=1}^{k_0} e^{\tilde{\mu}_k t} \sum_{j=0}^{\tilde{m}_k - 1} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_{k,j}^\ell \tilde{u}_{k,s}^\ell(x) + \sum_{k=k_0+1}^{\infty} e^{\tilde{\mu}_k t} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_k^\ell \tilde{u}_k^\ell(x) = 0,$$
$$(t,x) \in \mathbb{R} \times (a,b). \tag{6.19}$$

Step 4. Let us denote

$$\tilde{\mu}_k = \xi_k + i\eta_k, \qquad \xi_k, \eta_k \in \mathbb{R}, \tag{6.20}$$

where $k = 1, 2, \cdots$. Recall $\widetilde{\mathcal{A}}$ is a compact operator. Thus 0 is the only accumulation point of $\{\tilde{\mu}_k\}_{k=1}^\infty$. Note that $\{\tilde{\mu}_k\}_{k=1}^\infty$ is bounded. Thus, if $\xi_{\ell_1} > 0$ for some $\ell_1 \in \mathbb{N}$, we can find a $\tilde{\ell}_1 \in \mathbb{N}$ such that

$$\xi_{\tilde{\ell}_1} = \sup\{\xi_k;\ k \in \mathbb{N}\} > 0.$$

To simplify the presentation we assume that $\tilde{\ell}_1 = 1$. The proof works in the same way with obvious changes in the notation when it is not that way.

We distinguish the following two cases.

Case 1. $\xi_1 > \sup\{\xi_2, \xi_3, \cdots\}$. In this case, multiplying both sides of (6.19) by

$$e^{-\tilde{\mu}_1 t} t^{1-\tilde{m}_1},$$

we have

$$\begin{aligned}
&\frac{1}{(\tilde{m}_1 - 1)!} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_{1,\tilde{m}_1-1}^\ell \tilde{u}_{1,0}^\ell(x) \\
&= -\sum_{j=0}^{\tilde{m}_1-2} t^{1-\tilde{m}_1} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} \sum_{\ell=1}^{\hat{m}_1} \tilde{a}_{1,j}^\ell \tilde{u}_{1,s}^\ell(x) \\
&\quad - \sum_{k=2}^{k_0} e^{(\tilde{\mu}_k - \tilde{\mu}_1)t} t^{1-\tilde{m}_1} \sum_{j=0}^{\tilde{m}_k-1} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_{k,j}^\ell \tilde{u}_{k,s}^\ell(x)) \\
&\quad - \sum_{k=k_0+1}^{\infty} e^{(\tilde{\mu}_k - \tilde{\mu}_1)t} t^{1-\tilde{m}_1} \sum_{\ell=1}^{\hat{m}_k} \tilde{a}_k^\ell \tilde{u}_k^\ell(x) = 0, \quad (t,x) \in \mathbb{R} \times (a,b).
\end{aligned} \tag{6.21}$$

Thus, letting $t \to +\infty$ in (6.21), we get

$$\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,\tilde{m}_1-1}\tilde{u}^{\ell}_{1,0}(x) = 0, \qquad x \in (a,b). \tag{6.22}$$

Repeating the above argument, we get

$$\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,\tilde{m}_1-j}\tilde{u}^{\ell}_{1,0}(x) = 0, \qquad x \in (a,b);\ \ j = 1,\cdots,\tilde{m}_1. \tag{6.23}$$

Note that $\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,j}\tilde{u}^{\ell}_{1,0}$ are eigenvectors of $\widetilde{\mathcal{A}}$ ($j = 0,\cdots,\tilde{m}_1 - 1$). Thus, by Lemma 2.5 and (6.23), we get

$$\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,\tilde{m}_1-j}\tilde{u}^{\ell}_{1,0}(x) \equiv 0, \qquad x \in (0,1);\ \ j = 1,\cdots,\tilde{m}_1. \tag{6.24}$$

However, $\{\tilde{u}^{\ell}_{1,0}\}_{\ell=1}^{\hat{m}_1}$ is a subset of a Riesz basis. Thus

$$\tilde{u}^{1}_{1,0},\cdots,\tilde{u}^{\hat{m}_1}_{1,0}$$

are linearly independent, which implies

$$\tilde{a}^{\ell}_{1,j} = 0, \qquad \ell = 1,\cdots,\hat{m}_1;\ \ j = 0,\cdots,\tilde{m}_1 - 1. \tag{6.25}$$

Case 2. $\xi_1 = \sup\{\xi_2,\xi_3,\cdots\}$. In this case, for simplicity, let us assume that

$$\xi_1 = \xi_2 > \sup\{\xi_3,\xi_4,\cdots\}$$

(the other cases can be treated similarly). Also, without loss of generality, we assume that

$$\tilde{m}_1 \geq \tilde{m}_2.$$

Multiplying both sides of (6.19) by

$$e^{-\tilde{\mu}_1 t}t^{1-\tilde{m}_1},$$

as in (6.21), we see that for $(t,x) \in (a,b) \times \mathbb{R}$ it holds

$$\frac{1}{(\tilde{m}_1-1)!}\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,\tilde{m}_1-1}\tilde{u}^{\ell}_{1,0}(x) = -\frac{e^{i(\zeta_2-\zeta_1)t}t^{\tilde{m}_2-\tilde{m}_1}}{(\tilde{m}_2-1)!}\sum_{\ell=1}^{\hat{m}_2} \tilde{a}^{\ell}_{2,\tilde{m}_2-1}\tilde{u}^{\ell}_{2,0}(x) + o(1) \quad \text{as } t \to +\infty. \tag{6.26}$$

Thus, if $\tilde{m}_1 > \tilde{m}_2$, taking $t \to +\infty$ in (6.26), we get

$$\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^{\ell}_{1,\tilde{m}_1-1}\tilde{u}^{\ell}_{1,0}(x) = 0, \qquad x \in (a,b). \tag{6.27}$$

If $\tilde{m}_1 = \tilde{m}_2$, noting that (6.13) gives

$$\zeta_2 \neq \zeta_1,$$

thus by (6.26), we get

$$\begin{aligned}\sum_{\ell=1}^{\hat{m}_k} \tilde{a}^\ell_{1,\tilde{m}_1-1}\tilde{u}^\ell_{1,0}(x) &= \frac{1}{S}\lim_{S\to+\infty}\int_0^S \sum_{\ell=1}^{\hat{m}_k} \tilde{a}^\ell_{1,\tilde{m}_1-1}\tilde{u}^\ell_{1,0}(x)dt \\ &= \frac{1}{S}\lim_{S\to+\infty}\int_0^S \left[-e^{i(\zeta_2-\zeta_1)t}\sum_{\ell=1}^{\hat{m}_2}\tilde{a}^\ell_{2,\tilde{m}_2-1}\tilde{u}^\ell_{2,0}(x)+o(1)\right]dt \\ &= 0, \qquad x\in(a,b).\end{aligned} \tag{6.28}$$

Consequently, similar to case 1, we obtain the same conclusion as (6.25).

Now, repeating the above argument, we see that

$$\tilde{a}^\ell_{k,j} = 0, \qquad \ell = 1,\cdots,\hat{m}_k;\ j = 0,\cdots,\tilde{m}_k - 1 \tag{6.29}$$

whenever $\xi_k > 0$. The same argument but by letting $t \to -\infty$ allows to deduce (6.29) whenever $\xi_k < 0$.

It remains to show that (6.29) holds for any k such that $\xi_k = 0$. For this purpose, we note that by the above argument we may write

$$u(t,x) = \sum_{k=1}^{k_0'} e^{i\zeta_k' t}\sum_{j=0}^{\tilde{m}_k'-1}\sum_{s=0}^{j}\frac{t^{j-s}}{(j-s)!}\sum_{\ell=1}^{\hat{m}_k'}\tilde{a}^\ell_{k,j}\tilde{u}^\ell_{k,s}(x) + \sum_{k=k_0'+1}^{\infty} e^{i\zeta_k' t}\sum_{\ell=1}^{\hat{m}_k'}\tilde{a}^\ell_k\tilde{u}^\ell_k(x) \tag{6.30}$$

for some integers k_0', $\tilde{m}_k'$, $\hat{m}_k'$ and real number ζ_k', where $(t,x) \in (0,1)\times\mathbb{R}$. Furthermore, by (6.13), we see that

$$\zeta_k' \neq \zeta_\ell' \qquad \text{whenever } k \neq \ell. \tag{6.31}$$

By (6.31) and (1.2) (with $F = (a,b)$), similar to the above argument, it is easy to conclude that

$$\tilde{a}^\ell_{k,j} = 0, \qquad \ell = 1,\cdots,\hat{m}_k';\ j = 0,\cdots,\tilde{m}_k'-1;\ k = 1,\cdots,k_0'$$

and

$$\tilde{a}^\ell_k = 0, \qquad \ell = 1,\cdots,\hat{m}_k';\ k = k_0'+1, k_0'+2,\cdots.$$

Thus, we arrive at

$$u(t,x) \equiv 0, \qquad (t,x)\in\mathbb{R}\times(0,1).$$

This completes the proof of Theorem 1.3. □

7. Application to approximate controllability and stabilization of Benjamin-Bona-Mahony like equation

First of all, let us consider the following controlled Benjamin-Bona- Mahony like equation

$$\begin{cases} u_t - u_{txx} = [\alpha(x)u]_x + f\chi_F & \text{in } (0,T)\times(0,1),\\ u(t,0) = u(t,1) = 0 & t\in(0,T),\\ u(0) = u_0 & \text{in } (0,1), \end{cases} \tag{7.1}$$

where $f \in L^2((0,T)\times F)$ denotes the control and χ_F denotes the characteristic function of the set F where the control is localized.

By means of Hahn-Banach theorem and using our unique continuation result in Theorem 1.3, we have the following controllability result (We refer to [13] for a good introduction to this subject).

Theorem 7.1. *Let $T > 0$ be given. Let $\alpha(\cdot)$ and F satisfy the assumptions in Theorem 1.3. Then (7.1) is approximately controllable in $L^2(0,1)$, i.e., for every pair of data u_0, u_1 in $L^2(0,1)$ and every $\varepsilon > 0$, there exists a control $f \in L^2((0,T)\times F)$ such that the solution of (7.1) satisfies*

$$|u(T) - u_1|_{L^2(0,1)} \le \varepsilon.$$

Next, let us give an application of our unique continuation theorem to the stabilization of the following Benjamin-Bona-Mahony like equation

$$\begin{cases} u_t - u_{txx} = [\alpha(x)u]_x - \chi_F(x)u & \text{in } (0,\infty)\times(0,1),\\ u(t,0) = u(t,1) = 0 & t\in(0,\infty),\\ u(0) = u_0 & \text{in } (0,1). \end{cases} \tag{7.2}$$

We denote the energy of solution of (7.2) by

$$E(u(t)) \triangleq \int_0^1 [u^2(t,x) + u_x^2(t,x)]dx. \tag{7.3}$$

We have the following results.

Theorem 7.2. *Let $\alpha \in W(0,1)\cap H^1(0,1)$, $1/\alpha \in L^1(0,1)$ and*

$$\alpha'(x) \le 0 \quad \text{a.e. } (0,1). \tag{7.4}$$

Let $0 < a < b < 1$, and $F = (0,a)\cup(b,1)$. Then for any $u_0 \in H_0^1(0,1)$, $u(t)$ tends to 0 in $H_0^1(0,1)$ weakly as $t\to\infty$.

Furthermore, if $u_0 \in U_2$, where U_2 is the subspace in $H_0^1(0,1)$ spanned by the following space

$$V_2 \triangleq \left\{ v_0 \in H_0^1(0,1)\cap H^2(0,1);\ \int_0^1 e^{-\int_0^x \frac{\chi_F(s)}{\alpha(s)}ds}[v_0(x) - v_{0,xx}(x)]dx = 0 \right\}, \tag{7.5}$$

then $E(u(t))$ tends to 0 as $t\to\infty$.

If we impose more regularity conditions on $\alpha(\cdot)$, we have the following better result.

Theorem 7.3. *Let $\alpha(\cdot)$ and F satisfy the assumptions in Theorem 1.3. Let (7.4) hold. Then for any $u_0 \in H_0^1(0,1)$, $E(u(t))$ tends to 0 as $t \to \infty$.*

Remark 7.1. We note that the condition (7.4) is almost necessary for stabilization. In fact, by (7.2), it is easy to check that

$$E(u(t)) = E(u_0) + \int_0^t \int_0^1 \alpha'(x) u^2(s,x)dxds - 2\int_0^t \int_F u^2(s,x)dxds. \quad (7.6)$$

Now, if we take

$$\alpha(x) = 2(1+x), \qquad F = (0,1), \quad (7.7)$$

we get

$$E(u(t)) \equiv E(u(0)).$$

Thus, the energy does not tend to zero.

Remark 7.2. Whether the second assertion in Theorem 7.2 holds for any $u_0 \in H_0^1(0,1)$ or not is an open problem.

In order to prove Theorem 7.2, we will use LaSalle's invariance principle. For this purpose, we need the following lemma, which has its independent interest.

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary Γ. Then for any $0 \neq \gamma(\cdot) \in L^2(\Omega)$, there exist a $1-d$ subspace V_1 and a subspace V_2 in $H_0^1(\Omega) \cap H^2(\Omega)$ such that $H_0^1(\Omega) \cap H^2(\Omega)$ is the direct sum of V_1 and V_2, and for any $u_0 \in V_2$, it holds*

$$\int_\Omega \gamma(x)[u_0(x) - \Delta u_0(x)]dx = 0. \quad (7.8)$$

Proof. We denote by L_1 the $1-d$ subspace spanned by $\gamma(\cdot)$ in $L^2(\Omega)$, and by L_2 its orthogonal complement in $L^2(\Omega)$. It is well-known that $L^2(\Omega)$ is the direct sum of L_1 and L_2. Now, for any $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, we denote

$$f \stackrel{\triangle}{=} u_0 - \Delta u_0 (\in L^2(\Omega)). \quad (7.9)$$

Therefore, we can find $f_i \in L_i$ $(i = 1, 2)$ such that

$$f = f_1 + f_2. \quad (7.10)$$

We solve the following elliptic equation of second order $(i = 1, 2)$:

$$\begin{cases} u_i - \Delta u_i = f_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \Gamma. \end{cases} \quad (7.11)$$

We then obtain a unique solution $u_i \in H_0^1(\Omega) \cap H^2(\Omega)$. It is easy to see that

$$u_0 = u_1 + u_2. \tag{7.12}$$

Denote

$$V_i = \{u_i \in H_0^1(\Omega) \cap H^2(\Omega);\ \exists f_i \in L_i \text{ such that (7.11) holds}\}. \tag{7.13}$$

Then the solution of (7.11) satisfy $u_i \in V_i$ $(i = 1, 2)$. It remains to check that the sum $H_0^1(\Omega) \cap H^2(\Omega) = V_1 + V_2$ is direct. For this purpose, we suppose $f \in V_1 \cap V_2$. Then, we get

$$f - \Delta f \in L_1 \cap L_2. \tag{7.14}$$

Thus

$$\begin{cases} f - \Delta f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma, \end{cases} \tag{7.15}$$

which gives $f = 0$. □

From now on, let us denote

$$\gamma = \gamma(x) \stackrel{\triangle}{=} e^{-\int_0^x \frac{\chi_F(s)}{\alpha(s)} ds}, \tag{7.16}$$

where α and F are given in Theorem 7.2. Then, by Lemma 7.1, we decompose $H_0^1(0,1) \cap H^2(0,1)$ as the direct sum of V_1 and V_2. It is easy to see that in this case V_2 is exactly the space given by (7.5). The following lemma is crucial in the proof of Theorem 7.2.

Lemma 7.2. *Let the assumptions on α and F in Theorem 7.2 hold. Let V_2 be given by (7.5). Then for any $u_0 \in V_2$, it holds*

$$\sup_{t\in[0,\infty)} |u(t)|_{H_0^1(0,1)\cap H^2(0,1)} < \infty, \tag{7.17}$$

where $u(\cdot)$ is the solution of (7.2).

Proof. First of all, let us solve the following ODE

$$\begin{cases} (\alpha\phi)_x - \chi_F \phi = u_0 - u_{0,xx}, & 0 < x < 1, \\ \phi(0) = 0. \end{cases} \tag{7.18}$$

We get

$$\phi(x) = \frac{1}{\alpha(x)} \int_0^x \gamma(u_0 - u_{0,xx}) ds, \tag{7.19}$$

where γ is given by (7.16). From the fact that $u_0 \in V_2$, we see that

$$\phi(1) = 0. \tag{7.20}$$

Next, we introduce the following key transform:

$$v = v(t, x) \triangleq \int_0^t u(s, x)ds + \phi(x). \tag{7.21}$$

Then, by (7.2), (7.18) and (7.20), we see that $v(\cdot)$ solves

$$\begin{cases} v_t - v_{txx} = [\alpha(x)v]_x - \chi_F(x)v & \text{in } (0, \infty) \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & t \in (0, \infty), \\ v(0) = \phi & \text{in } (0, 1). \end{cases} \tag{7.22}$$

Thus, we have

$$\sup_{t\in[0,\infty)} |v(t)|_{H_0^1(0,1)} < \infty. \tag{7.23}$$

Consequently, by $u = v_t$ and (7.22)–(7.23), we get (recall (2.1) for A)

$$\begin{aligned} &\sup_{t\in[0,\infty)} |u(t)|_{H_0^1(0,1)\cap H^2(0,1)} = \sup_{t\in[0,\infty)} |v_t(t)|_{H_0^1(0,1)\cap H^2(0,1)} \\ &= \sup_{t\in[0,\infty)} |(I - A)^{-1}\Big((\alpha v)_x - \chi_F v\Big)|_{H_0^1(0,1)\cap H^2(0,1)} \\ &\leq C \sup_{t\in[0,\infty)} |v(t)|_{H_0^1(0,1)} < \infty, \end{aligned} \tag{7.24}$$

which completes the proof of Lemma 7.2. □

Proof of Theorem 7.2. We prove only the second assertion (The first assertion follows almost immediately from the boundedness of trajectories and the unique continuation result in Theorem 1.2). By (7.4) and (7.6), it is obvious that the corresponding C_0-semigroup $S(t)$ (in $H_0^1(0, 1)$) of (7.2) is contractive. It suffices to consider $u_0 \in V_2$ and to prove that the $\omega-$limit set of u_0 defined by

$$\omega(u_0) \triangleq \{v \in H_0^1(0, 1);\ \lim_{n\to\infty} |S(t_n)u_0 - v|_{H_0^1(0,1)} = 0 \text{ for some } t_n \to \infty\} \tag{7.25}$$

is equal to $\{0\}$. First of all, by Lemma 7.2, it is easy to see that $\cup_{t\geq 0}S(t)u_0$ is precompact set in $H_0^1(0, 1)$. Hence $\omega(u_0) \neq \emptyset$. Next, let us choose $v_0 \in \omega(u_0)$. Then

$$|v_0|_{H_0^1(0,1)} = |S(t)u_0|_{H_0^1(0,1)} \qquad \text{for all } t \geq 0. \tag{7.26}$$

Thus, combining (7.6) and (7.26), and noting (7.4), we get

$$v(t, x) = 0 \qquad (t, x) \in (0, \infty) \times F, \tag{7.27}$$

where v is the solution of (7.2) with u_0 replaced by v_0. Therefore, by (7.27), we get v satisfies

$$\begin{cases} v_t - v_{txx} = [\alpha(x)v]_x & \text{in } (0, \infty) \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & t \in (0, \infty), \\ v(0) = v_0 & \text{in } (0, 1). \end{cases} \tag{7.28}$$

However, by (7.27)–(7.28) and Theorem 1.2, we get

$$v \equiv 0. \tag{7.29}$$

Thus $v_0 = 0$, which completes the proof of Theorem 7.2. □

Proof of Theorem 7.3. We use Theorem 5.4 with $\beta(x) = -\chi_F(x)$. Recall the operator $\widetilde{\mathcal{A}}$ defined by (2.11). By Theorem 5.4, we know that one can find a sequence of generalized eigenvectors $\{u_{n,0}, \cdots, u_{n,m_n-1}\}_{n=1}^{\infty}$ of $\widetilde{\mathcal{A}}$ which forms a Riesz basis of $H_0^1(0,1)$, where $u_{n,0}, \cdots, u_{n,m_n-1}$ is the associated Jordan chain of $\widetilde{\mathcal{A}}$ with respect to eigenvalue μ_n, m_n is its algebraic multiplicity which is uniformly bounded with respect to $n \in \mathbb{N}$. Thus, for any $u_0 \in H_0^1(0,1)$, we may decompose u_0 as

$$u_0 = \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} a_{n,j} u_{n,j}, \qquad a_{n,j} \in \mathbb{C} \text{ for } j = 0, \cdots, m_n - 1; n \in \mathbb{N}.$$

Therefore, the solution of (7.2) can be expressed as follows

$$u(t) = \sum_{n=1}^{\infty} e^{\mu_n t} \sum_{j=0}^{m_n-1} a_{n,j} \sum_{s=0}^{j} \frac{t^{j-s}}{(j-s)!} u_{n,s}. \tag{7.30}$$

However, by (7.4) and (7.6), it is obvious that the energy $E(u(t))$ of (7.2) is decreasing. Thus, by (7.30), we see that

$$\operatorname{Re} \mu_n < 0 \qquad \text{for all } n \in \mathbb{N}. \tag{7.31}$$

Now, by (7.30)–(7.31), and noting again that $\{u_{n,0}, \cdots, u_{n,m_n-1}\}_{n=1}^{\infty}$ is a Riesz basis of $H_0^1(0,1)$, we obtain the desired result immediately. □

Remark 7.3. (7.31) and the asymptotic formula (4.40) in Theorem 4.4 show that the energy decay rate of system (7.2) is not uniform. But they allow to prove polynomial decay rates for the energy $E(u(t))$ of smooth initial data. In order to get an uniform exponentially decay rate for $E(u(t))$, one should choose another damping mechanism of the form

$$\left(\chi_F(x) u_x\right)_x$$

rather that $-\chi_F(x)u$ in (7.2) in order to guarantee that the energy of the solution is dissipated at a rate which is proportional to the H^1 norm of the restriction of the solution to F. But this is also an open problem except the trivial case $F = (0,1)$.

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