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OBSERVABILITY OF HEAT PROCESSES BY TRANSMUTATION WITHOUT GEOMETRIC RESTRICTIONS

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Abstract. The goal of this note is to explain how transmutation techniques (originally introduced in [14] in the context of the control of the heat equation, inspired on the classical Kannai transform, and recently revisited in [4] and adapted to deal with observability problems) can be applied to derive observability results for the heat equation without any geometric restriction on the subset in which the control is being applied, from a good understanding of the wave equation. Our arguments are based on the recent results in [15] on the frequency depending observability inequalities for waves without geometric restrictions, an iteration argument recently developed in [13] and the new representation formulas in [4] allowing to make a link between heat and wave trajectories.

1. Introduction.

Let Ω be a bounded smooth domain and ω be an open subset of Ω. We consider the heat equation with state z

$$\left\{ \begin{array}{ll} \partial_t z - \Delta_x z = 0, & (t,x) \in \mathbb{R}_+^* \times \Omega, \\ z = 0, & (t,x) \in \mathbb{R}_+^* \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega. \end{array} \right. \tag{1.1}$$

Our goal is to develop an alternate proof of the following well known result:

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Theorem 1.1. *Let Ω be a bounded smooth domain and ω be an open subset of Ω.*

Then for any time $T > 0$, there exists a constant C_T such that any solution z of (1.1) with initial data in $L^2(\Omega)$ satisfies

$$\|z(T)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \int_\omega |z(t,x)|^2 \, dtdx. \tag{1.2}$$

The well known estimate (1.2) ([5, 8]) is the so-called observability inequality for the heat equation. Such estimate is of primary importance when dealing with controllability properties of heat equations with controls in $L^2((0,T) \times \omega)$ acting in ω, see e.g. [12].

Here, our main goal consists in deriving a new proof complementing the existing results making the link between the observability of wave and heat equations. Hence, before describing our approach, we shall first present the proofs of Theorem 1.1 in [5] and [8]. We shall also mention and comment the approach in [19] which consists in seeing the heat equation as a singular limit of dissipative wave equations.

The article [5] uses a global Carleman estimate derived directly on the parabolic operator, that we shall not comment extensively here.

The other approach developed in [8] (see also [10]), consists in estimating the cost of controllability on the first eigenfunctions of the laplacian, and then using the strong dissipativity of the heat semigroup to guarantee the existence of a control for the time evolution heat equation. The proof in [8] uses an integral transform making the link between finite eigenfunction clusters of the laplacian and solutions of the elliptic equation

$$\begin{cases} -\partial_{\tau\tau} w - \Delta_x w = 0, & (\tau, x) \in \mathbb{R}_+^* \times \Omega, \\ w(0,x) = 0, & x \in \Omega, \\ w(\tau, x) = 0 & (\tau, x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{1.3}$$

A quantification of the unique continuation property for (1.3), depending on the frequency function, obtained through Carleman estimates, allows then to estimate the cost of controlling the first modes for the heat equation.

Let us be more precise on that point, which is closely related to the approach we develop here. First, since $A = -\Delta$ defined on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H^2 \cap H^1_0(\Omega)$ is a self-adjoint positive definite operator with compact resolvent, we can write its spectral decomposition $A\Phi_j = \mu_j \Phi_j$, where the set of $(\Phi_j)_{j\in\mathbb{N}}$ forms an orthonormal basis of $L^2(\Omega)$ and μ_j is the increasing positive sequence (with multiplicity) formed by the eigenvalues of the operator A. Now, for $\lambda > 0$, we introduce the low frequency subspace

$$V_\lambda = \text{Span}\{\Phi_j, \quad \text{such that } \sqrt{\mu_j} \leq \lambda\}. \tag{1.4}$$

The results in [8] (revisited in [10]) show the following estimate: There exist positive constants C, a such that, for all $\lambda > 0$, all functions $\phi \in V_\lambda$ satisfy

$$\int_\Omega |\phi|^2 \, dx \leq C e^{a\lambda} \int_\omega |\phi|^2 \, dx, \quad \phi \in V_\lambda. \tag{1.5}$$

As explained in [8], this non-trivial estimate, obtained by Carleman estimates for (1.3), shows that, for the heat equation (1.1), controlling the projection of solutions over V_λ can be done with a cost of order $\exp(a\lambda)/T$ which, of course, diverges as $\lambda \to \infty$. But then the dissipation mechanism of the heat equation damps out the solution with a multiplicative factor $\exp(-C\lambda^2 T)$ and an iteration argument can be developed, dividing the time interval $(0,T)$ into subintervals and controlling uniformly an increasing number of frequencies, to eventually prove the uniform control of the whole heat flow in any time T and without any constraint on the geometry of the control subdomain ω, as stated in the main Theorem above.

In some sense, the approach in [19] (see also [11]) lies in between the direct approach based on Carleman estimates developed in [5] and the iteration argument developed in [8]. The idea is to consider the heat equation as the singular limit of dissipative wave equation, and to distinguish between low-frequencies, that are controlled in the beginning of the time interval, and high-frequencies, that are controlled at the end of the time interval, after having been damped out significantly due to the dissipation mechanism.

As mentioned earlier the main object of this paper is to make the link of the existing observability results for the wave and the heat equation in a way so to produce a new proof of the main Theorem above. This has been done previously in various manners but always under the condition that the wave equation is also observable, a fact that does not hold in the general context we are considering here, without imposing some conditions on the control subregion.

For instance, in [4] (see also [14] for the dual control point of view) the observability of the heat equation has been shown to be a consequence of the property of observability of the wave equation

$$\begin{cases} \partial_{ss}y - \Delta_x y = 0, & (s,x) \in \mathbb{R}\times\Omega, \\ y = 0, & (s,x) \in \mathbb{R}\times\partial\Omega, \\ y(0,x) = y_0(x), \partial_s y(0,x) = y_1(x), & x\in\Omega \end{cases} \tag{1.6}$$

which reads as follows: There exist a time $S > 0$ and a positive constant C such that all solutions y satisfy

$$\|(y(0,\cdot), \partial_s y(0,\cdot))\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \le C \int_{-S}^{S}\int_\omega |\partial_s y|^2\, ds dx. \tag{1.7}$$

Here, we have chosen to denote the time variable for the waves by s, as it will be interesting in the sequel to distinguish between the time of the heat process and that of the wave equation.

Note however that, for (1.7) to hold, some geometric restrictions have to be imposed on the observation subdomain ω, the so called Geometric Control Condition (GCC) (see [1, 3]). Thus this approach can not be applied directly in the present setting to derive the result for the heat equation on the generality of the main Theorem above. The method developed in [4] is inspired by the transmutation technique developed in [14] linking the control properties of the wave equation and those of the heat equation. These two techniques, though they might seem reverse one from another, can also be seen as dual versions one from another.

Also note that the first result linking control/observation properties for heat and wave equations is due to Russell [18] who applied the method of moments.

Roughly speaking, all the existing results and methods linking control/observation properties of wave and heat equations require the wave equation (1.6) to be observable in some time $2S$, a fact which is well-known to hold if and only if the GCC is satisfied so that all the rays of Geometric Optics meet the domain ω in a time strictly less than $2S$. Note that, in our simple context of waves with velocity of propagation normalized to one, the rays of Geometric Optics simply are straight lines bouncing on the boundary according to Descartes-Snell's laws. We refer to [1] for a more precise definition of these rays.

Our goal is to provide a new way to deduce (1.2) from the observability properties of the wave equation (1.6), allowing to get rid of those geometric assumptions and yielding an alternate proof to the main Theorem above. Our approach uses three ingredients that have been developed very recently and that we briefly present now.

The first one is the representation formula in [4], allowing to transform the solutions of the heat equation (1.1) into solutions of the free wave equation (1.6). This is the reverse version of the classical Kannai formula that has been systematically developed in [14] in the control setting. The approach in [4] has already allowed us to prove some new estimates on the cost of observability of the heat equation when spectral observability holds. The goal of this paper is to derive such estimates even in those cases in which this spectral observability inequality for the wave equation is unknown and, in this way, to some extent, to fully clarify the connections between the wave and the heat equations at the level of the observability properties.

The second one is the existing observability results for the wave equation in general geometries, and in particular without the GCC. Of course, (1.7) cannot hold in such a general setting, and the known weaker observability inequalities depend on the frequency function as proved in the pioneer works in that direction: [16, 7, 17]. Here we shall rather use the more recent improved version in [15]. All these results use the Fourier Bros Iagoniltzer (FBI) transform making the link between the wave equation (1.6) and the elliptic equation (1.3).

The third one is the iteration argument developed in [13] for deducing the observability (1.2) of the heat equation (1.1) from (1.5). This can be seen as a dual formulation of the iteration argument originally developed in [8] for the control problem.

This note is organized as follows. In Section 2 we recall the results in [15] on the observability of waves in general situations, the transmutation technique developed in [4] and a lemma derived in [13]. In Section 3, we show how these ingredients can be combined to prove the observability inequality (1.2) for solutions of the heat equation (1.1). We finally provide the reader with some further comments in Section 4.

2. **Ingredients of our proof.**

In this section, we recall the results of [15] on the observability of the wave equation (1.6), the transmutation technique developed in [4] and a useful lemma obtained in [13].

2.1. **An observability result for the wave equation in general geometries.**

According to [15], we have the following:

Theorem 2.1 ([15])**.** *Let Ω be a bounded smooth domain, ω be an open subset of Ω and $\varepsilon > 0$.*

Then there exist a time $S > 0$ and constants C and b so that every solutions y of (1.6) with initial data in $H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$ satisfy

$$\|(y_0, y_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C e^{b\Lambda^{1+\varepsilon}} \int_{-S}^{S} \int_{\omega} |\partial_s y|^2 \, ds dx, \tag{2.1}$$

where Λ is the frequency function, given by

$$\Lambda = \frac{\|(y_0, y_1)\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}}{\|(y_0, y_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}. \tag{2.2}$$

As we have said, the proof of Theorem 2.1 is based on the Fourier Bros Iagoniltzer (FBI) transform of [9], on the three-spheres inequality for the elliptic equation (1.3) and some interpolation arguments on the elliptic equation (1.3).

Remark 1. Note that, with $\varepsilon = 1$, this result has already been stated in [17, Theorem 1] for the boundary case and later, in [2, Proposition 2.1] using a more direct proof based on the interpolation estimates in [8] for the elliptic equation (1.3).

In these works, the approach is based on the FBI transform corresponding to a quadratic phase, whereas the proof of Theorem 2.1 in [15] uses the FBI transform corresponding to a polynomial phase, namely the one given in [9, p.473–474].

Remark 2. Note that the results in [15] hold for bounded domains Ω being either C^2 or convex.

Remark 3. The time $2S$ in Theorem 2.1 is *a priori* much larger than the time of unique continuation for waves, which, by Holmgren's Uniqueness Theorem (see [6]), corresponds to

$$2S^* = 2\max\{d(x,\omega),\, x \in \Omega\}.$$

Whether the same estimates hold for this sharp value of time is an open problem.

2.2. **A transmutation technique.** In [4], we have built an integral transform associating to any solution z of the heat equation (1.1), a solution of the wave equation (1.6). Let us briefly explain how this was done. The first step is the construction of the following heat kernel:

Proposition 1 ([4]). *Given $T > 0$ and $S > 0$, for any $\alpha > 2S^2$, there exists a function $k_T = k_T(t,s)$ such that*

$$\begin{cases} \partial_t k_T(t,s) + \partial_{ss} k_T(t,s) = 0, & t \in (0,T),\ s \in (-S,S) \\ k_T(0,s) = 0, & s \in (-S,S) \\ k_T(T,s) = 0, & s \in (-S,S), \end{cases} \tag{2.3}$$

and

$$k_T(t,0) = 0, \quad t \in (0,T), \qquad \partial_s k_T(t,0) = \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right). \tag{2.4}$$

Moreover, for all $\delta \in (0,1)$, k_T satisfies the following estimates for $(t,s) \in (0,T) \times (-S,S)$

$$|k_T(t,s)| \leq |s| \exp\left(\frac{1}{\min\{t, T-t\}}\left(\frac{s^2}{\delta} - \frac{\alpha}{(1+\delta)}\right)\right), \tag{2.5}$$

$$|\partial_s k_T(t,s)| \leq \exp\left(\frac{1}{\min\{t, T-t\}}\left(\frac{s^2}{\delta} - \frac{\alpha}{(1+\delta)}\right)\right). \tag{2.6}$$

Then, according to [4],

Proposition 2 ([4]). *Given $\alpha > 0$ and k_T the kernel function given by Proposition 1, if z is a solution of the heat equation (1.1), the function*

$$y(s) = \int_0^T k_T(t,s) z(t)\, dt \tag{2.7}$$

is a solution of the wave equation (1.6) *on* $(-S, S)$ *for* $S < \sqrt{\alpha/2}$ *with initial data*

$$\begin{aligned}(y_0, y_1) &= \left(0, \int_0^T \partial_s k_T(t,0) z(t)\, dt\right) \\ &= \left(0, \int_0^T \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right) z(t)\, dt\right).\end{aligned} \tag{2.8}$$

2.3. **A useful lemma.** We now recall the following lemma:

Lemma 2.2 ([13])**.** *Let* $f = f(t)$ *be a strictly positive function of time* t *satisfying*

$$\lim_{t\to 0} f(t) = 0. \tag{2.9}$$

Further assume that there exist a constant C_* *and a time* $T^* > 0$ *such that for all time* $T \in (0, T^*)$*, for all* $z_0 \in L^2(\Omega)$ *and* z *the corresponding solution of* (1.1)*,*

$$f(T)\, \|z(T)\|^2_{L^2(\Omega)} - f\left(\frac{T}{2}\right) \|z_0\|^2_{L^2(\Omega)} \le C_* \int_0^T \int_\omega |z(t,x)|^2\, dtdx. \tag{2.10}$$

Then for all time $T \in (0, T^*)$ *and* $z_0 \in L^2(\Omega)$*,*

$$f\left(\frac{T}{2}\right) \|z(T)\|^2_{L^2(\Omega)} \le C_* \int_0^T \int_\omega |z(t,x)|^2\, dtdx. \tag{2.11}$$

Note that Lemma 2.2 is a special case of Lemma 2.1 in [13], which has been derived there with a lot of generality to improve existing constants on the cost of controllability for the heat equation in small time.

For the sake of completeness let us briefly indicate the proof of this simplified version.

Proof. Let $T < T^*$. Let $T_0 = T$ and set, for $k \in \mathbb{N}$,

$$\tau_k = \frac{T}{2^{k+1}} \quad \text{and} \quad T_{k+1} = T_k - \tau_k = \frac{T}{2^{k+1}}.$$

Applying (2.10) to z between the times T_{k+1} and T_k, we obtain

$$f(\tau_k)\, \|z(T_k)\|^2_{L^2(\Omega)} - f\left(\frac{\tau_k}{2}\right) \|z(T_{k+1})\|^2_{L^2(\Omega)} \le C_* \int_{T_{k+1}}^{T_k} \int_\omega |z(t,x)|^2\, dtdx.$$

But $\tau_k/2 = \tau_{k+1}$. Hence, since $f(\tau_{k+1})\, \|z(T_{k+1})\|^2_{L^2(\Omega)}$ goes to zero by (2.9), summing up these estimates for k from 0 to ∞, we obtain

$$f(\tau_0)\, \|z(T)\|^2_{L^2(\Omega)} \le C_* \int_0^T \int_\omega |z(t,x)|^2\, dtdx,$$

which proves (2.11). □

3. **A new proof on the observability estimate for the heat equation.** We shall begin with the following lemma:

Lemma 3.1. *Let Ω be a bounded smooth domain and ω an open subset of Ω.*

For any $\varepsilon > 0$ and $\lambda > 0$, there exist positive constants C, γ and b (independent of time T) such that for all $T > 0$ and all solutions z of (1.1) with initial data $z_0 \in V_\lambda$,

$$\|z(T)\|^2_{L^2(\Omega)} \leq \frac{C}{T^2} \exp\left(b\lambda^{1+\varepsilon} + \frac{\gamma}{T}\right) \int_0^T \int_\omega |z(t,x)|^2 \, dtdx. \tag{3.1}$$

Proof. Applying Theorem 2.1 we deduce that there exists a time S and some constants C and b so that the frequency depending inequality (2.1) holds for the wave equation. Let $\alpha > 2S^2$ and k_T be the kernel given by Proposition 1.

Let $z_0 \in V_\lambda$. Applying the transmutation technique, according to Proposition 2 and (2.8), we obtain a trajectory y of the wave equation (1.6) on $(-S, S)$ with initial data $y_0 = 0$ and

$$y_1 = \int_0^T \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right) z(t) \, dt. \tag{3.2}$$

Hence

$$\begin{aligned}
\|y_1\|^2_{L^2(\Omega)} &= \sum_j \left(\int_0^T \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right) e^{-\mu_j t} \, dt\right)^2 |a_j|^2 \\
&\geq \sum_j \left(\int_0^T \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right) dt\right)^2 e^{-2\mu_j T} |a_j|^2 \\
&\geq \|z(T)\|^2_{L^2(\Omega)} \left(\int_0^T \exp\left(-\alpha\left(\frac{1}{t} + \frac{1}{T-t}\right)\right) dt\right)^2 \\
&\geq \|z(T)\|^2_{L^2(\Omega)} \frac{T^2}{9} \exp\left(-\frac{9\alpha}{T}\right).
\end{aligned} \tag{3.3}$$

Besides, computing Λ defined in (2.2), we obtain

$$\Lambda = \frac{\|y_1\|_{H^1_0(\Omega)}}{\|y_1\|_{L^2(\Omega)}} \leq \lambda, \tag{3.4}$$

since y_1 belongs to V_λ.

Finally, using the estimate (2.6), one easily checks that there exists some constant C such that

$$\int_{-S}^S \int_\omega |\partial_s y(s,x)|^2 \, dsdx \leq C \int_0^T \int_\omega |z(t,x)|^2 \, dtdx. \tag{3.5}$$

Combining estimates (3.3)-(3.4)-(3.5), we deduce (3.1) immediately from (2.1). □

We are now in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $T \in (0,1)$ and $\varepsilon \in (0,1)$.

Then, according to Lemma 3.1, estimate (3.1) holds for all solutions of the heat equation (1.1) with initial data $z_0 \in V_\lambda$. Since $T \in (0,1)$, let us remark that estimate (3.1) implies that there exists a constant C independent of $T \in (0,1)$ such that for all $T \in (0,1)$, for all $\lambda > 0$, for all solutions z of (1.1) with initial data $z_0 \in V_\lambda$,

$$\|z(T)\|^2_{L^2(\Omega)} \leq C \exp\left(b\lambda^{1+\varepsilon} + \frac{\gamma}{T^\beta}\right) \int_0^T \int_\omega |z(t,x)|^2 \, dtdx, \tag{3.6}$$

where $\beta = (1+\varepsilon)/(1-\varepsilon)(>1)$.

Let $z_0 \in L^2(\Omega)$ and z be the corresponding solution of (1.1). For $\lambda > 0$, denote by $\mathbb{P}_\lambda$ the $L^2(\Omega)$-orthogonal projection on V_λ.

For $\lambda > 0$ that we will chose later, set

$$z_\lambda(t) = \mathbb{P}_\lambda z(t), \quad w_\lambda(t) = z - z_\lambda(t).$$

Then z_λ is a solution of the heat equation (1.1) with initial data lying in V_λ. Therefore, applying (3.6) between the times $T/2$ and T, we deduce

$$\|z_\lambda(T)\|^2_{L^2(\Omega)} \leq C \exp\left(b\lambda^{1+\varepsilon} + \frac{2^\beta\gamma}{T^\beta}\right) \int_{T/2}^T \int_\omega |z_\lambda(t,x)|^2 \, dtdx. \tag{3.7}$$

Of course,

$$\begin{aligned}
&\int_{T/2}^T \int_\omega |z_\lambda(t,x)|^2 \, dtdx \\
&\quad\leq 2\int_{T/2}^T \int_\omega |z(t,x)|^2 \, dtdx + 2\int_{T/2}^T \int_\omega |w_\lambda(t,x)|^2 \, dtdx. \\
&\quad\leq 2\int_0^T \int_\omega |z(t,x)|^2 \, dtdx + 2\int_{T/2}^T \exp(-2\lambda^2 t) \|w_\lambda(0)\|^2_{L^2(\Omega)} \, dt \\
&\quad\leq 2\int_0^T \int_\omega |z(t,x)|^2 \, dtdx + \frac{1}{\lambda^2}\exp(-\lambda^2 T) \|w_\lambda(0)\|^2_{L^2(\Omega)} \\
&\quad\leq 2\int_0^T \int_\omega |z(t,x)|^2 \, dtdx + \frac{1}{\lambda^2}\exp(-\lambda^2 T) \|z_0\|^2_{L^2(\Omega)}, \qquad (3.8)
\end{aligned}$$

where we have used successively that $w_\lambda(t)$ lies in $V_\lambda^\perp$ and hence

$$\|w_\lambda(t)\|_{L^2(\Omega)} \leq \exp(-\lambda^2 t) \|w_\lambda(0)\|_{L^2(\Omega)}, \quad t \geq 0,$$

and that $\|w_\lambda(0)\|_{L^2(\Omega)} \leq \|z_0\|_{L^2(\Omega)}$.

Besides, we obviously have

$$\begin{aligned}
\|z(T)\|^2_{L^2(\Omega)} &\leq 2\|z_\lambda(T)\|^2_{L^2(\Omega)} + 2\|w_\lambda(T)\|^2_{L^2(\Omega)} \\
&\leq 2\|z_\lambda(T)\|^2_{L^2(\Omega)} + 2\exp(-2\lambda^2 T)\|z(0)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, plugging (3.8) in (3.7), we obtain, for some C independent of time $T \in (0,1)$,

$$\begin{aligned}
&\exp\left(-b\lambda^{1+\varepsilon} - \frac{2^\beta\gamma}{T^\beta}\right)\|z(T)\|^2_{L^2(\Omega)} \leq C\int_0^T \int_\omega |z(t,x)|^2 \, dtdx \\
&\quad + C\exp(-\lambda^2 T)\left(\frac{1}{\lambda^2} + \exp(-\lambda^2 T)\exp\left(-b\lambda^{1+\varepsilon} - \frac{2^\beta\gamma}{T^\beta}\right)\right)\|z_0\|^2_{L^2(\Omega)}. \qquad (3.9)
\end{aligned}$$

Of course, using that λ is necessarily larger than $\mu_0^2 > 0$ (otherwise V_λ is empty), we obtain a constant C_0 independent of $T \in (0,1)$ such that

$$\exp\left(-b\lambda^{1+\varepsilon} - \frac{2^\beta \gamma}{T^\beta}\right) \|z(T)\|^2_{L^2(\Omega)} - C_0 \exp(-\lambda^2 T) \|z_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |z(t,x)|^2 \, dt dx. \quad (3.10)$$

Let us then choose λ to prove estimate (2.10) for some function f.

For doing this, we set the threshold λ as

$$\lambda^{1+\varepsilon} = \frac{\delta}{T^\beta}, \quad (3.11)$$

where δ will be chosen later on. Then

$$\exp\left(-b\lambda^{1+\varepsilon} - \frac{2^\beta \gamma}{T^\beta}\right) = \exp\left(-\frac{1}{T^\beta}\left(b\delta + 2^\beta \gamma\right)\right)$$

whereas, since $T \in (0,1)$,

$$C_0 \exp(-\lambda^2 T) \leq \exp\left(-\frac{\delta^{2/(1+\varepsilon)}}{T^\beta} + \frac{\ln(C_0)}{T^\beta}\right).$$

We then choose $\delta > 0$ large enough such that

$$2^{-\beta}\left(\delta^{2/(1+\varepsilon)} - \ln(C_0)\right) = \left(b\delta + 2^\beta\gamma\right) := A.$$

Note that this requirement defines δ independently of the time $T \in (0,1)$, thus making the restriction $\lambda \geq \mu_0^2$ and the identity (3.11) compatible for $T \in (0, T^*)$, T^* small enough.

Then (3.10) yields that for all $T \in (0, T^*)$, all solutions z of (1.1) satisfy

$$\exp\left(-\frac{A}{T^\beta}\right) \|z(T)\|^2_{L^2(\Omega)} - \exp\left(-\frac{A}{(T/2)^\beta}\right) \|z_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |z(t,x)|^2 \, dt dx, \quad (3.12)$$

which coincides with (2.10) for

$$f(t) = \exp\left(-\frac{A}{t^\beta}\right), \quad t > 0.$$

We then deduce the result (1.2) from Lemma 2.2 for $T \in (0, T^*)$ and then for any time T by a semigroup argument. □

Remark 4. The above proof is very close to the one in [13], in which the estimate (1.2) is deduced from (1.5). This is not so surprising since estimate (3.1) can be seen as a time-integrated version of (1.5).

4. **Further comments.**

1. The result stated in Theorem 1.1 is not an easy one. All proofs involve quite sophisticated arguments. Except for the direct proof using Carleman inequalities

for the heat equation, the others use the links with elliptic and wave equations that are represented in the following diagram:

$$\begin{array}{ccc} \text{Heat equation} & \underset{[8,\,10]}{\overset{(1)}{\longrightarrow}} & \text{Elliptic equation} \\ (3),\text{ cf }[14] \uparrow\downarrow (4),\text{ cf }[4] & & \\ \text{Wave equation} & \underset{[16,\,17,\,15]}{\overset{(2)\,FBI}{\longrightarrow}} & \text{Elliptic equation.} \end{array} \tag{4.1}$$

Here, we emphasize that the arrow (4) is given by our transmutation technique developed in [4] and that this diagram is "commutative", at least for what concerns the observability inequality (1.2) for the heat equation (1.1).

2. According to the spectral estimates (1.5), the choice $\varepsilon = 0$ in (2.1) for the quantification of the unique continuation property for waves should also be true but this is still an open problem. When looking at the proof in [15], this seems to be a consequence of the use of the FBI transform in [15], thus already indicating some possible limitations to the above diagram (4.1) and of our approach which relies on a result for the wave equation which might not be sharp.

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