Controllability of tree-shaped networks of vibrating strings *

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Abstract. We consider a tree-shaped network of vibrating strings. The problem of controllability when the control acts on the root of the tree is analyzed. We give a necessary and sufficient condition for the approximate controllability of the system in time $2(\ell_1 + \cdots + \ell_M)$, where ℓ_1,...,ℓ_M denote the lengths of the strings of the network. This is a non-degeneracy condition guaranteeing that the spectra of any two subtrees of the network connected at a multiple node are mutually disjoint.

Contrôlabilité de réseaux de cordes vibrantes en forme d'arbre

Résumé. *Nous étudions la contrôlabilité d'un réseau de cordes vibrantes en forme d'arbre, contrôlé dans sa racine. On donne une condition nécessaire et suffisante de contrôlabilité approchée en temps* $2(\ell_1 + \cdots + \ell_M)$, *où* ℓ_1,...,ℓ_M *dénotent les longueurs des cordes du réseau. Il s'agit d'une condition de non-dégénérecence du réseau qui exige que les spectres de chaque couple de sous-arbres connéctés en un nœud multiple du réseau soient mutuellement disjoints.*

Version française abrégée

On considère un réseau de cordes vibrantes en forme d'arbre contrôlé dans un extrême libre, appellé racine. Les déformations verticales de ce réseau sont décrites par un système d'équations d'ondes verifiées sur chaque corde du réseau et couplées dans les nœuds multiples. Les conditions de couplage imposent la continuité des déplacements et l'équilibre des tensions. Sur les extrêmes simples du réseau on impose des conditions aux limites de Dirichlet homogènes sauf sur la racine où le contrôle agit.

En absence de contrôle il s'agit d'un système conservatif. En fait, il s'agit de l'équation de D'Alembert sur le réseau.

On s'intéresse au problème de la contrôlabilité. En particulier, on cherche à établir des conditions garantissant la contrôlabilité approchée du système, i.e., le fait que le système est contrôlable à zéro en un temps uniforme $T_0 > 0$ dans un sous-espace de données initiales dense dans l'espace d'énergie.

On donne une condition nécessaire et suffisante de contrôlabilité approchée en temps $T_0 = 2(\ell_1 + \cdots + \ell_M)$, ℓ_1,...,ℓ_M étant les longueurs des cordes du réseau. Il s'agit d'une condition de non-dégénérecence du réseau exigeant que les spectres de chaque couple de sous-arbres connectés en un nœud multiple du réseau soient mutuellement disjoints.

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Il est facile de voir que ceci est une condition nécessaire de contrôlabilité approchée. En effet, si deux sous-arbres connectés en un nœud multiple ont une valeur propre en commun, il est possible de construire une vibration propre du réseau, à support dans ces deux sous-arbres, qui s'annule sur la corde partant de la racine du reseau. Ceci entraîne de manière immédiate le manque de densité de l'espace de données contrôlables.

La contribution principale de cette Note est de montrer que cette condition de non-dégénérecence est aussi suffisante pour la contrôlabilité approchée.

Au fait, sous la condition de non-dégénérecence, on donne une inégalité d'observabilité pour les solutions du système homogène sans contrôle avec des poids sur les coefficients de Fourier des solutions qui entraîne, en particulier, la contrôlabilité approchée mais qui permet aussi de décrire, en termes des séries de Fourier, l'espace des données contrôlables.

Ce résultat est une généralisation de ceux de [1], [2] et [4] sur le contrôle de réseaux étoilés. Il est convenable d'observer que, dans le cas des réseaux étoilés, la condition de non-dégénérecence dit simplement que les longueurs des cordes du réseau son mutuellement irrationnelles, condition nécessaire et suffisante de contrôlabilité approchée déjà connue.

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1. Introduction and statement of the problem

We consider a vibrating network of elastic homogeneous strings whose rest configuration coincides with a planar tree, i.e., a simple, connected graph without closed paths. The network performs transversal vibrations and is clamped at its simple nodes, except at one, where a control is applied. At the multiple nodes the strings are connected elastically: the sum of the forces is equal to zero and the displacements of the strings coincide.

Our goal is to drive the network to rest in finite time by means of an appropriate choice of the control.

First of all we introduce some notation that is needed to formulate the problem under consideration precisely.

Let $\mathcal{A}$ be a tree. Assume that the vertices of $\mathcal{A}$ have been numbered by the index $n \in I = \{1, ...N\}$ and the edges by $m \in J = \{1, ...M\}$. We choose an arbitrary orientation in every edge and denote by ℓ_m the length of the edge with index m. Once the orientation is chosen, every point of the m-th string is identified with a real number $x \in [0, \ell_m]$. The points $x = 0$ and $x = \ell_m$ are called the initial and final points of the m-th string, respectively.

The vertex with index $n = 1$ is supposed to be a simple one. It will be called the root of $\mathcal{A}$ and will be denoted by $\mathcal{R}$.

Let $J_n \subset J$, $n = 1, ..., N$, be the set of indices of those edges having the vertex with index n as one of their ends. Set, for $m \in J_n$

$$\varepsilon_{m,n} = \begin{cases} 0 & \text{if the vertex with index } n \text{ is the initial point of the edge with index } m, \\ 1 & \text{if the vertex with index } n \text{ is the final point of the edge with index } m. \end{cases}$$

We also assume that the edge with the root as an end has index $m = 1$ and that the root is the initial point of such edge, i.e., $\varepsilon_{1,1} = 0$. Further, the sets of indices of simple and multiple nodes are denoted by $I_{\mathcal{S}}$ and $I_{\mathcal{M}}$, respectively.

Let $u_m = u_m(x, t) : [0, \ell_m] \times \mathbb{R} \to \mathbb{R}$ denote the deformation of the m-th string and $v = v(t) : \mathbb{R} \to \mathbb{R}$ the control. Then the motion of the network is described by the system

$$u_{m,tt} - u_{m,xx} = 0 \quad \text{in } [0, \ell_m] \times \mathbb{R}, \tag{1}$$

$$u_m(x,0) = u_m^0(x), \qquad u_{m,t}(x,0) = u_m^1(x) \quad \text{in } [0,\ell_m], \tag{2}$$

for $m = 1,...,M$, where $u_m^0(x)$ and $u_m^1(x)$ are the initial deformation and velocity of the m-th string, respectively, and the boundary conditions at the multiple nodes

$$u_m(\varepsilon_{m,n}\ell_m, t) = u_{m'}(\varepsilon_{m,'n}\ell_{m'}, t), \quad t \in \mathbb{R}, \tag{3}$$

for $m, m' \in J_n$ and

$$\sum_{m \in J_n} (-1)^{\varepsilon_{m,n}} u_{m,x}(\varepsilon_{m,n}\ell_m, t) = 0, \quad t \in \mathbb{R} \tag{4}$$

for every $n \in I_{\mathcal{M}}$ and, at the simple ones,

$$u_1(0,t) = v(t), \quad u_m(\varepsilon_{m,n}\ell_m, t) = 0, \quad t \in \mathbb{R} \tag{5}$$

for $m \geq 2$ and $n \in I_{\mathcal{S}}$. Observe that these coupling conditions allow to assume that the multiplicity of the multiple nodes is at least three. Indeed, when the multiplicity of a node is two, the two strings may be identified with a single one with length the sum of the lengths of the two strings.

In order to provide a precise functional framework for this problem, we introduce the Hilbert spaces

$$V = \{\bar{\psi} : \ \psi_m \in H^1(0,\ell_m), \ \psi_m(\varepsilon_{m,n}\ell_m) = 0 \text{ for } n \in I_{\mathcal{S}}, \ \psi_m(\varepsilon_{m,n}\ell_m) = \psi_{m'}(\varepsilon_{m',n}\ell_{m'}) \text{ for } n \in I_{\mathcal{M}}\},$$

$$H = \{\bar{\psi} : \ \psi_m \in L^2(0,\ell_m)\}$$

with the Hilbert structures inherited from H^1 and L^2, respectively. Here and in the sequel, $\bar{\psi}$ denotes a vector function $(\psi_1,...,\psi_M)$ where each ψ_m is a function depending on $x \in (0,\ell_m)$ and possibly also on time. Let V' be the dual space of V.

It is known that for any $T > 0$, problem (1)-(5) is well posed for $\bar{u}^0 \in H$, $\bar{u}^1 \in V'$ and $v \in L^2(0,T)$ (see [3]).

Given initial states $\bar{u}^0 \in H$, $\bar{u}^1 \in V'$ and $T > 0$, we shall say that the initial data $(\bar{u}^0, \bar{u}^1) \in H \times V'$ are controllable in time T , if the control function $v \in L^2(0,T)$ may be chosen such that the solution of (1)-(5) with these initial data verifies

$$\bar{u}(T) = \bar{u}_t(T) = 0.$$

When the set of such controllable initial data is dense in $H \times V'$ the system is said to be *approximately controllable in time T*.

The approximate controllability of (1)-(5) is equivalent to the following unique continuation property

$$If \quad v \equiv 0 \ \ and \ \ u_{1,x}(0,t) = 0 \ in \ [0,T] \ \ then \quad \bar{u}(0) = \bar{u}_t(0) = 0, \tag{6}$$

for the solutions $\bar{u}$ of (1)-(5) with $(\bar{u}^0, \bar{u}^1) \in V \times H$ (see [4]).

Our first goal is to determine under what conditions the network is approximately controllable in some finite time from any simple node, i.e., for an arbitrary choice of the root.

In order to do that, it is convenient to introduce the eigenvalues $(\lambda_k)_{k\in\mathbb{Z}_+}$ and the eigenfunctions $\bar{\theta}_k = \bar{\theta}_k(x)$ of system (1)-(5). They are determined by the condition that $e^{\pm i\sqrt{\lambda_k}t}\,\bar{\theta}_k(x)$ solves (1), (3), (4), (5) with $v \equiv 0$.

Given a multiple node $\mathcal{O}$ of multiplicity s, there are s subtrees of $\mathcal{A}$ having $\mathcal{O}$ as the root and that are not contained in any larger subtree of $\mathcal{A}$ with this property. They will be called $\mathcal{O}$-subtrees.

For an $\mathcal{O}$-subtree we consider the eigenvalue problem inherited from the eigenvalue problem for the whole tree $\mathcal{A}$ with homogeneous Dirichlet boundary condition at the new root $\mathcal{O}$. This eigenvalue problem is similar to that for $\mathcal{A}$. Its spectrum will be called the *spectrum of the $\mathcal{O}$-subtree.*

2. Main results

We shall say that the tree $\mathcal{A}$ is *non-degenerate* if for any multiple node $\mathcal{O}$ the spectra of the $\mathcal{O}$-subtrees are pairwise disjoint.

It is easy to see that the condition of being non-degenerate is necessary for a tree to be approximately controllable from any simple vertex. Indeed, if for some multiple vertex $\mathcal{O}$, two $\mathcal{O}$-subtrees share an eigenvalue, then it is possible to construct an eigenvibration of the tree, supported on these two subtrees, vanishing elsewhere. In particular, such solution vanishes, together with its normal derivative, at the simple nodes of $\mathcal{A}$ that belong to the remaining $\mathcal{O}$-subtrees. Thus, (6) fails to hold. The main result of this Note is that this non-degeneracy condition turns out to be also sufficient. The main ingredient of the proof is the following weighted observability inequality.

THEOREM 1. – *There exists a sequence d_k, $k \in \mathbb{Z}_+$, of real numbers such that*

$$\int_0^{T_0} |u_{1,x}(0,t)|^2 dt \geq \sum_{k\in\mathbb{Z}_+} d_k^2(\lambda_k u_{0,k}^2 + u_{1,k}^2) \tag{7}$$

for every solution of (1)-(5) with $v \equiv 0$ and initial data

$$\bar{u}^0 = \sum_{k\in\mathbb{Z}_+} u_{0,k}\bar{\theta}_k \in V, \qquad \bar{u}^1 = \sum_{k\in\mathbb{Z}_+} u_{1,k}\bar{\theta}_k \in H, \tag{8}$$

where T_0 is twice the sum of the lengths of all the edges of the tree, i.e., $T_0 = 2(\ell_1 + \cdots + \ell_M)$.

The weights d_k depend only on the lengths ℓ_m of the strings and on the choice of the root. They may be computed explicitly. Note that Theorem 1 holds for all trees, regardless they are degenerate or not. Moreover, the following holds:

THEOREM 2. – *If the tree $\mathcal{A}$ is non-degenerate, all the coefficients d_k are different from zero.*

This implies that for non-degenerate trees the unique continuation property (6) holds for $T \geq T_0$. Consequently, we obtain a complete characterization of the approximately controllable trees:

THEOREM 3. – *The system (1)-(5) is approximately controllable from any simple node in some time $T > 0$ if, and only if, the tree $\mathcal{A}$ is non-degenerate. Moreover, non-degenerate trees are approximately controllable from any simple node in time T_0.*

Remark 1. – If we restrict ourselves to controlling the network from a single fixed end, then the non-degeneracy condition may be slightly weakened. In this case, the controllability holds if, and only if, the spectra of those $\mathcal{O}$-subtrees that do not contain that end are pairwise disjoint. □

Remark 2. – This result extends those in [1], [2] and [4] on the controllability of stars. Note that a star is a tree formed by strings connected at a single multiple node. In this simple case, the non-degeneracy condition coincides with the property of mutual irrationality of the lengths of all the strings of the star introduced in those papers. □

Remark 3. – According to the inequality (7), one can even characterize a space of controllable data with control in $L^2(0, T_0)$. Indeed, it is the space of data of the form (8) with Fourier coefficients satisfying the summability condition

$$\sum_{k\in\mathbb{Z}_+} \frac{1}{d_k^2}(|u_{0,k}|^2 + \frac{1}{\lambda_k}|u_{1,k}|^2) < \infty.$$

Given initial data satisfying this property, the control of minimal L^2 norm may be obtained by Lions' HUM method (see [5]).

If we were able to establish uniform lower estimates of the form

$$|d_k| \geq C\lambda_k^{-\alpha},$$

inequality (7) would imply the controllability of (1)-(5) in the space $W^\alpha \times W^{\alpha-\frac{1}{2}}$, where W^α is the domain of the α-power of the Laplacian with the boundary conditions corresponding to (3)-(5). This may be done in some particular cases, e.g., for star-shaped trees (see [2]). However, the controllability in the whole energy space $H \times V'$ is never reached. □

Remark 4. – Due to the analytic character of the functions involved in the equations from which the eigenvalues of the subtrees are determined, the non-degeneracy property may be shown to be satisfied generically in the set of trees having the same topological configuration. □

Remark 5. – The control time T_0 may be shown to be optimal. Indeed, for any $T < T_0$ it is possible to construct a smooth non-zero solution of (1), (3)-(5) with $v \equiv 0$ such that the unique continuation property (6) fails. □

3. Sketch of the proof of Theorem 1

The proof is based on the fact that the components u_m of the solution $\bar{u}$ of (1)-(5) may be expressed by the D'Alembert formula.

Let $\ell > 0$. We shall denote by $\hat{\ell}^+$, $\hat{\ell}^-$ the operators acting on a time-dependent function f by

$$\hat{\ell}^\pm f(t) = \frac{1}{2}[f(t+\ell) \pm f(t-\ell)].$$

If $u = u(x,t) : (0,\ell) \times \mathbb{R} \to \mathbb{R}$ is a solution of the wave equation $u_{tt} - u_{xx} = 0$ then, as a consequence of the D'Alembert formula, the following energy estimate holds:

$$E_u(t) = \frac{1}{2}\int_0^\ell (|u_t(x,t)|^2 + |u_x(x,t)|^2)dx \leq 4\int_{t-\ell}^{t+\ell} (|u_t(0,\tau)|^2 + |u_x(0,\tau)|^2)d\tau \tag{9}$$

and the values of the tension u_x and the velocity u_t at the initial and final points of the string satisfy the relations

$$u_x(\ell,t) = \hat{\ell}^+ u_x(0,t) + \hat{\ell}^- u_t(0,t), \tag{10}$$

$$u_t(\ell,t) = \hat{\ell}^- u_x(0,t) + \hat{\ell}^+ u_t(0,t). \tag{11}$$

The formulas (10), (11) together with the coupling conditions (3)-(5) allow to construct, for every $n \in I_{\mathcal{M}}$ and $m \in J_n$, linear operators D, B_m^1 and B_m^2 such that

$$Du_{m,x}(\varepsilon_{m,n}\ell_m, t) = B_m^1 u_{1,x}(0,t), \qquad Du_{m,t}(\varepsilon_{m,n}\ell_m, t) = B_m^2 u_{1,x}(0,t). \tag{12}$$

The operators D, B_m^1 and B_m^2 are linear combinations of products of operators $\hat{\ell_m}^{\pm}$. Formally, they may be computed using the Cramer's formula for the solution of the linear system of equations for $u_{m,t}(0,t)$, $u_{m,x}(0,t)$, $u_{m,t}(\ell_m,t)$, $u_{m,x}(\ell_m,t)$, $m = 1,..,M$, given by relations (3)-(5), (10), (11).

Further, there exists a function $d(\lambda)$, which is either even or odd, real or imaginary valued, depending on the structure of the tree, such that

$$De^{i\lambda t} = d(\lambda)e^{i\lambda t}. \tag{13}$$

Observe that the function d may be easily obtained by replacing in D the operators $\hat{\ell}_m^+$, $\hat{\ell}_m^-$ by $\cos\lambda\ell_m$, $i\sin\lambda\ell_m$, respectively. Besides, d has the property that $d(\mu)=0$ for some $\mu\in\mathbb{R}$ if, and only if, μ^2 is a common eigenvalue of two subtrees of $\mathcal{A}$ with the same root.

Set now $\bar{w}=D\bar{u}$. The operator D being linear, $D\bar{u}$ is also a solution of (1), (3)-(5).

As the solution $\bar{u}$ may be expressed in terms of the Fourier coefficients $(u_{0,k})$, $(u_{1,k})$ of its initial data $\bar{u}^0$, $\bar{u}^1$ by

$$\bar{u}(x,t)=\sum_{k\in\mathbb{Z}_+}(u_{0,k}\cos\sqrt{\lambda_k}t+\frac{u_{1,k}}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t)\bar{\theta}_k(x)$$

then, taking into account (13), we get

$$\bar{w}(x,t)=\sum_{k\in\mathbb{Z}_+}d(\sqrt{\lambda_k})(u_{0,k}\cos\sqrt{\lambda_k}t+\frac{u_{1,k}}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t)\bar{\theta}_k(x)\quad\text{or}$$

$$w(x,t)=\sum_{k\in\mathbb{Z}_+}id(\sqrt{\lambda_k})(u_{0,k}\sin\sqrt{\lambda_k}t-\frac{u_{1,k}}{\sqrt{\lambda_k}}\cos\sqrt{\lambda_k}t)\bar{\theta}_k(x),$$

depending on whether d is even or odd. This implies that the energy $E_{\bar{w}}$ of $\bar{w}$, which is preserved in time, is given by

$$E_{\bar{w}}=\sum_{k\in\mathbb{Z}_+}|d(\sqrt{\lambda_k})|^2(\lambda_k u_{0,k}^2+u_{1,k}^2).\tag{14}$$

On the other hand, from (12) it follows that

$$w_{m,x}(\varepsilon_{m,n}\ell_m,t)=B_m^1u_{1,x}(0,t),\qquad w_{m,t}(\varepsilon_{m,n}\ell_m,t)=B_m^2u_{1,x}(0,t).$$

Then, (9) gives

$$E_{w_m}(t)\le 4\int_{t-\ell_m}^{t+\ell_m}(|B_m^1u_{1,x}(0,\tau)|^2+|B_m^2u_{1,x}(0,\tau)|^2)d\tau\le C\int_{t-\frac{T_0}{2}}^{t+\frac{T_0}{2}}|u_{1,x}(0,\tau)|^2d\tau$$

for some positive constant C. In particular,

$$E_{w_m}(\frac{T_0}{2})\le C\int_0^{T_0}|u_{1,x}(0,\tau)|^2d\tau.\tag{15}$$

Combining (14), (15) and taking into account that the energy $E_{\bar{w}}=\sum_{m\in J}E_{w_m}$ is conserved, inequality (7) is obtained. Observe that the coefficients d_k are defined by $d_k=|d(\sqrt{\lambda_k})|$. □

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