NULL CONTROLLABILITY IN UNBOUNDED DOMAINS FOR THE SEMILINEAR HEAT EQUATION WITH NONLINEARITIES INVOLVING GRADIENT TERMS

Victor R. Cabanillas (*)
Instituto de Matemática
Universidade Federal do Rio de Janeiro
CP 68530, 21945-970, RJ, Brazil
victorcz@pg.im.ufrj.br

Silvano B. de Menezes (†)
Departamento de Matemática
Universidade Federal do Pará
66075-110, Belém, Pa, Brazil
silvano@ufpa.br

and

Enrique Zuazua (**)

Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
zuazua@eucmax.sim.ucm.es

ABSTRACT. In this paper, we consider the null controllability problem for the semilinear heat equation with nonlinearities involving gradient terms in an unbounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. The control is assumed to be distributed along a subdomain ω such that the uncontrolled region $\Omega\backslash\omega$ is bounded. Using Carleman inequalities we first prove the null controllability of the linearized equation. Then, by a fixed point method, we obtain the main result for the semilinear case. This result asserts that, when the nonlinearity is C^1 and globally Lipschitz, the system is null controllable.

Key words: null controllability, approximate controllability, unbounded domains, Carleman inequalities, observability inequality.

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1. Introduction and main results

This paper is devoted to the study of the null controllability of the semilinear heat equation

$$
(1.1) \qquad \begin{cases} u_t - \Delta u + f(u, \nabla u) = h 1_\omega \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \end{cases}
$$

where Ω is an open and unbounded set of $\mathbb{R}^N$ of class C^2 uniformly, with boundary $\partial\Omega$ (see Section 2 for a precise definition) and ω is an open and nonempty subset of Ω. In (1.1) $u = u(x,t)$ is the state, $h = h(x,t)$ is the control function and 1_ω denotes the characteristic function of the subset ω.

Therefore, the control h acts on the system through the subset ω. We shall assume that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a C^1 and globally Lipschitz function such that $f(0,0) = 0$.

We also assume that the unbounded sets Ω and ω satisfy that

$$
(1.2) \qquad \Omega \backslash \omega \quad \text{is bounded.}
$$

According to this, the control acts on a large subset of Ω and it only leaves a bounded subset of Ω without control. Therefore, the problem we are addressing is close to the classical one of controlling to zero the heat equation in bounded domains. We shall describe below the state of the art on this topic.

Let $u_0 \in L^2(\Omega)$, $h \in L^2(0,T;L^2(\Omega))$, $T > 0$ and f be a C^1 and globally Lipschitz function such that $f(0,0) = 0$. Then there exists a unique solution

$$
u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))
$$

of problem (1.1).

The null controllability problem for (1.1) can be formulated as follows: *Given* $T > 0$ *and* $u_0 \in L^2(\Omega)$ *to find a control* $h \in L^2(0, T; L^2(\Omega))$ *such that the solution of* (1.1) *satisfies*

$$u(T) = 0 \quad in \quad \Omega, \tag{1.3}$$

with an estimate of the form

$$|h|_{L^2(0,T;L^2(\Omega))} \le c|u_0|_{L^2(\Omega)} \quad for\ all \quad u_0 \in L^2(\Omega). \tag{1.4}$$

Note that, in view of the condition $f(0, 0) = 0$, $u \equiv 0$ is an equilibrium solution of system (1.1) in the absence of control, i.e., with $h \equiv 0$. Thus, in the null controllability problem under consideration, we intend to drive the solution to the equilibrium in time T. Of course, if (1.3) is achieved, extending the control by zero for all $t \ge T$, we obtain a globally defined solution of (1.1) such that $u(t) \equiv 0$ for all $t \ge T$.

There is a large literature on the null controllability of heat equations in bounded domains. Let us briefly mention some of the existing works.

In the context of linear heat equations with time independent coefficients, D.L. Russell [R] proved that the null controllability of the heat equation for all $T > 0$ is a consequence of the exact controllability of the wave equation for some T. More recently, G. Lebeau and L. Robbiano [LR] proved the null controllability without any geometric restrictions on the open subset ω where the control acts using Fourier series and sharp estimates on the eigenfunctions of the Laplacian obtained by means of Carleman's inequalities. Similar results but in a more general context including time-dependent coefficients were prove by A. Fursikov and O. Yu. Imanuvilov [FI] using global Carleman's inequalities for the heat equation. In [FI] local null-controllability results were also proved for semilinear heat equations (see also [IY]). More recently, the connections between null and approximate controllability were investigated in [FCZ1].

In [FCZ2] the null controllability of (1.1) was proved for a class of non-linearities for which blow-up phenomena may arise.

There is a large literature on the so-called *approximate controllability* problem as well. System (1.1) is said to be approximately controllable in time T if the reachable set $\{u(\cdot, T);\text{ with } h \in L^2(0,T;L^2(\Omega))\}$ is dense in $L^2(\Omega)$ for any initial datum $u_0 \in L^2(\Omega)$. In [FPZ] the approximate controllability was proved to hold in bounded domains in the particular case in which $f = f(y)$, f being globally Lipschitz. This result was extended to the case of unbounded domains in [TZ]. The case $f = f(y, \nabla y)$ was addressed in [Z3]. However the property of null-controllability is much stronger and very little is known when the domain Ω is unbounded.

Recently, in [MZ1], the one-dimensional linear heat equation was considered in $\Omega = \mathbb{R}_+ = (0, \infty)$ with control at the extreme $x > 0$. It was proved that, within the class of solutions defined by transposition, there is no smooth, compactly supported initial data that might be driven to zero in finite time. This result shows how differently the null controllability property behaves in bounded and unbounded domains. Note that, as indicated above, approximate controllability does hold even in unbounded domains due to infinite speed of propagation. But null controllability does not!. We refer to [MZ2] for the extension of this result to the multidimensional case.

Analyzing the proofs of [MZ1, 2] it becomes clear that such a negative result holds since the controlled heat equation holds in an unbounded domain while the control is localized in a bounded domain. Thus, an unbounded region is left without control and this is the cause of the lack of null controllability.

However, in the present paper, even if the domain Ω is unbounded, the control acts on a large subdomain that only leaves a bounded subset uncontrolled. It is then natural to expect the positive results of the case where Ω is bounded to hold. The main result of the paper is the following:

Theorem 1.1. *Assume that* f *is a* C^1 *and globally Lipschitz function, such that* $f(0,0) = 0$*, and let* Ω *be an unbounded domain of class* C^2 *uniformly, and* ω *is an open nonempty subset of* Ω *such that* $\Omega \backslash \omega$ *is bounded. Then, for all* $T > 0$ *and for every* $u_0 \in L^2(\Omega)$*, there exists* $h \in L^2(0,T; L^2(\omega))$ *such that the solution of* (1.1) *satisfies* (1.3). *Moreover*, (1.4) *holds for a suitable* $c > 0$*, independent of* u_0 *.*

In other words, system (1.1) *is null controllable for all* $T > 0$.

Remark 1.1.: Several remarks are in order:

(a) This result extends those in [IY] that provide the null controllability when Ω is bounded, under the same assumptions on f.

(b) Combining the methods of this paper with those developed in [FCZ1,2] the following additional results may be proved under the assumptions of Theorem 1.1:

- Let v be any solution of system (1.1) corresponding to initial data $v_0 \in L^2(\Omega)$ and a control $g \in L^2(\omega \times (0,T))$. Then, the solutions of (1.1) may be driven to the final state $v(T)$, i.e. for any $u_0 \in L^2(\Omega)$ there exists a control such that the solution of (1.1) satisfies $u(T) = v(T)$.
- System (1.1) is approximately controllable in any time $T > 0$. More precisely, for any $u_0, u_1 \in L^2(\Omega)$ and $\varepsilon > 0$ there exists a control $h_\varepsilon \in L^2(\Omega \times (0,T))$ such that the solution of (1.1) satisfies

$$|u(T) - u_1|_{L^2(\Omega)} \leq \varepsilon.$$

- System (1.1) is finite-approximately controllable. In other words, given any finite-dimensional subspace E of $L^2(\Omega)$ and denoting by π_E the orthogonal projection over E, for any $u_0, u_1 \in L^2(\Omega)$ and $\varepsilon > 0$ there exists a control function $h_{E,\varepsilon} \in$

$L^2(\Omega \times (0,T))$ such that the solution of (1.1) satisfies

$$\pi_E(u(T)) = \pi_E(u_1) \quad \text{and} \quad |u(T) - u_1|_{L^2(\Omega)} \leq \varepsilon.$$

(c) Combining the methods in [FCZ2] and the techniques of this paper one may expect Theorem 1.1 to hold for a class of nonlinearities that grow at infinity in a superlinear way. According to the results in [FCZ2] one may conjecture that the growth condition

$$\frac{|f(x)|}{|s| \log^{3/2}|s|} \longrightarrow 0 \ , \quad \text{as} \quad |s| \to \infty$$

suffices to guarantee null controllability.

(d) The negative results in [FCZ2] apply in this case in a straightforward way. Accordingly, there exist nonlinearities that grow at infinity as $|s| \log^p |s|$ with $p > 2$ and for which null-controllability of (1.1) does not hold, except for the trivial case where the control is distributed everywhere in the domain Ω. □

The proof of the result is based on the by now well known fixed point method (see, for instance, [Z3]). There is however, a new difficulty, related to the fact that Ω is unbounded. Indeed, in this case, the compactness of the Sobolev's imbeddings fails and Schauder's fixed point theorem can not be applied directly. However, we observe that, since the control acts on a "large" subdomain ω so that $\Omega \backslash \omega$ is bounded, the problem may be reduced to the case where the nonlinearity is supported in this bounded region. This remark allows us to reduce the fixed point argument to functions defined in $[\Omega \backslash \omega] \times (0,T)$ and this allows to apply Schauder's fixed point Theorem. The first step on the application of the fixed point method is to analyze the null control of the linearized system. The problem is reduced, by duality, to the obtention of suitable observability inequalities for the adjoint system. These inequalities are obtained cutting-of the solutions in two pieces. Roughly, the first piece is the one in ω where the observed quantity provides estimates immediately, and the second one is the one in $\Omega\backslash\omega$ where the estimates are obtained using Carleman's inequalities as in [IY].

The paper is organized as follows: Section 2 is devoted to prove the null controllability of the linearized system. In Section 3 we prove Theorem 1.1 by a fixed point method.

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2. Preliminaries and analysis of the linearized system

2.1. A basic definition

As we said in the introduction, we assume the domain Ω to be of class C^2 uniformly. For the sake of completeness we recall the definition of domain of class C^s uniformly (see [B]). We say that a domain (bounded or not) is uniformly regular of class C^s $(s \geq 1)$, if there exists an integer $r > 0$ and a sequence $\{N_j\}$ of open subsets of $\mathbb{R}^N$ and homeomorphisms $\{\psi_j\}$ from N_j to the unit ball in $\mathbb{R}^N$ such that:

i) Any $(r+1)$ distinct sets N_j have empty intersection;

ii) $\psi_j(N_j \cap \Omega) = \{x\colon |x| < 1, x_n > 0\}$, $\psi_j(N_j \cap \partial\Omega) = \{x\colon |x| < 1, x_n = 0\}$;

iii) If $N_j' = \psi_j^{-1}(|x| < 1/2)$, $\bigcap_j = N_j'$ contains the $(1/r)$-neighborhood of $\partial\Omega$;

iv) For $y \in N_j$, $x \in \psi_j(N_j)$ we have $|(D^\alpha \psi_j)(y)| \leq r$, $|(D^\alpha \psi_j^{-1})(x)| \leq r$, for all $|\alpha| \leq s$.

2.2. Observability of the linearized adjoint system.

The main result will be proved by means of a fixed point argument. Therefore, first, we need to analyze the null controllability of the following linearized system:

$$
\begin{cases} u_t - \Delta u + au + b \cdot \nabla u = h 1_\omega \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{2.1}
$$

where the potentials $a = a(x,t)$, $b = b(x,t)$ are assumed to be in $L^\infty(\Omega \times (0,T))$ and $[L^\infty(\Omega \times (0,T))]^N$ respectively.

From now on, $||.||_\infty$ will denote so much the norm in $L^\infty(\Omega \times (0,T))$ as in $[L^\infty(\Omega \times (0,T))]^N$.

As usual, this property will be equivalent to a suitable observability property for the adjoint system.

Thus, let us consider the adjoint system

$$
\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - \operatorname{div}(b\varphi) = 0 \quad \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = \varphi^0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.2}
$$

We have to prove the following observability property:

Proposition 2.1. *For all $T > 0$ and $R > 0$ there exists a positive constant C such that*

$$
|\varphi(0)|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 \, dxdt \tag{2.3}
$$

for every solution of (2.2) *and for any $a \in L^\infty([\Omega\backslash\omega] \times (0,T))$ and $b \in [L^\infty([\Omega\backslash w] \times (0,T))]^N$ satisfying $||a||_{L^\infty([\Omega\backslash\omega]\times(0,T))} \leq R$, $||b||_{[L^\infty([\Omega\backslash w]\times(0,T))]^N} \leq R$.*

Remark 2.1. The constant C in inequality (2.3) will be refered to as the observability constant. It depends on Ω, ω, the time T and the size R of the potentials, but it is independent of the solution φ of (2.2). □

Remark 2.2. From now on we will denote by C a generic constant whose value varies of a line for another one. $\square$

In order for (2.3) to be true the fact that $\Omega\backslash\omega$ is bounded is essential. Thus, inequality (2.3) is a natural extension of the existing observability inequalities in bounded domains ([FCZ1], [FI], [IY],...).

Note that in (2.3) we get an upper bound on the norm of φ at time $t = 0$, which is the final time for the adjoint system (2.2). Due to the regularizing effect of the heat equation one can not expect to get such a bound when $|\varphi(0)|_{L^2(\Omega)}$ is replaced by $|\varphi(T)|_{L^2(\Omega)}$ in (2.3).

Proof of Proposition 2.1: In order to prove (2.3), given $\varepsilon > 0$ small enough, we introduce a "cut off" function $\rho \in C^\infty(\Omega)$ such that

$$\begin{cases} \rho \geq 0 \quad \text{in} \quad \Omega \\ \rho = 0 \quad \text{on} \quad \omega_\varepsilon = \{x \in \omega\colon\ d(x, \partial\omega) > \varepsilon\} \\ \rho = 1 \quad \text{in} \quad \Omega\backslash\omega. \end{cases} \tag{2.4}$$

We define

$$\theta = \rho\,\varphi. \tag{2.5}$$

Then θ satisfies

$$\begin{cases} -\theta_t - \Delta\theta + a\theta - \operatorname{div}(b\theta) = g \quad \text{in} \quad \Theta \times (0,T) \\ \theta = 0 \quad \text{on} \quad \partial\Theta \times (0,T) \\ \theta(T) = \varphi^0\rho \quad \text{in} \quad \Theta, \end{cases} \tag{2.6}$$

where $g = -2\nabla\rho \cdot \nabla\varphi - (\Delta\rho)\varphi - (b \cdot \nabla\rho)\varphi \in L^2(\Theta \times (0,T))$, and

$$\Theta = \{x \in \Omega;\ \rho(x) > 0\}. \tag{2.7}$$

Note that, according to the hypothesis above on Ω and ω, Θ is a bounded open set.

We now apply the global Carleman inequalities of O. Imanuvilov and M. Yamamoto (see [IY]) to system (2.6). To do this, following [IY], we introduce a function $\psi = \psi(x)$ such that

$$
\begin{cases} \psi \in C^2(\overline{\Theta}) \\ \psi > 0 \quad \text{in} \quad \Theta, \quad \psi = 0 \quad \text{on} \quad \partial\Theta \\ \nabla\psi \neq 0 \quad \text{in} \quad \overline{\Theta\backslash\omega}. \end{cases} \tag{2.8}
$$

We refer to [FI] for the proof of the existence of a function satisfying (2.8).

Now, using the function ψ we introduce weight functions:

$$
\xi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}\,, \quad \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||_{C(\overline{\Theta})}}}{t(T-t)} \tag{2.9}
$$

where $\lambda > 0$ is a parameter.

The following Global Carleman Inequality holds, (see [IY]):

Proposition 2.2, ([IY]). *There exists a number $\hat{\lambda} > 0$ such that for an arbitrary $\lambda \geq \hat{\lambda}$, we can choose $s_0(\lambda) > 0$ such that: there exists a constant $C_1 > 0$ such that for each $s \geq s_0(\lambda)$ the solution $\theta \in L^2(\Theta \times (0,T))$ of* (2.6), *satisfies the following inequality:*

$$
\begin{aligned} &\int_0^T \int_\Theta \left(\frac{1}{s\xi} |\nabla\theta|^2 + s\xi|\theta|^2 \right) e^{2s\alpha}\, dxdt \\ &\leq C_1 \left[||ge^{s\alpha}||^2_{L^2(0,T;H^{-1}(\Theta))} + \int_0^T \int_{\Theta\cap\omega} s\xi|\theta|^2\, e^{2s\alpha}\, dxdt \right] \end{aligned} \tag{2.10}
$$

for all $s \geq s_0(\lambda)$. Here, the constant C_1 depends continuously on λ, $||a||_\infty$ and $||b||_\infty$ and is independent of s. $\square$

We observe that $ge^{s\alpha} = -2(\nabla\rho \cdot \nabla\varphi)e^{s\alpha} - (\varphi\Delta\rho + (b \cdot \nabla\rho)\varphi)e^{s\alpha}$, and note that

Θ is a bounded subset of Ω and $\nabla\rho = 0$, $\Delta\rho = 0$ in $\Omega\backslash\omega \subset \Theta$, since $\rho = 1$ in $\Omega\backslash\omega$, and taking into account that $e^{s\alpha} < 1$, we have

$$||(\varphi\Delta\rho + (b\cdot\nabla\rho)\varphi)e^{s\alpha}||^2_{L^2(0,T;H^{-1}(\Theta))} \leq C||\varphi||^2_{L^2(0,T;L^2(\omega))}, \tag{2.11}$$

for some positive constant C, which depends on $||b||_\infty$.

On other hand, we fix $\hat{\lambda} > 0$, for which Proposition 2.2 holds and observe that

$$-2(\nabla\rho\cdot\nabla\varphi)e^{s\alpha} = -2\nabla\cdot(\varphi e^{s\alpha}\nabla\rho) + 2\varphi e^{s\alpha}\Delta\rho + 2\varphi s e^{s\alpha}\nabla\rho\cdot\nabla\alpha.$$

Thus, there exists a positive constant C such that

$$||2(\nabla\rho\cdot\nabla\varphi)e^{s\alpha}||^2_{L^2(0,T;H^{-1}(\Theta))} \leq C\Big[||\varphi e^{s\alpha}\nabla\rho||^2_{L^2(\Theta\times(0,T))^N} + ||\varphi e^{s\alpha}\Delta\rho + \varphi s e^{s\alpha}\nabla\rho\cdot\nabla\alpha||^2_{L^2(\Theta\times(0,T))}\Big].$$

Now, we observe again that $\nabla\rho = 0$, $\Delta\rho = 0$ in $\Omega\backslash\omega \subset \Theta$, that $e^{s\alpha} < 1$, and that there exists $s_1 > 0$ such that $|se^{s\alpha}\frac{\partial\alpha}{\partial x_i}| < 1$, $1 \leq i \leq 1$ for all $s \geq s_1$. Then, there exists a positive constant C such that

$$||2(\nabla\rho\cdot\nabla\varphi)e^{s\alpha}||^2_{L^2(0,T;H^{-1}(\Theta))} \leq C||\varphi||^2_{L^2(0,T;L^2(\omega))}, \quad \text{for all} \quad s \geq s_1. \tag{2.12}$$

By (2.11) and (2.12) we conclude that there exists a positive constant C such that

$$||ge^{s\alpha}||^2_{L^2(0,T;H^{-1})} \leq C||\varphi||^2_{L^2(0,T;L^2(\omega))}. \tag{2.13}$$

According to Proposition 2.2, for $\hat{\lambda} > 0$ fixed above, we can choose $s_0(\hat{\lambda}) > 0$ such that the inequality (2.10) holds for all $s \geq s_0(\hat{\lambda})$ and for some positive constant C_1.

We consider the inequality (2.10) with $\hat{\lambda}$ and $\hat{s} = \max\{s_0(\hat{\lambda}), s_1\}$ fixed, and we denote by $\hat{\alpha}$ and $\hat{\xi}$ the functions α and ξ in (2.9) with $\lambda = \hat{\lambda}$.

Since $\psi \in C^2(\overline{\Theta})$, there are constants $\eta_1, \eta_2 > 0$ such that

$$\frac{\eta_1}{t(T-t)} \quad \leq \quad \hat{\xi} \quad \leq \quad \frac{\eta_2}{t(T-t)}.$$

Then, we can substitute $\hat{\xi}$ by $(t(T-t))^{-1}$. Thus, there exists a positive constant C_2 such that the inequality (2.10) can be written as follows

$$\begin{aligned}
&\int_0^T \int_\Theta \left(\frac{t(T-t)}{\hat{s}} \, |\nabla\theta|^2 + \frac{\hat{s}}{t(T-t)} \, |\theta|^2 \right) e^{2\hat{s}\hat{\alpha}} \, dxdt \\
&\le C_2 \bigg[||g e^{\hat{s}\hat{\alpha}}||^2_{L^2(0,T;H^{-1}(\Theta))} + \int_0^T \int_{\Theta\cap\omega} \frac{\hat{s}}{t(T-t)} \, |\theta|^2 \, e^{2\hat{s}\hat{\alpha}} \, dxdt \bigg].
\end{aligned} \tag{2.14}$$

Let us estimate the weights appearing in (2.14). We have the following:

Lemma 2.1. *Let k and K be a positive constants such that*

$$k \le e^{2\hat{\lambda}||\psi||_{C(\overline{\Theta})}} - e^{\hat{\lambda}\psi(x)} \le K, \quad x \in \overline{\Theta}. \tag{2.15}$$

Then, for $x \in \overline{\Theta}$, we have

$$||\hat{s}t^{-1}(T-t)^{-1} \, e^{2\hat{s}\hat{\alpha}}||_{L^\infty((\Theta\cap\omega)\times(0,T))} \le \frac{1}{2k} \, e^{-1} \tag{2.16}$$

and

$$\frac{1}{\hat{s}} \, t(T-t) e^{2\hat{s}\hat{\alpha}} \ge \frac{3T^2}{16\hat{s}} \, \exp\left(- \frac{32\hat{s}K}{3T^2} \right), \quad t \in \left[\frac{T}{4}, \frac{3T}{4} \right]. \tag{2.17}$$

Proof of Lemma 2.1: We note that $\Theta \cap \omega \subset \overline{\Theta}$. We fix $x \in \Theta \cap \omega$, and denote

$$\phi_x(t) = \frac{\hat{s}}{t(T-t)} e^{2\hat{s}\hat{\alpha}} \quad \text{and} \quad r(x) = e^{2\hat{\lambda}||\psi||_{C(\overline{\Theta})}} - e^{\hat{\lambda}\psi(x)}.$$

Then $\hat{\alpha}(t,x) = -(t(T-t))^{-1} r(x) < 0$. Thus,

$$\phi_x(t) = \frac{\hat{s}}{t(T-t)} \exp\left(\frac{-2\hat{s}r(x)}{t(T-t)} \right) = \frac{\hat{s}}{\tau} \exp\left(\frac{-2\hat{s}r(x)}{\tau} \right) = \eta_x(\tau),$$

where $\tau = t(T-t) \in [0, T^2/4]$.

The maximum of η_x is achieved at $\hat{\tau} = 2\hat{s}r(x)$. On the other hand, η_x is increasing for $\tau \in (0, \hat{\tau})$ and decreasing for $\tau > \hat{\tau}$. Thus,

$$\max_{t\in[0,T]} \phi_x(t) = \max_{\tau\in[0,\frac{T^2}{4}]} \eta_x(\tau) = \begin{cases} \eta_x(\hat{\tau}) = \left(\dfrac{1}{2r(x)}\right)e^{-1}, & \text{if } \hat{\tau} < \dfrac{T^2}{4} \\ \eta_x(T^2/4) = \dfrac{4\hat{s}}{T^2}\exp\left(\dfrac{-8\hat{s}r(x)}{T^2}\right), & \text{if } \hat{\tau} > \dfrac{T^2}{4}. \end{cases}$$

In any case we have

$$\max_{t\in[0,T]} \phi_x(t) = \max_{\tau\in[0,\frac{T^2}{4}]} \eta_x(\tau) \leq \eta_x(\hat{\tau}) = \frac{1}{2r(x)}e^{-1}, \tag{2.18}$$

and applying (2.15) in (2.18), (2.16) it follows.

Let us prove (2.17). Fixed $x \in \overline{\Theta}$, we denote

$$p_x(t) = \frac{1}{\hat{s}}t(T-t)e^{2\hat{s}\hat{\alpha}} = \frac{1}{\hat{s}}\tau\exp\left(\frac{-2\hat{s}r(x)}{\tau}\right) = q_x(\tau),$$

where $\tau = t(T-t)$. When $t \in [\frac{T}{4}, \frac{3T}{4}]$, one has $\tau \in [\frac{3T^2}{16}, \frac{T^2}{4}]$.

Proceeding as in the proof of (2.16), we deduce that

$$p_x(t) = q_x(\tau) \geq q_x(\frac{3T^2}{16}) = \frac{3T^2}{16\hat{s}}\exp\left(\frac{-32\hat{s}r(x)}{3T^2}\right). \tag{2.19}$$

Therefore, (2.17) follows after apply (2.15) in (2.19). This concludes the proof. □

Coming back to (2.14) and applying (2.13) and the Lemma 2.1 we deduce that there exists a constant $C > 0$ such that

$$\int_0^T \int_\Theta |\nabla\theta|^2\, dxdt \leq C \int_0^T \int_\omega |\varphi|^2\, dxdt. \tag{2.20}$$

Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Theta)$. In view of (2.20), the fact that $\theta = \rho\varphi$ and also according to the choice of ρ, we deduce

$$\int_{T/4}^{3T/4} \int_{\Omega\setminus\omega} |\varphi|^2 \, dxdt \leq C \int_0^T \int_\omega |\varphi|^2 \, dxdt.$$

Thus, for some constant $C > 0$ we have

$$\int_{T/4}^{3T/4} \int_{\Omega} |\varphi|^2 \, dxdt \leq C \int_0^T \int_\omega |\varphi|^2 \, dxdt. \tag{2.21}$$

Let $\chi = \chi(t) \in C^\infty([0,T])$ be a function such that $\chi(t) \geq 0$ for $t \in [0,T]$, $\chi(t) = 1$ for $t \in \left[0, \frac{T}{2}\right]$ and $\chi(t) = 0$ for $t \in \left[\frac{3T}{4}, T\right]$.

Multiplying (2.2) by $\varphi\chi$ and integrating in Q, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \varphi^2\chi(t)dx + \frac{1}{2}\int_\Omega \varphi^2 \frac{d\chi(t)}{dt}\, dx + \int_\Omega |\nabla\varphi|^2 \chi(t)dx +$$
$$+ \int_\Omega a\varphi^2 \chi(t)dx + \int_\Omega (b\cdot\nabla\varphi)\varphi\chi(t)dx = 0.$$

Using Hölder's inequality and the elementar inequality: $2|pq| \leq \varepsilon p^2 + \varepsilon^{-1}q^2$, we obtain

$$-\frac{d}{dt}\int_\Omega \varphi^2\chi(t)dx + \int_\Omega |\nabla\varphi|^2\chi(t)dx \leq C\int_\Omega \left\{\left|\frac{d\chi(t)}{dt}\right| + \chi(t)\right\}\varphi^2 dx, \tag{2.22}$$

for some constant $C > 0$ which depends of $||a||_\infty$ and $||b||_\infty$.

Thus,

$$-\frac{d}{dt}\, w(t) \leq Cw(t) + C\int_\Omega \left|\frac{d\chi(t)}{dt}\right| \varphi^2 dx, \tag{2.23}$$

where

$$w(t) = \int_\Omega \chi(t)\varphi^2 dx.$$

Let us consider the inequality (2.23) on the time interval $[0, 3T/4]$. Note that $\chi(\frac{3T}{4}) = 0$. Applying the Gronwall's inequality to (2.23), we obtain

$$w(t) \leq C \int_0^{3T/4} \int_\Omega \left| \frac{d\chi(t)}{dt} \right| \varphi^2 \, dxdt, \quad 0 \leq t \leq T. \tag{2.24}$$

Then, since $|\frac{d\chi(t)}{dt}|$ is bounded in $[0, T]$ and $\chi(t) = 1$ in $[0, \frac{T}{2}]$, there exists $C > 0$ such that (2.24) takes the form

$$w(t) \leq C \int_{T/2}^{3T/4} \int_\Omega \varphi^2 \, dxdt.$$

In particular, for $t = 0$, by the definition of w, we have

$$|\varphi(0)|^2_{L^2(\Omega)} \leq C \int_{T/2}^{3T/4} \int_\Omega \varphi^2 \, dxdt. \tag{2.25}$$

Finally, applying (2.25) in (2.21), we have that there exists a constant $C > 0$ such that

$$|\varphi(0)|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\varphi|^2 \, dxdt, \tag{2.26}$$

which is the observability inequality (2.3). This concludes the proof of Proposition 2.1. □

2.3. Approximate controllability of the linearized system.

In view of the uniform observability inequality (2.3), the null controllability result of the linearized system can be proved as the limit of an approximate controllability property. Let us first discuss the approximate controllability property.

Given $u_0 \in L^2(\Omega)$ and $\delta > 0$ we introduce the quadratic functional

$$J_\delta(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 \, dxdt + \delta |\varphi^0|_{L^2(\Omega)} + \int_\Omega u_0 \, \varphi(0) \, dx, \tag{2.27}$$

where φ denotes the solution of (2.2) with initial data φ^0. The functional J_δ is continuous and strictly convex in $L^2(\Omega)$. Moreover, J_δ is coercive. More precisely, in view of (2.3), one has

$$\liminf_{|\varphi^0|_{L^2(\Omega)} \to \infty} \frac{J_\delta(\varphi^0)}{|\varphi^0|_{L^2(\Omega)}} \geq \delta. \tag{2.28}$$

Then J_δ has a unique minimizer in $L^2(\Omega)$. Let us denote it by $\hat{\varphi}^{0,\delta}$. It is easy see that the control $h_\delta = \hat{\varphi}^\delta$, where $\hat{\varphi}^\delta$ is the solution of (2.2) associated to the minimizer $\hat{\varphi}^{0,\delta}$, is such that the solution of (2.1) satisfies

$$|u_\delta(T)|_{L^2(\Omega)} \leq \delta. \tag{2.29}$$

We refer to [FPZ] for the details of the proof.

2.4. Null controllability of the linearized system.

Null controllability may be obtained as the limit when δ tends to zero of the approximate controllability property above. However, to pass to the limit we need a uniform bound on the control. To get this bound we observe that, by (2.3),

$$J_\delta(\varphi^0) \geq \frac{1}{2}\int_0^T \int_\omega \varphi^2 \, dxdt \; - c\left[\int_0^T \int_\omega \varphi^2 \, dxdt\right]^{1/2} |u_0|_{L^2(\Omega)} \tag{2.30}$$

with $c > 0$ independent of δ. On the other hand,

$$J_\delta(\hat{\varphi}^{0,\delta}) \leq J_\delta(0) = 0. \tag{2.31}$$

Writing (2.30) for $\hat{\varphi}^{0,\delta}$, being $\hat{\varphi}^{0,\delta}$ the minimizer of J_δ in $L^2(\Omega)$, and combining with (2.31), we deduce that

$$|h_\delta|_{L^2(0,T;L^2(\Omega))} \leq 2c\,|u_0|_{L^2(\Omega)}\,, \text{ for all } \delta > 0. \tag{2.32}$$

In other words, h_δ remains bounded in $L^2(0,T;L^2(\Omega))$ as $\delta \to 0$.

Extracting subsequences we deduce that

$$h_\delta \rightharpoonup h \quad \text{as} \quad \delta \to 0 \quad \text{weakly in} \quad L^2(\Omega \times (0,T)), \tag{2.33}$$

for some $h \in L^2(\Omega \times (0,T))$.

It is easy to see that the limit h is such that the solution u of (2.1) satisfies (1.3). Moreover, by lower semicontinuity of the norm with respect to the weak topology and in view of (2.33) we deduce that:

$$|h|_{L^2(\Omega\times(0,T))} \leq \liminf_{\delta\to 0} |h_\delta|_{L^2(\Omega\times(0,T))} \leq 2c\,|u_0|_{L^2(\Omega)}\,. \tag{3.34}$$

In this way we have completed the proof of the following result:

Theorem 2.1. *Assume that the hypotheses of Theorem 1.1 on Ω and ω are satisfied. Then, for every $T > 0$ and $u_0 \in L^2(\Omega)$, there exists $h \in L^2(0,T;L^2(\Omega))$ such that the solution of* (2.1) *satisfies* (1.3). *Moreover, there exists a constant $c > 0$, depending on $R > 0$, but independent of u_0 such that* (1.4) *holds for every potentials a, b such that $||a||_\infty \leq R$ and $||b||_\infty \leq R$.*

3. Proof of the main result

This section is devoted to prove Theorem 1.1. As we said in the introduction, it will be a consequence of Theorem 2.1 above and of a fixed point argument. We note however that the fixed point method may not be applied directly since the domain where the equation holds is unbounded and therefore the Sobolev's imbeddings are not compact.

Thus, we note that system (1.1) can be written as follows:

$$
\begin{cases} u_t - \Delta u + (1 - 1_\omega) f(u, \nabla u) = h 1_\omega - f(u, \nabla u) 1_\omega & \text{in} \quad \Omega \times (0, T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{3.1}
$$

Therefore it is sufficient to analyze the controllability of the system:

$$
\begin{cases} u_t - \Delta u + (1 - 1_\omega) f(u, \nabla u) = q(x,t) 1_\omega & \text{in} \quad \Omega \times (0, T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{3.2}
$$

Indeed, if q is the control for (3.2), $h = f(u, \nabla u) + q$ is the control for (1.1) and vice-versa. The advantage of writing system (1.1) in the form (3.2) is that the nonlinearity is now localized in a bounded subdomain of Ω, according to hypothesis (1.2). This fact is important to guarantee the compactness properties that are needed to apply the fixed point argument.

We observe that, for any $y \in L^2(0, T; H^1(\Omega \backslash \omega))$ the following identity holds:

$$
f(y, \nabla y) = F(y) y + G(y) \cdot \nabla y \tag{3.3}
$$

where

$$
F(y) = \int_0^1 \frac{\partial f}{\partial y} (\sigma y, \sigma \nabla y) d\sigma, \quad G(y) = \int_0^1 \frac{\partial f}{\partial \eta} (\sigma y, \sigma \nabla y) d\sigma.
$$

In view of the assumptions on f, (F, G) map $L^2(0, T; H^1(\Omega \backslash \omega))$ into $L^\infty((\Omega \backslash \omega) \times (0, T))^{N+1}$. Moreover

$$
||F(y)||_\infty \leq L, \quad ||G(y)||_\infty \leq L, \quad \forall\, y \in L^2(0, T; H^1(\Omega \backslash \omega)),
$$

where L is the Lipschitz constant of the function f.

Given any $v \in L^2(0,T;H^1(\Omega\backslash\omega))$ we consider the linearized system:

$$(3.4) \qquad \begin{cases} u_t - \Delta u + 1_{\Omega\backslash\omega}\, F(v)u + 1_{\Omega\backslash\omega}\, G(v) \cdot \nabla u = q\, 1_\omega \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

Observe that (3.4) is a linear system on the state u with potentials $a = F(v) \in L^\infty((\Omega\backslash\omega) \times (0,T))$ and $b = G(v) \in [L^\infty((\Omega\backslash\omega) \times (0,T))]^N$.

With this notation the system (3.4) can be rewritten as

$$(3.5) \qquad \begin{cases} u_t - \Delta u + 1_{\Omega\backslash\omega}\, au + 1_{\Omega\backslash\omega}\, b \cdot \nabla u = q\, 1_\omega \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

As we saw in section 2.3, $\delta > 0$ being fixed, for any $v \in L^2(0,T;H^1(\Omega\backslash\omega))$ we can define a control $q_\delta = q_\delta(x,t;v) \in L^2(0,T;L^2(\Omega))$ such that the solution u_δ of (3.5) satisfies

$$(3.6) \qquad |u_\delta(T)|_{L^2(\Omega)} \leq \delta.$$

Moreover, for every $R > 0$ and every potentials satisfying $||a||_{L^\infty([\Omega\backslash\omega]\times(0,T))} \leq R$, and $||b||_{L^\infty([\Omega\backslash\omega]\times(0,T))^N} \leq R$, we have:

$$(3.7) \qquad |q_\delta|_{L^2(0,T;L^2(\Omega))} \leq c|u_0|_{L^2(\Omega)}\,,$$

for all $\delta > 0$. Therefore, the controls q_δ are uniformly (with respect to v and δ) bounded in $L^2(\omega \times (0,T))$.

This allows to build a nonlinear mapping

$$N_\delta\colon L^2(0,T;H^1(\Omega\backslash\omega)) \to L^2(0,T;H^1(\Omega\backslash\omega)),\ \ N_\delta(v)=u, \tag{3.8}$$

where u satisfies (3.5) and (3.6).

Thus, the approximate control problem for system (3.2) is reduced to find a fixed point for the map N_δ. Indeed, if $v \in L^2(0,T;H^1(\Omega\backslash\omega))$ is such that $N_\delta(v)=u=v,\ u$ solution of (3.5) is actually solution of (3.2). Then, the control $q_\delta = q_\delta(v)$ is the one we were looking for since, by construction, $u_\delta = u_\delta(v)$ satisfies (3.6).

As we shall see, the nonlinear map N_δ satisfies the following two properties:

$$N_\delta \quad \text{is continuous and compact,} \tag{3.9}$$

$$\begin{cases} \text{the range of } N_\delta \text{ is bounded, i.e.,} \\ \exists\, M>0:\ |N_\delta(v)|_{L^2(0,T;H^1(\Omega\backslash\omega))} \le M, \quad \forall\, v\in L^2(0,T;H^1(\Omega\backslash\omega)). \end{cases} \tag{3.10}$$

In view of these two properties and as a consequence of Schauder's fixed point Theorem, the existence of a fixed point of N_δ follows immediately.

We shall return later to the proof of (3.9) and (3.10). By the moment let us assume that these properties hold. Then, by Schauder's fixed point Theorem, we have found a control q_δ in $L^2(0,T;L^2(\Omega))$ such that the solution u_δ of

$$\begin{cases} u_\delta' - \Delta u_\delta + 1_{\Omega\backslash\omega} f(u_\delta, \nabla u_\delta) = q_\delta(x,t)1_\omega \quad \text{in} \quad \Omega\times(0,T) \\ u_\delta = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u_\delta(x,0)=u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{3.11}$$

satisfies

$$|u_\delta(T)| \le \delta \tag{3.12}$$

with an estimate of the form

$$|q_\delta|_{L^2(0,T;L^2(\Omega))} \leq c|u_0|_{L^2(\Omega)} \tag{3.13}$$

with c indepedent of δ.

Passing to the limit as $\delta \to 0$, as in section 2, we deduce the existence of a limit control $q \in L^2(0,T;L^2(\Omega))$ such that the solution u of (3.2) satisfies (1.3) and (1.4).

Let us now return to the proof of (3.9) and (3.10).

Continuity of N_δ. Assume that $v_j \to v$ in $L^2(0,T;H^1(\Omega\backslash\omega))$. Then the potentials $a_j = F(v_j)$ and $b_j = G(v_j)$ are such that

$$\begin{cases} a_j = F(v_j) \to a = F(v) \quad \text{in} \quad L^p([\Omega\backslash\omega]\times(0,T)) \\ b_j = G(v_j) \to b = G(v) \quad \text{in} \quad L^p([\Omega\backslash\omega]\times(0,T)) \end{cases} \tag{3.14}$$

for all $1 \leq p < \infty$ and

$$||a_j||_{L^\infty([\Omega\backslash\omega]\times(0,T))} \leq L, \quad ||b_j||_{L^\infty([\Omega\backslash\omega]\times(0,T))^N} \leq L, \tag{3.15}$$

L being the Lipschitz constant of f. According to Theorem 2.1 the corresponding controls are uniformly bounded:

$$|q_j|_{L^2(0,T;L^2(\Omega))} \leq c, \quad \forall j \geq 1 \tag{3.16}$$

and, more precisely,

$$q_j = \hat{\varphi}_j \quad \text{in} \quad \omega\times(0,T) \tag{3.17}$$

where $\hat{\varphi}_j$ solves

$$\begin{cases} -\varphi_t - \Delta\varphi + 1_{\Omega\backslash\omega}F(v_j)\varphi - \operatorname{div}(1_{\Omega\backslash\omega}G(v_j)\varphi) = 0 \quad \text{in} \quad \Omega\times(0,T) \\ \varphi = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ \varphi(T) = \hat{\varphi}_j^0 \quad \text{in} \quad \Omega \end{cases} \tag{3.18}$$

with the datum $\hat{\varphi}_j^0$ minimizing the corresponding functional in $L^2(\Omega)$. We also have

$$|\hat{\varphi}_j^0|_{L^2(\Omega)} \leq c. \tag{3.19}$$

By extracting subsequences we have

$$\hat{\varphi}_j^0 \rightharpoonup \hat{\varphi}^0 \quad \text{weakly in} \quad L^2(\Omega) \tag{3.20}$$

and in view of (3.14)-(3.15), we deduce that

$$\hat{\varphi}_j \rightharpoonup \hat{\varphi} \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)) \tag{3.21}$$

where $\hat{\varphi}$ solves

$$\begin{cases} -\varphi_t - \Delta\varphi + 1_{\Omega\backslash\omega} F(v)\varphi - \operatorname{div}(1_{\Omega\backslash\omega} G(v)\varphi) = 0 \quad \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = \varphi^0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.22}$$

We also have that

$$\partial_t \hat{\varphi}_j \quad \text{is bounded in} \quad L^2(0,T;H^{-1}(\Omega)), \tag{3.23}$$

and, once again, by Aubin-Lions compactness lemma, it follows that

$$\hat{\varphi}_j \to \hat{\varphi} \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega\backslash\omega)). \tag{3.24}$$

Consequently

$$q_j \to q \quad \text{in} \quad L^2(0,T;L^2(\Omega\backslash\omega)) \tag{3.25}$$

where

$$q = \hat{\varphi} \quad \text{in} \quad \omega \times (0,T). \tag{3.26}$$

It is then easy to see that

$$u_j \to u \quad \text{in} \quad L^2(0,T,L^2(\Omega\backslash\omega)), \tag{3.27}$$

where

$$\begin{cases} u_t - \Delta u + 1_{\Omega\backslash\omega}\, F(v)u + 1_{\Omega\backslash\omega} G(v)\cdot\nabla u = q\, 1_\omega & \text{in} \quad \Omega\times(0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{3.28}$$

and

$$|u(T)|_{L^2(\Omega)} \le \delta. \tag{3.29}$$

To conclude the continuity of N_δ, it is sufficient to check that the limit $\hat{\varphi}^0$ in (3.20) is the minimizer of the function J_δ associated to the limit control problem (3.28)-(3.29). To do this, given $\psi^0 \in L^2(\Omega)$ we have to show that

$$J_\delta(\hat{\varphi}^0) \le J_\delta(\psi^0). \tag{3.30}$$

But this is immediate since, by lower semicontinuity, we have

$$J_\delta(\hat{\varphi}^0) \le \liminf_{j\to\infty} J_{\delta,j}(\hat{\varphi}^0_j),$$

on one hand,

$$J_\delta(\psi^0) = \liminf_{j\to\infty} J_{\delta,j}(\psi^0), \quad \forall\, \psi^0 \in L^2(\Omega)$$

on the other one, and finally

$$J_{\delta,j}(\hat{\varphi}^0_j) \le J_{\delta,j}(\psi^0), \quad \forall\, \psi^0 \in L^2(\Omega), \;\; j \ge 1$$

since $\hat{\varphi}_j^0$ is the minimizer of $J_{\delta,j}$.

Compactness of N_δ. The arguments above show that when v lies in a bounded set B of $L^2(0,T;H^1(\Omega\backslash\omega))$, $u = N_\delta(v)$ also lies in an bounded set of $L^2(0,T;H^1(\Omega\backslash\omega))$. We have to show that $N_\delta(B)$ is relatively compact in $L^2(0,T;H^1(\Omega\backslash\omega))$. Indeed, we have

$$(3.31)\qquad \begin{cases} u_t - \Delta u = \beta \quad \text{in} \quad \Omega\times(0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases}$$

with $\beta = q1_\omega - 1_{\Omega\backslash\omega}\,F(v)u - 1_{\Omega\backslash\omega}G(v)\cdot\nabla u$ which is uniformly bounded in $L^2(0,T;L^2(\Omega))$. By means of the regularizing effect of the heat equation we can obtain that the restriction u to $\Omega\backslash\omega$ belongs to $W(0,T;H^2(\Omega\backslash\omega),H^1(\Omega\backslash\omega))$. We remember that $W(0,T;X,Y) = \{u \in L^2(0,T;X); u_t \in L^2(0,T;Y)\}$. Thus, the compactness of N_δ is a consequence of the compactness of the embedding of $W(0,T;H^2(\Omega\backslash\omega),H^1(\Omega\backslash\omega))$ into $L^2(0,T;H^1(\Omega\backslash\omega))$.

Boundedness of the range of N_δ. According to (3.13), there exists $c > 0$ such that the control $q = q(v)$ satisfies $|q(v)|_{L^2(0,T;L^2(\Omega\backslash\omega))} \leq c$. Classical energy estimates for the system (3.28) show that $|u(v)|_{L^2(0,T;H^1(\Omega\backslash\omega))} \leq c$ as well, since the potentials involved in it are uniformly bounded.

This concludes the proof of Theorem 1.1. $\square$

Remark 3.1. We have proceeded as follows: First, we have applied the fixed point argument on the approximate controllability property with $\delta > 0$ to get the approximate controllability of the semilinear system and then we have let $\delta \to 0$ to obtain the null controllability of the semilinear system. We could have also proceeded applying directly the fixed point argument on the null controllability property that Theorem 2.1 provides for the linearized system. □

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