From averaged to simultaneous controllability of parameter dependent finite-dimensional systems*

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Abstract

We consider a linear finite dimensional control system depending on unknown parameters. We aim to design controls, independent of the parameters, to control the system in some optimal sense. We discuss the notions of averaged control, according to which one aims to control only the average of the states with respect to the unknown parameters, and the notion of simultaneous control in which the goal is to control the system for all values of these parameters. We show how these notions are connected through a penalization process. Roughly, averaged control is a relaxed version of the simultaneous control property, in which the differences of the states with respect to the various parameters are left free, while simultaneous control can be achieved by reinforcing the averaged control property by penalizing these differences. We show however that these two notions require of different rank conditions on the matrices determining the dynamics and the control. When the stronger conditions for simultaneous control are fulfilled, one can obtain the later as a limit, through this penalization process, out of the averaged control property.

AMS subject classification:
Key words: Controllability, parameter dependent system, averaged control, simultaneous control, penalization.

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1 Introduction

We consider a parameter dependent control system:

$$\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u \qquad (t \in (0,T))\,, \tag{1.1a}$$

$$y_\zeta(0) = \mathrm{y}^i_\zeta\,. \tag{1.1b}$$

In order to fix the notation, all along this paper, $\zeta \in \Omega$ is a random parameter (the system's parameter) following a probability law μ, with $(\Omega, \mathcal{F}, \mu)$ a probability space, $X = \mathbb{R}^n$ is the state space and $U = \mathbb{R}^m$ the control one. We assume that for every $\zeta \in \Omega$, $A_\zeta \in \mathcal{L}(X)$ and $B_\zeta \in \mathcal{L}(U, X)$.

The control $t \mapsto u(t) \in U$ is assumed to be independent of the parameter ζ whereas the state $y_\zeta(t) = y_\zeta(t; u) \in X$ is time and parameter dependent. In addition, by Duhamel formula, y_ζ can be represented as follows:

$$y_\zeta(t; u) = e^{tA_\zeta}\mathrm{y}^i_\zeta + \int_0^T e^{(t-s)A_\zeta} B_\zeta u(s)\,\mathrm{d}s \qquad (\zeta \in \Omega\,,\ t \geqslant 0\,,\ u \in L^2_{loc}(\mathbb{R}_+, U))\,. \tag{1.2}$$

Let us also define the space:

$$L^2(\Omega, X; \mu) = \left\{ (\mathrm{y}_\zeta)_\zeta \in X^\Omega\,,\ \int_\Omega \|\mathrm{y}_\zeta\|^2_X \,\mathrm{d}\mu_\zeta \right\}\,, \tag{1.3}$$

which is an Hilbert space endowed with the scalar product:

$$\langle \mathrm{y}_\zeta, \mathrm{z}_\zeta \rangle_{L^2(\Omega, X;\mu)} = \int_\Omega \langle \mathrm{y}_\zeta, \mathrm{z}_\zeta \rangle_X \,\mathrm{d}\mu_\zeta \qquad ((\mathrm{y}_\zeta)_\zeta, (\mathrm{z}_\zeta)_\zeta \in L^2(\Omega, X; \mu))\,.$$

In section 2 we introduce precise conditions on $\zeta \mapsto (A_\zeta, B_\zeta)$ ensuring that for every $t \geqslant 0$ and every $u \in L^2_{loc}(\mathbb{R}_+, U)$, $(y_\zeta(t; u))_\zeta \in L^2(\Omega, X; \mu)$ whenever the parameter-dependent initial data satisfy $(\mathrm{y}^i_\zeta)_\zeta \in L^2(\Omega, X; \mu)$.

This paper is devoted to analyse the following controllability problems.

- **Averaged controllability:** The system is said to be averaged controllable in time $T > 0$ if, for every $(\mathrm{y}^i_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ and every $\mathrm{y}^f \in X$, there exists $u \in L^2([0,T], U)$ such that:

$$\int_\Omega y_\zeta(T; u)\,\mathrm{d}\mu_\zeta = \mathrm{y}^f\,, \tag{1.4}$$

  In other words, averaged controllability is the control of the expectation of the system's output. This notion is illustrated on Figure 1a.

- **Exact simultaneous controllability:** The system is said to be exactly simultaneously controllable in time $T > 0$ if, for every $(\mathrm{y}^i_\zeta)_\zeta, (\mathrm{y}^f_\zeta)_\zeta \in L^2(\Omega, X; \mu)$, there exists $u \in L^2([0,T], U)$ such that:

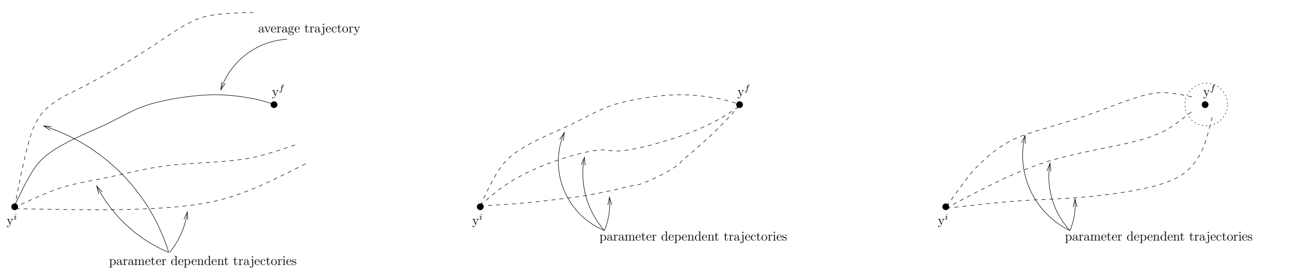
$$y_\zeta(T; u) = \mathrm{y}^f_\zeta \qquad (\zeta \in \Omega \quad \mu\text{−a.e.})\,, \tag{1.5}$$

  This notion is illustrated on Figure 1b.

- **Approximate simultaneous controllability:** The system is said to be approximately simultaneously controllable in time $T > 0$ if, for every $(\mathrm{y}^i_\zeta)_\zeta, (\mathrm{y}^f_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ and every $\varepsilon > 0$, there exists $u \in L^2([0,T], U)$ such that:

$$\int_\Omega \left\| y_\zeta(T; u) - \mathrm{y}^f_\zeta \right\|^2_X \mathrm{d}\mu_\zeta \leqslant \varepsilon \,. \tag{1.6}$$

  This notion is illustrated on Figure 1c.



(a) Averaged controllability. (b) Simultaneous controllability. (c) Approximate Simultaneous controllability.

Figure 1: Different controllability notions, introduced in (1.4), (1.5) and (1.6), for parameter dependent systems, with initial condition and target independent of ζ.

Remark 1.1. *1. Even if the system* (1.1) *is controllable in average, this fact does not give any information on the variance of the outputs.*

2. It is obvious that the exact simultaneous controllability property implies the averaged controllability and the approximate simultaneous controllability ones. In addition, one can find systems which are controllable in average (resp. approximatively simultaneously controllable) which are not exactly simultaneously controllable.

Moreover, the approximate simultaneous controllability property implies the averaged controllability one. In fact, the approximate simultaneous controllability property ensures that given $T > 0$, $(\mathrm{y}^i_\zeta)_\zeta \in L^2(\Omega, X; \mu)$, $\mathrm{y}^f \in X$ *and* $\varepsilon > 0$, *there exists* $u^\varepsilon \in L^2([0,T], U)$ *such that*

$$\left\| y_\zeta(T; u^\varepsilon) - \mathrm{y}^f \right\|^2_{L^2(\Omega, X;\mu)} \leqslant \varepsilon.$$

But, by Cauchy-Schwarz inequality,

$$\left\| \int_\Omega \left(y_\zeta(T; u^\varepsilon) - \mathrm{y}^f \right) \mathrm{d}\mu_\zeta \right\|^2_X \leqslant \int_\Omega \left\| y_\zeta(T; u^\varepsilon) - \mathrm{y}^f \right\|^2_X \mathrm{d}\mu_\zeta.$$

Thus, the system is approximatively controllable in average i.e. the linear and continuous map $\Phi : u \in L^2([0,T], U) \mapsto \int_\Omega \int_0^T e^{(t-t)A_\zeta} B_\zeta u(t) \,\mathrm{d}t\mathrm{d}\mu_\zeta \in X$ *has a dense image in* X. *But since* X *is a finite dimensional vector space, we obtain* $\operatorname{Im} \Phi = X$, *i.e. the system is controllable in average.*

3. *There is no natural ordinary differential equation describing the average* $Y(t) = \int_\Omega y_\zeta(t)\,\mathrm{d}\mu_\zeta$*, except when* A_ζ *is independent of* ζ *for which we have:* $\dot{Y} = AY + \left(\int_\Omega B_\zeta\,\mathrm{d}\mu_\zeta\right)u$*. In this particular case, the averaged controllability property is equivalent to the controllability of the pair* $(A, \int_\Omega B_\zeta\,\mathrm{d}\mu_\zeta)$*.*

4. *When* $\Omega = \{\zeta_1, \cdots, \zeta_K\}$ *is of finite cardinal, the simultaneous controllability is equivalent to the classical controllability one for the augmented system:*

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{y} + \mathbf{B}u\,,$$

with:

$$\boldsymbol{y} = \begin{pmatrix} y_{\zeta_1} \\ \vdots \\ y_{\zeta_K} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} A_{\zeta_1} & & 0 \\ & \ddots & \\ 0 & & A_{\zeta_K} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} B_{\zeta_1} \\ \vdots \\ B_{\zeta_K} \end{pmatrix}.$$

And the controllability of this system is equivalent to the Kalman rank condition:

$$\operatorname{rank}\left[\mathbf{B}\ \mathbf{AB}\ \cdots\ \mathbf{A}^{K\dim X-1}\mathbf{B}\right] = K\dim X\,.$$

5. *In the previous item, we have seen that the simultaneous controllability property when the cardinal of* Ω *is finite can be interpreted in terms of a classical rank condition. But, when* Ω *is infinite, the output of the system is the function* $\zeta \in \Omega \mapsto y_\zeta(T) \in X$*, living in an infinite-dimensional space. The first issue to be addressed is the choice of the norm in that space.*
In the following, we choose the L^2*-norm. Accordingly, the fact that* $y_\zeta(T) = \mathrm{y}^f_\zeta$ *holds for almost every* $\zeta \in \Omega$ *with respect to the measure* μ *is guaranteed by the fact that* $\displaystyle\int_\Omega \left\|y_\zeta(T) - \mathrm{y}^f_\zeta\right\|^2_X \mathrm{d}\mu_\zeta = 0$*.*
This choice is natural, since in the particular case where $\mathrm{y}^f_\zeta = \mathrm{y}^f$ *is independent of* ζ *and* $\displaystyle\int_\Omega y_\zeta(T)\,\mathrm{d}\mu_\zeta = \mathrm{y}^f$*, the integral* $\displaystyle\int_\Omega \left\|y_\zeta(T) - \mathrm{y}^f_\zeta\right\|^2_X \mathrm{d}\mu_\zeta$ *is the variance of the system's output.*
Thus, the L^2*-norm approach is natural from a probabilistic point of view but one could also use any* $L^p(\Omega, X; \mu)$*-norm. In the next item, we mention some existing literature when considering the* L^∞*-norm.*

6. *For parameter dependent systems, J.-S. Li and N. Khaneja [5] (see also J.-S. Li [4]) introduced the notion of ensemble controllability: The system is said to be ensemble controllable in time* $T > 0$ *if, for every* ε *and every* $\mathrm{y}^i_\zeta, \mathrm{y}^f_\zeta \in X$*, there exists* $u \in L^2([0,T], U)$ *such that:*

$$\left\|y_\zeta(T;u) - \mathrm{y}^f_\zeta\right\|_X \leqslant \varepsilon \qquad (\zeta \in \Omega)\,. \tag{1.7}$$

This notion of ensemble controllability, which does not seem to have a probabilistic interpretation, is similar to our notion of approximate simultaneous controllability *above, where the* $L^2(\Omega, X; \mu)$*-norm is replaced by the* $L^\infty(\Omega, X)$ *one.*

Controlling the average (or the expectation) of a parameter dependent system is not a new problem. It has been previously studied when a classical control system is perturbed by an additional drift (V. A. Ugrinovskii [13], A. V. Savkin and I. R. Petersen [11], I. R. Petersen [9]). We present here a different frame for which the uncertainty is inside the system itself, and not due to some external noise. Taking into account that we only know the probability distribution of the unknown parameter, it is natural to try to control the expectation of the output of the system.

In [14], it has been shown that the averaged controllability property is equivalent to a Kalman rank condition of infinite order. However, even if the average of the system is controlled, this fact does not ensure that the output of system is close to the desired target for any specific realisation of the parameter. Of course, the ideal situation arises when all the parameter dependent trajectories exactly reach the desired target. This corresponds, precisely, to the notion of simultaneous controllability.

Classically, the simultaneous exact controllability property corresponds, by duality, to the one of simultaneous exact observability (see § 3.2). However, when Ω is an infinite dimensional set, those properties are difficult to check in practice. This is why, in this article, we show that, if the simultaneous controllability property holds, then the approximate simultaneous control can be achieved from the averaged controls by means of a penalisation procedure and at the limit, when the penalizing parameter goes to ∞, we recover the simultaneous control.

The notion of simultaneous controllability was introduced by D. L. Russell [10] (see also J.-L. Lions [6, Chapter 5]) for partial differential equations. As mentioned above, when dealing with finite dimensional systems and when the parameter ranges over a finite set, the problem can be handled through classical rank conditions. However, the issue is much more complex when the parameter ranges over an infinite set.

The averaged controllability property has already been tackled by E. Zuazua et al [14, 3, 8] for some relevant PDE models. However, the link between the averaged and simultaneous controllability in that setting has not been yet developed. The tools developed here could be used to handle PDE and, in general, infinite-dimensional systems, but this requires further efforts.

In general, the simultaneous controllability problem is set in an infinite dimensional space (this holds when the cardinal of Ω is infinite). In infinite dimensional spaces the choice of the norm is important and an appropriate choice has to be done. According to the 5^{th} item of Remark 1.1, we chose the weighted L^2-norm, that corresponds to the variance. More precisely, the simultaneous controllability property (1.5) holds if:

$$\int_\Omega \|y_\zeta(T) - \mathrm{y}_\zeta^f\|_X^2 \,\mathrm{d}\mu_\zeta = 0\,. \tag{1.8}$$

Consequently, in section 4, we introduce the parametrized optimal control problems:

$$\begin{array}{ll} \min & \mathcal{J}_\kappa(u) = \dfrac{1}{2}\displaystyle\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t + \kappa \int_\Omega \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_X^2 \,\mathrm{d}\mu_\zeta \\ \Big| & \displaystyle\int_\Omega y_\zeta(T;u)\,\mathrm{d}\mu_\zeta = \int_\Omega \mathrm{y}_\zeta^f \,\mathrm{d}\mu_\zeta \end{array} \qquad (\kappa \geqslant 0)\,,$$

with y_ζ the solution of (1.1) with control u and initial condition y_ζ^i.
We will see in Theorem 4.1 that, at the limit $\kappa \to \infty$, the minimum u_κ is a control which minimizes the variance of the system's outputs. For instance, we will see that if the sequence $(\mathcal{J}_\kappa(u_\kappa))_\kappa$ is bounded then the sequence $(u_\kappa)_\kappa$ converges to a control u_∞ which solves the minimisation problem:

$$\begin{array}{ll} \min & \dfrac{1}{2}\displaystyle\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t \\ \Big| & \displaystyle\int_\Omega \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_X^2 \,\mathrm{d}\mu_\zeta = 0\,. \end{array}$$

In other words, u_∞ is the HUM control (the control obtained from the Hilbert Uniqueness Method) for the simultaneous control problem.
More generally, the result of Theorem 4.1 can be summarized in Table 1, where we have defined $(\mathrm{y}^\star_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ as the minimizer of $\|\mathrm{y}_\zeta - \mathrm{y}^f_\zeta\|_{L^2(\Omega,X;\mu)}$ under the constraints $\int_\Omega \mathrm{y}_\zeta \,\mathrm{d}\mu_\zeta = \int_\Omega \mathrm{y}^f_\zeta \,\mathrm{d}\mu_\zeta$ and $(\mathrm{y}_\zeta)_\zeta \in \overline{\{y_\zeta(T; u)\,,\ u \in L^2([0, T], U)\}}$.

		$\left(\lVert y_\zeta(T; u_\kappa) - \mathrm{y}^f_\zeta\rVert_{L^2(\Omega,X;\mu)}\right)_\kappa$	
		converge to 0	do not converge to 0
$(\lVert u_\kappa\rVert_{L^2})_\kappa$	bounded	simultaneous exact controllability	simultaneous exact controllability to $\mathrm{y}^\star_\zeta$
	unbounded	simultaneous approximate controllability	simultaneous approximate controllability to $\mathrm{y}^\star_\zeta$

Table 1: Possible behaviors as $\kappa \to \infty$, $\mathrm{y}^\star_\zeta$ beeing defined by (4.4).

This penalty argument is natural and has already been used in control theory. In J.-L. Lions [7] it was used to achieve approximate controllability as the limit of a sequence of optimal control problems (see also L. A. Fernández and E. Zuazua [2] for semi-linear heat equations). This penalty method has also been used numerically, for the numerical approximation of null controls for parabolic problems (see F. Boyer [1]).

This paper is organized as follows.
In section 2, we give some conditions on A_ζ, B_ζ and y^i_ζ ensuring that the problem we are considering is well defined. Then, in section 3, we recall some known results about averaged controllability and we describe the duality approach for simultaneous controllability. In section 4, we present the penalty method and give some convergence results. More precisely, in this section we prove the main theorem (Theorem 4.1) of this article. Then, in section 5, we present some results for a further numerical development of the case where Ω is a countable set. Finally, in section 6, we conclude this work by some general remarks and open problems.

2 Admissibility conditions

In this section, we give some conditions ensuring that $\int_\Omega y_\zeta(t)\,\mathrm{d}\mu_\zeta$ and $\int_\Omega \|y_\zeta(t)\|^2\,\mathrm{d}\mu_\zeta$ are well defined.
Let us consider the Hilbert space $L^2(\Omega, X; \mu)$ defined by (1.3). Using Cauchy-Schwarz together with $\int_\Omega \mathrm{d}\mu_\zeta = 1$, leads to:

$$\left\| \int_\Omega \mathrm{y}_\zeta \,\mathrm{d}\mu_\zeta \right\|_X^2 \leqslant \|\mathrm{y}_\zeta\|^2_{L^2(\Omega,X;\mu)} \qquad ((\mathrm{y}_\zeta)_\zeta \in L^2(\Omega, X; \mu))\,.$$

Thus, in this paragraph, we only give conditions on A_ζ, B_ζ and μ such that $\|y_\zeta(t)\|_{L^2(\Omega,X;\mu)} < \infty$ and in all this article, we assume that initial and final condition are elements of $L^2(\Omega, X; \mu)$.
By Duhamel formula, the solution $y_\zeta(t) = y_\zeta(t; u)$ of (1.1) is given by (1.2), i.e.,

$$y_\zeta(t; u) = e^{tA_\zeta}\mathrm{y}^i_\zeta + \int_0^t e^{(t-s)A_\zeta} B_\zeta u(s)\,\mathrm{d}s \qquad (\zeta \in \Omega\,,\ t \geqslant 0)\,.$$

Lemma 2.1. *Set $(A_\zeta)_{\zeta\in\Omega} \in \mathcal{L}(X)^\Omega$. For every $T>0$ and every $\zeta\in\Omega$, there exists $\varsigma_\zeta(T)>0$ such that:*

$$\left\|e^{TA_\zeta^*}e^{TA_\zeta}\mathrm{y}\right\|_X \leqslant \varsigma_\zeta(T)\|\mathrm{y}\|_X \qquad (\mathrm{y}\in X)\,.$$

Assume:

$$\varsigma_\zeta(T)<\infty \qquad (\zeta\in\Omega \quad \mu-a.e.). \tag{2.1}$$

Then for every $T>0$, there exists $\varsigma(T)>0$ ($\varsigma(T)=\sup\limits_{\zeta\in\Omega}\varsigma_\zeta(T)$) such that:

$$\left\|e^{TA_\zeta}\mathrm{y}^i_\zeta\right\|_{L^2(\Omega,X;\mu)} \leqslant \varsigma(T)\|\mathrm{y}^i_\zeta\|_{L^2(\Omega,X;\mu)} \qquad ((\mathrm{y}^i_\zeta)_\zeta\in L^2(\Omega,X;\mu))\,. \tag{2.2}$$

Proof. The existence of $\varsigma_\zeta(T)$ is clear. The result follows from Cauchy-Schwarz inequality. □

Example 2.1. *Let $(\Omega,\mathcal{F},\mu)$ be a probability space. If for every $\zeta\in\Omega$, A_ζ is skew-adjoint, then* (2.2) *holds with $\varsigma(T)=1$. (In this case, we have $\varsigma_\zeta(T)=1$ for every $\zeta\in\Omega$.)*

Lemma 2.2. *Set $(A_\zeta)_{\zeta\in\Omega}\in\mathcal{L}(X)^\Omega$ and $(B_\zeta)_{\zeta\in\Omega}\in\mathcal{L}(U,X)^\Omega$. For every $T>0$ and every $\zeta\in\Omega$, there exists a constant $C_\zeta(T)>0$ such that:*

$$\left\|\int_0^T e^{(T-t)A_\zeta}B_\zeta u(t)\,\mathrm{d}t\right\|_X^2 \leqslant C_\zeta(T)\|u\|^2_{L^2([0,T],U)}\,.$$

Assume that:

$$\int_\Omega C_\zeta(T)\,\mathrm{d}\mu_\zeta<\infty\,. \tag{2.3}$$

Then for every $T>0$, there exists $C(T)>0$ such that:

$$\int_\Omega\left\|\int_0^T e^{(T-t)A_\zeta}B_\zeta u(t)\,\mathrm{d}t\right\|_X^2\mathrm{d}\mu_\zeta \leqslant C(T)\|u\|^2_{L^2([0,T],U)} \qquad (u\in L^2([0,T],U))\,.$$

Proof. The existence of $C_\zeta(T)>0$ independent of u is classical. The result follows from Minkowski and Cauchy-Schwarz inequalities. □

Thus, if A_ζ and B_ζ satisfies the assumption of lemmas 2.1 and 2.2, then for every $(\mathrm{y}^i_\zeta)_\zeta\in L^2(\Omega,X;\mu)$, $y(T;u)$ defined by (1.2) is an element of $L^2(\Omega,X;\mu)$.
From these two lemmas, we can derive the following corollaries:

Corollary 2.1. *Assume* $\mathrm{Card}\,\Omega<\infty$ *and set $\zeta\in\Omega\mapsto(A_\zeta,B_\zeta)\in\mathcal{L}(X)\times\mathcal{L}(U,X)$, then for every $(\mathrm{y}^i_\zeta)_\zeta\in L^2(\Omega,X;\mu)$, and every $u\in L^2_{loc}(\mathbb{R}_+,U)$, the solution $y_\zeta(t;u)$ of* (1.1) *belongs to $L^2(\Omega,X;\mu)$ for every $t\geqslant 0$.*

Corollary 2.2. *Assume $\Omega\subset\mathbb{R}^d$ is a bounded set and assume the map $\zeta\mapsto(A_\zeta,B_\zeta)$ is continuous on $\overline{\mathrm{co}(\Omega)}$, with $\mathrm{co}(\Omega)$ the smallest convex set containing Ω.*

Then for every $(\mathrm{y}^i_\zeta)_\zeta\in L^2(\Omega,X;\mu)$, every $u\in L^2_{loc}(\mathbb{R}_+,U)$ and every $t\geqslant 0$, the solution $y_\zeta(t;u)$ of (1.1) *belongs to $L^2(\Omega,X;\mu)$.*

Proof. Since X and U are finite dimensional spaces, for every $\zeta \in \overline{\mathrm{co}(\Omega)}$,

$$\varsigma_\zeta(T) := \sup_{\substack{\mathrm{y}\in X\,,\\ \|\mathrm{y}\|_X=1}} \left\| e^{TA_\zeta^*} e^{TA_\zeta} \mathrm{y} \right\|_X \qquad \text{and} \qquad C_\zeta(T) := \sup_{\substack{u\in L^2([0,T],U)\,,\\ \|u\|_{L^2([0,T],U)}=1}} \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta u(t)\,\mathrm{d}t \right\|_X^2$$

are well defined for every $\zeta \in \overline{\mathrm{co}(\Omega)}$ and every $T \geqslant 0$.
Moreover, since $\zeta \in \overline{\mathrm{co}(\Omega)} \mapsto (A_\zeta, B_\zeta) \in \mathcal{L}(X) \times \mathcal{L}(U,X)$ is continuous, the map $\zeta \in \overline{\mathrm{co}(\Omega)} \mapsto (\varsigma_\zeta(T), C_\zeta(T)) \in \mathbb{R}^2$ is continuous, thus bounded.
The result follows from lemmas 2.1 and 2.2. ☐

Remark 2.1. *Even if Corollary 2.1 can be proved directly, it can also be seen as a consequence of Corollary 2.2.*

Corollary 2.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and assume A_ζ skew-adjoint for every $\zeta \in \Omega$. If $\int_\Omega \|B_\zeta\|^2_{\mathcal{L}(U,X)}\,\mathrm{d}\mu_\zeta < \infty$, then for every $(\mathrm{y}^i_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ and every $u \in L^2_{loc}(\mathbb{R}_+, U)$, the solution $y_\zeta(t;u)$ of (1.1) belongs to $L^2(\Omega, X; \mu)$ for every $t \geqslant 0$.*

Proof. According to Lemma 2.1 and Example 2.1, we have $\left(e^{tA_\zeta}\mathrm{y}^i_\zeta\right)_\zeta \in L^2(\Omega, X; \mu)$. In addition, we have:

$$\left\| \int_0^t e^{(t-s)A_\zeta} B_\zeta u(s)\,\mathrm{d}s \right\|_X \leqslant \int_0^t \left\| e^{(t-s)A_\zeta} B_\zeta u(s) \right\|_X \mathrm{d}s = \int_0^t \|B_\zeta u(s)\|_X \,\mathrm{d}s \leqslant \sqrt{t}\, \|B_\zeta\|_{\mathcal{L}(U,X)} \|u\|_{L^2([0,t],U)}\,.$$

Thus the assumptions of Lemma 2.2 are fulfilled. ☐

3 Duality approach and Kalman rank conditions

Here and in the sequel we assume that the hypotheses of lemmas 2.1 and 2.2 are satisfied.

3.1 State of the art for averaged controllability

Let us recall some known results on averaged controllability for finite dimensional systems. These results are taken from [14].

Theorem 3.1 ([14] Theorem 1)**.** *System* (1.1) *fulfills the averaged controllability property* (1.4) *if and only if the following rank condition is satisfied:*

$$\operatorname{rank}\left[\int_\Omega A_\zeta^j B_\zeta\,\mathrm{d}\mu_\zeta\,,\ j \geqslant 0\right] = \dim X\,. \tag{3.1}$$

This result is based on duality arguments. More precisely, we introduce the (parameter dependent) adjoint system:

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta \qquad (t \in (0,T))\,, \tag{3.2a}$$

$$z_\zeta(T) = \mathrm{z}^f\,. \tag{3.2b}$$

Notice that even if this system depends of the parameter ζ the final condition z^f is independent of ζ.
The next result makes the link between averaged controllability, and averaged observability and gives also a link between the adjoint system and the control of minimal L^2-norm.

Theorem 3.2 ([14] Theorem 2)**.** *System* (1.1) *fulfills the averaged controllability property* (1.4) *if and only if the adjoint system* (3.2) *satisfies the averaged observability inequality:*

$$\bar{c}(T)\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \qquad (\mathrm{z}^f \in X)\,, \tag{3.3}$$

where $\bar{c}(T) > 0$ *is a constant independent of* z^f*.*
In addition, both conditions are equivalent to the rank condition (3.1)*.*

When these properties hold, the averaged control of minimal $L^2([0,T],U)$*-norm is given by:*

$$u(t) = \int_\Omega B_\zeta^* \bar{z}_\zeta(t)\,\mathrm{d}\mu_\zeta \qquad (t \in (0,T))\,, \tag{3.4}$$

where $\{\bar{z}_\zeta\}_\zeta$ *is the solution of the adjoint system* (3.2) *corresponding to the datum* $\mathrm{z}^f \in X$ *minimizing the functional:*

$$\begin{array}{rccl} J: & X & \longrightarrow & \mathbb{R} \\ & \mathrm{z}^f & \longmapsto & \dfrac{1}{2}\displaystyle\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t - \left\langle \mathrm{y}^f, \mathrm{z}^f \right\rangle_X + \int_\Omega \left\langle \mathrm{y}_\zeta^i,\, z_\zeta(0) \right\rangle_X \mathrm{d}\mu_\zeta\,. \end{array} \tag{3.5}$$

3.2 Observability inequality for exact simultaneous controllability

Let us define for every $\zeta \in \Omega$ the adjoint system of (1.1):

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta \qquad (t \in (0,T))\,, \tag{3.6a}$$

$$z_\zeta(T) = \mathrm{z}_\zeta^f\,. \tag{3.6b}$$

If the system (1.1) is simultaneously controllable then for every $(\mathrm{z}_\zeta^f)_\zeta \in L^2(\Omega, X; \mu)$,

$$\left\langle \left(y_\zeta(T) - \mathrm{y}_\zeta^f\right)_\zeta, \left(\mathrm{z}_\zeta^f\right)_\zeta \right\rangle_{L^2(\Omega,X;\mu)} = 0\,.$$

That is to say:

$$\int_0^T \left\langle u(t), \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\rangle_U \mathrm{d}t = \left\langle \left(\mathrm{y}_\zeta^f\right)_\zeta, \left(\mathrm{z}_\zeta^f\right)_\zeta \right\rangle_{L^2(\Omega,X;\mu)} - \left\langle \left(\mathrm{y}_\zeta^i\right)_\zeta, \left(z_\zeta(0)\right)_\zeta \right\rangle_{L^2(\Omega,X;\mu)}\,.$$

Let us then define the cost function $\mathfrak{J}$ by:

$$\begin{array}{rccl} \mathfrak{J}: & L^2(\Omega, X; \mu) & \longrightarrow & \mathbb{R} \\ & \left(\mathrm{z}_\zeta^f\right)_\zeta & \longmapsto & \dfrac{1}{2}\displaystyle\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t - \int_\Omega \left\langle \mathrm{y}_\zeta^f, \mathrm{z}_\zeta^f \right\rangle_X \mathrm{d}\mu_\zeta + \int_\Omega \left\langle \mathrm{y}_\zeta^i, z_\zeta(0) \right\rangle_X \mathrm{d}\mu_\zeta\,, \end{array} \tag{3.7}$$

where z_ζ is the solution of (3.6).
The only difference between the cost functions defined by (3.5) for averaged controllability and (3.7) for simultaneous controllability is that, for simultaneous controllability, we allowed the final condition of the adjoint system to depend on the parameter ζ.

Assuming that $\mathfrak{J}$ has a minimizer $(\hat{z}_\zeta)_\zeta \in L^2(\Omega, X; \mu)$, we obtain, by computing the first variation of $\mathfrak{J}$, that:

$$\hat{u}(t) = \int_\Omega B_\zeta^* \hat{z}_\zeta(t)\,\mathrm{d}\mu_\zeta \qquad (t \in [0,T] \quad \text{a.e.})\,. \tag{3.8}$$

It is clear that $\mathfrak{J}$ is convex. Thus, proving the existence of a minimizer $\left(\mathrm{z}_\zeta^f\right)_\zeta \in L^2(\Omega, X; \mu)$ for $\mathfrak{J}$ is equivalent to showing that $\mathfrak{J}$ is coercive, i.e. to the existence of a constant $\hat{c}(T) > 0$ such that:

$$\hat{c}(T) \int_\Omega \left\|\mathrm{z}_\zeta^f\right\|_X^2 \mathrm{d}\mu_\zeta \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\, \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \qquad \left(\left(\mathrm{z}_\zeta^f\right)_\zeta \in L^2(\Omega, X; \mu)\right). \tag{3.9}$$

where z_ζ is the solution of (3.6) with final condition z_ζ^f.

Summarizing this discussion, we end up with:

Theorem 3.3. *System* (1.1) *fulfills the exact simultaneous controllability property* (1.8) *if and only if the adjoint system* (3.6) *satisfies the exact simultaneous observability inequality* (3.9).

When these properties hold, the exact simultaneous control of minimal norm is given by (3.8)*, where* $\hat{z}_\zeta$ *is the solution of* (3.6) *with final condition* $\hat{\mathrm{z}}_\zeta^f$ *and* $(\hat{\mathrm{z}}_\zeta^f)_\zeta \in L^2(\Omega, X; \mu)$ *is the minimizer of* $\mathfrak{J}$ *defined by* (3.7).

3.3 The case where exact simultaneous controllability fails

Let us notice that very few systems have the property of simultaneous controllability. When $\operatorname{card}\Omega$ is finite, the situation is clear since simultaneous controllability follows from a Kalman rank on an augmented system (see 4^{th} item of Remark 1.1). But when $\operatorname{card}\Omega$ is infinite, the situation more complex. In propositions 3.1 and 3.2 we present two situations where simultaneous controllability cannot hold.

Proposition 3.1. *Set* $(\Omega, \mathcal{F}, \mu)$ *with* $\Omega = \{\zeta_j\,,\ j \in \mathbb{N}\} \subset \mathbb{R}$ *where* $\zeta_i \neq \zeta_j$ *for every* $i \neq j$ *and* $\mathcal{F} = \mathcal{P}(\Omega)$. *Assume there exists an open interval* I *of* $\mathbb{R}$ *such that* $\zeta \in \Omega \mapsto (A_\zeta, B_\zeta)$ *admits an analytic extension on* I *and assume there exists* $\zeta_\infty \in \overline{I \cap \operatorname{supp}\mu}$ *and a sequence* $(\zeta_{j_k})_{k\in\mathbb{N}} \in (I \cap \operatorname{supp}\mu)^{\mathbb{N}}$ *such that* $\lim\limits_{k\to\infty} \zeta_{j_k} = \zeta_\infty$ *and* $\zeta_{j_k} \neq \zeta_\infty$ *for every* $k \in \mathbb{N}$.
Then the system (1.1) *cannot be exactly simultaneously controllable.*

Example 3.1. *Consider the system* $\dot{y}_\zeta = (\zeta + 1)y_\zeta + u$ *with* $\zeta \in \left\{\frac{1}{n}\,,\ n \in \mathbb{N}^*\right\} = \Omega$ *with a probability measure* μ *such that* $\mu(\{\zeta\}) > 0$ *for every* $\zeta \in \Omega$. *This system is not exactly simultaneously controllable although the truncated system in which we consider* $\zeta \in \left\{\frac{1}{n}\,,\ n \in \{1, \cdots, N\}\right\}$ *with the probability measure* μ_N *given by* $\mu_N(\{\zeta\}) = \frac{\mu(\{\zeta\})}{\mu\left(\left\{\frac{1}{N}, \frac{1}{N-1}, \cdots, 1\right\}\right)}$, *for* $\zeta \in \left\{\frac{1}{n}\,,\ n \in \{1, \cdots, N\}\right\}$ *is simultaneously controllable, whatever* $N \in \mathbb{N}^*$ *is.*

In fact, for this truncated system, the precise values of the measure μ_N *are not important since its simultaneous controllability can be understood though the augmented system:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ \vdots \\ y_{\frac{1}{N}} \end{pmatrix} = \begin{pmatrix} 1+1 & & 0 \\ & \ddots & \\ 0 & & 1+\frac{1}{N} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\frac{1}{N}} \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} u\,.$$

The Kalman matrix of this system is:

$$\begin{pmatrix} 1 & 1+1 & \dots & (1+1)^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & 1+\frac{1}{N} & \dots & (1+\frac{1}{N})^{N-1} \end{pmatrix},$$

which is a Vandermonde matrix of determinant $\displaystyle\prod_{1\leqslant i<j\leqslant N}\left(\frac{1}{i}-\frac{1}{j}\right)\neq 0$.
In addition, since this determinant goes to 0 *as* N *goes to* ∞*, it is not surprising that the system with the full set of parameters* $\left\{\frac{1}{n}\,,\ n\in\mathbb{N}^*\right\}$ *is not simultaneously controllable.*

Proof of Proposition 3.1. First of all, the solution of (1.1) with initial condition $y_\zeta^i=0$ and control u is:

$$y_\zeta(t)=\int_0^t e^{(t-s)A_\zeta}B_\zeta u(s)\,\mathrm{d}s \qquad (\zeta\in\Omega\,,\ t\geqslant 0)\,.$$

Since $\zeta\mapsto(A_\zeta,B_\zeta)$ admits an analytic extension in I, $\zeta\mapsto y_\zeta(t)$ can also be extend to an analytic function on I for every $t>0$. Then, by analytic unique continuation, if for some $K\in\mathbb{N}$ we have $y_{\zeta_{j_k}}(T)=0$ for every $k>K$, then $y_\zeta(T)=0$ for every $\zeta\in I\cap\Omega$. Thus, any final state such that $\mathrm{y}^f_{\zeta_{j_k}}=0$ for $k>K$ and $\mathrm{y}^f_\zeta\neq 0$ for $\zeta\in\Omega\setminus\{\zeta_{j_k}\,,\ k>K\}$ cannot be reached from the initial state $\mathrm{y}^i_\zeta=0$. ☐

Let us notice that a similar result holds when dealing with Lipschitz functions.

Proposition 3.2. *Set* $(\Omega,\mathcal{F},\mu)$ *a probability space. Assume* $\Omega\subset\mathbb{R}^d$*, with* $\overset{\circ}{\Omega}\neq\varnothing$ *and* $\mathcal{F}=\mathcal{B}(\Omega)$*.*
Set $\zeta_0\in\overset{\circ}{\Omega}$ *and* $\varepsilon_0>0$ *such that* $B_{\varepsilon_0}(\zeta_0)\subset\Omega$*, with* $B_{\varepsilon_0}(\zeta_0)$ *the ball of* $\mathbb{R}^d$ *centered on* ζ_0 *of radius* ε_0*.*

Assume $\mu|_{B_{\varepsilon_0}(\zeta_0)}$ *is a nonnegative regular measure and the map* $\zeta\mapsto(A_\zeta,B_\zeta)$ *is Lipschitz-continuous on* $B_{\varepsilon_0}(\zeta_0)$*. Then the system* (1.1) *cannot be exactly simultaneously controllable.*

Example 3.2. *Let us consider the system* $\dot y_\zeta=(\zeta+1)y_\zeta+u$ *with* $\zeta\in[0,1]=\Omega$ *is the random parameter and for which we consider the probability measure* μ *given by* $\mathrm{d}\mu_\zeta=\mathrm{d}\zeta$*. Due to Proposition 3.2, this system is not simultaneously controllable since* $\zeta\in[0,1]\mapsto(\zeta+1,1)\in\mathbb{R}\times\mathbb{R}$ *is Liepschitz-continuous and* μ *is a nonnegative regular measure.*

The proof of this result is based on the three following lemmas:

Lemma 3.1. *Set* $(\Omega,\mathcal{F},\mu)$ *a probability space. Then the system* $\dot y_\zeta=Ay_\zeta+Bu$ *cannot be exactly simultaneously controllable unless for every* $e\in\mathcal{F}$*,* $\mu(\Omega\cap e)$ *is either* 1 *or* 0*.*

Remark 3.1. *Note that in this Lemma we are considering identical copies of the same system. The Lemma ensures that simultaneous controllability necessarily fails if we have more than one copy of the same system.*

Proof. Assume by contradiction there exists $e\in\mathcal{F}$ so that $\mu(\Omega\cap e)\in(0,1)$ and the system $\dot y_\zeta=Ay_\zeta+Bu$ is simultaneously controllable. Let us now consider the final target y^f_ζ such that $\mathrm{y}^f_\zeta=0$ for every $\zeta\in e$ and $\mathrm{y}^f_\zeta\neq 0$ for every $\zeta\in\Omega\setminus e$.
Then it is clear that this final target cannot be reached from the initial point $\mathrm{y}^i_\zeta=0$. ☐

Lemma 3.2. *Set* $T>0$*,* $(\Omega,\mathcal{F},\mu)$ *a probability space and let* $\zeta\mapsto(A_\zeta,B_\zeta)$ *be a bounded and integrable map and assume there exists* $\hat c(T)>0$ *such that:*

$$\int_0^T\left\|\int_\Omega B_\zeta^* e^{tA_\zeta^*}\mathrm{z}_\zeta\,\mathrm{d}\mu_\zeta\right\|_U^2\mathrm{d}t\geqslant\hat c(T)\int_\Omega\|\mathrm{z}_\zeta\|_X^2\,\mathrm{d}\mu_\zeta \qquad ((\mathrm{z}_\zeta)_\zeta\in L^2(\Omega,X;\mu))\,.$$

Let $\varepsilon > 0$ and define an integrable map $\zeta \mapsto (A^\varepsilon_\zeta, B^\varepsilon_\zeta)$ such that:

$$\operatorname*{ess\,sup}_{\zeta\in\Omega} \left(\|A_\zeta - A^\varepsilon_\zeta\|_{\mathcal{L}(X)} + \|B_\zeta - B^\varepsilon_\zeta\|_{\mathcal{L}(U,X)}\right) \leqslant \varepsilon\,.$$

Then, for small enough $\varepsilon > 0$, there exists a constant $\hat{c}^\varepsilon(T) > 0$ such that:

$$\int_0^T \left\| \int_\Omega B^{\varepsilon *}_\zeta e^{tA^{\varepsilon *}_\zeta} \mathrm{z}_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \geqslant \hat{c}^\varepsilon(T) \int_\Omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\Omega, X;\mu))\,.$$

This lemma ensures that, if system $\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u$ is exactly simultaneously controllable then, under small enough perturbation assumptions, the perturbed system A_ζ and B_ζ is still exactly simultaneously controllable.

Proof. Since $(A^\varepsilon_\zeta, B^\varepsilon_\zeta)$ is uniformly convergent to (A_ζ, B_ζ) as $\varepsilon \to 0$, we obtain:

$$\int_0^T \left\| \int_\Omega \left(B^*_\zeta e^{tA^*_\zeta} \mathrm{z}_\zeta - B^{\varepsilon *}_\zeta e^{tA^{\varepsilon *}_\zeta} \mathrm{z}_\zeta \right) \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \leqslant C(\varepsilon) \int_\Omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\Omega, X;\mu))\,,$$

with $C(\varepsilon) > 0$ and $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$.

But, using Minkowski inequality, we obtain, for every $\varepsilon > 0$,

$$\begin{aligned}
\left(\int_0^T \left\| \int_\Omega B^{\varepsilon *}_\zeta e^{tA^{\varepsilon *}_\zeta} z_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} &\geqslant \left(\int_0^T \left\| \int_\Omega B^*_\zeta e^{tA^*_\zeta} z_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad - \left(\int_0^T \left\| \int_\Omega \left(B^*_\zeta e^{tA^*_\zeta} z_\zeta - B^{\varepsilon *}_\zeta e^{tA^{\varepsilon *}_\zeta} z_\zeta \right) \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\geqslant \left(\sqrt{\hat{c}(T)} - \sqrt{C(\varepsilon)} \right) \left(\int_\Omega \|z_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta \right)^{\frac{1}{2}} \qquad ((z_\zeta)_\zeta \in L^2(\Omega, X;\mu))\,.
\end{aligned}$$

This proves the desired result for small enough ε. ☐

Let us finally show that the exact simultaneous controllability property is inherited by reduction of the set of parameters.

Lemma 3.3. *Set $T > 0$, $(\Omega, \mathcal{F}, \mu)$ a probability space and $(A_\zeta, B_\zeta)_\zeta \in (\mathcal{L}(X) \times \mathcal{L}(U,X))^\Omega$. Assume there exists $\hat{c}(T) > 0$ such that:*

$$\int_0^T \left\| \int_\Omega B^*_\zeta e^{tA^*_\zeta} \mathrm{z}_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \geqslant \hat{c}(T) \int_\Omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\Omega, X;\mu))\,.$$

Then for every $\omega \in \mathcal{F}$, such that $\mu(\omega) \neq 0$, we have:

$$\int_0^T \left\| \int_\omega B^*_\zeta e^{tA^*_\zeta} \mathrm{z}_\zeta \,\mathrm{d}\mu^\omega_\zeta \right\|_U^2 \mathrm{d}t \geqslant \mu(\omega)\hat{c}(T) \int_\omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu^\omega_\zeta \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\omega, X;\mu))\,,$$

with $\mu^\omega = \dfrac{\mu|_\omega}{\mu(\omega)}$.

Proof. For every $(\mathrm{z}_\zeta)_\zeta \in L^2(\omega, X; \mu)$, we define $\tilde{z}_\zeta \in L^2(\Omega, X; \mu)$ by $\tilde{\mathrm{z}}_\zeta = \begin{cases} \mathrm{z}_\zeta & \text{if } \zeta \in \omega\,, \\ 0 & \text{if } \zeta \in \Omega \setminus \omega\,. \end{cases}$ We have, using the simultaneous controllability property (3.9),

$$\int_0^T \left\| \int_\omega B_\zeta^* e^{tA_\zeta^*} \mathrm{z}_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t = \int_0^T \left\| \int_\Omega B_\zeta^* e^{tA_\zeta^*} \tilde{\mathrm{z}}_\zeta \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \geqslant \hat{c}(T) \int_\Omega \|\tilde{\mathrm{z}}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta = \hat{c}(T) \int_\omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta\,.$$

□

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. According to Lemma 3.3, in order to prove that the system is not exactly simultaneously controllable, it is enough to prove it on the reduced set of parameters $\omega = B_{\varepsilon_0}(\zeta_0)$ with probability measure $\mu^\omega = \frac{\mu|_\omega}{\mu(\omega)}$ (by assumption, $\mu(\omega) > 0$ and μ^ω is nonnegative and regular).

Let us assume by contradiction that the system (1.1) is exactly simultaneously controllable in a time $T > 0$. Hence, by duality, there exists a constant $\hat{c}^\omega(T) > 0$ such that:

$$\int_0^T \left\| \int_\omega B_\zeta^* e^{tA_\zeta^*} \mathrm{z}_\zeta \,\mathrm{d}\mu_\zeta^\omega \right\|_U^2 \mathrm{d}t \geqslant \hat{c}^\omega(T) \int_\omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta^\omega \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\omega, X; \mu^\omega))\,.$$

Furthermore, $\zeta \in \omega \mapsto (A_\zeta, B_\zeta)$ is Lipschitz-continuous on the bounded domain ω of $\mathbb{R}^d$ and thus it can be uniformly approximated by piecewise constant functions, i.e. for every $\varepsilon > 0$, there exists a piecewise constant map $\zeta \in \omega \mapsto (A_\zeta^\varepsilon, B_\zeta^\varepsilon)$ such that:

$$\operatorname*{ess\,sup}_{\zeta \in \omega} \left(\|A_\zeta - A_\zeta^\varepsilon\|_{\mathcal{L}(X)} + \|B_\zeta - B_\zeta^\varepsilon\|_{\mathcal{L}(U,X)} \right) \leqslant \varepsilon\,.$$

Thus from Lemma 3.2, for $\varepsilon > 0$ small enough, there exists $\hat{c}^{\omega,\varepsilon}(T) > 0$ so that:

$$\int_0^T \left\| \int_\omega B_\zeta^{\varepsilon *} e^{tA_\zeta^{\varepsilon *}} \mathrm{z}_\zeta \,\mathrm{d}\mu_\zeta^\omega \right\|_U^2 \mathrm{d}t \geqslant \hat{c}^{\omega,\varepsilon}(T) \int_\omega \|\mathrm{z}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta^\omega \qquad ((\mathrm{z}_\zeta)_\zeta \in L^2(\Omega, X; \mu^\omega))\,.$$

That is to say that for $\varepsilon > 0$ small enough, the system $\dot{y}_\zeta = A_\zeta^\varepsilon y_\zeta + B_\zeta^\varepsilon u$ is exactly simultaneously controllable. But, since μ^ω is nonnegative and regular, there exists a domain $D_\varepsilon \subset \omega$ with nonempty interior such that $\mu^\omega(D_\varepsilon) > 0$ and $\zeta \mapsto (A_\zeta^\varepsilon, B_\zeta^\varepsilon)$ is constant on D_ε. Using again Lemma 3.3, the system $\dot{y}_\zeta = A_\zeta^\varepsilon y_\zeta + B_\zeta^\varepsilon u$ has to be exactly simultaneously controllable for the probability space $(D_\varepsilon, \mathcal{B}(D_\varepsilon), \frac{\mu^\omega|_{D_\varepsilon}}{\mu^\omega(D_\varepsilon)})$. (Notice that $\frac{\mu^\omega|_{D_\varepsilon}}{\mu^\omega(D_\varepsilon)}$ is a regular measure.)
Since $\zeta \mapsto (A_\zeta^\varepsilon, B_\zeta^\varepsilon)$ is constant on D_ε, from Lemma 3.1 we have $\mu^\omega(e)$ is either $\mu^\omega(D_\varepsilon)$ or 0, for every $e \in \mathcal{B}(D_\varepsilon)$. This contradicts $\frac{\mu^\omega|_{D_\varepsilon}}{\mu^\omega(D_\varepsilon)}$ regular. □

Propositions 3.1 and 3.2 told us that it is hard to build continuously-dependent parameter systems which are exactly simultaneously controllable. However, as we have seen, the averaged controllability property holds for a variety of models. Consequently, it is natural to look for averaged controls which are optimal in the sense that they minimize the output's variance. This is the core of section 4.

3.4 Momentum approach for simultaneous controllability

In § 3.2, we gave a necessary and sufficient condition, (3.9), for simultaneous controllability. However, even on simple problems, it is difficult to check whether this condition is satisfied or not. In this paragraph, we present an iterative approach to check whether the observability inequality (3.9) is fulfilled or not. The method presented here can also be seen as an alternative method to the one we proposed in the rest of this paper (see section 4) in order to link averaged controllability to exact simultaneous controllability.

To simplify the notation we define the operator $\mathbb{E} \in \mathcal{L}(L^2(\Omega, X; \mu), X)$ by:

$$\mathbb{E}(\mathrm{y}_\zeta)_\zeta = \int_\Omega \mathrm{y}_\zeta \, \mathrm{d}\mu_\zeta \qquad ((\mathrm{y}_\zeta)_\zeta \in L^2(\Omega, X; \mu)) \,. \tag{3.10}$$

Notice that we have $\mathbb{E}^*\mathrm{z} = (\mathrm{z})_\zeta$ and $\mathbb{E}\,\mathbb{E}^* = \mathrm{Id}_X$.

Let us first remind that proving the averaged controllability property is equivalent as proving that the cost function J defined by (3.5) is coercive and proving the exact simultaneous controllability is equivalent as proving that the cost function $\mathfrak{J}$ defined by (3.7) is coercive. In addition, we have also noticed that we have $J = \mathfrak{J} \circ \mathbb{E}^*$, where $\mathbb{E}$ is given by (3.10). Thus, proving that J is coercive means proving that the restriction of $\mathfrak{J}$ to the subset $\mathbb{E}^*(X) = \{\zeta \in \Omega \mapsto \mathrm{y} \in X \,,\ \mathrm{y} \in X\}$ of $L^2(\Omega, X; \mu)$ is coercive.

Let us also notice that since $L^2(\Omega, \mathbb{R}; \mu)$ is an Hilbert space, one can define an orthonormal basis $(\varphi_i)_{i\in I}$ (with the convention $0 \in I$ and $\varphi_0 = 1$) of this space. Based on the above construction of $\mathbb{E}$, we define for every $i \in I$, the operator $\mathbb{E}_i \in \mathcal{L}(L^2(\Omega, X; \mu), X)$ by:

$$\mathbb{E}_i(\mathrm{y}_\zeta)_\zeta = \int_\Omega \mathrm{y}_\zeta \varphi_i(\zeta) \, \mathrm{d}\mu_\zeta \qquad ((\mathrm{y}_\zeta)_\zeta \in L^2(\Omega, X; \mu)) \,, \tag{3.11}$$

so that $L^2(\Omega, X; \mu) = \displaystyle\bigoplus_{i \in I} \mathbb{E}_i^*(X)$.

Let us assume that $L^2(\Omega, \mathbb{R}; \mu)$ is a separable Hilbert space, that is to say that we can choose $I = \mathbb{N}$ (if $L^2(\Omega, \mathbb{R}; \mu)$ is of infinite dimension) or $I = \{0, \cdots, d\} \subset \mathbb{N}$ (if $L^2(\Omega, \mathbb{R}; \mu)$ is of dimension d). For every $k \in \mathbb{N}$, we define the finite dimensional subspace V_k of $L^2(\Omega, X; \mu)$ by:

$$V_k = \bigoplus_{\substack{i \in I \\ i \leqslant k}}^{k} \mathbb{E}_i^*(X) \ \subset L^2(\Omega, X; \mu) \,. \tag{3.12}$$

Let us also define the constant $\hat{c}_k(T) \geqslant 0$ by:

$$\hat{c}_k(T) = \inf_{(\mathrm{z}_\zeta^f)_\zeta \in V_k \setminus \{0\}} \frac{\displaystyle\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \, \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t}{\left\| \mathrm{z}_\zeta^f \right\|_{L^2(\Omega, X : \mu)}^2} \qquad (k \in \mathbb{N}) \,, \tag{3.13}$$

that is to say:

$$\hat{c}_k(T) \int_\Omega \left\| \mathrm{z}_\zeta^f \right\|_X^2 \mathrm{d}\mu_\zeta \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \, \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \qquad \left(k \in \mathbb{N} \quad \left(\mathrm{z}_\zeta^f\right)_\zeta \in V_k\right) ,$$

with $z_\zeta(t)$ the solution of the adjoint problem (3.6) with final condition $z_\zeta(T) = \mathrm{z}_\zeta^f \in V_k$.
Thus, if $\hat{c}_k(T) > 0$, $\mathfrak{J}$ is convex and coercive on V_k.

Since $V_k \subset V_{k+1}$, it remains clear that the sequence $(\hat{c}_k(T))_{k\in\mathbb{N}}$ is decreasing. In addition, one can easily convince that if $\lim\limits_{k\to\infty} \hat{c}_k(T) > 0$, there exists $\hat{c}(T) > 0$ ($\hat{c}(T) = \lim\limits_{k\to\infty} \hat{c}_k(T)$) such that:

$$\hat{c}(T) \int_\Omega \|\mathrm{z}_\zeta^f\|_X^2 \,\mathrm{d}\mu_\zeta \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \qquad \left(\left(\mathrm{z}_\zeta^f\right)_\zeta \in L^2(\Omega, X; \mu)\right).$$

That is to say that $\mathfrak{J}$ is convex and coercive on $L^2(\Omega, X; \mu)$ and hence we have exact simultaneous controllability. Moreover, the sequence $(\mathrm{z}_{k,\zeta}^f)_\zeta \in V_k$ of minimizer of $\mathfrak{J}$ on the finite dimensional subspace V_k of $L^2(\Omega, X; \mu)$ is convergent to a minimizer $(\hat{\mathrm{z}}_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ of $\mathfrak{J}$ on $L^2(\Omega, X; \mu)$.

Summarizing the above discussion, leads to the following:

Theorem 3.4. *Assume that $L^2(\Omega, \mathbb{R}; \mu)$ is a separable Hilbert space and let $(\varphi_i)_{i\in I}$ (with $\{0\} \subset I \subset \mathbb{N}$ and the convention $\varphi_0 = 1$) be an orthonormal basis of this space. For every $j \in I$, we define the map $\mathbb{E}_j \in \mathcal{L}(L^2(\Omega, X; \mu), X)$ by (3.11) and, for every $k \in \mathbb{N}$, we define the finite dimensional subspace V_k of $L^2(\Omega, X; \mu)$ by (3.12) and $\mathfrak{J}_k$ the restriction of $\mathfrak{J}$ (defined by (3.7)) on V_k. Let us also introduce for every $k \in \mathbb{N}$, the constant $\hat{c}_k(T) \geqslant 0$ defined by (3.13).*

Then, the system (1.1) is exact simultaneously controllable if and only if $\inf\limits_{k\in I} \hat{c}_k(T) > 0$, .

Moreover, if this property holds, for every $k \in \mathbb{N}$, $\mathfrak{J}_k$ admits a minimum $(\hat{\mathrm{z}}_{k,\zeta}^f)_\zeta \in V_k$ and as k goes to ∞, $(\hat{\mathrm{z}}_{k,\zeta}^f)_\zeta$ is convergent in $L^2(\Omega, X; \mu)$ to an element $(\hat{\mathrm{z}}_\zeta^f)_\zeta$ which minimize $\mathfrak{J}$ and the exact simultaneous control of minimal L^2-norm is given by:

$$u(t) = \int_\Omega B_\zeta^* \hat{z}_\zeta(t) \,\mathrm{d}\mu_\zeta \qquad (t \in [0, T] \quad a.e.),$$

where $\hat{z}_\zeta$ is the solution of (3.6) with final condition $\hat{\mathrm{z}}_\zeta^f$.

Remark 3.2.

1. *It is obvious that if there exists $k \in \mathbb{N}$ such that $\hat{c}_k(T) = 0$ the system (1.1) cannot be exactly simultaneously controllable.*

2. *The case $\hat{c}_k(T) > 0$ for every $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} \hat{c}_k(T) = 0$ is undetermined. However, for this case, we might recover the property of approximate simultaneous controllability.*

3. *Let us mention that the property:*

$$\exists\, \hat{c}_k(T) > 0 \quad s.t. \quad \forall \left(\mathrm{z}_\zeta^f\right)_\zeta \in V_k\,, \quad \hat{c}_k(T) \int_\Omega \|\mathrm{z}_\zeta^f\|_X^2 \,\mathrm{d}\mu_\zeta \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \,\mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t\,, \tag{3.14}$$

correspond to a Kalman rank condition.

More precisely, $\left(\mathrm{z}_\zeta^f\right)_\zeta \in V_k$ means there exists $(\mathrm{z}^{j,f})_{j=0,\cdots,k} \in X^{k+1}$ such that $\mathrm{z}_\zeta^f = \sum\limits_{j=0}^k \varphi_j(\zeta)\mathrm{z}^{j,f}$. Let us then denote by z_ζ^j the solution of (3.2) with final condition $\mathrm{z}^{j,f}$. Due to linearity, the solution z_ζ of (3.6) with final condition z_ζ^f is:

$$z_\zeta(t) = \sum_{j=0}^k \varphi_j(\zeta) z_\zeta^j(t) = \sum_{j=0}^k \varphi_j(\zeta) e^{(T-t)A_\zeta^*}\, \mathrm{z}^{j,f}\,.$$

Finally, since we are in a finite dimensional space the coercive property (3.14) *is equivalent to the uniqueness property:*

$$\int_\Omega B_\zeta^* \sum_{j=0}^{k} \varphi_j(\zeta) e^{(T-t)A_\zeta^*} \mathrm{z}^{j,f} \, \mathrm{d}\mu_\zeta = 0 \qquad (t \in [0,T] \quad a.e.) \qquad \Longrightarrow \qquad \mathrm{z}^{j,f} = 0 \qquad (j \in \{0, \cdots, k\}) \, .$$

We conclude by time analyticity that (3.14) *holds if and only if:*

$$\mathrm{rank}\left[\int_\Omega \hat{A}_\zeta^l \hat{B}_\zeta \, \mathrm{d}\mu_\zeta \, , \quad l \in \mathbb{N}\right] = (k+1) \dim X \, ,$$

where we have defined:

$$\hat{A}_\zeta = \begin{pmatrix} A_\zeta & & 0 \\ & \ddots & \\ 0 & & A_\zeta \end{pmatrix} \in \mathcal{L}(X^{k+1}) \qquad \text{and} \qquad \hat{B}_\zeta = \begin{pmatrix} \varphi_0(\zeta) B_\zeta \\ \vdots \\ \varphi_k(\zeta) B_\zeta \end{pmatrix} \in \mathcal{L}(U, X^{k+1}) \qquad (\zeta \in \Omega) \, .$$

Even if (3.14) *can be easily obtained by the use of Kalman rank condition, in order to conclude that the system satisfies the exact simultaneous controllability, we need an estimate on the constants* $\hat{c}_k(T)$ *which is hard to obtain. This is why in section 4 we present another approach based on a penalty problem.*

4. *When* $\mathrm{Card}\,\Omega < \infty$*, the moments are solution of an ordinary differential equation.*
More precisely, consider $\Omega = \{1, \cdots, K\}$ *with measure* μ *given by* $\mu(\{k\}) = \theta_k$ *with* $\theta_k \in (0,1)$ *and* $\sum_{k=1}^K \theta_k = 1$*. Let us consider an orthonormal basis* $\{\varphi_0, \cdots, \varphi_{K-1}\}$ *of* $L^2(\Omega, \mathbb{R}; \mu)$ *(with the convention,* $\varphi_0(k) = 1$*). Then the* i^{th}*-momentum is:*

$$Y_i = \sum_{k=1}^{K} \theta_k \varphi_i(k) y_k = \mathbb{M}_i \mathbb{I} \, y \qquad (i \in \{0, \cdots, K-1\}) \, ,$$

with:

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_K \end{pmatrix} \in X^K \, , \qquad \mathbb{M}_i = \begin{pmatrix} \varphi_i(1)\sqrt{\theta_1}\mathrm{Id}_X & \cdots & \varphi_i(K)\sqrt{\theta_K}\mathrm{Id}_X \end{pmatrix} \in \mathcal{L}(X^K, X)$$

$$\text{and} \quad \mathbb{I} = \begin{pmatrix} \sqrt{\theta_1}\mathrm{Id}_X & & 0 \\ & \ddots & \\ 0 & & \sqrt{\theta_K}\mathrm{Id}_X \end{pmatrix} \in \mathcal{L}(X^K) \, .$$

Thus, setting:

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_0 \\ \vdots \\ \mathbb{M}_{K-1} \end{pmatrix} \in \mathcal{L}(X^K) \, , \qquad \mathbb{A} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_K \end{pmatrix} \in \mathcal{L}(X^K)$$

$$\text{and} \quad \mathbb{B} = \begin{pmatrix} B_1 \\ \vdots \\ B_K \end{pmatrix} \in \mathcal{L}(U, X^K) \, ,$$

the momentums $Y = \begin{pmatrix} Y_0 \\ \vdots \\ Y_{K-1} \end{pmatrix}$ *satisfies (noticing that* $\mathbb{M}\mathbb{M}^\top = \mathrm{Id}_{X^K}$*):*

$$\dot{Y} = \mathbb{M}\mathbb{I}\mathbb{A}\mathbb{I}^{-1}\mathbb{M}^\top Y + \mathbb{M}\mathbb{I}\mathbb{B}u\,. \tag{3.15}$$

Controlling the first k *momentums of* $(y_k)_k$ *means controlling the first* $k \dim X$ *components of* Y*, solution of* (3.15)*.*
Since the basis $\varphi_0, \varphi_1, \cdots, \varphi_{K-1}$ *is free (except* $\varphi_0 = 1$*) one can consider the problem of finding the best possible basis. For instance we can wonder if there exists* $\varphi_1, \cdots, \varphi_{K-1}$ *such that the pair* $\big(\mathbb{M}\mathbb{I}\mathbb{A}\mathbb{I}^{-1}\mathbb{M}^\top\,,\ \mathbb{M}\mathbb{I}\mathbb{B}\big)$ *has a normal form (see [12, Proposition 2.2.6]). That is to say find* $\varphi_1, \cdots, \varphi_{K-1}$ *such that* $\mathbb{M}\mathbb{I}\mathbb{A}\mathbb{I}^{-1}\mathbb{M}^\top$ *has the structure* $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ *and* $\mathbb{M}\mathbb{I}\mathbb{B}$ *the structure* $\begin{pmatrix} * \\ 0 \end{pmatrix}$*.*

4 A penalty method linking averaged and simultaneous controllability

As in all this paper, we assume in this section that the assumptions of lemmas 2.1 and 2.2 are satisfied.

In this section, we will present our strategy to link averaged controllability to exact simultaneous controllability. First of all, solving the averaged control problem, can be done with the Hilbert Uniqueness Method, that is to say minimize the L^2-norm of the control with the constraint $\displaystyle\int_\Omega y_\zeta(T)\,\mathrm{d}\mu_\zeta = \int_\Omega \mathrm{y}_\zeta^f\,\mathrm{d}\mu_\zeta$. Thus, using Euler-Lagrange formulation (or directly Theorem 3.2), on can see that the averaged control of minimal L^2-norm is given by (3.4).

In order to reduce the output's variance, one can think to penalise the cost function $\mathcal{J}_0$ (given by $\mathcal{J}_0(u) = \frac{1}{2}\|u\|^2_{L^2([0,T],U)}$ with the output's variance, $\displaystyle\int_\Omega \big\|y_\zeta(T) - \mathrm{y}_\zeta^f\big\|_X^2\,\mathrm{d}\mu_\zeta$. Thus, we introduce the penalty problem of optimisation:

$$\begin{array}{ll} \min & \mathcal{J}_\kappa(u) := \frac{1}{2}\|u(t)\|^2_{L^2([0,T],U)} + \kappa\big\|y_\zeta(T;u) - \mathrm{y}_\zeta^f\big\|^2_{L^2(\Omega,X;\mu)} \\ \big| & \mathbb{E}\big(y_\zeta(T;u) - \mathrm{y}_\zeta^f\big) = 0\,. \end{array} \qquad (\kappa \geqslant 0)\,, \tag{4.1}$$

where in the above, y_ζ is the solution of (1.1) defined by (1.2) with control u, $L^2(\Omega, X;\mu)$ is the Hilbert space introduced in (1.3) and $\mathbb{E}$ is the expectation defined by (3.10).

Let us give an existence result.

Proposition 4.1. *If system* (1.1) *satisfies the averaged controllability property* (1.4) *then for every* $T > 0$*,* $(\mathrm{y}_\zeta^i)_\zeta, (\mathrm{y}_\zeta^f)_\zeta \in L^2(\Omega, X;\mu)$ *and* $\kappa \geqslant 0$*, the minimisation problem* (4.1) *admits one and only one solution* $u_\kappa \in L^2([0,T],U)$*.*

In addition, the optimal control u_κ *satisfies:*

$$u_\kappa(t) = \int_\Omega B_\zeta^\star z_\zeta(t)\,\mathrm{d}\mu_\zeta \qquad (t \in [0,T])\,, \tag{4.2a}$$

where, z_ζ *is solution of:*

$$\dot{z}_\zeta = -A_\zeta^* z_\zeta\,, \qquad z_\zeta(T) = \mathrm{z} + \mathrm{y}_\zeta^f - y_\zeta(T;u_\kappa)\,, \tag{4.2b}$$

with $\mathrm{z} \in X$ *unknown.*

Proof. For every $\kappa \geqslant 0$, it is clear that $\mathcal{J}_\kappa$ is convex. Since we have assumed that the system (1.1) satisfies the averaged controllability property (1.4), this ensure that the set:

$$\Big\{u \in L^2([0,T],U)\,,\ \mathbb{E}\big(y_\zeta(T;u) - \mathrm{y}_\zeta^f\big) = 0\Big\}$$

is non empty and in addition, this set is a convex and closed set of $L^2([0,T],U)$. Moreover, the averaged controllability property ensure that $\mathcal{J}_0$ is coercive on this set and consequently $\mathcal{J}_\kappa$ is also coercive on this set. Thus, there exists a unique minimizer $u_\kappa \in L^2([0,T],U)$ for the minimisation problem (4.1).

Let us now prove the optimality conditions. Let us define the Lagrangian of the system:

$$L(u,\mathrm{z}) = \mathcal{J}_\kappa(u) + \langle \mathrm{z},\, \mathbb{E}\big(y_\zeta(T;u) - \mathrm{y}_\zeta^f\big)\rangle_X \qquad (u \in L^2([0,T],U)\,,\ \mathrm{z} \in X)\,.$$

The optimality conditions are:

$$\partial_{\mathrm{z}} L = 0 \quad \text{and} \quad \partial_u L = 0\ .$$

But we have, $\partial_u L(u,\mathrm{z}) = u + \displaystyle\int_\Omega B_\zeta^* e^{(T-t)A_\zeta^*}\big(\mathrm{z} + y_\zeta(T;u) - \mathrm{y}_\zeta^f\big)\,\mathrm{d}\mu_\zeta$. That is to say that, the optimal control u_κ should satisfy (4.2). □

Of course, we have introduced the cost functions $\mathcal{J}_\kappa$ in order to pass to the limit $\kappa \to \infty$. Let us first state a trivial statement:

Lemma 4.1. *Set $T > 0$ and assume the system (1.1) is controllable in average.*
For every $\kappa \geqslant 0$, let us define u_κ the minimum of $\mathcal{J}_\kappa$ under the constraint $\mathbb{E}\big(y_\zeta(T,u_\kappa) - \mathrm{y}_\zeta^f\big) = 0$.

Then, we have:

$$\|u_\kappa\|_{L^2([0,T],U)} \leqslant \|u_{\kappa+\varepsilon}\|_{L^2([0,T],U)} \quad \text{and} \quad \|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \geqslant \|y_\zeta(T;u_{\kappa+\varepsilon}) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \qquad (\kappa,\varepsilon \geqslant 0)\,.$$

In addition, for every $\kappa \geqslant 0$, we have:

$$\begin{aligned}\big\|y_\zeta(T,u_\kappa) - \mathrm{y}_\zeta^f\big\|_{L^2(\Omega,X;\mu)} = \min\Big\{&\big\|y_\zeta(T;u) - \mathrm{y}_\zeta^f\big\|_{L^2(\Omega,X;\mu)}\,,\ u \in L^2([0,T],U)\,,\ \|u\|_{L^2([0,T],U)} \leqslant \|u_\kappa\|_{L^2([0,T],U)} \\ &\text{and}\ \ \mathbb{E}\big(y_\zeta(T;u) - \mathrm{y}_\zeta^f\big) = 0\Big\}\,. \end{aligned} \tag{4.3}$$

Proof. It remains clear that for every $\kappa,\varepsilon \geqslant 0$, we have:

$$\mathcal{J}_\kappa(u_\kappa) \leqslant \mathcal{J}_\kappa(u_{\kappa+\varepsilon}) \leqslant \mathcal{J}_{\kappa+\varepsilon}(u_{\kappa+\varepsilon}) \leqslant \mathcal{J}_{\kappa+\varepsilon}(u_\kappa)\,.$$

Thus from, $\mathcal{J}_\kappa(u_\kappa)+\mathcal{J}_{\kappa+\varepsilon}(u_{\kappa+\varepsilon}) \leqslant \mathcal{J}_\kappa(u_{\kappa+\varepsilon})+\mathcal{J}_{\kappa+\varepsilon}(u_\kappa)$, it is easy to see that $\Big(\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\Big)_{\kappa\geqslant 0}$ is decreasing and then, form $\mathcal{J}_\kappa(u_\kappa) \leqslant \mathcal{J}_\kappa(u_{\kappa+\varepsilon})$, we obtain that $\big(\|u_\kappa\|_{L^2([0,T],U)}\big)_{\kappa\geqslant 0}$ is increasing.

Let us now prove (4.3). To this end, we assume by contradiction that there exists $u \in L^2([0,T],U)$ such that:

$$\|u\|_{L^2([0,T],U)} \leqslant \|u_\kappa\|_{L^2([0,T],U)}\,, \qquad \mathbb{E}\big(y_\zeta(T;u) - \mathrm{y}_\zeta^f\big) = 0 \quad \text{and} \quad \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} < \|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\,.$$

Then we have $\mathcal{J}_\kappa(u) < \mathcal{J}_\kappa(u_\kappa)$ which is in contradiction with u_κ minimize $\mathcal{J}_\kappa$. □

Various situations could hold as $\kappa \to \infty$. These different situations, reported on Table 1, are given by the following theorem.

Theorem 4.1. *Set $T > 0$ and assume that the system (1.1) in controllable in average in time T. For every $\kappa \geqslant 0$, let us define u_κ the minimum of $\mathcal{J}_\kappa$ under the constraint $\mathbb{E}\left(y_\zeta(T, u_\kappa) - \mathrm{y}_\zeta^f\right) = 0$. Define $(\mathrm{y}_\zeta^\star)_\zeta \in L^2(\Omega, X; \mu)$ as the minimizer of:*

$$\begin{array}{l|l} \min & \|\mathrm{y}_\zeta - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \\ & (\mathrm{y}_\zeta)_\zeta \in \overline{\{y_\zeta(T;u)\,,\ u \in L^2([0,T],U)\}}\,, \\ & \mathbb{E}(\mathrm{y}_\zeta)_\zeta = \mathbb{E}(\mathrm{y}_\zeta^f)_\zeta\,. \end{array} \tag{4.4}$$

Then, the following alternative holds:

- *If $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ is bounded, then up to a subsequence, $(u_\kappa)_\kappa$ converges to a control which steers exactly y_ζ^i to $\mathrm{y}_\zeta^\star$ and realises the minimum of:*

$$\begin{array}{l|l} \min & \frac{1}{2}\|u\|^2_{L^2([0,T],U)} \\ & \|y_\zeta(T;u) - \mathrm{y}_\zeta^\star\|_{L^2(\Omega,X;\mu)} = 0\,. \end{array}$$

- *If $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ is unbounded, then y_ζ^i can be approximatively steered to $\mathrm{y}_\zeta^\star$.*

In addition, if $\lim\limits_{\kappa\to\infty}\left(\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\right) = 0$, then we have $\mathrm{y}_\zeta^\star = \mathrm{y}_\zeta^f$.

Proof. Without loss of generality, we can assume that $\mathrm{y}_\zeta^i = 0$.
Let us first notice that $(\mathrm{y}_\zeta^\star)_\zeta \in L^2(\Omega, X; \mu)$ is well defined. In fact, $(\mathrm{y}_\zeta^\star)_\zeta$ is the orthogonal projection of $(\mathrm{y}_\zeta^f)_\zeta$ in $L^2(\Omega, X; \mu)$ on the closed vector space $\overline{\{y_\zeta(T;u)\,,\ u \in L^2([0,T],U)\}} \cap \left\{(\mathrm{y}_\zeta)_\zeta\,,\ \mathbb{E}(\mathrm{y}_\zeta)_\zeta = \mathbb{E}(\mathrm{y}_\zeta^f)_\zeta\right\}$.

- Let us assume $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ bounded.
From Lemma 4.1, the sequence $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ is increasing, hence there exists $u_\infty \in L^2([0,T],U)$ such that up to a subsequence, $(u_\kappa)_{\kappa\geqslant 0}$ is weakly convergent to u_∞ and in addition, we have:

$$\|u_\infty\|_{L^2([0,T],U)} \leqslant \lim_{\kappa\to\infty} \|u_\kappa\|_{L^2([0,T],U)}\,.$$

  Since $(u_\kappa)_{\kappa\geqslant 0}$ is weakly convergent to u_∞, it is easy to obtain that $((y_\zeta(T;u_\kappa))_\zeta)_{\kappa\geqslant 0}$ is weakly convergent to $(y_\zeta(T;u_\infty))_\zeta \in L^2(\Omega, X; \mu)$. Hence,

$$\mathbb{E}\big(y_\zeta(T;u_\infty)\big)_\zeta = \mathbb{E}\big(y_\zeta^f\big)_\zeta \quad \text{and} \quad \|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \leqslant \liminf_{\kappa\to\infty}\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\,.$$

  In addition, from Lemma 4.1, the sequence $\left(\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\right)_\kappa$ is decreasing thus, we have:

$$\|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \leqslant \|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \qquad (\kappa > 0)$$

  and hence, from relation (4.3) of Lemma 4.1, we obtain $\|u_\infty\|_{L^2([0,T],U)} \geqslant \|u_\kappa\|_{L^2([0,T],U)}$ that is to say, $\|u_\infty\|_{L^2([0,T],U)} = \lim\limits_{\kappa\to\infty} \|u_\kappa\|_{L^2([0,T],U)}$ and (up to a subsequence, $(u_\kappa)_\kappa$ is strongly convergent to u_∞ in $L^2([0,T],U)$. Consequently, $\big((y_\zeta(T;u_\kappa))_\zeta\big)_\kappa$ is strongly convergent to $(y_\zeta(T;u_\infty))_\zeta$ in $L^2(\Omega,X;\mu)$.

Let us now prove that $y_\zeta(T,u_\infty) = \mathrm{y}_\zeta^\star$. Assume by contradiction that it is not the case. That is to say there exists $\bar u \in L^2([0,T],U)$ such that:

$$\mathbb{E}\big(y_\zeta(T;\bar u)\big)_\zeta = \mathbb{E}\big(\mathrm{y}_\zeta^f\big)_\zeta \quad \text{and} \quad \big\|y_\zeta(T;\bar u) - \mathrm{y}_\zeta^f\big\|_{L^2(\Omega,X;\mu)} < \big\|y_\zeta(T,u_\infty) - \mathrm{y}_\zeta^f\big\|_{L^2(\Omega,X;\mu)}\,.$$

On the other hand, we have $\mathcal{J}_\kappa(u_\kappa) \leqslant \mathcal{J}_\kappa(\bar u)$ for every $\kappa \geqslant 0$, i.e.:

$$\frac{1}{2\kappa}\Big(\|\bar u\|^2_{L^2([0,T],U)} - \|u_\kappa\|_{L^2([0,T],U)}\Big) \\ \geqslant \big\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\big\|^2_{L^2(\Omega,X;\mu)} - \big\|y_\zeta(T;\bar u) - \mathrm{y}_\zeta^f\big\|^2_{L^2(\Omega,X;\mu)} \qquad (\kappa > 0)\,.$$

Taking the limit $\kappa \to \infty$ comes the contradiction:

$$\big\|y_\zeta(T;\bar u) - \mathrm{y}_\zeta^f\big\|^2_{L^2(\Omega,X;\mu)} \geqslant \big\|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\big\|^2_{L^2(\Omega,X;\mu)}\,.$$

Finally, it remains clear that $\lim_{\kappa\to\infty}\big\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\big\|_{L^2(\Omega,X;\mu)} = 0$ is equivalent as $\mathrm{y}_\zeta^\star = \mathrm{y}_\zeta^f$.

- Let us assume that $\big(\|u_\kappa\|_{L^2([0,T],U)}\big)_{\kappa\geqslant 0}$ is not bounded.
  The results of this point are direct consequences of (4.3) given in Lemma 4.1.

□

If the system (1.1) is simultaneously controllable, then we do not need the extraction of a subsequence procedure and the convergence rates to the simultaneous control and the variance are linked.

Proposition 4.2. *Assume system* (1.1) *is exactly simultaneously controllable in time* $T > 0$. *Let* $u_\infty \in L^2([0,T],U)$ *be the exact simultaneous control of minimal norm steering* y_ζ^i *to* y_ζ^f *and let* $u_\kappa \in L^2([0,T],U)$ *be the minimizer of* (4.1).

Then, $(u_\kappa)_{\kappa\geqslant 0}$ *is strongly convergent to* u_∞ *and, in addition,*

$$\|y_\zeta(T,u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} \leqslant \|u_\infty\|_{L^2([0,T],U)}\,\kappa^{-1}\|u_\kappa - u_\infty\|_{L^2([0,T],U)}\,. \tag{4.5}$$

Proof. Let us first prove that the sequence $(u_\kappa)_\kappa$ is strongly convergent to u_∞.
First of all, since we assumed that the system (1.1) is exactly simultaneously controllable, the minimisation problem:

$$\min \ \tfrac{1}{2}\|u\|_{L^2([0,T],U)} \\ \Big| \quad \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} = 0\,,$$

admits one and only one minimizer $u_\infty \in L^2([0,T],U)$. Regarding to the proof of Theorem 4.1, there exist a nondecreasing sequence $(\kappa_n)_{n\in\mathbb{N}}$ such that $(u_{\kappa_n})_{n\in\mathbb{N}}$ is strongly convergent to u_∞. Let us assume by contradiction that there exists another sequence $(\tilde\kappa_n)_{n\in\mathbb{N}}$ such that $(u_{\tilde\kappa_n})_{n\in\mathbb{N}}$ is not convergent to u_∞. But from Lemma 4.1, we have $\|u_{\tilde\kappa_n}\|_{L^2([0,T],U)} \leqslant \|u_\infty\|_{L^2([0,T],U)}$ and hence, $(u_{\tilde\kappa_n})_{n\in\mathbb{N}}$ is weakly convergent to some $\tilde u \in L^2([0,T],U)$, with $\tilde u \neq u_\infty$. As in the proof of Theorem 4.1, we can prove that this convergence is strong. Consequently, $\|\tilde u\|_{L^2([0,T],U)} = \|u_\infty\|_{L^2([0,T],U)}$ and since $\tilde u \neq u_\infty$, $\|y_\zeta(T;\tilde u) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)} > 0$. But since $(\|y_\zeta(T;u_\kappa) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)})_\kappa$ is decreasing, for every $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that:

$$\|y_\zeta(T;\tilde u) - \mathrm{y}^f\|_{L^2(\Omega,X:\mu)} \leqslant \|y_\zeta(T;u_{\tilde\kappa_n}) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X:\mu)} \leqslant \|y_\zeta(T,u_{\kappa_{n_0}}) - \mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\mu)}\,.$$

But when $n \to \infty$, we n_0 can also be choosen so that it goes to infinity. Thus, $\|y_\zeta(T;\tilde{u}) - \mathrm{y}^f\|_{L^2(\Omega,X:\mu)} = \|y_\zeta(T;u_\infty) - \mathrm{y}^f\|_{L^2(\Omega,X:\mu)} = 0$ leading to a contradiction.

Let us now prove (4.5). First of all, changing y^f_ζ in $\mathrm{y}^f - e^{TA_\zeta}\mathrm{y}^i_\zeta$, we can assume without loss of generality that $\mathrm{y}^i_\zeta = 0$.
Set $u_\kappa = u_\infty + v_\kappa$, then v_κ is a minimizer of:

$$\min \quad \mathcal{G}_\kappa(v) = \frac{1}{2}\|v\|^2_{L^2([0,T],U)} + \langle v, u_\infty\rangle_{L^2([0,T],U)} + \kappa \int_\Omega \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta v(t)\,\mathrm{d}t \right\|_X^2 \mathrm{d}\mu_\zeta$$
$$\Big| \quad \mathbb{E}\left(\int_0^T e^{(T-t)A_\zeta} B_\zeta v(t)\,\mathrm{d}t \right) = 0\,.$$

We have:

$$\mathcal{G}_\kappa(v_\kappa) \leqslant \mathcal{G}_\kappa(0) = 0\,.$$

Thus, for every $\kappa \geqslant 0$,

$$\kappa \int_\Omega \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta v_\kappa(t)\,\mathrm{d}t \right\|^2 \mathrm{d}\mu_\zeta \leqslant \frac{1}{2}\|v_\kappa\|^2_{L^2([0,T],U)} + \kappa \int_\Omega \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta v_\kappa(t)\,\mathrm{d}t \right\|^2 \mathrm{d}\mu_\zeta$$
$$\leqslant -\langle v_\kappa, u_\infty\rangle_{L^2([0,T],U)} \leqslant \|v_\kappa\|_{L^2([0,T],U)}\|u_\infty\|_{L^2([0,T],U)}\,.$$

□

Let us now give the consequences of Theorem 4.1 in the case where the cardinal of Ω is finite.

Corollary 4.1. *Assume $L^2(\Omega, X;\mu)$ is of finite dimension.*
Then the sequence of minimizers $(\hat{u}_\kappa)_\kappa$ of the optimisation problem (4.1) is strongly convergent (up to the extraction of a subsequence) to an element $\hat{u}_\infty \in L^2([0,T],U)$ satisfying the minimisation problem:

$$\min \quad \tfrac{1}{2}\|u\|^2_{L^2([0,T],U)}$$
$$\Big| \quad y_\zeta(T) = \mathrm{y}^\star_\zeta \qquad (\zeta \in \Omega \quad \mu - a.e.)\,,$$

where $\mathrm{y}^\star_\zeta$ is defined by Theorem 4.1, i.e. is the minimizer of (4.4).

A graphical interpretation of this result is given on Figure 2.

Proof. Let us use the notations introduced in Theorem 4.1. Since $L^2(\Omega, X;\mu)$ is a finite dimensional space, $\{y_\zeta(T;u)\,,\ u \in L^2([0,T],U)\} \cap \{(\mathrm{y}_\zeta)_\zeta\,,\ \mathbb{E}(\mathrm{y}_\zeta)_\zeta = \mathbb{E}(\mathrm{y}^f_\zeta)_\zeta\}$ is a closed affine subspace of $L^2(\Omega, X;\mu)$. Consequently, there exists $u^\star \in L^2([0,T],U)$ such that $\mathrm{y}^\star_\zeta = y_\zeta(T;u^\star)$. □

Example 4.1. *This example illustrates the result of Corollary 4.1 in the exact simultaneous controllability case.*
Consider the probability space $\Omega = \{1,2\}$ and the probability measure μ is given by $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{2}$. The parameter dependent system under consideration is:

$$\dot{y}_\zeta = \zeta A y_\zeta + Bu \qquad y_\zeta(0) = \mathrm{y}^i\,,$$

with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{y}^i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
We fix the final target to $\mathrm{y}^f = (0\ 0)^\top$ and the final time T to be 1.

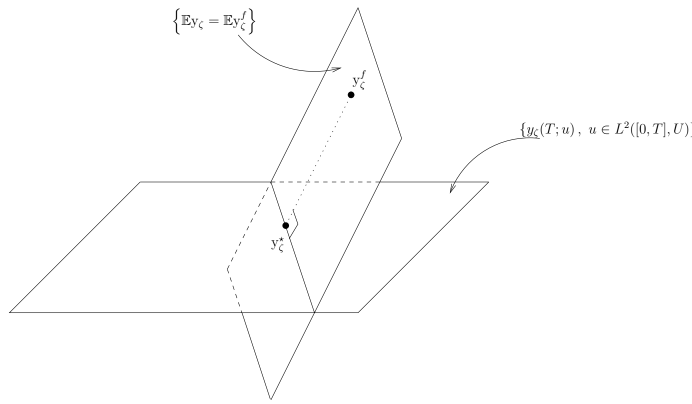


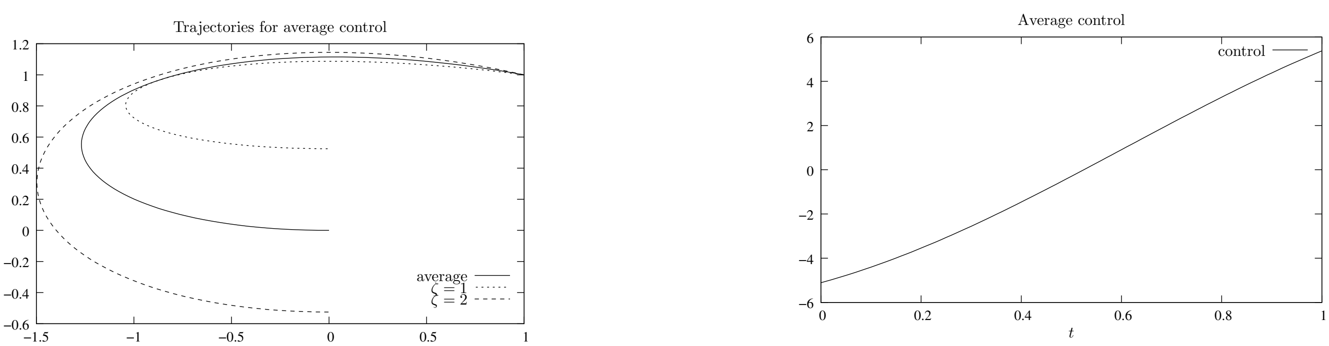
Figure 2: Under the assumptions of Corollary 4.1, at the limit $\kappa \to \infty$, the emergent control will be a control steering $(y^i_\zeta)_\zeta$ to $(y^\star_\zeta)_\zeta$.

The corresponding augmented system is:

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{y} + \mathbf{B}u \qquad \boldsymbol{y}(0) = \mathbf{y}^i\,,$$

with $\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & 2A \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} B \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ *and* $\mathbf{y}^i = \begin{pmatrix} y^i \\ y^i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.
Using the Kalman rank condition, it is easy to see that this system is controllable (in the classical sense) and controlling the average means controlling $\frac{1}{2}\left(\boldsymbol{y}_1 + \boldsymbol{y}_3\,,\ \boldsymbol{y}_2 + \boldsymbol{y}_4\right)^\top$.

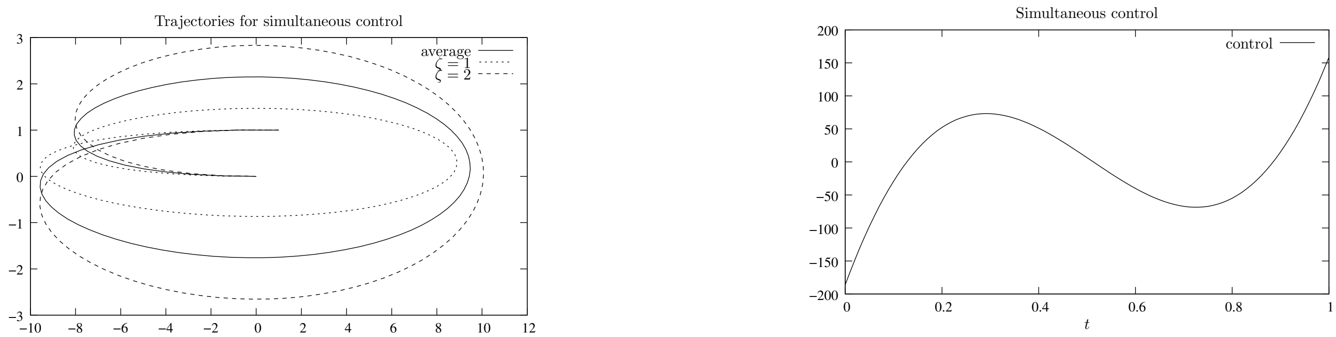
On figures 3, 4 and 5, we plot the numerical results dealing with the averaged control, the exact simultaneous control and the solution of the penalisation problem, when letting the parameter κ growing.



(a) Controlled trajectories in the phase plan using the averaged control. The variance at final time is `2.75e-01`.

(b) Averaged control, the norm of the control is `3.19`.

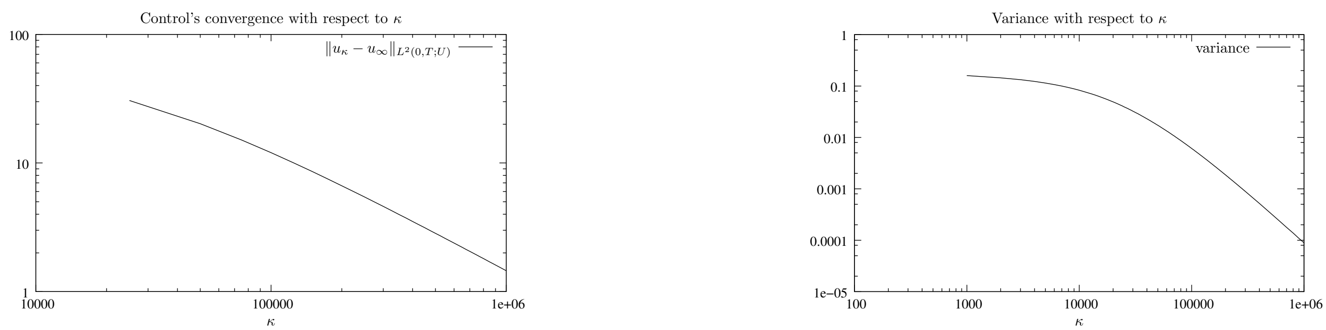
Figure 3: On left, we plotted the trajectories obtained by the averaged control (right) which is of minimal L^2-norm.



(a) Controlled trajectories in the phase plan using the simultaneous control.

(b) Simultaneous control, the norm of the control is `6.34e+01`.

Figure 4: On left, we plotted the trajectories obtained by the simultaneous control (right) which is of minimal L^2-norm.



(a) Plot of the L^2-distance between the exact simultaneous control and the optimal control of the minimisation problem indexed with κ. This distance behaves as $C\kappa^{-\alpha}$ with $\alpha \simeq$ `0.98`.

(b) Plot of the variance at final state ($\int \|y_\zeta(T) - \mathrm{y}^f\|_X^2 \mathrm{d}\mu_\zeta$) with respect to κ. The variance behaves as $C\kappa^{-\alpha}$ with $\alpha \simeq$ `1.95`.

Figure 5: Plots in $\log - \log$ scale of the L^2-distance between the solution of the optimal control with parameter κ and the exact simultaneous control (left) and of the variance at final state (right) as κ grows. The decay rates obtained are coherent with the results of Proposition 4.2.

Example 4.2. *This example illustrate the result of Corollary 4.1 when the is no simultaneous controllability. For this example, we consider again the probability space $\Omega = \{1, 2\}$ and the probability density μ given by $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{2}$. The parameter dependent system under consideration is:*

$$\dot{y_\zeta} = A_\zeta y_\zeta + Bu \qquad y_\zeta(0) = \mathrm{y}^i\,,$$

with $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathrm{y}^i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $A_\zeta = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{if } \zeta = 1\,, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \zeta = 2\,. \end{cases}$

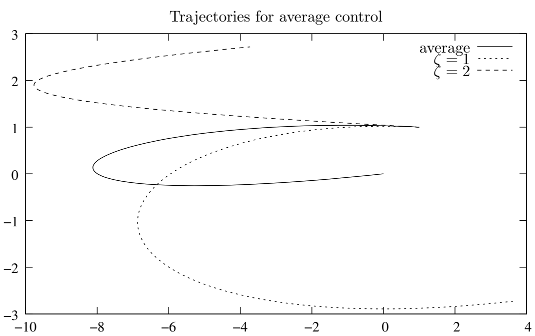
Using the Kalman rank condition, introduced by E. Zuazua (see Theorem 3.1), one can see that this system is controllable in average. On the other hand, the simultaneous controllability of this system reduce to *prove the classical controllability of the augmented system:*

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{y} + \mathbf{B}u\,,$$

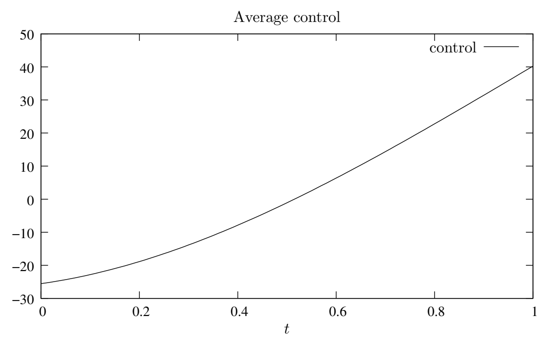
with $\mathbf{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} B \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

One can easily see that $\operatorname{rank}\left[\mathbf{B},\, \mathbf{AB},\, \mathbf{A}^2\mathbf{B},\, \mathbf{A}^3\mathbf{B}\right] = 3 < 4$ *and hence, the Kalman rank condition is not satisfied.*

On figures 6, 7 and 8, we present the numerical results for this system. As in Example 4.1, the final time T *is set to* 1 *and the target* y^f *is* $(0\ 0)^\top$.

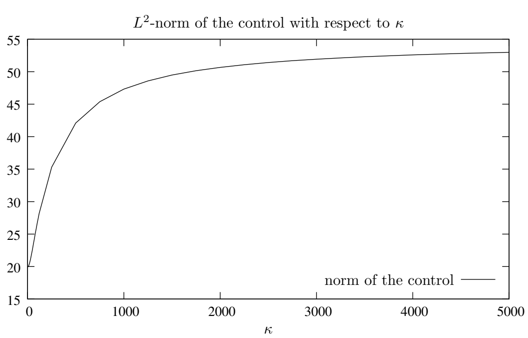


(a) Controlled trajectories in the phase plan using the averaged control. The variance at final time is `2.13e+01`.

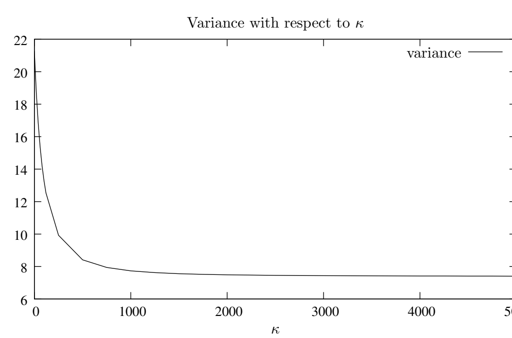


(b) Averaged control, the L^2-norm of the control is `1.99e+01`.

Figure 6: On left, we plotted the trajectories obtained by the averaged control (right) which is of minimal L^2-norm.

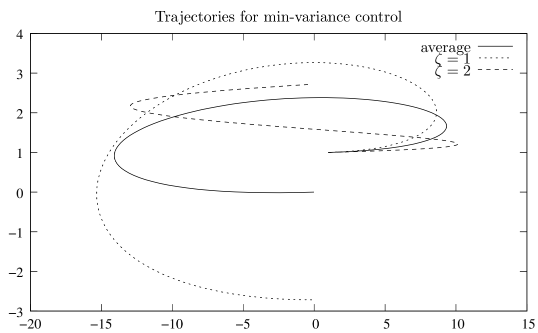


(a) Plot of the norm of the control with respect to κ.

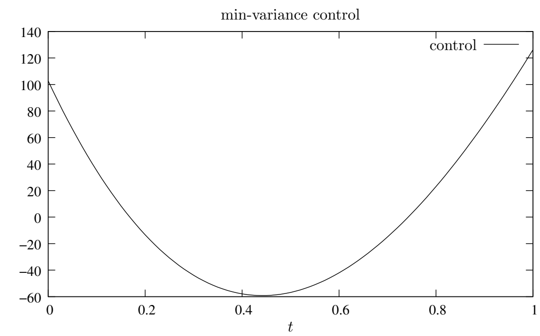


(b) Plot of the variance at final state with respect to κ.

Figure 7: Plots of the norm of the control (left) and of the variance at final state (right) as κ grows.



(a) Controlled trajectories in the phase plan using the optimal control for $\kappa = 5.10^3$. The variance at final time is `7.41`.



(b) Optimal control for $\kappa = 5.10^3$ its L^2-norm is `5.30e+01`.

Figure 8: On left, we plotted the trajectories obtained by the optimal control (right) for $\kappa = 5.10^3$.

5 Numerical realisation when $\operatorname{Card}\Omega$ is infinite

In this section we will study the discrete event case ($\Omega = \mathbb{N}^*$).

For this case, we consider the probability space $(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu)$. A natural way to deal with this problem is to truncate it. More precisely, instead of considering the probability space $(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu)$, we consider the probability space $(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \chi_Z\mu)$ with the measure $\chi_Z\mu$ given by

$$\chi_Z\mu(\{\zeta\}) = \begin{cases} \dfrac{\mu(\{\zeta\})}{\mu(\{1,\cdots,Z\})} & \text{if } \zeta \leqslant Z\,, \\ 0 & \text{otherwise,} \end{cases} \qquad (Z \in \mathbb{N}^*\,,\ \zeta \in \mathbb{N}^*)\,, \tag{5.1}$$

for $Z \in \mathbb{N}^*$ large enough so that $\mu(\{1,\cdots,Z\}) > 0$.

Since our penalisation procedure needs the system $\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u$ to be controllable in average the first question we should answer is whether this averaged controllability property is stable or not through the truncation procedure.

Proposition 5.1. *Assume the system* (1.1) *is controllable in average for the measure* μ.

Then there exists $Z_0 \in \mathbb{N}^*$ *such that for every* $Z \geqslant Z_0$, *this system is controllable in average for the measure* $\chi_Z\mu$ *given by* (5.1).

Let us also notice that this *truncation procedure* does not affect the simultaneous controllability property for Z large enough. More precisely, by direct application of Lemma 3.3, we have:

Proposition 5.2. *Assume the system* (1.1) *is exactly or approximatively simultaneously controllable for the measure* μ.

Then for every $Z \in \mathbb{N}^*$ *such that* $\mu(\{1,\cdots,Z\}) > 0$, *this system is simultaneously controllable for the measure* $\chi_Z\mu$ *given by* (5.1).

Remark 5.1. *Notice that by truncation, one can lose the averaged controllability property. This is for instance the case of the system considered in Example 4.2.*

In opposition the simultaneous controllability property cannot be lost by truncation. This is natural since if the system in simultaneously controllable, all the events $y_1, \cdots, y_Z$ *can be exactly controlled.*

Consequently, if a system is simultaneously controllable, then it is controllable in average and each of its truncation is controllable in average.

Proof of Proposition 5.1. Set $\theta_\zeta = \mu(\{\zeta\})$ without loss of generality, we can assume that $\theta_\zeta > 0$ for every $\zeta \in \mathbb{N}^*$. Set $\theta_\zeta^Z = \chi_Z \mu(\{\zeta\}) = \begin{cases} \dfrac{\theta_\zeta}{\sum_{\zeta=1}^Z \theta_\zeta} & \text{if } \zeta \leqslant Z\,, \\ 0 & \text{otherwise.} \end{cases}$

Let us remind that due to Theorem 3.2, the pairs $(A_\zeta, B_\zeta)_\zeta$ being controllable in average, is equivalent as (3.3):

$$c\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\| \sum_{\zeta\in\mathbb{N}^*} B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t \qquad (\mathrm{z}^f \in X)\,.$$

with $c = c(T) > 0$ independent of z^f.
But,

$$\left(\int_0^T \left\| \sum_{\zeta=1}^Z B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta^Z \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} = \frac{1}{\sum_{\zeta=1}^Z \theta_\zeta} \left(\int_0^T \left\| \sum_{\zeta=1}^Z B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}$$

and using Minkowsky inequality,

$$\left(\sum_{\zeta=1}^Z \theta_\zeta \right) \left(\int_0^T \left\| \sum_{\zeta=1}^Z B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta^Z \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}$$
$$\geqslant \left(\int_0^T \left\| \sum_{\zeta=1}^\infty B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} - \left(\int_0^T \left\| \sum_{\zeta=Z+1}^\infty B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}.$$

From the averaged controllability property, there exists $c > 0$ such that:

$$c\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\| \sum_{\zeta=1}^\infty B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t$$

and due to the admissibility condition, there exists $C > 0$ such that:

$$\int_0^T \left\| \sum_{\zeta=Z+1}^\infty B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta \right\|_U^2 \mathrm{d}t \leqslant C\|\mathrm{z}^f\|_X^2 \sum_{\zeta=Z+1}^\infty \theta_\zeta\,.$$

Consequently,

$$\left(\int_0^T \left\| \sum_{\zeta=1}^Z B_\zeta^* e^{tA_\zeta^*} \mathrm{z}^f \theta_\zeta^Z \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} \geqslant \frac{\sqrt{c} - \sqrt{C\left(1 - \sum_{\zeta=1}^Z \theta_\zeta\right)}}{\sum_{\zeta=1}^Z \theta_\zeta} \|\mathrm{z}^f\|_X\,.$$

Since $\lim\limits_{Z\to\infty} \dfrac{\sqrt{c} - \sqrt{C\left(1 - \sum_{\zeta=1}^Z \theta_\zeta\right)}}{\sum_{\zeta=1}^Z \theta_\zeta} = \sqrt{c} > 0$, we obtain the result. ☐

Let us finally study the error between the initial minimisation problem:

$$\begin{array}{l} \min \quad \mathcal{J}_\kappa^\infty(u) := \frac{1}{2}\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t + \kappa \sum_{\zeta=1}^{\infty} \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_X^2 \mu(\{\zeta\}) \\ \Big| \quad \sum_{\zeta=1}^{\infty} \left(y_\zeta(T;u) - \mathrm{y}_\zeta^f\right)\mu(\{\zeta\}) = 0 \end{array} \qquad (\kappa \geqslant 0) \tag{5.2}$$

and the truncated minimisation problem:

$$\begin{array}{l} \min \quad \mathcal{J}_\kappa^Z(u) := \frac{1}{2}\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t + \kappa \sum_{\zeta=1}^{Z} \|y_\zeta(T;u) - \mathrm{y}_\zeta^f\|_X^2 \chi_Z \mu(\{\zeta\}) \\ \Big| \quad \sum_{\zeta=1}^{Z} \left(y_\zeta(T;u) - \mathrm{y}_\zeta^f\right)\chi_Z\mu(\{\zeta\}) = 0 \end{array} \qquad (\kappa \geqslant 0\,,\ Z \geqslant Z_0)\,, \tag{5.3}$$

with $Z_0 \in \mathbb{N}^*$ given by Proposition 5.1.

Proposition 5.3. *Assume that the system* (1.1) *is controllable in average for the probability measure* μ*. Set* $\kappa \geqslant 0$*. Let* u_κ^Z *be a minimizer of the truncated minimisation problem* (5.3)*.*

Then, as $Z \to \infty$*, the sequence* $(u_\kappa^Z)_Z$ *is strongly convergent in* $L^2([0,T],U)$ *to the minimizer* u_κ *of the initial minimisation problem* (5.2)*.*

Proof. Without loss of generality, we can assume that $\mu(\{\zeta\}) > 0$ for every $\zeta \in \mathbb{N}^*$ and for convenience, we set $\mu(\{\zeta\}) = \theta_\zeta = \theta_\zeta^\infty$ and as previously, $\theta_\zeta^Z = \chi_Z\mu(\{\zeta\}) = \begin{cases} \dfrac{\theta_\zeta}{\sum_{\zeta=1}^{Z}\theta_\zeta} & \text{if } \zeta \leqslant Z\,, \\ 0 & \text{otherwise.} \end{cases}$

Without loss of generality, we can also assume that for every $Z \in \mathbb{N}^*$, the system (1.1) in controllable in average for the probability measure $\chi_Z\mu$.

Let us introduce for every $Z \in \mathbb{N}^* \cup \{\infty\}$ the map $\mathcal{I}^Z : L^2([0,T],U) \to \{0,\infty\}$ defined by:

$$\mathcal{I}^Z(u) = \begin{cases} 0 & \text{if } \sum_{\zeta=1}^{Z}\left(y_\zeta(T;u) - \mathrm{y}_\zeta^f\right)\theta_\zeta^Z = 0\,, \\ \infty & \text{otherwise.} \end{cases}$$

Thus minimizing $\mathcal{J}_\kappa^Z$ under the constraint $\sum_{\zeta=1}^{Z}\left(y_\zeta(T;u) - \mathrm{y}_\zeta^f\right)\theta_\zeta^Z = 0$ is equivalent as minimizing $\mathcal{J}_\kappa^Z + \mathcal{I}^Z$.

The proof of this result is based on Γ-convergence. More precisely, we will prove that the sequence $\left(\mathcal{J}_\kappa^Z + \mathcal{I}^Z\right)_{Z\in\mathbb{N}^*}$ Γ-converge to $\mathcal{J}_\kappa^\infty + \mathcal{I}^\infty$.

- **Upper bound**:
  Let $(u_Z)_{Z\in\mathbb{N}^*} \in L^2([0,T],U)^{\mathbb{N}^*}$ be strongly convergent to an element $u_\infty \in L^2([0,T],U)$.
  The aim of this point is to prove:

$$\mathcal{J}_\kappa^\infty(u_\infty) + \mathcal{I}^\infty(u_\infty) \leqslant \liminf_{Z\to\infty}\left(\mathcal{J}_\kappa^Z(u_Z) + \mathcal{I}^Z(u_Z)\right)\,. \tag{5.4}$$

If $\liminf_{Z\to\infty} \mathcal{I}^Z(u_Z) = \infty$, then, it is clear that (5.4) is true.
Otherwise, we can assume up to the extraction of a subsequence that for every $Z \in \mathbb{N}^*$, we have $\mathcal{I}^Z(u_Z) = 0$. Under this assumption, let us prove:

$$\mathcal{I}^\infty(u_\infty) = 0 \qquad \text{and} \qquad \lim_{Z\to\infty} \sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z = \sum_{\zeta=1}^{\infty} \|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta .$$

This ensure (5.4).

1. Let us prove that $\mathcal{I}^\infty(u_\infty) = 0$:
   To this end, let us notice:

$$\begin{aligned}
\sum_{\zeta=1}^{\infty} \left(y_\zeta(T,u_\infty) - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty &= \sum_{\zeta=1}^{\infty} \left(y_\zeta(T,u_\infty) - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty - \sum_{\zeta=1}^{\infty} \left(y_\zeta(T,u_Z) - \mathrm{y}_\zeta^f\right)\theta_\zeta^Z \\
&= \sum_{\zeta=1}^{\infty} \left(y_\zeta(T,u_\infty) - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty - \frac{1}{\mu(\{1,\cdots,Z\})} \sum_{\zeta=1}^{\infty} \left(y_\zeta(T,u_Z) - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty \\
&\quad + \frac{1}{\mu(\{1,\cdots,Z\})} \sum_{\zeta=Z+1}^{\infty} \left(y_\zeta(T,u_Z) - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty \\
&= \sum_{\zeta=1}^{\infty} \left(\int_0^T e^{(T-t)A_\zeta} B_\zeta \left(u_\infty(t) - \frac{u_Z(t)}{\mu(\{1,\cdots,Z\})}\right) \mathrm{d}t\right)\theta_\zeta^\infty \\
&\quad + \left(1 - \frac{1}{\mu(\{1,\cdots,Z\})}\right) \sum_{\zeta=1}^{\infty} \left(e^{TA_\zeta}\mathrm{y}_\zeta^i - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty \\
&\quad + \frac{1}{\mu(\{1,\cdots,Z\})} \sum_{\zeta=Z+1}^{\infty} \left(\int_0^T e^{(T-t)A_\zeta} B_\zeta u_Z(t)\,\mathrm{d}t\right)\theta_\zeta^\infty \\
&\quad + \frac{1}{\mu(\{1,\cdots,Z\})} \sum_{\zeta=Z+1}^{\infty} \left(e^{TA_\zeta}\mathrm{y}_\zeta^i - \mathrm{y}_\zeta^f\right)\theta_\zeta^\infty
\end{aligned}$$

   The admissibility condition, ensure:

$$\begin{aligned}
&\left\|\sum_{\zeta=1}^{\infty} \left(\int_0^T e^{(T-t)A_\zeta} B_\zeta \left(u_\infty(t) - \frac{u_Z(t)}{\mu(\{1,\cdots,Z\})}\right) \mathrm{d}t\right)\theta_\zeta^\infty\right\|_X^2 \\
&\qquad\qquad \leqslant C \left\|u_\infty - \frac{u_Z}{\mu(\{1,\cdots,Z\})}\right\|_{L^2([0,T],U)}^2 ,
\end{aligned}$$

   with $C > 0$ a constant.

Using Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\left\| \sum_{\zeta=Z+1}^{\infty} \left(\int_0^T e^{(T-t)A_\zeta} B_\zeta u_Z(t)\,\mathrm{d}t \right) \theta_\zeta^\infty \right\|_X^2 & \\ & \leqslant (1-\mu(\{1,\cdots,Z\})) \sum_{\zeta=Z+1}^{\infty} \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta u_Z(t)\,\mathrm{d}t \right\|_X^2 \theta_\zeta^\infty \\ & \leqslant (1-\mu(\{1,\cdots,Z\})) \sum_{\zeta=1}^{\infty} \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta u_Z(t)\,\mathrm{d}t \right\|_X^2 \theta_\zeta^\infty\end{aligned}$$

But, according to the admissibility conditions (see Lemma 2.2), there exists a constant $\hat{C} > 0$ such that:

$$\left\| \sum_{\zeta=Z+1}^{\infty} \left(\int_0^T e^{(T-t)A_\zeta} B_\zeta u_Z(t)\,\mathrm{d}t \right) \theta_\zeta^\infty \right\|_X^2 \leqslant (1-\mu(\{1,\cdots,Z\}))\,\hat{C} \|u_Z\|_{L^2([0,T],U)}^2 .$$

Thus, taking the limit $Z \to \infty$, we obtain $\left\| \sum_{\zeta=1}^{\infty} \left(y_\zeta(T, u_\infty) - \mathrm{y}_\zeta^f \right) \theta_\zeta^\infty \right\|_X = 0$, i.e. $\mathcal{I}^\infty(u_\infty) = 0$.

2. Let us prove $\lim_{Z\to\infty} \sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z = \sum_{\zeta=1}^{\infty} \|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^\infty$:

   For every $Z \in \mathbb{N}^*$, we have, by Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z &= \sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - y_\zeta(T;u_\infty)\|_X^2 \theta_\zeta^Z + \sum_{\zeta=1}^{Z} \|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z \\ &\quad + 2 \sum_{\zeta=1}^{Z} \langle y_\zeta(T;u_Z) - y_\zeta(T;u_\infty), y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f \rangle_X \theta_\zeta^Z \\ &\leqslant \left(\left(\sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - y_\zeta(T;u_\infty)\|_X^2 \theta_\zeta^Z \right)^{\frac{1}{2}} \right. \\ &\qquad \left. + \left(\sum_{\zeta=1}^{Z} \|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z \right)^{\frac{1}{2}} \right)^2 .\end{aligned}$$

   Using the admissibility of every system indexed by ζ, for every $\zeta \in \mathbb{N}^*$, there exists $C_\zeta > 0$ such that:

$$\sum_{\zeta=1}^{Z} \|y_\zeta(T;u_Z) - y_\zeta(T;u_\infty)\|_X^2 \theta_\zeta^Z \leqslant \sum_{\zeta=1}^{Z} C_\zeta \theta_\zeta^Z \|u_Z - u_\infty\|_{L^2([0,T],U)}^2 .$$

In addition, due to assumption (2.3) made in Lemma 2.2, we have $\lim_{Z\to\infty}\sum_{\zeta=1}^{Z} C_\zeta \theta_\zeta^Z < \infty$ and hence, since $(u_Z)_Z$ is strongly convergent to u_∞,

$$\lim_{Z\to\infty}\sum_{\zeta=1}^{Z}\left\|y_\zeta(T;u_Z) - y_\zeta(T;u_\infty)\right\|_X^2 \theta_\zeta^Z = 0\,.$$

On the other hand, it remains clear, due to the construction of θ_ζ^Z that:

$$\lim_{Z\to\infty}\sum_{\zeta=1}^{Z}\left\|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\right\|_X^2 \theta_\zeta^Z = \sum_{\zeta=1}^{\infty}\left\|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\right\|_X^2 \theta_\zeta^\infty\,.$$

Thus,

$$\lim_{Z\to\infty}\sum_{\zeta=1}^{Z}\left\|y_\zeta(T;u_Z) - \mathrm{y}_\zeta^f\right\|_X^2 \theta_\zeta^Z = \sum_{\zeta=1}^{\infty}\left\|y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\right\|_X^2 \theta_\zeta^\infty\,.$$

- **Lower bound**:
Set $u_\infty \in L^2([0,T],U)$. The aim is to prove that there exists a sequence $(u_Z)_{Z\in\mathbb{N}^*}$ strongly convergent to u_∞ such that:

$$\mathcal{J}_\kappa^\infty(u_\infty) + \mathcal{I}^\infty(u_\infty) \geqslant \limsup_{Z\to\infty}\left(\mathcal{J}_\kappa^Z(u_\infty) + \mathcal{I}^Z(u_\infty)\right)\,.$$

If $\mathcal{I}^\infty(u_\infty) = \infty$ then this result can be easily obtained with $u_Z = u_\infty$.
Let us now assume that $\mathcal{I}^\infty(u_\infty) = 0$. From the previous point, it remains clear that if the sequence $(u_Z)_Z$ is converging to u_∞ and if for every $Z \in \mathbb{N}^*$, $\mathcal{I}^Z(u_Z) = 0$ then:

$$\mathcal{J}^\infty(u_\infty) = \lim_{Z\to\infty}\mathcal{J}_\kappa^Z(u_Z)\,.$$

Thus we only need to prove that such a sequence $(u_Z)_Z$ exists.

Let us write $u_Z = u_\infty + v_Z$. Then $\mathcal{I}^Z(u_Z) = 0$ means:

$$\sum_{\zeta=1}^{Z}\int_0^T e^{(T-t)A_\zeta} B_\zeta v_Z(t)\,\mathrm{d}t\,\theta_\zeta^Z = -\sum_{\zeta=1}^{Z}\left(y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\right)\theta_\zeta^Z\,.$$

Since we assumed that the system (1.1) is controllable in average, such a v_Z exists and in addition, there exists a constant $C > 0$ independent of v_Z such that:

$$\|v_Z\|_{L^2([0,T],U)}^2 \leqslant C\left\|\sum_{\zeta=1}^{Z}\left(y_\zeta(T;u_\infty) - \mathrm{y}_\zeta^f\right)\theta_\zeta^Z\right\|_X^2\,.$$

But since $\sum_{\zeta\in\mathbb{N}^*}\left(y_\zeta(T;u_\infty)-\mathrm{y}_\zeta^f\right)\theta_\zeta^\infty=0$, we have:

$$\begin{aligned}\left\|\sum_{\zeta=1}^{Z}\left(y_\zeta(T;u_\infty)-\mathrm{y}_\zeta^f\right)\theta_\zeta^Z\right\|_X &= \left\|\sum_{\zeta=1}^{\infty}\left(y_\zeta(T;u_\infty)-\mathrm{y}_\zeta^f\right)(\theta_\zeta^\infty-\theta_\zeta^Z)\right\|_X\\ &\leqslant \frac{1-\mu(\{1,\cdots,Z\})}{\mu(\{1,\cdots,Z\})}\left\|\sum_{\zeta=1}^{Z}\left(y_\zeta(T;u_\infty)-\mathrm{y}_\zeta^f\right)\theta_\zeta^\infty\right\|_X+\left\|\sum_{\zeta=Z+1}^{\infty}\left(y_\zeta(T;u_\infty)-\mathrm{y}_\zeta^f\right)\theta_\zeta^\infty\right\|_X,\end{aligned}$$

which is going to 0 when $Z\to\infty$. Consequently, $(v_Z)_Z$ converges to 0, that is to say, there exists a sequence $(u_Z)_Z$ convergent to u_∞ such that $\mathcal{I}^Z(u_Z)=0$ for every $Z\geqslant 1$.

The final result follows from Γ-convergence property and $\mathcal{J}_\kappa^\infty+\mathcal{I}^\infty$ admits one and only one minimizer. □

Let us denote by u_κ^Z (resp. u_κ^∞) the minimizer of the truncated (resp. initial) minimisation problem. We proved here that $\lim_{Z\to\infty}u_\kappa^Z=u_\kappa^\infty$. Thus, if $\lim_{\kappa\to\infty}u_\kappa^\infty=u_\infty^\infty$ exists, we have: $\lim_{\kappa\to\infty}\lim_{Z\to\infty}u_\kappa^Z=u_\infty^\infty$. But, do we have $\lim_{Z\to\infty}\lim_{\kappa\to\infty}u_\kappa^Z=u_\infty^\infty$?

This question is the aim of the next proposition.

Proposition 5.4. *Let us assume that the system* (1.1) *is controllable in average for the probability measure* μ.
For every $\kappa\geqslant 0$ *and every large enough* $Z\in\mathbb{N}^*$, *there exists a minimizer* $u_\kappa^Z\in L^2([0,T],U)$ *of the truncated minimisation problem* (5.3).

Up to a subsequence, the sequence $(u_\kappa^Z)_\kappa$ *is strongly convergent to an element* $u_\infty^Z\in L^2([0,T],U)$ *which is a solution of the minimisation problem:*

$$\min\ \tfrac{1}{2}\|u\|^2_{L^2([0,T],U)}\quad\Big|\quad y_\zeta(T)=\mathrm{y}_\zeta^{Z,\star}\qquad(\zeta\in\mathbb{N}^*\quad \chi_Z\mu-a.e.)\,,\tag{5.5}$$

where $\mathrm{y}_\zeta^{Z,\star}$ *is a minimizer of the minimization problem:*

$$\min\ \|\mathrm{y}_\zeta-\mathrm{y}_\zeta^f\|_{L^2(\Omega,X;\chi_Z\mu)}\quad\Big|\quad \begin{array}{l}(\mathrm{y}_\zeta)_\zeta\in\overline{\{y_\zeta(T;u)\,,\ u\in L^2([0,T],U)\}}\,,\\ \mathbb{E}^Z(\mathrm{y}_\zeta)_\zeta=\mathbb{E}^Z(\mathrm{y}_\zeta^f)_\zeta\end{array}\tag{5.6}$$

and where we have set:

$$\mathbb{E}^Z(\mathrm{y}_\zeta)=\sum_{\zeta\in\mathbb{N}^*}\mathrm{y}_\zeta\chi_Z\mu(\{\zeta\})\qquad((\mathrm{y}_\zeta)_\zeta\in L^2(\mathbb{N}^*,X;\chi_Z\mu))\,.$$

Then we have:

1. $\lim_{Z\to\infty}\mathrm{y}_\zeta^{Z,\star}=\mathrm{y}_\zeta^\star$, *with* $\mathrm{y}_\zeta^\star\in L^2(\Omega,X;\mu)$ *given by Theorem 4.1.*
2. *if the sequence* $(u_\infty^Z)_Z$ *is bounded, then the system* (1.1) *can be exactly steered from* y_ζ^i *to* $\mathrm{y}_\zeta^\star$ *and up to a subsequence* $(u_\infty^Z)_Z$ *is weakly convergent to such a control, otherwise, the system* (1.1) *can be approximatively steered from* y_ζ^i *to* $\mathrm{y}_\zeta^\star$.

Proof. Without loss of generality, we can assume $\mu(\{\zeta\}) > 0$ for every $\zeta \in \mathbb{N}^*$ and the system (1.1) is controllable in average for the measure $\chi_Z\mu$ for every $Z \in \mathbb{N}^*$. As in the previous proofs, we set for convenience, $\mu(\{\zeta\}) = \theta_\zeta = \theta_\zeta^\infty$ and $\theta_\zeta^Z = \chi_Z\mu(\{\zeta\}) = \begin{cases} \dfrac{\theta_\zeta}{\sum_{\zeta=1}^Z \theta_\zeta} & \text{if } \zeta \leqslant Z\,, \\ 0 & \text{otherwise.} \end{cases}$

Finally, changing y_ζ^f in $\mathrm{y}_\zeta^f - e^{TA_\zeta}\mathrm{y}_\zeta^i$, we can assume without loss of generality that $\mathrm{y}_\zeta^i = 0$.

Let us notice that for every $Z \geqslant 1$, the control system (1.1) endowed with the measure $\chi_Z\mu$ can be recast as a parameter dependent system whose parameters take place in a set of finite cardinal. Consequently, Corollary 4.1 ensure that the sequence of minimizers $(u_\kappa^Z)_{\kappa\geqslant 0}$ is convergent to $u_\infty^Z \in L^2([0,T],U)$ solution of the minimisation problem (5.5).

Let us prove the 1st item.
For every $Z \in \mathbb{N}^* \cup \infty$, the minimisations problems (5.6) is:

$$\min \quad \mathcal{G}^Z(\mathrm{y}_\zeta) = \sum_{\zeta=1}^Z \|\mathrm{y}_\zeta - \mathrm{y}_\zeta^f\|_X^2 \theta_\zeta^Z$$
$$\mathrm{y}_\zeta \in \overline{\{(y_\zeta(T;u))_\zeta\,,\ u \in L^2([0,T],U)\}}\,,$$
$$\sum_{\zeta=1}^Z (\mathrm{y}_\zeta - \mathrm{y}_\zeta^f)\theta_\zeta^Z = 0\,.$$

Let us also define

$$\mathcal{I}_0(\mathrm{y}_\zeta) = \begin{cases} 0 & \text{if } \mathrm{y}_\zeta \in \overline{\{(y_\zeta(T;u))_\zeta\,,\ u \in L^2([0,T],U)\}}\,, \\ \infty & \text{otherwise} \end{cases}$$
$$\text{and} \qquad \mathcal{I}_1^Z(\mathrm{y}_\zeta) = \begin{cases} 0 & \text{if } \sum_{\zeta=1}^Z (\mathrm{y}_\zeta - \mathrm{y}_\zeta^f)\theta_\zeta^Z = 0\,, \\ \infty & \text{otherwise.} \end{cases}$$

So that the above minimisation problem is:

$$\min_{\mathrm{y}_\zeta \in L^2(\mathbb{N}^*,X;\mu)} \mathcal{G}^Z(\mathrm{y}_\zeta) + \mathcal{I}_0(\mathrm{y}_\zeta) + \mathcal{I}_1^Z(\mathrm{y}_\zeta)\,.$$

In the next points, we will prove that $\left(\mathcal{G}^Z + \mathcal{I}_0 + \mathcal{I}_1^Z\right)_Z$ Γ-convergence to $\mathcal{G}^\infty + \mathcal{I}_0 + \mathcal{I}_1^\infty$.

- **Lower bound**:
  Let $((\mathrm{y}_\zeta^Z)_\zeta)_Z \in L^2(\mathbb{N}^*,X;\mu)^{N^*}$ be a convergent sequence in $L^2(\mathbb{N}^*,X;\mu)$ to $(\mathrm{y}_\zeta^\infty)_\zeta$. The aim is to prove:
  $$\mathcal{G}^Z(\mathrm{y}_\zeta^\infty) + \mathcal{I}_0(\mathrm{y}_\zeta^\infty) + \mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) \leqslant \liminf_{Z\to\infty} \mathcal{G}^Z(\mathrm{y}_\zeta^Z) + \mathcal{I}_0(\mathrm{y}_\zeta^Z) + \mathcal{I}_1^Z(\mathrm{y}_\zeta^Z)\,. \tag{5.7}$$

  First of all, if $\liminf_{Z\to\infty} \mathcal{G}^Z(\mathrm{y}_\zeta^Z) + \mathcal{I}_0(\mathrm{y}_\zeta^Z) + \mathcal{I}_1^Z(\mathrm{y}_\zeta^Z) = \infty$, the result is obvious. Consequently, we can assume $\mathcal{I}_0(\mathrm{y}_\zeta^Z) + \mathcal{I}_1^Z(\mathrm{y}_\zeta^Z) = 0$ for every $Z \in \mathbb{N}^*$.

  Since $\mathcal{I}_0(\mathrm{y}_\zeta^Z) = 0$ for every $Z \in \mathbb{N}^*$ and since $\overline{\{(y_\zeta(T;u))_\zeta\,,\ u \in L^2([0,T],U)\}}$ is a closed set, then $\mathcal{I}_0(\mathrm{y}_\zeta^\infty) = 0$.

Let us now prove that $\mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) = 0$. To this end, we notice that:

$$\begin{aligned}\left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X &= \left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^Z)\theta_\zeta^\infty + \frac{1}{\sum_{\zeta=1}^Z \theta_\zeta^\infty}\sum_{\zeta=1}^{Z}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^Z + \sum_{\zeta=Z+1}^{\infty}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X \\ &= \left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^Z)\theta_\zeta^\infty + \sum_{\zeta=Z+1}^{\infty}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X \\ &\leqslant \left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^Z)\theta_\zeta^\infty\right\|_X + \left\|\sum_{\zeta=Z+1}^{\infty}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X .\end{aligned}$$

Thus, taking the limit $Z \to \infty$, we obtain $\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^f)\theta_\zeta^\infty = 0$, i.e. $\mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) = 0$.

To conclude, it is obvious that $\lim_{Z\to\infty} \mathcal{G}^Z(\mathrm{y}_\zeta^Z) = \mathcal{G}^\infty(\mathrm{y}_\zeta^\infty)$.

- **Upper bound**:
  Let $(\mathrm{y}_\zeta^\infty)_\zeta \in L^2(\mathbb{N}^*, X; \mu)$, the aim is to prove that there exists $((\mathrm{y}_\zeta^Z)_\zeta)_Z \in L^2(\mathbb{N}^*, X; \mu)^{N^*}$, a sequence converging to $(\mathrm{y}_\zeta^\infty)_\zeta$ such that:

$$\mathcal{G}^Z(\mathrm{y}_\zeta^\infty) + \mathcal{I}_0(\mathrm{y}_\zeta^\infty) + \mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) \geqslant \limsup_{Z\to\infty} \mathcal{G}^Z(\mathrm{y}_\zeta^Z) + \mathcal{I}_0(\mathrm{y}_\zeta^Z) + \mathcal{I}_1^Z(\mathrm{y}_\zeta^Z) .$$

If $\mathcal{I}_0(\mathrm{y}_\zeta^\infty) = \infty$, the result is clear with $\mathrm{y}_\zeta^Z = \mathrm{y}_\zeta^\infty$.

If $\mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) = \infty$, i.e. there exists $\varepsilon > 0$ such that $\left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X \geqslant \varepsilon$. Consider the sequence y_ζ^Z given by $\mathrm{y}_\zeta^Z = \begin{cases}\mathrm{y}_\zeta^\infty & \text{if } \zeta \leqslant Z\,,\\ 0 & \text{otherwise.}\end{cases}$ Then $((\mathrm{y}_\zeta^Z)_\zeta)_Z$ converges to $(\mathrm{y}_\zeta^\infty)_\zeta$ in $L^2(\mathbb{N}^*, X; \mu)$ as $Z \to \infty$ and

$$\begin{aligned}\left\|\sum_{\zeta=1}^{Z}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^Z\right\|_X &= \frac{1}{\sum_{\zeta=1}^Z \theta_\zeta^\infty}\left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^\infty + \mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X \\ &\geqslant \frac{1}{\sum_{\zeta=1}^Z \theta_\zeta^\infty}\left(\left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^f)\theta_\zeta^\infty\right\|_X - \left\|\sum_{\zeta=1}^{\infty}(\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^Z)\theta_\zeta^\infty\right\|_X\right) \\ &\geqslant \frac{1}{\sum_{\zeta=1}^Z \theta_\zeta^\infty}\left(\varepsilon - \sqrt{\sum_{\zeta=1}^{\infty}\|\mathrm{y}_\zeta^\infty - \mathrm{y}_\zeta^Z\|_X^2\theta_\zeta^\infty}\right) .\end{aligned}$$

But, since $((\mathrm{y}_\zeta^Z)_\zeta)_Z$ converges to $(\mathrm{y}_\zeta^\infty)_\zeta$, we have for Z large enough, $\left\|\sum_{\zeta=1}^{Z}(\mathrm{y}_\zeta^Z - \mathrm{y}_\zeta^f)\theta_\zeta^Z\right\|_X \geqslant \frac{\varepsilon}{2}$, that is to say $\mathcal{I}_1^Z(\mathrm{y}_\zeta^Z) = \infty$.

Now assume that $\mathcal{I}_0(\mathrm{y}_\zeta^\infty) = \mathcal{I}_1^\infty(\mathrm{y}_\zeta^\infty) = 0$. First of all, it is easy to show that if the sequence $((\mathrm{y}_\zeta^Z)_\zeta)_Z$ converges to $(\mathrm{y}_\zeta^\infty)_\zeta$ then $\lim\limits_{Z\to\infty} \mathcal{G}^Z(\mathrm{y}_\zeta^Z) = \mathcal{G}^\infty(\mathrm{y}_\zeta^\infty)$. Consequently, in order to prove (5.7), we only need to prove the existence of a sequence $((\mathrm{y}_\zeta^Z)_\zeta)_Z \in L^2(\mathbb{N}^*, X; \mu)^{\mathbb{N}^*}$ convergent to $(\mathrm{y}_\zeta^\infty)_\zeta$ such that $\mathcal{I}_0(\mathrm{y}_\zeta^Z) = \mathcal{I}_1^Z(\mathrm{y}_\zeta^Z) = 0$ for every large enough Z.
Since $\mathrm{y}_\zeta^\infty \in \overline{\{(y_\zeta(T;u))_\zeta\,,\ u \in L^2([0,T],U)\}}$, there exists a sequence $(u_Z)_Z \in L^2([0,T],U)^{\mathbb{N}^*}$ such that $\lim\limits_{Z\to\infty} \sum\limits_{\zeta=1}^{\infty} \|y_\zeta(T,u_Z) - \mathrm{y}_\zeta^\infty\|\theta_\zeta^\infty = 0$ and in addition, since $\mathbb{E}\mathrm{y}_\zeta^\infty = \mathbb{E}\mathrm{y}_\zeta^f$, we have $\lim\limits_{Z\to\infty} \sum\limits_{\zeta=1}^{\infty} (y_\zeta(T;u_Z) - \mathrm{y}_\zeta^f)\theta_\zeta^\infty = 0$. Moreover, the system (1.1) is controllable in average for the measure $\chi_Z\mu$, thus the minimisation problem:

$$\begin{array}{ll} \min & \frac{1}{2}\|v\|^2_{L^2([0,T],U)} \\ \Big| & \mathbb{E}^Z\big(y_\zeta(T;v)\big)_\zeta = \mathbb{E}^Z\big(\mathrm{y}_\zeta^f - y_\zeta(T;u_Z)\big)_\zeta\,, \end{array}$$

admits a minimum which is obtained for $v = v_Z$. In addition, since $\lim\limits_{Z\to\infty} \mathbb{E}^Z\big(\mathrm{y}_\zeta^f - y_\zeta(T;u_Z)\big)_\zeta = 0$, we obtain $\lim\limits_{Z\to\infty} \|v_Z\|_{L^2([0,T],U)} = 0$. Consequently, we have build a sequence $((y_\zeta(T;u_Z + v_Z))_\zeta)_Z$, satisfying $\mathcal{I}_0(y_\zeta(T;u_Z+v_Z)) = \mathcal{I}_1^Z(y_\zeta(T;u_Z+v_Z)) = 0$ for every $Z \in \mathbb{N}^*$ and convergent to $(\mathrm{y}_\zeta^\infty)_\zeta$, since,

$$\|\mathrm{y}_\zeta^\infty - y_\zeta(T;u_Z+v_Z)\|_{L^2(\mathbb{N}^*,X;\mu)} \leqslant \|\mathrm{y}_\zeta^\infty - y_\zeta(T;u_Z)\|_{L^2(\mathbb{N}^*,X;\mu)} + \|y_\zeta(T;v_Z)\|_{L^2(\mathbb{N}^*,X;\mu)}$$

is going to 0 as $Z \to \infty$.

All in all, from Γ-convergence tools and the fact that $\mathcal{G}^\infty + \mathcal{I}_0 + \mathcal{I}_1^\infty$ admits one and only one minimizer, we obtain $\lim\limits_{Z\to\infty} \big\|\mathrm{y}_\zeta^{Z,\star} - \mathrm{y}_\zeta^\star\big\|_{L^2(\mathbb{N}^*,X;\mu)} = 0$.

Let us now prove the 2^{nd} item.
Firstly, we have for every $Z \in \mathbb{N}^*$, $y_\zeta(T;u_\infty^Z) = \mathrm{y}_\zeta^{Z,\star}$ and hence, from the above point, the sequence $\big((y_\zeta(T;u_\infty^Z))_\zeta\big)_Z$ is strongly convergent to $(\mathrm{y}_\zeta^\star)_\zeta$ in $L^2(\mathbb{N}^*,X;\mu)$.
In addition, if the sequence $(u_\infty^Z)_Z$ is bounded, then up to a subsequence, this sequence is weakly convergent to a control u_∞^∞ and hence the sequence $\big((y_\zeta(T;u_\infty^Z))_\zeta\big)_Z$ is weackly convergent to $(y_\zeta(T;u_\infty^\infty))_\zeta$ in $L^2(\mathbb{N}^*,X;\mu)$. But from the above point, the sequence $\big((\mathrm{y}_\zeta^{Z,\star})_\zeta\big)_Z = \big((y_\zeta(T;u_\infty^Z))_\zeta\big)_Z$ is convergent to $(\mathrm{y}_\zeta^\star)_\zeta$. Thus, $y_\zeta(T;u_\infty^\infty) = \mathrm{y}_\zeta^\star$. □

6 Concluding remarks

In this paper, we have presented a theoretical link between the averaged controllability and the exact simultaneous controllability. But there still exist many practical questions to be addressed. We list here some of them:

- The problem of convergence rates both for variances and controls as $\kappa \to \infty$ is open. Such results would be helpful in order to validate numerical simulations, since from a computational viewpoint, it is hard to determine what the decay or convergence rate is or even if the limit vanishes or not.

- When the probability space Ω is of infinite cardinal, we have introduced a truncation parameter Z. In that case, we have to parameters (Z and κ) going to infinity. Propositions 5.3 and 5.4 show that the limits in κ and Z commute. But, in practice, it would be interesting to be in condition to bound the analysis and simulations to deal with a single parameter. To this end, we should establish some explicit relation between both of them, for instance, find a function $Z \mapsto \kappa(Z)$ such that when letting $Z \to \infty$, the correct asymptotic behavior in ensured. This problem is related to the one of convergence rates mentioned in the previous item.

- Similar results as those in section 5 could be obtained with a continuous measure and under Lipschitz-regularity assumptions on $\zeta \mapsto (A_\zeta, B_\zeta)$. In this situation, instead of truncating the system, one could use the approximation of Lipschitz functions by piecewise constant functions.

- Finally, the penalization procedure proposed here could be extended in the PDE context.

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