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The Energy Decay Rate for the Modified Von Kármán System of Thermoelastic Plates: An Improvement

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Abstract—We prove that the energy of solutions of the modified von Kármán system of a thermoelastic plate decays with the rate

$$E(t) \leq CE(0) \exp\left(\frac{-\omega t}{1+E(0)}\right),$$

as $t \to +\infty$ where C and ω are positive constants which are independent of the solution. This improves an earlier result in which we claimed the decay rate to be of the order of $\exp(-\omega t/(1+E^2(0))$ and provides a simpler and complete proof.

Keywords—Modified von Kármán system, Exponential rate of decay.

1. INTRODUCTION AND MAIN RESULT

In this note, we want to clarify the incomplete proof of the main result in [1] on the decay of the total energy of the solutions of the von Kármán system of thermoelastic plates and, at the same time, to improve it.

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Given a bounded smooth domain Ω of the plane, the modified von Kármán system reads as follows:

$$
\begin{aligned}
u_{tt} + \Delta^2 u - h\Delta u_{tt} + \Delta\theta &= [u,v], && \text{in } \Omega \times (0,\infty),\\
\Delta^2 v &= -[u,u], && \text{in } \Omega \times (0,\infty),\\
\theta_t - \Delta\theta - \Delta u_t &= 0, && \text{in } \Omega \times (0,\infty),\\
u = \frac{\partial u}{\partial \eta} = \theta &= 0, && \text{on } \partial\Omega \times (0,\infty),
\end{aligned} \tag{1.1}
$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \qquad \text{in } \Omega,$$

where $u = u(x,t)$ is the displacement, $v = v(x,t)$ the Airy stress function, and $\theta = \theta(x,t)$ the temperature. The bracket $[\cdot,\cdot]$ is defined as follows:

$$[u,v] = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}. \tag{1.2}$$

The energy of the system reads as follows:

$$E(t) = \frac{1}{2}\int_\Omega \left[u_t^2 + |\Delta u|^2 + h|\nabla u_t|^2 + \theta^2\right] dx + \frac{1}{4}\int_\Omega |\Delta v|^2\, dx, \tag{1.3}$$

and decreases along trajectories according to the following law:

$$\frac{dE(t)}{dt} = -\int_\Omega |\nabla\theta|^2\, dx. \tag{1.4}$$

The main result of this note is as follows.

THEOREM. *Let $h > 0$. Then there exist positive constants c and $\omega > 0$ such that*

$$E(t) \le c \exp\left(-\frac{\omega t}{1+E(0)}\right) E(0), \qquad \forall\, t \ge 0, \tag{1.5}$$

for any solution of (1.1) with $(u_0,u_1,\theta_0) \in H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$.

REMARK (a). We recall that the result in [1] provided a weaker decay rate of the form

$$E(t) \le c \exp\left(-\frac{\omega t}{1+E(0)^2}\right) E(0), \qquad \forall\, t \ge 0. \tag{1.6}$$

The exponential decay rate we obtain here is stronger by a factor of $E(0)$. Actually, the proof we give in this note is much simpler than the one in [1].

REMARK (b). In fact, the proof in [1] was incomplete. Indeed, in [1], we considered $H(t)$ (a suitable perturbation of the energy $E(t)$) and proved that it satisfies the inequality

$$\frac{dH}{dt} \le -c_3\varepsilon H(t) + c_2\varepsilon \int_{\partial\Omega} |\Delta u|^2\, d\sigma, \tag{1.7}$$

for some positive constants c_2 and c_3 and $\varepsilon > 0$ sufficiently small. Then, we used (1.7) to deduce the exponential decay of $H(t)$ (and consequently, of $E(t)$). However, we did not know *a priori* if c_2 was small with respect to c_3. This subtleness has as a consequence that the existence of the constants c_2, c_3, ρ, and ε claimed and needed in the proof of Theorem 1 in [1] may not be guaranteed. Thus, the proof in [1] was incomplete.

2. PROOF OF THE MAIN RESULT

The proof of the result in this note follows the notations in [1] and we only present the new developments. We consider $H = E + \varepsilon F + (\varepsilon/2)G$, where

$$F = \int_\Omega \left[h u_t \theta - \frac{h}{2}\theta^2 + u_t(-\Delta)^{-1}\theta \right] dx, \qquad G = \int_\Omega \left[u u_t + h \nabla u \cdot \nabla u_t \right] dx. \tag{2.1}$$

As shown in Lemma 4 of [1], by taking $\varepsilon > 0$ sufficiently small independently of the initial data, one may guarantee that $H/2 \le E \le 2H$. Thus, E and H are equivalent in what concerns the decay rate. Using the estimate of the terms I_j, $1 \le j \le 3$ as in the proof of Lemma 5 of [1], we know that

$$\frac{dH}{dt} \le -c_3 \varepsilon H(t) + \varepsilon \int_{\partial\Omega} \Delta u \frac{\partial}{\partial \eta}(-\Delta)^{-1}\theta \, d\sigma, \tag{2.2}$$

for a suitable $c_3 > 0$ and any $\varepsilon > 0$ small satisfying

$$\varepsilon < \left[h + \frac{13}{4\lambda_1} + \frac{1}{2} + 2CH(0) \right]^{-1}, \tag{2.3}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions and C is a positive constant independent of the data. Now, for any $\delta > 0$, we have that

$$\left| \int_{\partial\Omega} \Delta u \frac{\partial}{\partial \eta}(-\Delta)^{-1}\theta \, d\sigma \right| \le \frac{\delta}{2} \int_{\partial\Omega} |\Delta u|^2 \, d\sigma + \frac{1}{2\delta} \int_{\partial\Omega} \left| \frac{\partial}{\partial \eta}(-\Delta)^{-1}\theta \right|^2 d\sigma. \tag{2.4}$$

Since $\frac{\partial(-\Delta)^{-1}}{\partial \eta}$ is a bounded linear operator from $H_0^1(\Omega)$ into $L^2(\partial\Omega)$, and in view of (1.4), we deduce the existence of a constant $\tilde{c} > 0$ such that

$$\left\| \frac{\partial}{\partial \eta}(-\Delta)^{-1}\theta \right\|^2_{L^2(\Omega)} \le \tilde{c} \|\nabla \theta\|^2_{L^2(\Omega)} = -\tilde{c}\frac{dE}{dt}\cdot \tag{2.5}$$

Using inequalities (2.2), (2.4), and (2.5), we deduce that

$$\frac{d}{dt}\left\{ H(t) + \frac{\varepsilon \tilde{c}}{2\delta} E(t) \right\} \le -c_3 \varepsilon H(t) + \frac{\varepsilon\delta}{2} \int_{\partial\Omega} |\Delta u|^2 \, d\sigma. \tag{2.6}$$

Integration of (2.6) from $t = 0$ up to $t = T$ (with $T > 0$ to be chosen later) and using Lemma 2 of [1] gives that

$$\begin{aligned} H(T) + \frac{\varepsilon \tilde{c}}{2\delta} E(T) &\le H(0) + \frac{\varepsilon \tilde{c}}{2\delta} E(0) - c_3 \varepsilon \int_0^T H(s)\, ds \\ &+ \frac{c\varepsilon\delta}{2} \left\{ E(0) + 2\left(1 + E^{1/2}(0)\right) \int_0^T H(s)\, ds \right\}. \end{aligned} \tag{2.7}$$

Now we choose $\delta > 0$ such that $\delta \le c_3[1 + E^{1/2}(0)]^{-1}$. With this choice, we deduce from (2.7) that

$$H(T) + \frac{\varepsilon \tilde{c}}{2\delta} E(T) \le H(0) + \frac{\varepsilon \tilde{c}}{2\delta} E(0) - \frac{c_3 \varepsilon}{2} \int_0^T H(s)\, ds + \frac{c\varepsilon\delta}{2} E(0). \tag{2.8}$$

Since $2H(t) \ge E(t)$ and $E(t)$ is decreasing, then $-\int_0^T H(s)\, ds \le -(T/2)E(T)$. Consequently, from (2.8), it follows that

$$\left[\frac{1}{2} + \frac{\varepsilon \tilde{c}}{2\delta} + \frac{c_3 \varepsilon T}{4} \right] E(T) \le \left[2 + \frac{\varepsilon \tilde{c}}{2\delta} + \frac{c\varepsilon\delta}{2} \right] E(0).$$

Given $0 < \gamma < 1$ fixed, we choose $T > 0$ large enough so that

$$\left(2 + \frac{\varepsilon\tilde{c}}{2\delta} + \frac{c\varepsilon\delta}{2}\right)\left(\frac{1}{2} + \frac{\varepsilon\tilde{c}}{2\delta} + \frac{c_3\varepsilon T}{4}\right)^{-1} \leq \gamma < 1.$$

With this choice, we have that $E(T) \leq \gamma E(0)$ with $0 < \gamma < 1$. Now, we use the semigroup property. Since problem (1.1) is globally well posed, we may write the solution as $S(t)\varphi_0 = (u, u_t, \theta)^\tau$, where $\varphi_0 = (u_0, u_1, \theta_0)^\tau$. Consequently, $S(t + r) = S(t)S(r)$ for all $t, r \geq 0$. It follows that

$$E(nT) = E(S(nT)\varphi_0) = E(S^n(T)\varphi_0) \leq \gamma^n E(S(T)\varphi_0) \leq \gamma^n E(0),$$

which implies that there exist positive constants c and w such that

$$E(t) \leq c \exp\left(\frac{-wt}{T}\right) E(0), \qquad \forall\, t \geq 0. \tag{2.9}$$

The constants in (2.9) are precisely $c = \gamma^{-1}$ and $\omega = -\log(\gamma)$. Finally, let us see how T depends on $E(0)$. Due to our previous discussion, we can take

$$\varepsilon = \frac{1}{2}\left[h + \frac{13}{4\lambda_1} + \frac{1}{2} + 4C\, E(0)\right]^{-1},$$

and δ can be taken as $\delta = k/(1 + E^{1/2}(0))$ for some positive constant k independent of the initial data. It follows that $0 < \gamma < 1$ will hold when T is chosen of the order of $C(1 + E(0))$ with $C > 0$ large enough, depending on γ but independent of the initial data. This completes the proof of the theorem.

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