

On the equivalence of minimal time and minimal norm controls for internally controlled heat equations

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Abstract

We prove the equivalence of the minimal time and minimal norm control problems for heat equations on bounded smooth domains of the euclidean space with homogeneous Dirichlet boundary conditions and controls distributed internally on an open subset of the domain where the equation evolves. We consider the problem of null controllability whose aim is to drive solutions to rest in a finite final time. As a consequence of this equivalence, using the well-known variational characterization of minimal norm controls, we establish necessary and sufficient conditions for the minimal time and the corresponding control.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$, and ω be an open and non-empty subset of Ω . Denote by χ_ω the characteristic function of the set ω . Let T be a positive number and write $\mathbb{R}^+ \equiv (0, +\infty)$.

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We consider the following two controlled heat equations:

$$\begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(x, 0) = y_0(x) & \text{in } \Omega \end{cases} \quad (1.1)$$

and

$$\begin{cases} y_t - \Delta y = \chi_\omega f & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases} \quad (1.2)$$

The two systems under consideration are actually the same but in the first one the time horizon is infinite while in the second one it is finite and coincides with the interval $(0, T)$. The initial state y_0 is assumed to be a non-trivial function in $L^2(\Omega)$, and u and f are the controls taken accordingly from the spaces $L^\infty(\mathbb{R}^+; L^2(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$.

Controls in the above two systems are active only on a subset ω of Ω . Such a control strategy, compared with the global one in which the control is applied over the whole domain Ω , is of bigger interest in applications because it is easier to be implemented and less expensive. On the other hand, this internal control strategy makes the problem much more challenging. There is in fact a wide literature on the control of heat processes and research in this topic is still ongoing.

The solutions of (1.1) and (1.2), denoted by $y(\cdot; u)$ and $y(\cdot; f)$, are considered to be functions of the time variable t , from $[0, +\infty)$ and $[0, T]$ to the space $L^2(\Omega)$. By $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ we denote the usual norm and inner product of $L^2(\Omega)$ respectively, while $\|\cdot\|_\omega$ and $\langle \cdot, \cdot \rangle_\omega$ stand for the usual norm and inner products of the space $L^2(\omega)$ respectively. By $B(0, r)$ we denote the closed ball in $L^2(\Omega)$, centered at the origin and of radius $r > 0$.

For each $T > 0$ and each $M > 0$, we define the following two admissible sets of controls:

- $\mathcal{F}_T = \{f \in L^\infty(0, T; L^2(\Omega)) : y(T; f) = 0\}$;
- $\mathcal{U}_M = \{u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) : u(\cdot) \in B(0, M) \text{ over } \mathbb{R}^+ \text{ and } \exists t > 0 \text{ s.t. } y(t; u) = 0\}$.

For each admissible control $u \in \mathcal{U}_M$ of the infinite horizon control problem, we define $\tilde{t}(u) = \inf \{t > 0; y(t; u) = 0\}$.

The two control problems under consideration are as follows:

- **Minimal norm control problem, (NOCP) $_T$** : $\inf_{f \in \mathcal{F}_T} \{\|f\|_{L^\infty(0, T; L^2(\Omega))}\}$;
- **Minimal time control problem, (TOCP) $_M$** : $\inf_{u \in \mathcal{U}_M} \{\tilde{t}(u)\}$.

These two problems provide the following two values:

$$\alpha(T) \equiv \inf_{f \in \mathcal{F}_T} \{\|f\|_{L^\infty(0, T; L^2(\Omega))}\} \quad \text{and} \quad \tau(M) \equiv \inf_{u \in \mathcal{U}_M} \{\tilde{t}(u)\},$$

the minimal (or optimal) norm and the minimal (or optimal) time for $(\text{NOCP})_T$ and $(\text{TOCP})_M$ respectively.

The optimal controls for these two problems are defined as follows:

- (i) f^* is a norm optimal control (or a minimal norm control) to $(\text{NOCP})_T$, if $y(T; f^*) = 0$ and $\|f^*\|_{L^\infty(0,T;L^2(\Omega))} = \alpha(T)$;
- (ii) u^* is a time optimal control (or a minimal time control) to $(\text{TOCP})_M$, if $u^* \in \mathcal{U}_M$ and $y(\tau(M); u^*) = 0$.

Problem $(\text{TOCP})_M$ is a special instance of free terminal time-problems (see, for instance, [30], [17] and [16]).

The admissible set \mathcal{F}_T is well known to be non empty as a consequence of the null controllability property of the heat equation for any $T > 0$ and from any open subset ω of Ω . This is also true when the controlled heat equation has added lower order terms (see, for instance, [1], [6], [9], [12], [13], [33], [37] and the references therein). However, the issue of whether the admissible set \mathcal{U}_M is non empty for all $M > 0$ is more subtle. This holds for the heat equation under consideration (see [29] and also [35]) since the cost of control is well-known to tend to zero as $T \rightarrow \infty$, but not necessarily for heat equations with lower order terms (see [29]).

In the present context of the pure heat equation, as a consequence of the non-emptiness of the admissible set \mathcal{F}_T and \mathcal{U}_M , one can easily derive the existence of both norm optimal and time optimal controls (see, for instance, [37] and [29]).

Clearly, $M \rightarrow \tau(M)$ defines a non-increasing function $\alpha(\cdot)$ over \mathbb{R}^+ . The same can be said about $T \rightarrow \alpha(T)$. This is so since we are dealing with the null control problem. Indeed, since the null target is an equilibrium of the system, once the solution $y(\cdot; u)$ reaches it at time t_1 , the solution $y(\cdot; \chi_{(0,t_1)}u)$ stays at the null state for each time $t \geq t_1$. This implies that the minimal time diminishes as the constraint on the size of control increases and vice versa.

The first main result of this paper ensures that *these two functions are strictly monotonically decreasing and continuous, and they are inverse one of each other*. More precisely, the following holds:

Theorem 1.1. *For each $T > 0$, the norm optimal control to $(\text{NOCP})_T$, when extended by zero to $(T, +\infty)$, is the time optimal control to $(\text{TOCP})_{\alpha(T)}$. Conversely, for each $M > 0$, the time optimal control to $(\text{TOCP})_M$, when restricted over $(0, \tau(M))$, is the norm optimal control to $(\text{NOCP})_{\tau(M)}$.*

It is worth mentioning that the above equivalence theorem still holds when ω is a subset of positive measure (see Remark 2.3).

This equivalence property allows to use the techniques normally employed to deal with the norm optimal controls, to analyze the time optimal ones. More precisely, with the aid of Theorem 1.1, we use the variational characterization of minimal norm controls to establish a sufficient and necessary condition for the optimal time and the time optimal control to the problem $(\text{TOCP})_M$.

To present this variational approach, for each $T > 0$ we introduce the following functional J^T :

$$J^T(\varphi_T) = \frac{1}{2} \left(\int_0^T \|\varphi(t; \varphi_T)\|_\omega dt \right)^2 + \langle \varphi(0; \varphi_T), y_0 \rangle, \quad \varphi_T \in L^2(\Omega), \quad (1.3)$$

where, $\varphi(\cdot; \varphi_T)$ denotes the solution of the adjoint heat equation:

$$\begin{cases} \varphi_t + \Delta \varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(T) = \varphi_T & \text{on } \Omega, \end{cases} \quad (1.4)$$

with φ_T as datum at $t = T$.

This functional can be extended over a normed space X_T (which is constructed in Section 3 and consists roughly on the completion of $L^2(\Omega)$ with respect to the norm $\int_0^T \|\varphi(t; \varphi_T)\|_\omega dt$) so that the extended functional, still denoted by J^T , has a minimizer in X_T . The coercivity of J^T is by now a well known consequence of the observability inequalities for the adjoint heat equation that can be derived by means of Carleman inequalities (see [37]). Using standard arguments, one can see that, for each $T \in (0, \infty)$ and each minimizer $\hat{\varphi}_T$ of J^T , the control

$$f^*(t) = \left(\int_0^T \|\varphi(t; \hat{\varphi}_T)\|_\omega dt \right) \frac{\varphi(t; \hat{\varphi}_T)}{\|\varphi(t; \hat{\varphi}_T)\|_\omega} \quad \text{for a.e. } t \in (0, T), \quad (1.5)$$

is the minimal norm control to $(\text{NOCP})_T$.

The uniqueness of the minimizer for J^T is not obvious since J^T is not strictly convex (the $L^1(0, T; L^2(\omega))$ -norm is not strictly convex). We use (1.5) and the uniqueness of the minimal norm control of $(\text{NOCP})_T$, as well as the strong unique continuation result for the heat equation, established in [20], to derive the uniqueness of the minimizer for J^T . The uniqueness of the minimal norm control follows from Theorem 1.1 and the bang-bang property for $(\text{TOCP})_M$ built up in [33] (see Remark 2.2).

In view of these we can derive the next relevant result of this paper as follows:

Theorem 1.2. *Let $M > 0$. Then $\tau(M)$, the optimal time for $(\text{TOCP})_M$, is characterized by the fact that*

$$M = \int_0^{\tau(M)} \|\hat{\varphi}(t)\|_\omega dt, \quad (1.6)$$

where $\hat{\varphi}$ solves the equation:

$$\begin{cases} \hat{\varphi}_t + \Delta \hat{\varphi} = 0 & \text{in } \Omega \times (0, \tau(M)), \\ \hat{\varphi} = 0 & \text{on } \partial\Omega \times (0, \tau(M)), \\ \hat{\varphi}(\tau(M)) = \hat{\varphi}_{\tau(M)} & \text{in } \Omega, \end{cases} \quad (1.7)$$

with $\hat{\varphi}_{\tau(M)}$ the minimizer of $J^{\tau(M)}$.

Furthermore, the time optimal control $u^* \in L^\infty(\mathbb{R}^+; L^2(\Omega))$, under the norm constraint M , is characterized by the condition

$$\langle u^*(t), \chi_\omega \hat{\varphi}(t) \rangle = \max_{v^0 \in B(0, M)} \langle v^0, \chi_\omega \hat{\varphi}(t) \rangle \quad \text{for a.e. } t \in [0, \tau(M)), \quad (1.8)$$

or, equivalently,

$$u^*(t) = M \chi_\omega \frac{\hat{\varphi}(t)}{\|\hat{\varphi}(t)\|_\omega} \quad \text{for a.e. } t \in [0, \tau(M)). \quad (1.9)$$

Several remarks are in order:

- The main results in Theorem 1.1, as well as Theorem 1.2, can also be used to build algorithms to compute the optimal time (see Remark 3.3).
- The necessary and sufficient condition presented in Theorem 1.2 seems to be new. It provides the following valuable information: (i) The equality (1.6), together with (1.7) gives a formula for the optimal time; (ii) The initial value $\hat{\varphi}(\tau(M))$ of the dual equation is exactly a minimizer of $J^{\tau(M)}$. Such an initial value plays the role of a multiplier from the perspective of the calculus of variations, while it is a normal vector of a hyperplane separating the target set and the attainable set, from a geometric point of view, in the context of the Pontryagin maximal principle for time optimal control problems. In the previous related literature (see, for instance, [10] and [19]), it is only guaranteed that there exists a non-trivial initial value for the dual system such that the maximal condition (i.e., the analog of (1.8) in our paper) is fulfilled.
- In the context of time optimal control problems for ordinary differential equations, $y'(t) = Ay(t) + Bu(t)$ where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the control constraint set of the ball type, $\{u \in L^\infty(\mathbb{R}^+; \mathbb{R}^m) : \|u(t)\|_{\mathbb{R}^m} \leq M \text{ for a.e. } t \in \mathbb{R}^+\}$, is one of the most important ones. Problem $(\text{TOCP})_M$ is a natural and simple extension to parabolic equations (see, for instance, [10]). This is why we consider controls in the spaces $L^\infty(\mathbb{R}^+; L^2(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$. Of course, the study of analogue problems with control spaces $L^p(\mathbb{R}^+; L^q(\Omega))$ and $L^p(0, T; L^q(\Omega))$ (where $1 \leq p \leq \infty$ and $1 \leq q < \infty$) is also interesting and can be done with similar techniques as the ones considered here.

- The connections between minimal time and minimal norm controls is not new and there is plenty that has been done in this context. This issue was addressed in F. Gozzi and P. Loreti ([14]) for abstract equations in Hilbert spaces obtaining the equality $M = M_{\tau(M)}$ for each $M > 0$ (see Theorem 4.1 in [14]). From this equality, it follows that any minimal time control to $(\text{TOCP})_M$ is a minimal norm control to $(\text{NOCP})_{\tau(M)}$. The novelty in our Theorem 1.1 is that we prove the equivalence between these two optimal control problems, based on the identities $T = \tau(\alpha(T))$ for each $T > 0$, and $M = \alpha(\tau(M))$ for each $M > 0$ (which are proved in Theorem 2.1 below). The book by H. O. Fattorini [10] is also worth mentioning. There the equivalence of time optimal control and norm optimal controls was established for abstract equations in Banach spaces. But in the context of the distributed control of the heat equation these results only apply for the very special case where the control is distributed everywhere in the domain, i.e. $\omega = \Omega$. Earlier, the relationship between time and norm optimal controls was studied for the heat equation with scalar controls, i.e., controls depending only on the time variable (see [18]). The dependence of the minimal time function with respect to the initial data has also been analyzed in a number of papers: [3], [4] and [32].

The rest of the paper is organized as follows: Section 2 presents some properties of the function $\tau(\cdot)$, followed by the proof of Theorem 1.1. Section 3 gives the variational characterization for minimal norm controls and the proof of Theorem 1.2. In the last section, some further comments and open problems are provided.

2 Equivalence between minimal time and minimal norm controls

We begin by studying the properties of the function $\tau(\cdot)$.

Theorem 2.1. *The function $\tau(\cdot)$ is strictly monotonically decreasing and continuous from \mathbb{R}^+ onto \mathbb{R}^+ . Moreover, it holds that*

$$\tau(\alpha(T)) = T \text{ for all } T > 0 \text{ and } \alpha(\tau(M)) = M \text{ for all } M > 0. \quad (2.1)$$

Consequently, the inverse of $\alpha(\cdot)$ is $\tau(\cdot)$.

Proof. The proof is organized in several steps as follows: *Step 1.* *The function $\tau(\cdot)$ is strictly monotonically decreasing.*

Let $M_1 > M_2 > 0$. It suffices to show that $\tau(M_1) < \tau(M_2)$. Seeking for a contradiction, suppose that $\tau(M_1) \geq \tau(M_2)$. Then the optimal control u_2 to the problem $(\text{TOCP})_{M_2}$ would verify that

$$\|\chi_{(0, \tau(M_2))} u_2\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq M_2 < M_1$$

and

$$y(\tau(M_1); \chi_{(0, \tau(M_2))} u_2) = y(\tau(M_2); u_2) = 0.$$

(The existence of optimal controls to this problem is provided by, for instance, Theorem 3.3 in [29].) These imply that $\chi_{(0, \tau(M_2))} u_2$ is an optimal control to $(\text{TOCP})_{M_1}$. By the bang-bang property of $(\text{TOCP})_{M_1}$ (established in [33], see also [24]), we find that

$$\|\chi_{(0, \tau(M_2))}(t) u_2(t)\| = M_1 \quad \text{over } (0, \tau(M_1)).$$

This leads to a contradiction, since $M_1 > M_2$.

Step 2. When $\{M_n\}$ is such that

$$M_1 \geq M_2 \geq \dots \geq M_n \rightarrow M \in (0, \infty),$$

it holds that

$$\lim_{n \rightarrow \infty} \tau(M_n) = \tau(M).$$

Consequently, the function $\tau(\cdot)$ is right-continuous.

If it did not hold, then, by the monotonicity of $\tau(\cdot)$, we would have

$$\tau(M_n) \nearrow (\tau(M) - \delta) \quad \text{for some } \delta > 0.$$

On the other hand, the optimal controls u_n for $(\text{TOCP})_{M_n}$, $n = 1, 2, \dots$, satisfy

$$y(\tau(M_n); u_n) = 0$$

and

$$\|u_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq M_n \leq M_1 \quad \text{for all } n.$$

Hence, there is a subsequence of $\{u_n\}$, still denoted in the same way, such that

$$\chi_{(0, \tau(M_n))} u_n \rightarrow \tilde{u} \quad \text{weakly star in } L^\infty(\mathbb{R}^+; L^2(\Omega))$$

and

$$0 = y(\tau(M_n); u_n) = y(\tau(M) - \delta; \chi_{(0, \tau(M_n))} u_n) \rightarrow y(\tau(M) - \delta; \tilde{u}) \quad \text{strongly in } L^2(\Omega),$$

from which, it follows that

$$\|\tilde{u}\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq \liminf_{n \rightarrow \infty} M_n = M$$

and

$$y(\tau(M) - \delta; \tilde{u}) = 0.$$

This contradicts the optimality of $\tau(M)$ for $(\text{TOCP})_M$.

Step 3. When $\{M_n\}$ is such that

$$M_1 \leq M_2 \leq \dots \leq M_n \rightarrow M \in (0, \infty),$$

it holds that

$$\lim_{n \rightarrow \infty} \tau(M_n) = \tau(M).$$

Consequently, the function $\tau(\cdot)$ is left-continuous.

If it did not hold, then by the monotonicity of the function $\tau(\cdot)$, we would have

$$\tau(M_n) \searrow (\tau(M) + \delta) \quad \text{for some } \delta > 0,$$

from which, it follows that

$$\tau(M_n) > \tau(M) + \delta \quad \text{for all } n. \quad (2.2)$$

Let u be the optimal control for $(\text{TOCP})_M$ and write $y(\cdot)$ for the solution $y(\cdot; u)$. Set

$$w_n(\cdot) = \frac{M_n}{M} y(\cdot) \quad \text{and} \quad v_n(\cdot) = \frac{M_n}{M} \chi_{(0, \tau(M))} u(\cdot), \quad n = 1, 2, \dots$$

It is clear that

$$\begin{cases} (w_n)_t - \Delta w_n = \chi_\omega v_n & \text{in } \Omega \times \mathbb{R}^+, \\ w_n = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ w_n(0) = \frac{M_n}{M} y_0, \quad w_n(\tau(M)) = 0 & \text{in } \Omega. \end{cases}$$

Since $v_n = 0$ over $(\tau(M), \infty)$, it follows that

$$w_n(\tau(M) + \delta) = 0 \quad \text{for all } n. \quad (2.3)$$

Besides, one can easily check that

$$\|v_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq M_n \quad \text{for all } n. \quad (2.4)$$

Consider the equation:

$$\begin{cases} (z_n)_t - \Delta z_n = \chi_\omega \chi_{(\tau(M), \tau(M) + \delta)} f_n & \text{in } \Omega \times \mathbb{R}^+, \\ z_n = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z_n(0) = (1 - \frac{M_n}{M}) y_0 & \text{in } \Omega. \end{cases}$$

According to the well-known null controllability properties for heat equations (see, for instance, [12]), there are a positive constant $C(\delta, \tau(M))$ independent of n , and a control f_n with

$$\|f_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq C(\delta, \tau(M)) \|y_0\| \left(1 - \frac{M_n}{M}\right), \quad (2.5)$$

such that

$$z_n\left(\tau(M) + \delta\right) = 0. \quad (2.6)$$

Since $M_n \nearrow M$, we can fix a natural number n_0 such that

$$C(\delta, \tau(M))\|y_0\|\left(1 - \frac{M_{n_0}}{M}\right) \leq M_{n_0}. \quad (2.7)$$

Now we construct another control g_{n_0} by setting

$$g_{n_0}(t) = v_{n_0}(t) + \chi_{(\tau(M), \tau(M)+\delta)}(t)f_{n_0}(t) \text{ for a.e. } t \in (0, \infty).$$

Then it follows from (2.4), (2.5) and (2.7), that

$$\|g_{n_0}\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq M_{n_0}. \quad (2.8)$$

On the other hand, one can easily check that

$$y(\cdot; g_{n_0}) = w_{n_0}(\cdot) + z_{n_0}(\cdot) \text{ over } [0, \infty).$$

Along with (2.3) and (2.6), this yields that

$$y\left(\tau(M) + \delta; g_{n_0}\right) = 0. \quad (2.9)$$

Now it follows from (2.8) and (2.9) that $g_{n_0} \in \mathcal{U}_{M_{n_0}}$. Then, by the optimality of $\tau(M_{n_0})$ for $(\text{TOCP})_{M_{n_0}}$ and by (2.9), we obtain that $\tau(M_{n_0}) \leq \tau(M) + \delta$, which contradicts (2.2).

Step 4. $\lim_{M \rightarrow 0} \tau(M) = \infty$.

If it did not hold, then there would be a sequence $\{M_n\}$, with $M_1 \geq M_2 \geq \dots \geq M_n \rightarrow 0$, such that

$$\lim_{n \rightarrow \infty} \tau(M_n) = T < \infty. \quad (2.10)$$

Write u_n for the optimal control to $(\text{TOCP})_{M_n}$. Then, it holds that

$$\|u_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq M_n \leq M_1 \text{ for all } n.$$

Thus, there is a subsequence of $\{u_n\}$, still denoted in this way, such that

$$u_n \rightarrow \tilde{u} \text{ weakly star in } L^\infty(\mathbb{R}^+; L^2(\Omega)). \quad (2.11)$$

On one hand, it follows from (2.11) that

$$\|\tilde{u}\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq \liminf_{n \rightarrow \infty} M_n = 0,$$

from which, we derive that $\tilde{u} = 0$. On the other hand, by making use of (2.11) again, we deduce that, along a subsequence,

$$y(\cdot; u_n) \rightarrow y(\cdot; \tilde{u}) \text{ in } C([0, T]; L^2(\Omega)).$$

This, together with (2.10) and the fact that $y(\tau(M_n); u_n) = 0$, indicates that $y(T; \tilde{u}) = 0$, namely, $y(T; 0) = 0$. But this is in contradiction with the fact that y_0 is not trivial by backward uniqueness. Indeed, from the backward uniqueness of the heat equation (see [22]) the solution $y(\cdot; 0)$ (to the heat equation with the initial value y_0 and without control) cannot vanish at any time $t > 0$.

Step 5. $\lim_{M \rightarrow \infty} \tau(M) = 0$.

If it did not stand, then we could make use of the monotonicity of the function $\tau(\cdot)$ to get a sequence $\{M_n\}$, with $M_1 \leq M_2 \leq \dots \leq M_n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} \tau(M_n) = 2T > 0.$$

Then it would hold that

$$\tau(M_n) > T > 0 \text{ for all } n. \quad (2.12)$$

By the null controllability for heat equations (see, for instance, [12]), there are a positive constant $C(T)$, independent of n , and a control u , with $\|u\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq C(T)\|y_0\|$, such that $y(T; u) = 0$. Since $M_n \rightarrow \infty$, we can fix n that $C(T)\|y_0\| \leq M_n$, from which, it follows that $u \in \mathcal{U}_{M_n}$. Then by the optimality of $\tau(M_n)$ for $(\text{TOCP})_{M_n}$, we see that $\tau(M_n) \leq T$. This contradicts (2.12).

Step 6. The proof of (2.1)

We begin by proving the first equality in (2.1). Fix a $T > 0$. By the null controllability for the heat equation, one can easily check that $(\text{NOCP})_T$ has an optimal control \hat{f} . We extend this control by setting it to be zero over (T, ∞) . Clearly, the extended control, still denoted by \hat{f} , satisfies that

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} = \alpha(T) \text{ and } y(T; \hat{f}) = 0, \quad (2.13)$$

from which, it follows that $\hat{f} \in \mathcal{U}_{\alpha(T)}$. By the optimality of $\tau(\alpha(T))$ for $(\text{TOCP})_{\alpha(T)}$, together with the second equality of (2.13), we deduce that $\tau(\alpha(T)) \leq T$.

Seeking a contradiction, suppose that $\tau(\alpha(T)) < T$. Since $\tau(\cdot)$ is continuous and strictly monotonically decreasing, we could find $M_1 < \alpha(T)$ with $\tau(M_1) = T$. Clearly, the optimal control u_1 to $(\text{TOCP})_{M_1}$ satisfies that

$$\|\chi_{(0, T)} u_1\|_{L^\infty(0, T; L^2(\Omega))} = \|u_1\|_{L^\infty(0, \tau(M_1); L^2(\Omega))} \leq M_1 < \alpha(T) \quad (2.14)$$

and

$$y(T; \chi_{(0,T)}u_1) = y(\tau(M_1); u_1) = 0,$$

from which it follows that $\chi_{(0,T)}u_1 \in \mathcal{F}_T$. This, along with the optimality of $\alpha(T)$ to $(\text{NOCP})_T$, yields that $\|u_1\|_{L^\infty(0,T;L^2(\Omega))} \geq \alpha(T)$, which contradicts (2.14), since $M_1 < \alpha(T)$. Therefore, the first equality in (2.1) stands.

Next, by the first equality in (2.1), we see that $\tau(\alpha(\tau(M))) = \tau(M)$ for all $M > 0$, which, along with the strict monotonicity of $\tau(\cdot)$, gives the second equality in (2.1).

The proof is now complete. □

Proof of Theorem 1.1. If \hat{f} is an optimal control to $(\text{NOCP})_T$, then by the definition of the norm optimal control, together with the first equality in (2.1), we find that $y(\tau(\alpha(T)); \hat{f}) = 0$ and $\|\hat{f}\|_{L^\infty(0,T;L^2(\Omega))} = \alpha(T)$, from which, it follows that \hat{f} is the optimal control to $(\text{TOCP})_{\alpha(T)}$.

Conversely, if u^* is the optimal control to $(\text{TOCP})_M$, then it follows from the definition of the time optimal control and the second equality in (2.1) that

$$y(\tau(M); u^*) = 0 \quad \text{and} \quad \|u^*\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} \leq \alpha(\tau(M)).$$

Consequently, u^* is the optimal control to $(\text{NOCP})_{\tau(M)}$. This completes the proof. □

Remark 2.2. Note that the above results can be used to show that, for each $T > 0$, the optimal control to the problem $(\text{NOCP})_T$ is unique.

Indeed, according to Theorem 1.1 and the bang-bang property for the time optimal control problem $(\text{TOCP})_{\alpha(T)}$ (see [33]), Problem $(\text{NOCP})_T$ has the bang-bang property, i.e., any norm optimal control f^* to this problem satisfies that $\|f^*(t)\| = M$ for a.e. $t \in (0, T)$. Seeking for a contradiction, suppose that there were two different optimal controls f_1 and f_2 to Problem $(\text{NOCP})_T$. Then we would find an $\varepsilon > 0$ and a subset $E_\varepsilon \subset (0, T)$ with a positive measure such that $\|(f_1(t) - f_2(t))/2\| \geq \varepsilon$ for each $t \in E_\varepsilon$. Write $f = (f_1 + f_2)/2$ and $g = (f_1 - f_2)/2$. Clearly, f is still an optimal control to Problem $(\text{NOCP})_T$. Now, it follows from the bang-bang property and the parallelogram law that

$$\alpha(t)^2 = \|f(t)\|^2 = -\|g(t)\|^2 + \frac{1}{2}(\|f_1(t)\|^2 + \|f_2(t)\|^2) \leq \alpha(t)^2 - \varepsilon^2 \quad \text{for each } t \in E_\varepsilon,$$

which leads to a contradiction and completes the proof.

Remark 2.3. Recently, the null controllability property for the heat equation, with controls supported in a subset of positive measure ω , has been established (see [1]). This, together

with the argument in the proof of Theorem 3.1, as well as Theorem 3.3 in [29], shows the existence of optimal controls for $(TOCP)_M$ for any subset ω of positive measure. On the other hand, this null controllability property, together with the argument in the proof of Theorem 1.1, and Theorem 1.2 in [33], gives the bang-bang property for $(TOCP)_M$ for any ω of positive measure. As a consequence of this, Theorem 1.1 above still holds when ω is a subset of positive measure.

Remark 2.4. *The essential ingredients to prove Theorem 1.1 are follows: (1) The bang-bang property for time optimal controls, i.e., any optimal control u^* to $(TOCP)_M$ verifies that $\|u^*(t)\| = M$ for a.e. $t \in (0, \tau(M))$; (2) The null-controllability in arbitrary short time. These two requirements also makes it impossible to use the similar way to get the corresponding equivalence theorem for the wave equation.*

The bang-bang property was first built up for global controlled operational differential equations in [11]. The pointwise bang-bang property: $|u^*(x, t)| = M$ for a.e. (x, t) , was obtained for global boundary controlled heat equations in [31]. The bang-bang property for the internally controlled heat equation was proved in [33] (see also [24] and [28]). Recently, the pointwise bang-bang property was established for the heat equation where controls are active on an open subset of $\partial\Omega$ with Ω a rectangular domain. More recently, the pointwise bang-bang property was built up for the heat equation with controls restricted over either a subset of positive surface measure on the boundary or a subset of positive measure in Ω (see [2]). The null-controllability of heat equations has been analyzed quite extensively, see, for instance, [13], [37] and the references therein.

3 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. For this purpose, we first derive necessary and sufficient conditions for the minimal norm and the minimal norm control to $(NOCP)_T$ by the variational methods described in the introduction. Then with the help of Theorem 1.1 we prove Theorem 1.2.

Throughout this section, we denote by $\varphi(\cdot; \varphi_T)$ the solution of the adjoint heat equation (1.4) with the initial condition $\varphi(T) = \varphi_T$ at time $t = T$. It follows from [12] that there is a positive function $C(\cdot) \in C^\infty([0, T))$, with $C(t) \rightarrow +\infty$ as $t \rightarrow T$, such that

$$\|\varphi(t; \varphi_T)\| \leq C(t) \int_t^T \|\varphi(s; \varphi_T)\|_\omega ds, \quad t \in [0, T), \quad \text{for each } \varphi_T \in L^2(\Omega). \quad (3.1)$$

Write

$$L_C^1(0, T; L^2(\Omega)) = \{\psi(\cdot) : C^{-1}(\cdot)\psi(\cdot) \in L^1(0, T; L^2(\Omega))\},$$

with

$$\|\psi(\cdot)\|_{L^1_C(0,T;L^2(\Omega))} = \|C^{-1}(\cdot)\psi(\cdot)\|_{L^1(0,T;L^2(\Omega))}.$$

Let X_T be the closure of $L^2(\Omega)$ under the norm:

$$\|\varphi_T\|_{X_T} = \|\varphi(\cdot; \varphi_T)\|_{L^1_C(0,T;L^2(\Omega))} + \|\varphi(\cdot; \varphi_T)\|_{L^1(0,T;L^2(\omega))}, \quad \varphi_T \in X_T. \quad (3.2)$$

Then, by standard density arguments, one can easily check that (3.1) holds for all $\varphi_T \in X_T$, i.e.,

$$\|\varphi(t; \varphi_T)\| \leq C(t) \int_t^T \|\varphi(s; \varphi_T)\|_{\omega} ds, \quad t \in [0, T], \quad \text{for each } \varphi_T \in X_T. \quad (3.3)$$

Now, consider the functional J^T over X_T as in (1.3).

Lemma 3.1. *The functional J^T has a minimizer over X_T .*

Proof. Clearly, the functional J^T is continuous, coercive and convex over X_T . Let $\{\varphi_T^n\} \subset X_T$ be a minimizing sequence. Write $\varphi^n(\cdot)$ for the solution $\varphi(\cdot; \varphi_T^n)$. Then, one can easily derive

$$\int_0^T \|\varphi^n(t)\|_{\omega} dt \leq C \quad \text{for all } n \quad (3.4)$$

and

$$C^{-1}(t)\|\varphi^n(t)\| \leq \int_t^T \|\varphi^n(s)\|_{\omega} ds \quad \text{for all } t \in [0, T].$$

Thus

$$\int_0^T C^{-1}(t)\|\varphi^n(t)\| dt \leq \int_0^T \int_t^T \|\varphi^n(s)\|_{\omega} ds dt \leq TC.$$

This, together with (3.4), indicates that $\{\varphi^n(\cdot)\}$ is bounded in both $M(0, T; L^2(\omega))$ (the dual of the space $C([0, T]; L^2(\omega))$ and $M_C(0, T; L^2(\Omega))$, where

$$M_C(0, T; L^2(\Omega)) = \{\psi(\cdot) : C^{-1}(\cdot)\psi(\cdot) \in M(0, T; L^2(\Omega))\},$$

with

$$\|\psi(\cdot)\|_{M_C(0,T;L^2(\Omega))} = \|C^{-1}(\cdot)\psi(\cdot)\|_{M(0,T;L^2(\Omega))}.$$

According to Alaoglu's Theorem, there are a subsequence of $\{\varphi^n(\cdot)\}$ (still denoted in this way) and an element $\varphi(\cdot) \in M_c(0, T; L^2(\Omega)) \cap M(0, T; L^2(\omega))$ such that

$$\varphi^n \rightarrow \varphi \quad \text{weak}^* \quad \text{in } M_C(0, T; L^2(\Omega)) \quad (3.5)$$

and

$$\varphi^n \rightarrow \varphi \text{ weak}^* \text{ in } M(0, T; L^2(\omega)). \quad (3.6)$$

Since φ^n solves the heat equation, one can easily verify that $\varphi(\cdot)$ is a solution of the heat equation (1.4) in the sense of distributions. Then, one can use the smoothing effect of the heat equation to deduce that

$$\varphi(\cdot) \in L^1_C(0, T; L^2(\Omega)) \cap C([0, T]; L^2(\Omega)) \cap L^1(0, T; L^2(\omega)) \quad (3.7)$$

and

$$\varphi(T) \in X_T. \quad (3.8)$$

(For the sake of the completeness of the paper, we will provide the proof of (3.7) and (3.8) in an Appendix in the end of the paper.)

Furthermore

$$\int_0^T \|\varphi(t)\|_\omega dt \leq \liminf_{n \rightarrow \infty} \int_0^T \|\varphi^n(\cdot)\|_\omega dt. \quad (3.9)$$

We claim that

$$\varphi^n(0) \rightarrow \varphi(0) \text{ strongly in } L^2(\Omega). \quad (3.10)$$

In fact, by (3.5), we see that

$$\varphi^n \rightarrow \varphi \text{ weak}^* \text{ in } M(0, T/2; L^2(\Omega)). \quad (3.11)$$

On the other hand, it follows from (3.3) that

$$\|\varphi^n(T/2)\| \leq C(T/2) \int_{T/2}^T \|\varphi^n(s)\|_\omega ds \leq C.$$

Hence, on a subsequence,

$$\varphi^n(T/2) \rightarrow \psi_{T/2} \text{ weakly in } L^2(\Omega). \quad (3.12)$$

Let ψ satisfy the equation:

$$\begin{cases} \partial_t \psi + \Delta \psi = 0 & \text{in } \Omega \times (0, T/2), \\ \psi = 0 & \text{on } \partial\Omega \times (0, T/2), \\ \psi(T/2) = \psi_{T/2} & \text{in } \Omega. \end{cases}$$

Then it holds that

$$\varphi^n \rightarrow \psi \text{ weakly in } L^2(0, T/2; L^2(\Omega)).$$

This, together with (3.11), yields that

$$\varphi = \psi \text{ over } (0, T/2). \quad (3.13)$$

By (3.12) and (3.13), we find that

$$\varphi^n(T/2) \rightarrow \varphi(T/2) \text{ weakly in } L^2(\Omega),$$

from which, (3.10) follows at once.

Finally, by (3.9) and (3.10), it follows that $\varphi_T \equiv \varphi(T)$ is a minimizer of J^T . This completes the proof. \square

Theorem 3.2. *The functional J^T has a unique minimizer $\hat{\varphi}_T$. Furthermore, the function*

$$f^*(t) = \left(\int_0^T \|\varphi(s; \hat{\varphi}_T)\|_\omega ds \right) \frac{\chi_\omega \varphi(t; \hat{\varphi}_T)}{\|\varphi(t; \hat{\varphi}_T)\|_\omega}, \quad t \in [0, T), \quad (3.14)$$

is the optimal control to $(NOCP)_T$. Consequently, it holds that

$$\alpha(T) = \int_0^T \|\varphi(s; \hat{\varphi}_T)\|_\omega ds.$$

Proof. From Lemma 3.1, J^T has minimizers. By a very similar argument as that in [9], one can verify that

$$0 \text{ is not a minimizer of } J^T. \quad (3.15)$$

In fact, if 0 was a minimizer of J^T , then we would have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{J^T(\varepsilon \varphi_T)}{\varepsilon} \geq \lim_{\varepsilon \rightarrow 0^+} \frac{J^T(0)}{\varepsilon} = 0 \text{ for all } \varphi_T \in L^2(\Omega).$$

Since $\varphi(\cdot; \varepsilon \varphi_T) = \varepsilon \varphi(\cdot; \varphi_T)$, the above implies

$$0 \leq \langle \varphi(0; \varphi_T), y_0 \rangle = \langle e^{\Delta T} \varphi_T, y_0 \rangle = \langle \varphi_T, e^{\Delta T} y_0 \rangle \text{ for all } \varphi_T \in L^2(\Omega),$$

from which, $e^{\Delta T} y_0 = 0$. Then, by the backward uniqueness of the heat equation (see [22]), we would have $y_0 = 0$, which contradicts the assumption that $y_0 \neq 0$. Hence, (3.15) holds.

Next, we claim that for each minimizer $\tilde{\varphi}_T$, the function

$$\tilde{f}(t) \triangleq \left(\int_0^T \|(\varphi(s; \tilde{\varphi}_T))\|_\omega ds \right) \frac{\chi_\omega \varphi(t; \tilde{\varphi}_T)}{\|\varphi(t; \tilde{\varphi}_T)\|_\omega}, \quad t \in [0, T), \quad (3.16)$$

is the optimal control to $(NOCP)_T$, and

$$\alpha(T) = \int_0^T \|\varphi(s; \tilde{\varphi}_T)\|_\omega ds. \quad (3.17)$$

The proof of this claim is carried out by the following two steps:

Step 1. The control $\tilde{f}(\cdot)$ belongs to \mathcal{F}_T .

Simply write $\tilde{\varphi}(\cdot)$ for the solution $\varphi(\cdot; \tilde{\varphi}_T)$. To show that \tilde{f} is well defined, we first notice that $\tilde{\varphi}(T - \delta) \in L^2(\Omega)$ for each $\delta \in (0, T)$ (see (3.3)). Then, we prove

$$\tilde{\varphi}(T - \delta) \neq 0 \text{ in } L^2(\Omega) \text{ for each } \delta \in (0, T). \quad (3.18)$$

Indeed, if there was a $\delta_0 \in (0, T)$ such that $\tilde{\varphi}(T - \delta_0) = 0$ in $L^2(\Omega)$, then by the backward uniqueness of the heat equation (see [22]) and the fact that $\tilde{\varphi}(T - \delta) \in L^2(\Omega)$ for each $\delta > 0$, we would have that for each $t \in [0, T)$, $\tilde{\varphi}(t) = 0$ over Ω . This, along with (3.2), indicates that $\tilde{\varphi}_T = 0$ in X_T , which contradicts (3.15). Hence, (3.18) stands. Next, by (3.18) and the strong unique continuation property of the heat equation (see [20]), we see that for each $\delta \in (0, T)$,

$$\|\tilde{\varphi}(t)\|_\omega \neq 0 \text{ for each } t \in [0, T - \delta).$$

Thus, $\|\tilde{\varphi}(t)\|_\omega \neq 0$ for each $t \in [0, T)$, and \tilde{f} is well defined.

Next, from (3.16), we find

$$\tilde{f} \in L^\infty(0, T; L^2(\Omega)) \text{ and } \|\tilde{f}\|_{L^\infty(0, T; L^2(\Omega))} = \int_0^T \|\tilde{\varphi}(s)\|_\omega ds. \quad (3.19)$$

The remainder is to show that $y(T; \tilde{f}) = 0$. Clearly, this will be done if the following holds:

$$\langle y_0, \varphi(0; \varphi_T) \rangle + \int_0^T \langle \tilde{f}(t), \chi_\omega \varphi(t; \varphi_T) \rangle dt = 0 \text{ for all } \varphi_T \in X_T. \quad (3.20)$$

Fortunately, (3.20) is exactly the Euler equation associated with the minimizer $\tilde{\varphi}_T$ of J^T .

Step 2. The control \tilde{f} verifies that $\|\tilde{f}\|_{L^\infty(0, T; L^2(\omega))} \leq \|g\|_{L^\infty(0, T; L^2(\omega))}$ for all $g \in \mathcal{F}_T$.

Let $\{\varphi_{T,n}\} \subset L^2(\Omega)$ be such that $\varphi_{T,n} \rightarrow \tilde{\varphi}_T$ in X_T . Write $\varphi_n(\cdot) = \varphi(\cdot; \varphi_{T,n})$. Then, it holds that

$$\varphi_n(\cdot) \rightarrow \tilde{\varphi}(\cdot) \text{ strongly in } L^1(0, T; L^2(\omega)). \quad (3.21)$$

By (3.3), we find that

$$\|\varphi_n(0) - \tilde{\varphi}(0)\| \leq C(0) \int_0^T \|\varphi_n(s) - \tilde{\varphi}(s)\|_\omega ds \rightarrow 0. \quad (3.22)$$

Let $g \in \mathcal{F}_T$. Then, $y(T; g) = 0$. Since $\varphi_n(T) \in L^2(\Omega)$, we can easily derive

$$\langle y_0, \varphi_n(0) \rangle + \int_0^T \langle g(t), \chi_\omega \varphi_n(t) \rangle dt = 0 \text{ for all } n.$$

This, together with (3.21) and (3.22), indicates that

$$\langle y_0, \tilde{\varphi}(0) \rangle + \int_0^T \langle g(t), \chi_\omega \tilde{\varphi}(t) \rangle dt = 0. \quad (3.23)$$

Now it follows from (3.20) and (3.23) that

$$\int_0^T \langle \tilde{f}(t), \chi_\omega \tilde{\varphi}(t) \rangle dt = \int_0^T \langle g(t), \chi_\omega \tilde{\varphi}(t) \rangle dt.$$

From this, (3.19) and the Hölder inequality, we get

$$\|\tilde{f}\|_{L^\infty(0,T;L^2(\omega))}^2 \leq \|g\|_{L^\infty(0,T;L^2(\omega))} \int_0^T \|\tilde{\varphi}(t)\|_\omega dt = \|g\|_{L^\infty(0,T;L^2(\omega))} \|\tilde{f}\|_{L^\infty(0,T;L^2(\omega))}.$$

Hence,

$$\|\tilde{f}\|_{L^\infty(0,T;L^2(\omega))} \leq \|g\|_{L^\infty(0,T;L^2(\omega))}.$$

Combining the conclusions in Step 1 and Step 2 leads to the previous claim.

Finally, we will prove that the minimizer of J^T is unique. Suppose that $\hat{\varphi}_T$ and $\hat{\psi}_T$ are two minimizers of J^T . Simply write $\hat{\varphi}(\cdot)$ and $\hat{\psi}(\cdot)$ for the solutions $\varphi(\cdot; \hat{\varphi}_T)$ and $\varphi(\cdot; \hat{\psi}_T)$ respectively. By the previous claim (see (3.16) and (3.17)) and the uniqueness of the optimal control to $(\text{NOCP})_T$ (see Remark 2.2), we see that

$$\frac{\chi_\omega \hat{\varphi}(t)}{\|\hat{\varphi}(t)\|_\omega} = \frac{\chi_\omega \hat{\psi}(t)}{\|\hat{\psi}(t)\|_\omega} \text{ for each } t \in [0, T] \quad (3.24)$$

and

$$\int_0^T \|\hat{\varphi}(s)\|_\omega ds = \int_0^T \|\hat{\psi}(s)\|_\omega ds = \alpha(T). \quad (3.25)$$

Since both $\|\hat{\varphi}(\cdot)\|_\omega$ and $\|\hat{\psi}(\cdot)\|_\omega$ are continuous over $(0, T)$ (see (3.7)), one can easily derive from (3.25) that there is a $t_0 \in (0, T)$ such that

$$\|\hat{\varphi}(t_0)\|_\omega = \|\hat{\psi}(t_0)\|_\omega. \quad (3.26)$$

From (3.24) and (3.26), it follows that

$$\hat{\varphi}(t_0) = \hat{\psi}(t_0) \text{ over } \omega. \quad (3.27)$$

Set $p(\cdot) = \hat{\varphi}(\cdot) - \hat{\psi}(\cdot)$. Clearly, $p(\cdot) = \varphi(\cdot; \hat{\varphi}_T - \hat{\psi}_T)$, i.e., $p(\cdot)$ solves the adjoint equation (1.4) with the initial condition $p(T) = \hat{\varphi}_T - \hat{\psi}_T \in X_T$. Let $\delta \in (0, T - t_0)$. Then, $p(T - \delta) \in L^2(\Omega)$ (see (3.3)). Hence, $p(\cdot)$ verifies

$$\begin{cases} p_t + \Delta p = 0 & \text{in } \Omega \times (0, T - \delta), \\ p = 0 & \text{on } \partial\Omega \times (0, T - \delta), \\ p(T - \delta) \in L^2(\Omega), \end{cases}$$

Then, by (3.27) and the strong unique continuation of the heat equation (see [20]),

$$p = 0 \text{ over } \Omega \times [0, T - \delta].$$

Since the above holds for all $\delta \in (0, T - t_0)$, we find that

$$p = 0 \text{ over } \Omega \times [0, T].$$

This, along with (3.2), yields that

$$0 = \|p(0)\|_{X_T} = \|\hat{\varphi}_T - \hat{\psi}_T\|_{X_T}$$

from which, it follows that

$$\hat{\varphi}_T = \hat{\psi}_T \text{ in } X_T.$$

This completes the proof. □

As a consequence of Theorem 3.2, we deduce that M and f^* are the minimal norm and the optimal control to $(\text{NOCP})_T$ if and only if $M > 0$ and $f^* \in L^\infty(0, T; L^2(\Omega))$ satisfy that

$$\langle f^*(t), \chi_\omega \hat{\varphi}(t) \rangle = \max_{v^0 \in B(0, \alpha(T))} \langle v^0, \chi_\omega \hat{\varphi}(t) \rangle \text{ for a.e. } t \in [0, T) \quad (3.28)$$

and

$$M = \int_0^T \|\hat{\varphi}(t)\|_\omega dt, \quad (3.29)$$

where $\hat{\varphi}(\cdot)$ solves the adjoint heat equation with the minimizer $\hat{\varphi}_T$ of J^T as datum at time $t = T$.

We end this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\tau(M)$ and u^* be accordingly the optimal time and the optimal control to $(\text{TOCP})_M$. Clearly, $\tau(M) > 0$ since $y_0 \neq 0$. It follows from Theorem 1.1 that u^* and $\alpha(\tau(M))$ are the norm optimal control and the minimal norm to $(\text{NOCP})_{\tau(M)}$. We have also shown (see (3.28) and (3.29)) that

$$\langle u^*(t), \chi_\omega \hat{\varphi}(t) \rangle = \max_{v^0 \in B(0, \alpha(\tau(M)))} \langle v^0, \chi_\omega \hat{\varphi}(t) \rangle \quad \text{for all } t \in [0, \tau(M)) \quad (3.30)$$

and

$$\alpha(\tau(M)) = \int_0^{\tau(M)} \|\hat{\varphi}(t)\|_\omega dt, \quad (3.31)$$

where $\hat{\varphi}$ satisfies (1.7) with $\hat{\varphi}_{\tau(M)}$ the minimizer of $J^{\tau(M)}$. By the second equality in (2.1), $\alpha(\tau(M)) = M$. Along with this, (3.30) and (3.31) imply (1.8) and (1.6), respectively.

Conversely, suppose that $\tilde{T} > 0$ and $\tilde{u} \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ satisfy that

$$M = \int_0^{\tilde{T}} \|\hat{\varphi}(t)\|_\omega dt, \quad (3.32)$$

and

$$\tilde{u}(t) = M \chi_\omega \frac{\hat{\varphi}(t)}{\|\hat{\varphi}(t)\|_\omega} \quad \text{for a.e. } t \in [0, \tilde{T}), \quad (3.33)$$

where $\hat{\varphi}$ solves the equation:

$$\begin{cases} \hat{\varphi}_t + \Delta \hat{\varphi} = 0 & \text{in } \Omega \times (0, \tilde{T}), \\ \hat{\varphi} = 0 & \text{on } \partial\Omega \times (0, \tilde{T}), \\ \hat{\varphi}(\tilde{T}) = \hat{\varphi}_{\tilde{T}} & \text{in } \Omega, \end{cases} \quad (3.34)$$

with $\hat{\varphi}_{\tilde{T}}$ the minimizer of $J^{\tilde{T}}$. We first show that \tilde{T} is the optimal time to $(\text{TOCP})_M$. In fact, since $\hat{\varphi}_{\tilde{T}}$ is the minimizer of $J^{\tilde{T}}$, it follows from Theorem 3.2 that

$$\alpha(\tilde{T}) = \int_0^{\tilde{T}} \|\hat{\varphi}(t)\|_\omega dt.$$

This, together with (3.32), yields that $\alpha(\tilde{T}) = M$. From this and the second equality in (2.1), we find that

$$\alpha(\tilde{T}) = \alpha(\tau(M)).$$

Then, by the strict monotonicity of the function $\alpha(\cdot)$ (see Theorem 2.1), we see that

$$\tilde{T} = \tau(M), \text{ namely, } \tilde{T} \text{ is the optimal time to } (\text{TOCP})_M.$$

We next prove that \tilde{u} is the optimal control to $(\text{TOCP})_M$. Indeed, since $\tilde{T} = \tau(M)$, from (3.32), (3.33), (3.34) and Theorem 3.2, we see that \tilde{u} is the optimal control to $(\text{NOCP})_{\tau(M)}$. Then, according to Theorem 1.1, it is the optimal control to $(\text{TOCP})_{\alpha(\tau(M))}$. Because $M = \alpha(\tau(M))$ (see (2.1)), \tilde{u} is the optimal control to $(\text{TOCP})_M$. This completes the proof.

Remark 3.3. *Since the function $\alpha(\cdot)$ is continuous and strictly decreasing, the well-known bisection algorithm (see, for instance, [5]), as well as Theorem 1.2 (More precisely, (1.6)), can be used for the computation of the optimal time to $(\text{TOCP})_M$. This algorithm requires computing the minimal norm control for a sequence of approximated time instances t_n . Note however that, as described in [27], this is an ill-posed and challenging issue. Our result shows that the minimal time can be interpreted as the root of a continuous optimal valued function. For the case of vibrating systems, an approach of this type has been studied in [15].*

4 Further comments and open problems

A number of interesting issues could be considered in connection with the results and methods developed in this paper. Here we briefly describe some of them.

- **Heat equations with time dependent potentials:** The extension of the results of this paper to heat equations with lower order potentials depending on space and time is an open problem. The null controllability results in (see [28]) from time measurable sets when Ω is convex could be of use in this context.
- **Semilinear heat equations:** The same questions of this paper can be addressed in the semilinear setting in which there are already a number of positive null controllability results (see [37]). This is however to be done.

- **L^∞ -control constraints:** The same problems of this paper can be extended to the case where the control spaces are $L^\infty(\Omega \times (0, T))$ and $L^\infty(\mathbb{R}^+ \times \Omega)$. In this setting, the bang-bang property for $(\text{TOCP})_M$ is as follows: any time optimal control u satisfies that $|u(x, t)| = M$ for a.e. $(x, t) \in \omega \times (0, T)$. This bang-bang property follows from the null controllability of the heat equation with controls restricted to subsets \mathcal{D} of positive measure in $\Omega \times (0, T)$ (see [2]).
- **Boundary controlled heat equation:** It could be interesting to extend the results of this paper to the boundary controlled heat equation with L^∞ -control constraint. The corresponding bang-bang property established in [2] could be used.
- **Rectangular-control constraints:** It would be interesting to study the equivalence problem for heat equations with control constraints of the rectangular type (see [25]). The corresponding equivalence for ODEs with control constraints of this type was proved in [36].
- **Approximate controllability:** In problems $(\text{NOCP})_T$ and $(\text{TOCP})_M$, if the target set is $B(z_d, r)$ (the ball in $L^2(\Omega)$ centered at z_d and of radius $r > 0$), where z_d is arbitrarily taken from the state space $L^2(\Omega)$, then the bang-bang property follows from the Pontryagin maximum principle and the unique continuation property. However, the function $\tau(\cdot)$ is no longer necessarily monotonically decreasing and then whether the equivalence theorem holds or not in this context is an open problem.
- **Other equations:** It would be interesting to study the corresponding equivalence theorem for the wave and Schrödinger like equations. We refer to [23] for a very interesting recent result on the bang-bang property of the time-optimal controls for the Schrödinger equations.
- **Abstract semigroup setting:** It would be interesting to extend the results of this paper to the abstract semigroup setting considering systems of the form $y'(t) + Ay(t) = Bu(t)$ where A is a generator of a C_0 -semigroup over a Hilbert or Banach space.
- **Boundary control as limit of the internal one:** It would also be interesting to analyze if the time optimal control of the boundary controlled heat equation is the

limit of the time optimal control of the heat equation with control on a thin neighborhood ω of the boundary. In [7] (see also [8]) this continuity result was proved in the context of the L^2 -norm optimal exact controllability of the wave equation. It would be interesting to analyze if the same techniques of proof apply in the context of the heat equation as well.

- **Numerical analysis issues:** It would also be interesting to analyze numerically the minimal time (or norm) function $\tau(\cdot)$ (or $\alpha(\cdot)$) through making use of Theorem 1.1, as well as Theorem 2.1. Note however that, in the context of the heat equation, the difficulty of passing to the limit from norm optimal controls of semi-discrete or fully-discrete approximations of heat equations towards those of the continuous one, is by now well known because of the ill-posedness of the backward problem. This was explained in detail in [27] where, using the Kannai transform, a different family of controls, not obeying a direct optimality condition, was developed. This approach, based on the exact controllability property of the wave equation, requires also the control subset ω to fulfill the so-called Geometric Control Condition (GCC). Therefore, the issue of passing to the limit from numerical controls to the true continuous ones and its consequences in the context of the issues addressed in this paper is still a widely open subject.

5 Appendix

Proof of (3.7). Let $\tau \in (0, T)$ be arbitrarily given and $n \in \mathbb{N}$ be sufficiently large. According to the interpolation theorem (see, for instance, Theorem 1.1 on Page 102, [21]),

$$H_0^1(0, T - \tau/n; L^2(\Omega)) \subset H^1(0, T - \tau/n; L^2(\Omega)) \subset C([0, T - \tau/n]; L^2(\Omega)).$$

Hence,

$$M(0, T - \tau/n; L^2(\Omega)) \subset H^{-1}(0, T - \tau/n; L^2(\Omega)).$$

Since $\varphi(\cdot) \in M_C(0, T; L^2(\Omega))$, it holds that

$$\varphi(\cdot) \in M(0, T - \tau/n; L^2(\Omega)) \subset H^{-1}(0, T - \tau/n; L^2(\Omega)) \quad \text{for all } n \geq 1.$$

Let $\chi_n(\cdot)$ be a $C_0^\infty(\mathbb{R})$ -function such that

$$\chi_n(t) = \begin{cases} 1, & t \in [0, T - \tau/(n-1)), \\ 0, & t \in [T - \tau/n, \infty). \end{cases}$$

Set $\psi_n(\cdot) = \chi_n(\cdot)\varphi(\cdot)$. Then it holds that

$$\begin{cases} \partial_t \psi_n + \Delta \psi_n = \chi'_n \varphi & \text{in } \Omega \times (0, T - \tau/n), \\ \psi_n = 0 & \text{on } \partial\Omega \times (0, T - \tau/n), \\ \psi_n(T - \tau/n) = 0 & \text{in } \Omega. \end{cases}$$

Because $\chi'_n(\cdot)\varphi(\cdot) \in H^{-1}(0, T - \tau/n; L^2(\Omega))$, it holds that

$$\psi_n(\cdot) \in L^2(0, T - \tau/n; L^2(\Omega)),$$

from which it follows that $\varphi(\cdot) \in L^2(0, T - \tau/(n-1); L^2(\Omega))$. Since $\chi'_{n-1}(\cdot)\varphi(\cdot)$ is in $L^2(0, T - \tau/(n-1); L^2(\Omega))$, we can use the same argument to deduce that

$$\varphi(\cdot) \in H^1(0, T - \tau/(n-2); L^2(\Omega)).$$

Hence,

$$\varphi(\cdot) \in H^1(0, T - \tau; L^2(\Omega)).$$

By using the interpolation theorem again, we see that $\varphi(\cdot) \in C([0, T - \tau]; L^2(\Omega))$. Since $\tau \in (0, T)$ is arbitrary, it holds that $\varphi(\cdot) \in C([0, T]; L^2(\Omega))$.

Next, fix some $\tau \in (0, T)$. Since $\varphi(\cdot) \in C([0, T]; L^2(\Omega))$, it follows that $\varphi(\cdot) \in L^1(0, T - \tau; L^2(\Omega))$. This, along with that $\varphi(\cdot) \in M_C(0, T; L^2(\Omega))$, indicates that

$$\|\varphi(\cdot)\|_{L^1_C(0, T - \tau; L^2(\Omega))} = \|\varphi(\cdot)\|_{M_C(0, T - \tau; L^2(\Omega))} \leq \|\varphi(\cdot)\|_{M_C(0, T; L^2(\Omega))} < +\infty.$$

Consequently, $\varphi(\cdot) \in L^1_C(0, T; L^2(\Omega))$.

Finally, given $\delta \in (0, T)$ we have

$$\varphi(\cdot) \in L^1((0, T - \delta); L^2(\Omega)) \cap M(0, T; L^2(\omega)).$$

Thus, it stands that

$$\int_0^{T-\delta} \|\varphi(t)\|_{\omega} dt = \|\varphi(\cdot)\|_{M(0, T-\delta; L^2(\omega))} \leq \|\varphi(\cdot)\|_{M(0, T; L^2(\omega))} < +\infty,$$

from which, it follows that $\varphi(\cdot) \in L^1(0, T; L^2(\omega))$. This completes the proof (3.7).

Proof of (3.8). Write $\varphi_T = \varphi(T)$. We extend $\varphi(\cdot)$ to be the solution of the equation (1.4) over $(-T, T)$, and denote this extension by $\varphi(\cdot)$ again. Let $\{T_n\} \in (0, T)$ be such that $T_n \nearrow T$ as $n \rightarrow \infty$. Then it holds that $\{\varphi(T_n)\} \subseteq L^2(\Omega)$. Write $\varphi_n(\cdot)$ for the solution $\varphi(\cdot; \varphi(T_n))$. One can easily check that

$$\varphi_n(t) = \varphi(t - T + T_n), \quad t \in (0, T). \quad (5.1)$$

By the energy decay property of the heat equation, we see that

$$\|\varphi_n(t)\| = \|\varphi(t - T + T_n)\| \leq \|\varphi(t)\| \quad \text{for all } t \in (0, T). \quad (5.2)$$

To finish the proof, it suffices to show that there is a subsequence of $\{\varphi(T_n)\}$ converging to φ_T in the norm $\|\cdot\|_{X_T}$. The proof will be organized in two steps as follows:

Step 1. On a subsequence, $\varphi_n(\cdot)$ converges to $\varphi(\cdot)$ in $L^1_C(0, T; L^2(\Omega))$.

Let $\{\sigma_m\}$ be a sequence of positive numbers such that $\sigma_m \nearrow T$ as $m \rightarrow \infty$. Write $A_m = \max_{t \in [0, \sigma_m]} C^{-1}(t)$, $m \geq 1$. It is obvious that

$$\int_0^{\sigma_m} C^{-1}(t) \|\varphi_n(t) - \varphi(t)\| dt \leq A_m \int_0^{\sigma_m} \|\varphi(t - T + T_n) - \varphi(t)\| dt \quad \text{for each } m \in \mathbb{N}.$$

Since for each m , $\varphi(\cdot): [0, \sigma_m] \rightarrow L^2(\Omega)$ is uniformly continuous, and because $T_n \nearrow T$, it follows from the above estimate that there is a sequence $\{n_m\} \subset \mathbb{N}$, with $n_{m+1} > n_m$, such that

$$\int_0^{\sigma_m} C^{-1}(t) \|\varphi_{n_m}(t) - \varphi(t)\| dt < \frac{1}{m} \quad \text{for all } m. \quad (5.3)$$

On the other hand, since $C^{-1}(\cdot)\|\varphi(\cdot)\| \in L^1(0, T)$, it follows from (5.2) and the absolute continuity of the integral that

$$\int_{\sigma_m}^T C^{-1}(t) \|\varphi_{n_m}(t) - \varphi(t)\| dt \leq \int_{\sigma_m}^T 2C^{-1}(t) \|\varphi(t)\| dt \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

This, along with (5.3), indicates that

$$\int_0^T C^{-1}(t) \|\varphi_{n_m}(t) - \varphi(t)\| dt \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Hence, there is a subsequence of $\{\varphi_n(\cdot)\}$, still denoted in the same way, such that

$$\varphi_n(\cdot) \rightarrow \varphi(\cdot) \quad \text{strongly in } L^1_C(0, T; L^2(\Omega)) \quad \text{as } n \rightarrow \infty. \quad (5.4)$$

Step 2. There is a subsequence of the subsequence $\{\varphi_n(\cdot)\}$ (which satisfies (5.4)) converging to $\varphi(\cdot)$ in $L^1(0, T; L^2(\omega))$.

By the similar argument as that used to prove (5.3), we can find two sequences $\{\delta_k\}$ and $\{n_k\}$, with $\delta_k \nearrow T$ and $n_k \nearrow +\infty$, respectively, such that

$$\int_0^{\delta_k} \|\varphi_{n_k}(t) - \varphi(t)\|_{\omega} dt < \frac{1}{k} \quad \text{for all } k. \quad (5.5)$$

Now, we claim that

$$\int_{\delta_k}^T \|\varphi_{n_k}(t) - \varphi(t)\|_{\omega} dt \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (5.6)$$

In fact, if (5.6) did not stand, then we could find a $C_0 > 0$ such that for any natural number N , there is a natural number $k(N) > N$ satisfying

$$\int_{\delta_{k(N)}}^T \|\varphi_{n_{k(N)}}(t) - \varphi(t)\|_{\omega} dt \geq C_0,$$

from which, it follows that

$$\int_{\delta_{k(N)}}^T \|\varphi_{n_{k(N)}}(t)\|_{\omega} dt + \int_{\delta_{k(N)}}^T \|\varphi(t)\|_{\omega} dt \geq C_0 \text{ for all } N. \quad (5.7)$$

On the other hand, we write $\delta'_{k(N)} = \delta_{k(N)} + T_{n_{k(N)}} - T$. Clearly, there is a natural number N_0 such that $0 < \delta'_{k(N)} < \delta_{k(N)}$ for all $N \geq N_0$. Then it follows from (5.1) that when $N \geq N_0$,

$$\int_{\delta_{k(N)}}^T \|\varphi_{n_{k(N)}}(t)\|_{\omega} dt = \int_{\delta_{k(N)}}^T \|\varphi(t - T + T_{n_{k(N)}})\|_{\omega} dt = \int_{\delta'_{k(N)}}^{T_{n_{k(N)}}} \|\varphi(t)\|_{\omega} dt \leq \int_{\delta'_{k(N)}}^T \|\varphi(t)\|_{\omega} dt.$$

Along with (5.7), this indicates that

$$\int_{\delta'_{k(N)}}^T \|\varphi(t)\|_{\omega} dt + \int_{\delta_{k(N)}}^T \|\varphi(t)\|_{\omega} dt \geq C_0 \text{ for all } N \geq N_0.$$

Letting $N \rightarrow +\infty$ (which implies that $k(N) \rightarrow +\infty$), we get a contradiction, since $\varphi(\cdot) \in L^1(0, T; L^2(\omega))$. Thus, (5.6) holds. Then, by (5.5) and (5.6), we see that

$$\varphi_{n_k}(\cdot) \rightarrow \varphi(\cdot) \text{ strongly in } L^1(0, T; L^2(\omega)) \text{ as } k \rightarrow +\infty.$$

Together with (5.4), this yields that

$$\varphi_{n_k}(\cdot) \rightarrow \varphi(\cdot) \text{ strongly in } L^1(0, T; L^2(\omega)) \cap L^1_C(0, T; L^2(\Omega)).$$

In summary, we complete the proof of (3.8).

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