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On a Constrained Approximate Controllability Problem for the Heat Equation[1]

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Abstract. In this work, we study an approximate control problem for the heat equation, with a nonstandard but rather natural restriction on the solution. It is well known that approximate controllability holds. On the other hand, the total mass of the solutions of the uncontrolled system is constant in time. Therefore, it is natural to analyze whether approximate controllability holds supposing the total mass of the solution to be a given constant along the trajectory. Under this additional restriction, approximate controllability is not always true. For instance, this property fails when Ω is a ball. We prove that the system is generically controllable; that is, given an open regular bounded domain Ω, there exists an arbitrarily small smooth deformation u, such that the system is approximately controllable in the new domain $\Omega + u$ under this constraint. We reduce our control problem to a nonstandard uniqueness problem. This uniqueness property is shown to hold generically with respect to the domain.

Key Words. Boundary controllability, heat equation, spectral theory, shape differentiation.

1. Introduction and Main Results

This work is devoted to studying a constrained approximate controllability problem for the heat equation.

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Let $\Omega\subset\mathbb{R}^d$ be an open bounded set of class C^2, with $d\geq 1$ and $T>0$ a given time. Let Γ_0 be a nonempty subset o the boundary $\partial\Omega$, and let χ_{Γ_0} be the characteristic function,

$$\chi_{\Gamma_0}=\begin{cases}1, & \text{on } \Gamma_0,\\ 0, & \text{on } \partial\Omega\backslash\Gamma_0.\end{cases}$$

We consider the following heat equation with Neumann control:

$$y_t-\Delta y=0, \qquad \text{in } \Omega\times(0,T), \tag{1a}$$

$$\partial y/\partial n=v\chi_{\Gamma_0}, \qquad \text{on } \partial\Omega\times(0,T), \tag{1b}$$

$$y(0)=y_0, \qquad \text{in } \Omega, \tag{1c}$$

where $y_0\in L^2(\Omega)$. We also fix a target $y_1\in L^2(\Omega)$ such that

$$\int_\Omega y_0=\int_\Omega y_1. \tag{2}$$

We are interested in the following constrained approximate controllability problem:

Let $\epsilon>0$. Is there a control function $v\in L^2\ (\Gamma_0\times(0,T))$ such that

$$\|y(T)-y_1\|_{L^2(\Omega)}\leq\epsilon \tag{3}$$

and

$$\int_\Omega y(x,t)\,dx=\int_\Omega y_0\,dx, \qquad \forall t\in[0,T]? \tag{4}$$

Observe that, when $v\equiv 0$, i.e. when no control acts on the system, the total mass of the solution is conserved, i.e. (4) holds. Then, when the initial and final data have the same mass, i.e. when y_0 and y_1 satisfy (2), it is natural to ask whether the controllability property (3) may be achieved with a solution that fulfills the constraint (4) for all time.

Notice that, if we do not impose the additional condition (4), this is the classical problem of approximate controllability. By now, it is well known that this approximate controllability property [without the constraint (4)] holds in any bounded domain Ω, for any $T>0$ and any open nonempty subset Γ_0 of $\partial\Omega$.

On the other hand, it is well known that each approximate control problem for a linear system can be reduced to a unique continuation problem for the adjoint system. This can be done in at least two different ways. First, as a direct consequence of the application of Hahn–Banach theorem and then by the variational approach developed in Ref. 1. Very often, this uniqueness problem is analyzed in the context of the theory of the classical noncharacteristic Cauchy problems.

However, when the restriction (4) is added to the trajectory, the uniqueness problem turns out to be nonstandard, i.e. it does not fit in the classical context of the noncharacteristic Cauchy problems mentioned above. More precisely, the uniqueness problem that arises when analyzing the constrained approximate controllability problem above is as follows:

Let φ be a solution of the adjoint system

$$\varphi_t + \Delta\varphi = 0, \qquad \text{in } \Omega \times (0, T), \tag{5a}$$

$$\partial\varphi/\partial n = 0, \qquad \text{on } \partial\Omega \times (0, T). \tag{5b}$$

Assume that

$$\varphi = c(t), \qquad \text{on } \Gamma_0 \times (0, T),$$

with $c = c(t)$ a function which is independent of x. Does this imply that $\varphi = c(t)$ everywhere?

In the sequel, this uniqueness property will be referred to as the enhanced uniqueness property.

Using the analyticity in time of the semigroup generated by the heat equation, this problem may be reduced to analyzing the spectrum of the Laplacian:

Let $w \in H^1(\Omega)$ be an eigenfunction of

$$-\Delta w = \lambda w, \qquad \text{in } \Omega, \tag{6a}$$

$$\partial w/\partial n = 0, \qquad \text{on } \partial\Omega, \tag{6b}$$

such that $w = c$ on Γ_0, c being a constant. Does this imply that w is constant everywhere?

Obviously, this property fails when Ω is a ball or an annulus. Indeed, let w be a radially symmetric eigenfunction of the Neumann Laplacian in the ball. Except for the first eigenfunction, it is not a constant. However, the trace of the eigenfunction is constant on the boundary.

Our first result asserts that this uniqueness property holds generically with respect to the domain Ω. More precisely, the following theorem holds.

Theorem 1.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set of class C^2, and let $\Gamma_0 \subset \partial\Omega$ be a nonempty open subset of the boundary. Then, the set of perturbations $u \in W^{3,\infty}(\Omega, \mathbb{R}^d)$, such that $u = 0$ on $\partial\Omega \backslash \Gamma_0$ and the uniqueness property

$$-\Delta y = \lambda y, \qquad \text{in } \Omega + u, \tag{7a}$$

$$\partial y/\partial n = 0, \qquad \text{on } \partial\Omega + u \Rightarrow y \equiv c, \tag{7b}$$

$$y = c, \qquad \text{on } \Gamma_0 + u, \tag{7c}$$

holds for every eigenvalue $\lambda \neq 0$ simultaneously, is residual in

$$W_0 = \{u \in W^{3,\infty}(\Omega, \mathbb{R}^d): u = 0 \text{ on } \partial\Omega \backslash \Gamma_0\}.$$

In other words, it is a countable intersection of dense open sets of W_0. In particular, it is dense in W_0. As a consequence of this generic spectral uniqueness result, we deduce that the enhanced uniqueness property above for the heat equation also holds.

Corollary 1.1. Let Ω and Γ_0 as in Theorem 1.1 above. Let the deformation $u \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ be such that the spectral uniqueness property (7) holds in $\Omega + u$. Then, the enhanced uniqueness property also holds in $\Omega + u$ for the adjoint heat equation (5).

Consequently, we get the following desired approximate controllability result.

Theorem 1.2. Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain of class C^2, and let $\Gamma_0 \subset \partial\Omega$ such that the spectral unique continuation property holds. Then, the problem (1)–(4) is approximately controllable for all $T > 0$. Moreover, if the boundary of Ω is analytic, the control may be chosen to be of bang–bang form.

In this case, we define the bang–bang controllability as follows:

Definition 1.1. We say that the control v is bang–bang iff $v(x, t) = \pm\mu + c$, almost everywhere in $\Gamma_0 \times (0, T)$, for some $\mu \in \mathbb{R}$ and a function $c = c(t)$, independent of x.

On the other hand, we say that the control v is quasi-bang–bang if

$$v \in \mu \,\text{sign}(\varphi) + c(t),$$

for a regular function φ.

Notice that a quasi-bang–bang control is bang–bang when the zero set of φ, that is, $\{(x, t): \varphi(x, t) = 0\}$, has zero measure.

Remark 1.1. Usually, the control is said to be of bang–bang form if there exists a constant μ such that

$$v = \pm\mu, \qquad \text{almost everywhere in } \Gamma_0 \times (0, T).$$

However, in our case, we have a slightly different definition, since we allow an extra term $c(t)$ on the control.

The constant μ and the function c will be chosen such that the control function verifies that

$$\int_{\Gamma_0} v\, d\Gamma = 0, \qquad \text{a.e. } t \in [0, T],$$

in order to guarantee that the mass of the solution is preserved.

The rest of this paper is organized as follows. In Section 2, we give some basic results on the variational formulation of the problem (1) and shape differentiation. In Section 3, we prove the existence of smooth branches of eigenvalues and eigenfunctions depending on the deformation u. In Section 4, we reduce our control problem to a unique continuation question. In Section 5, we prove Theorem 1.1 and Corollary 1.1. In Section 6, we prove Theorem 1.2. Finally, in Section 7, we discuss some related problems.

2. Preliminaries

2.1. Baire Lemma. Firstly, we recall the Baire lemma, which will be a useful tool.

Lemma 2.1. Baire Lemma. Let X be a complete metric space, and let A_n be an open dense subset of X for all $n \in \mathbb{N}$. Then, $\bigcap_{n \in \mathbb{N}} A_n$ is dense in X.

A direct consequence of the Baire lemma is the following result.

Lemma 2.2. Let X be a complete metric space, and let $\{A_n\}_{n \geq 0}$ be a sequence of open subsets of X such that:

(ii) $A_0 = X$.
(ii) A_{n+1} is a dense subset of A_n, for all $n \geq 0$.

Then, $\bigcap_{n=1}^{\infty} A_n$ is dense in X.

2.2. Basic Results on the Laplace and Heat Equation with Neumann Boundary Conditions. First, let us consider the uncontrolled system

$$y_t - \Delta y = 0, \qquad \text{in } Q_T = \Omega \times (0, T), \tag{8a}$$

$$\partial y / \partial n = 0, \qquad \text{on } \Sigma_T = \partial\Omega \times (0, T), \tag{8b}$$

$$y(0) = y_0, \qquad \text{in } \Omega. \tag{8c}$$

By the classical semigroup theory, we know that, for any $y_0 \in L^2(\Omega)$, there exists a unique solution of (8) in the class $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

The adjoint problem associated to (8) is

$$-\varphi_t - \Delta\varphi = 0, \qquad \text{in } Q_T, \tag{9a}$$

$$\partial\varphi/\partial n = 0, \qquad \text{on } \Sigma_T, \tag{9b}$$

$$\varphi(T) = \varphi_0, \qquad \text{in } \Omega. \tag{9c}$$

The change of variables $t \to T - t$ reduces (9) to (8). Thus, (9) is well posed in the same spaces. Moreover, by the regularizing effect of the heat equation, we have

$$\|\varphi(t)\|_{H^s(\Omega)} \le [C/(T-t)^{s/2}]\|\varphi^0\|_{L^2(\Omega)}, \quad \text{for } t \in [0,T) \text{ and } s > 0, \tag{10}$$

and consequently,

$$\|\varphi(t)\|_{H^{s-1/2}(\partial\Omega)} \le [C_s/(T-t)^{s/2}]\|\varphi^0\|_{L^2(\Omega)}, \qquad \text{when } s > 1/2. \tag{11}$$

Hence,

$$\|\varphi(t)\|_{L^2(\partial\Omega)} \le [C_s/(T-t)^{s/2}]\|\varphi^0\|_{L^2(\Omega)}, \qquad \forall s > 1/2. \tag{12}$$

Thus, applying (12) with $s \in (1/2, 1)$ we deduce that, if the final datum $\varphi^0 \in L^2(\Omega)$, then $\varphi \in L^2(\partial\Omega \times (0,T))$.

Now notice that, from (10) with $s = 1$, we do not obtain that $\varphi \in L^2(0,T; H^1(\Omega))$. However, multiplying the equation by φ, we obtain that $\varphi \in L^2(0,T; H^1(\Omega))$. In fact,

$$\begin{aligned}
&(1/2)(d/dt)\int_\Omega \varphi^2 - \int_\Omega |\nabla\varphi|^2 = 0 \\
&\Rightarrow \int_0^T \int_\Omega |\nabla\varphi|^2 \\
&\quad = \int_\Omega \varphi^2(T) - \int_\Omega \varphi^2(0) \\
&\quad \le \int_\Omega \varphi^2(T) \\
&\quad = \int_\Omega \varphi_0^2.
\end{aligned} \tag{13}$$

On the other hand, for each $f \in L^2(\Omega)$ with

$$\int_\Omega f(x)\,dx = 0,$$

there exists a unique solution $y \in H^2(\Omega)$ of the elliptic problem

$$-\Delta y = f, \qquad \text{in } \Omega, \tag{14a}$$

$$\partial y/\partial n = 0, \qquad \text{on } \partial\Omega, \tag{14b}$$

with zero average. From the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we can see that the problem

$$-\Delta y = \lambda y, \qquad \text{in } \Omega, \tag{15a}$$

$$\partial y/\partial n = 0, \qquad \text{on } \partial\Omega, \tag{15b}$$

admits a sequence of eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Moreover, each eigenvalue has finite multiplicity and the eigenfunctions $y_i \in H^2(\Omega)$ can be chosen to form an orthonormal basis of $L^2(\Omega)$. The eigenfunctions associated to the first eigenvalue $\lambda_1 = 0$ are constant.

2.3. Shape Differentiation. Shape differentiation is an important tool to study generic properties with respect to the domain. Given a domain Ω and a function $u: \Omega \to \mathbb{R}^d$, we define the new domain $\Omega + u$ by

$$\Omega + u = \{y \in \mathbb{R}^d : y = x + u(x),\, x \in \Omega\}. \tag{16}$$

Let us consider perturbations u in the space $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with norm

$$\|u\|_{k,\infty} = \sup_{0 \leq |\alpha| \leq k,\, x \in \Omega} \operatorname{ess} |D^\alpha u(x)|.$$

The following results are well known. For more details, we refer the reader to Refs. 2, 4, 5, and their references.

Lemma 2.3. See Ref. 2. Let $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ and $k \geq 1$ be such that $\|u\|_{k,\infty} \leq 1/2$. Then, the map $(I + u): \Omega \to \Omega + u$ is invertible. Furthermore, there exists $w \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ such that $(I + u)^{-1} = I + w$ and $\|w\|_{k,\infty} \leq C_k \|u\|_{k,\infty}$, where C_k is a constant independent on u.

Lemma 2.4. See Ref. 2. Let $k \geq 1$ and consider the function

$$\gamma: W^{k,\infty}(\Omega, \mathbb{R}^d) \to W^{k-1,\infty}(\Omega, \mathbb{R}),$$

$$u \to \gamma(u) = \operatorname{Jac}(I + u) = |\det[\partial_j (I + u)_i]|.$$

This function γ is differentiable at $u=0$. Furthermore, the directional derivative in the direction w at the point $u=0$ is $\operatorname{div} w$, that is,

$$\langle D\gamma(0), w\rangle = \operatorname{div} w, \qquad \forall w \in W^{k,\infty}(\Omega, \mathbb{R}^d).$$

Lemma 2.5. See Ref. 2. Let $k \geq 1$. The map

$$\beta: \mathscr{W} \subset W^{k,\infty}(\Omega, \mathbb{R}^d) \to \mathscr{M}_{d\times d}(W^{k-1,\infty}(\Omega, \mathbb{R})),$$
$$u \to {}^t[\partial_j(I+u)_i]^{-1},$$

where $\mathscr{W}$ is a neighborhood of $u=0$ on $W^{k,\infty}(\Omega, \mathbb{R}^d)$, is differentiable on $u=0$. Its directional derivative on $u=0$ in the direction w is given by the matrix $-{}^t[\partial_j w_i]$, where ${}^t[\partial_j w_i]$ denotes the adjoint of $[\partial_j w_i]$,

$${}^t[\partial_j(I+u)_i]^{-1} = [I] - {}^t[\partial_j u_i] + \theta(u),$$

where the matrix $\theta(u)$ satisfies

$$\|\theta(u)\|_{k-1,\infty}/\|u\|_{k,\infty} \to 0, \qquad \text{as } \|u\|_{k,\infty} \to 0.$$

Now, we consider a function

$$y: W^{k,\infty}(\Omega, \mathbb{R}^d) \to W^{m,r}(\Omega+u), \qquad u \to y(u),$$

where $m \leq k$ is an integer to be chosen later and $y(u)$ is the solution of a suitable problem, which depends on the perturbation function u (for instance, a solution of our eigenvalue problem in a new domain $\Omega+u$).

We are interested in the study of the regularity of the function $y(u)$ with respect to the perturbation parameter u.

Definition 2.1. See Ref. 2. Let $k \geq m \geq 1$. We say that the function $y(u)$ has a first-order local variation at $u=0$ in $W^{m-1,r}_{\text{loc}}(\Omega)$, if $y(u) \in W^{m,r}(\Omega+u)$, for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$, and if there exists a linear map $y'(\Omega; u)$ defined from $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ to $W^{m-1,r}_{\text{loc}}(\Omega)$ such that, for all open sets $\omega \subset\subset \Omega$,

$$y(u) = y(0) + y'(\Omega; u) + \theta(u), \qquad \text{in } \omega,$$

when $\|u\|_{k,\infty}$ is small enough and

$$\theta(u)/\|u\|_{k,\infty} \to 0, \qquad \text{in } W^{m-1,r}(\omega), \qquad \text{as } \|u\|_{k,\infty} \to 0.$$

Remark 2.1. From Definition 2.1, it follows that the first local variation can be defined as

$$y'(\Omega; u) = \lim_{t\to 0} [y(tu)|_\omega - y(0)|_\omega]/t, \qquad \text{in } \omega, \tag{17}$$

where $\omega \subset\subset \Omega$ and $y(tu)|_\omega$, $y(0)|_\omega$ are the restrictions of the functions $y(tu)$, $y(0)$ to ω.

In what follows, to simplify the notation, we will write $y'(u) = y'(\Omega; u)$.

The following theorem provides sufficient conditions for the existence of the first local variation for functions which depend on the deformation u. Furthermore, it provides an expression for the local variation on the boundary in terms of the normal derivative of $y(0)$.

Theorem 2.1. See Ref. 3. Let Ω be a $C^{0,1}$ domain. Consider a map $u \rightarrow y(u) \in W^{m,r}(\Omega + u)$, defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$, with $k \geq m \geq 1$ and $1 \leq r < \infty$. Let us assume that there exists a linear continuous map $u \rightarrow \dot{y}(u)$ defined on $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m,r}(\Omega)$ such that

$$y(u) \circ (I + u) = y(0) + \dot{y}(u) + \theta(u), \qquad \text{in } W^{m,r}(\Omega),$$

for all $u \in {}^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough, where

$$\theta(u)/\|u\|_{k,\infty} \rightarrow 0, \qquad \text{on } W^{m,r}(\Omega), \qquad \text{as } \|u\|_{k,\infty} \rightarrow 0.$$

Furthermore, assume that, for each $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough,

$$y(u) = 0, \quad \text{on } \partial(\Omega + u).$$

Then, for each $\omega \subset\subset \Omega$, the function $u \rightarrow y_\omega(u) = y(u)|_\omega$, defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m-1,r}(\omega)$, is differentiable at $u = 0$. Moreover, the map $u \rightarrow y(u)$ has a local derivative at $u = 0$ (see Definition 2.1) and the local derivative at $u = 0$, in the direction u, denoted by $y'(u)$, verifies $y'(u) \in W^{m-1,r}(\Omega)$ and

$$y'(u) = -(u \cdot n)[\partial y(0)/\partial n], \qquad \text{on } \partial\Omega,$$

where n is the outward unit normal vector to Ω.

In what follows, we will use the notation

$$\mathscr{W} = \{u \in W^{k,\infty}(\Omega, \mathbb{R}^d): \|u\|_{k,\infty} < c_\Omega\},$$

where $k \geq 1$ and $c_\Omega < 1/2$ is small enough such that all the previous results hold.

Lemma 2.6. See Ref. 4, Lemma 9. Let $u \in \mathscr{W}$. If $f \in H^1(\Omega + u)$, then there exists a unique $g \in H^1(\Omega)$, such that $f \circ (I + u) = g$. Moreover,

$$(\partial f/\partial z_i) \circ (I + u) = \sum_j M_{ij}(u)(\partial g/\partial x_j) = D_i(u)g, \tag{18}$$

where the matrix $M(u)$ is defined as

$$M(u) = [M_{ij}(u)] = {}^t[(\partial/\partial x_j)(I+u)_i]^{-1}$$

and

$$z_i = x_i + u_i(x), \qquad \forall x \in \Omega.$$

Since we consider Neumann boundary conditions, we need some results of shape differentiation for the normal unit outward vector $n(\Omega+u)$. Notice that an expansion such as

$$n(\Omega+u) = n(\Omega) + n'(\Omega; u) + \theta(u)$$

does not have sense, because the vectors $n(\Omega+u)$ and $n(\Omega)$ are defined in different sets. To solve this problem, for each vector field $n(\Omega+u)$, we define an extension to $\mathbb{R}^d$ that depends regularly on the deformation u (see Ref. 5).

Definition 2.2. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2, and let $w \in C^1(\partial\Omega)$. Let $\tilde{w} \in C^1(\mathbb{R}^d)$ be an extension of w to $\mathbb{R}^d$. We define the tangential gradient of w as

$$\operatorname{grad}_{\partial\Omega} w = \operatorname{grad} \tilde{w} - (\operatorname{grad} \tilde{w} \cdot n)n, \qquad \text{on } \partial\Omega.$$

We can extend this definition by continuity for all $w \in W^{1,1}(\partial\Omega)$ so that $\operatorname{grad}_{\partial\Omega} w \in (L^1(\partial\Omega))^d$. Moreover, $w \to \operatorname{grad}_{\partial\Omega} w$ is a linear continuous map from $W^{1,1}(\partial\Omega)$ into $(L^1(\partial\Omega))^d$.

Thus, we have the following result.

Theorem 2.2. See Ref. 2. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2. Let $u \to y(u)$ be a map defined from a neighborhood of $u=0$ in $W^{2,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{2,1}(\Omega)$. Assume that the map $u \to y(u) \circ (I+u)$, defined in a neighorhood of $u=0$ in $W^{2,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{2,1}(\Omega)$, is differentiable at $u=0$ and verifies

$$y(0) \in W^{3,1}(\Omega), \qquad \partial y(u)/\partial n(\Omega+u) = g, \qquad \text{on } \partial\Omega + u,$$

for all $u \in W^{2,\infty}(\Omega, \mathbb{R}^d)$ in a neighborhood of $u=0$, where $g \in W^{2,1}(\mathbb{R}^d)$. Then, for each open set $\omega \subset\subset \Omega$, the map $u \to y(u)$ has a local derivative at $u=0$. The local derivative $y'(u)$ in the direction u verifies

$$y'(u) = \dot{y}(u) - u \cdot \nabla y(0), \qquad y'(u) \in W^{2,1}(\Omega), \tag{19}$$

where $\dot{y}(u)$ is the total derivative of the map $u \to y(u)$ at $u = 0$ in the direction u, defined by

$$\dot{y}(u) = \lim_{t \to 0} [y(u) \circ (I + u) - y(0)]/y.$$

Furthermore,

$$\partial y'(u)/\partial n = (u \cdot n)(\partial g/\partial n - \partial^2 y(0)/\partial n^2) + \nabla y(0) \cdot \mathrm{grad}_{\partial\Omega}(u \cdot n), \quad \text{on } \partial\Omega. \tag{20}$$

Lemma 2.7. See Ref. 6, Chapter 2, Lemma 12. Let $u \in \mathscr{W}$. Consider the map that associates, to each $k \in L_0^2(\Omega + u)$,

$$K(x) = k(x + u(x)) - (1/|\Omega|) \int_\Omega k(x + u(x))\, dx.$$

This map is a bijection between the spaces $L_0^2(\Omega + u)$ and $L_0^2(\Omega)$, where

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega): \int_\Omega f(x)\, dx = 0 \right\}.$$

3. Smooth Dependence of the Spectrum on the Perturbation

In this section, we study the regularity of the eigenvalues and eigenfunctions of the Laplacian with Neumann boundary conditions with respect to the deformation function u.

Lemma 3.1. See Ref. 7, Lemma 4.1. Suppose that X and Z are Hilbert spaces and that $A: X \to Z$ is a continuous linear operator. Let $U: X \to N(A)$, $E: Z \to R(A)$ be the orthogonal projections from X and Z on the kernel and range of A, respectively. Then, there exists a bounded linear operator $Q: R(A) \to N(A)^\perp$, called the right inverse of A, such that

$$AQ = I: R(A) \to R(A), \qquad QA = I - U: Z \to N(A)^\perp.$$

Now, we have the following result, which is a slight variation of a theorem due to Albert; see Ref. 8.

Theorem 3.1. See Ref. 9. Let E be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let Λ be a Banach space. Let $P: D(P) \subset E \to E$ be a self-adjoint map densely defined in E. Assume that λ is an eigenvalue of multiplicity h of P, and let $\phi_1, \ldots, \phi_h$ be the orthonormal eigenfunctions associated to λ.

Moreover, assume that there exists a bounded linear map $Q\colon E\to E$ such that $Q\Pi_N=0$ and $Q(P+\lambda)=I-\Pi_N$, where Π_N is the orthogonal projection in $N=\mathrm{Ker}(P+\lambda)$.

Let $R(u)$ be an analytic self-adjoint map in $B(E,F)$ for every u in a neighborhood of $u=0$ in Λ, such that $R(0)=0$ and $P(u)=P+R(u)$. Then, there exist h analytic functions, defined in a neighborhood of $u=0$ in Λ with values in $\mathbb{R}$, $u\to\lambda_i(u)$, and h analytic functions $u\to\phi_i(u)$, with values in E, $i=1,\dots,h$, defined in a neighborhood of $u=0$ in Λ, such that:

(i) $\lambda_j(0)=\lambda, \qquad j=1,\dots,h.$

(ii) For all u small enough, $(\lambda_j(u),\ \phi_j(u))$ is a solution of the eigenvalue problem

$$P(u)\phi_j(u)=\lambda_j(u)\phi_j(u).$$

(iii) For all u small enough, the set $\{\phi_1(u),\dots,\phi_h(u)\}$ is orthonormal in E.

(iv) For each interval $I\subset\mathbb{R}$ such that $\bar{I}$ contains only the eigenvalue λ of P, there exists a neighborhood U of $u=0$ such that there are exactly h eigenvalues (counting the multiplicity) $\lambda_1(u),\dots,\lambda_h(u)$ of P_u contained on I.

As a consequence of these abstract results, we have the following result for the Neumann eigenvalues of the Laplacian.

Theorem 3.2. Let $\Omega\subset\mathbb{R}^d$ be an open bounded domain of class C^2. Let λ be an eigenvalue of multiplicity h of the Laplacian with Neumann boundary conditions (15) with associated eigenfunctions $y_1,\dots,y_h$. Then, there exist h analytic functions with values in $\mathbb{R}$, $u\to\lambda_i(u)$, and h analytic functions $u\to y_i(u)$, with values in $H^2(\Omega+u)$, $i=1,\dots,h$, defined in a neighborhood of $u=0$ in $W^{3,\infty}(\Omega,\mathbb{R}^d)$, such that:

(i) $\lambda_j(0)=\lambda, \qquad j=1,\dots,h.$

(ii) For all u small enough, $\lambda_j(u)$ is an eigenvalue with associated eigenfunction $y_j(u)$ in the new domain $\Omega+u$, i.e.,

$$-\Delta y_j(u)=\lambda_j(u)y_j(u), \qquad \text{in } \Omega+u, \tag{21a}$$

$$\partial y_j(u)/\partial n=0, \qquad \text{on } \partial(\Omega+u). \tag{21b}$$

(iii) For all u small enough, the set $\{y_1(u),\dots,y_h(u)\}$ is orthonormal in $H^2(\Omega+u)$.

(iv) For each interval $I\subset\mathbb{R}$ such that $\bar{I}$ contains only the eigenvalue λ of (21), with $u=0$, there exists a neighborhood U of $u=0$ such that there are exactly h eigenvalues (counting the multiplicity) $\lambda_1(u), \ldots, \lambda_h(u)$ of (21) contained on I.

Proof of Theorem 3.2. We can see clearly that the first eigenvalue is $\lambda_1(u) = 0$, for all u in U and that $y_1(u) = |\Omega + u|^{-1/2}$ is an associated eigenfunction. Moreover, $y_j(u)$, $j = 2, \ldots,$ verifies

$$\int_{\Omega+u} y_j(u) = \int_{\Omega} y_j(u) \circ (I+u)\, \mathrm{Jac}(I+u) = 0,$$

because it is orthogonal to the first eigenfunction. Thus, from Lemma 2.7, for each $\varphi \in L_0^2(\Omega + u)$, we define the map

$$v_u(\varphi) = \hat{\varphi} = \varphi \circ (I+u) - (1/|\Omega|) \int_{\Omega} \varphi \circ (I+u) \in L_0^2(\Omega), \tag{22}$$

which is a bijective map from $L_0^2(\Omega + u)$ into $L_0^2(\Omega)$ and is analytic in a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$.

Firstly, we observe that finding $\varphi \in H^2(\Omega + u) \cap L_0^2(\Omega + u)$ and $\lambda \in \mathbb{R}$ such that

$$-\Delta\varphi = \lambda\varphi, \qquad \text{in } \Omega + u, \tag{23a}$$

$$\partial\varphi/\partial n = 0, \qquad \text{on } \partial(\Omega + u), \tag{23b}$$

is equivalent to finding $\hat{\varphi} \in H^2(\Omega) \cap L_0^2(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$-\sum_{ik} \partial_k (M_{ik}(u) D_i(u) \hat{\varphi}\, \mathrm{Jac}(I+u)) = \lambda \hat{\varphi}\, \mathrm{Jac}(I+u), \qquad \text{in } \Omega, \tag{24a}$$

$$\sum_{ik} D_i(u) \hat{\varphi} M_{ik}(u) n_k = 0, \qquad \text{on } \partial\Omega. \tag{24b}$$

Thus, we can define, in a neighborhood $\mathscr{W}$ of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$, the map

$$P: \mathscr{W} \to \mathscr{L}(H^2(\Omega) \cap L_0^2(\Omega); L_0^2(\Omega)), \tag{25a}$$

$$P(u)(\hat{\varphi} \circ (I+u)^{-1}) = \Delta\varphi, \tag{25b}$$

where Δ denotes the Neumann Laplacian operator defined in $\Omega + u$, that is,

$$\langle P(u)\hat{\varphi}, \hat{\psi} \rangle = \int_{\Omega} D(u)\hat{\varphi} \cdot D(u)\hat{\psi}\, \mathrm{Jac}(I+u), \tag{26}$$

$D(u)$ being the operator defined in (18), which satisfies

$$(\nabla\varphi) \circ (I+u) = D(u)[\varphi \circ (I+u)] = D(u)\hat{\varphi},$$

for every $\varphi \in H^1(\Omega + u)$.

Notice that

$$\langle P(u)\hat{\varphi}, \hat{\psi}\rangle = \int_{\Omega+u} \nabla\varphi \cdot \nabla\psi,$$

where $\hat{\varphi} = v(\varphi)$ and $\hat{\psi} = v(\psi)$ are defined in (22).

From (26), it is clear that $P(u)$ is self-adjoint with compact inverse in $L_0^2(\Omega)$. Moreover, if λ is an eigenvalue of the Neumann Laplacian, in the domain Ω, with multiplicity h, we consider the eigenspace N associated to λ, and let Π_N be the orthogonal projection from N into $E = L_0^2(\Omega)$. From Lemma 3.1, we have that the map $(P(0)+\lambda)$ has a right inverse Q: $L_0^2(\Omega) \to L_0^2(\Omega)$, which verifies

$$Q(P+\lambda) = I - \Pi_N, \qquad Q\Pi_N = 0.$$

The proof of the analyticity of the operator $P(u)$ is analogous to the one of the Dirichlet Laplacian (see Ref. 9), and it is a consequence of the analyticity of the mappings $u \to D(u)$ and $u \to \mathrm{Jac}(I+u)$ in a neighborhood U of $u = 0$ in $\Lambda = W^{3,\infty}(\Omega, \mathbb{R}^d)$. □

4. Reduction to the Spectral Unique Continuation Property

4.1. Preliminaries. When analyzing the constrained approximate controllability problem, without loss of generality we may assume that $y_0 \equiv 0$, and therefore,

$$\int_\Omega y_1 = 0.$$

Indeed, if $y_0 \not\equiv 0$, let z be the solution of the uncontrolled problem

$$z_t - \Delta z = 0, \qquad \text{in } \Omega \times (0, T), \tag{27a}$$

$$\partial z/\partial n = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{27b}$$

$$z(0) = y_0, \qquad \text{in } \Omega. \tag{27c}$$

Then, y satisfies (1) if and only if $w = y - z$ solves

$$w_t - \Delta w = 0, \qquad \text{in } \Omega \times (0, T), \tag{28a}$$

$$\partial w/\partial n = v\chi_{\Gamma_0}, \qquad \text{on } \partial\Omega \times (0, T), \tag{28b}$$

$$w(0) = 0, \qquad \text{in } \Omega. \tag{28c}$$

On the other hand, y satisfies (3) if and only if w satisfies

$$\|w(T) - (y_1 - z(T))\|_{L^2(\Omega)} \leq \epsilon.$$

However,

$$\int_\Omega (z(T) - y_0)\, dx = 0,$$

or in other words,

$$\int_\Omega z(T)\, dx = \int_\Omega y_0\, dx,$$

since z satisfies (27) and the mass is conserved for the solutions of the uncontrolled system. This argument shows that, without loss of generality, we may assume that $y_0 \equiv 0$ and

$$\int_\Omega y_1(x)\, dx = 0. \tag{29}$$

Let y be the unique solution of the problem (1), and let φ be the solution of the adjoint system (9) with final datum $\varphi_0 \in L_0^2(\Omega)$. Then, we have that

$$0 = \int_0^T \int_\Omega (y_t - \Delta y)\varphi = \int_\Omega y(T)\varphi_0 - \int_0^T \int_{\Gamma_0} \varphi v. \tag{30}$$

On the other hand, integrating (1) in Ω, we see that (4) is equivalent to

$$\int_{\Gamma_0} v(t)\, d\Gamma = 0, \qquad \text{a.e. } t \in (0, T). \tag{31}$$

Thus, (30) can be written as

$$\int_0^T \int_{\Gamma_0} v\left[\varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi\right] d\Gamma\, dt - \int_\Omega y(T)\varphi_0\, dx = 0. \tag{32}$$

We are going to build the controls v satisfying (32) as minimizers of suitable quadratic functionals, viewing (32) as the Euler–Lagrange equations associated to these minimization problems. To do this, we consider the functional

$$J: L_0^2(\Omega) \to \mathbb{R}, \qquad \varphi_0 \to J(\varphi_0), \tag{33}$$

such that

$$J(\varphi_0) = (1/2) \int_0^T \int_{\Gamma_0} \left[\varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi\right]^2 d\Gamma\, dt + \epsilon \|\varphi_0\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi_0\, dx, \tag{34}$$

or its bang–bang version

$$\hat{J}: L_0^2(\Omega) \to \mathbb{R}, \qquad \varphi_0 \to \hat{J}(\varphi_0), \tag{35}$$

where

$$\hat{J}(\varphi_0) = (1/2)\left[\int_0^T \int_{\Gamma_0} \left| \varphi - (1/|\Gamma|) \int_{\Gamma_0} \varphi \, d\Gamma \, dt \right]^2 + \epsilon \|\varphi_0\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi_0 \, dx, \tag{36}$$

φ being the unique solution of the adjoint problem (9) with final datum φ_0.

4.2. Analysis of the Functionals *J* and $\hat{J}$. Notice that the functionals J and $\hat{J}$ are well defined, since the map $\varphi_0 \in L_0^2(\Omega) \to \varphi \in L^2(\partial\Omega \times (0, T))$ is continuous as shown in (12). Moreover, if the enhanced unique continuation property holds, then J strictly convex and coercive.

Let us first analyze the functional J.

Lemma 4.1. Let $\Omega \subset \mathbb{R}^d$ be an open regular bounded domain, and let $\Gamma_0 \subset \partial\Omega$ be such that the enhanced uniqueness property holds. Then, given $\epsilon > 0$ and $y_1 \in L_0^2(\Omega)$, the functional J defined in (33) and (34) is continuous, strictly convex in $L_0^2(\Omega)$, and satisfies

$$\liminf J(\varphi_0)/\|\varphi_0\|_{L^2(\Omega)} \geq \epsilon, \tag{37a}$$

$$\|\varphi_0\|_{L^2(\Omega)} \to \infty, \tag{37b}$$

$$\varphi_0 \in L_0^2(\Omega). \tag{37c}$$

Moreover, J achieves its minimum at a unique point $\hat{\varphi}_0 \in L_0^2(\Omega)$ and

$$\hat{\varphi}_0 = 0 \Leftrightarrow \|y_1\|_{L^2(\Omega)} \leq \epsilon.$$

Proof. To simplify the notation, we define the map

$$L: L_0^2(\Omega) \to L^2(\partial\Omega \times (0, T)), \qquad \varphi_0 \to L(\varphi_0 = \varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi, \tag{38}$$

φ being the solution of the adjoint problem (9) with final datum φ_0. Therefore,

$$J(\varphi_0) = \int_0^T \int_{\Gamma_0} (L(\varphi_0))^2 + \epsilon \|\varphi_0\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi_0.$$

It is clear from the definition that the functional J is convex; moreover, it is strictly convex. Indeed, since J is the sum of three convex functionals, in order to show that J is strictly convex, it is sufficient to check that the first term is strictly convex. This is easy to check, due to the strict convexity of the norm in $L_0^2(\Gamma_0 \times (0, T))$ and the fact that the following unique continuation property holds for the solution of the adjoint system as a consequence of the enhanced uniqueness property:

$$\text{If } L(\varphi_0) = 0 \text{ on } \Gamma_0 \times (0, T) \text{ and } \varphi_0 \in L_0^2(\Omega), \text{ then } \varphi_0 \equiv 0.$$

From the continuity of the map

$$\varphi_0 \in L_0^2(\Omega) \to \varphi|_{\partial\Omega \times (0,T)} \in L^2(\partial\Omega \times (0, T)),$$

we obtain that the functional J is continuous.

Now, we prove the coercivity inequality (37). We proceed by contradiction. If (37) does not hold, there exists a sequence of functions $\{\varphi_0^n\}_n \subset L_0^2(\Omega)$ such that

$$\|\varphi_0^n\|_{L^2(\Omega)} \to \infty \quad \text{and} \quad \liminf_{n\to\infty} J(\varphi_0^n)/\|\varphi_0^n\|_{L^2(\Omega)} < \epsilon. \tag{39}$$

Let

$$\tilde{\varphi}_0^n = \varphi_0^n / \|\varphi_0^n\|_{L^2(\Omega)},$$

and let $\tilde{\varphi}^n$ be the solution of (9) such that $\tilde{\varphi}^n(T) = \tilde{\varphi}_0^n$. Since

$$\|\tilde{\varphi}_0^n\|_{L^2(\Omega)} = 1.$$

there exists a subsequence which converges weakly to $\tilde{\varphi}_0 \in L_0^2(\Omega)$. Since $\{\tilde{\varphi}_0^n\}_n$ converges weakly to $\tilde{\varphi}_0$ in $L_0^2(\Omega)$, the sequence of solutions $\{\tilde{\varphi}^n\}_n$ associated to the final data $\tilde{\varphi}_0^n$, converges weakly in $L^2(0, T; H^1(\Omega))$ to a function $\tilde{\varphi}$, which is a solution of (9) with final datum $\tilde{\varphi}_0$. Thus, we have that $\{\tilde{\varphi}^n\}_n$ converges weakly to $\tilde{\varphi}$ in $L^2\,(0, T; H^{1/2}(\partial\Omega))$. On the other hand, from (39) and since

$$\left| \left[\epsilon \|\varphi_0^n\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi_0^n \right] \Big/ \|\varphi_0^n\|_{L^2(\Omega)} \right| \leq \epsilon + \|y_1\|_{L^2(\Omega)},$$

we deduce that the quotient

$$(1/2)\left[\int_0^T \int_{\Gamma_0} \left| \varphi^n - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi^n \right|^2 d\Gamma\, dt \right] \Big/ \|\varphi_0^n\|_{L^2(\Omega)} \tag{40}$$

is uniformly bounded. From (40), we deduce that

$$(1/2)\left[\int_0^T \int_{\Gamma_0} \left| \tilde{\varphi}^n - (1/|\Gamma_0|) \int_{\Gamma_0} \tilde{\varphi}^n \right|^2 d\Gamma\, dt \right]$$

$$= [1/2\|\varphi_0^n\|_{L^2(\Omega)}^2]\left[\int_0^T \int_{\Gamma_0} \left| \varphi^n - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi^n \right|^2 d\Gamma\, dt \right] \to 0,$$

as $n \to \infty$. Hence,

$$(1/2)\int_0^T \int_{\Gamma_0} \left| \tilde{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \tilde{\varphi} \right|^2 d\Gamma\, dt = 0,$$

and from the enhanced uniqueness property, we have that

$$\tilde{\varphi} = c(t),$$

and therefore,

$$\tilde{\varphi}_0 = c(T).$$

Since $\tilde{\varphi}_0 \in L_0^2(\Omega)$, we deduce that

$$\tilde{\varphi}_0 = c(T) \equiv 0.$$

On the other hand, it is clear that

$$J(\varphi_0^n) \geq \|\varphi_0^n\|_{L^2(\Omega)} \left[\epsilon - \int_\Omega y_1 \tilde{\varphi}_0^n \right].$$

Since $\tilde{\varphi}_0^n$ converges weakly to zero, we have that

$$\liminf_{n \to \infty} J(\varphi_0^n)/\|\varphi_0^n\|_{L^2(\Omega)} \geq \epsilon,$$

which is in contradiction with (39). Thus, (37) holds and there exists a minimum point of J in $L_0^2(\Omega)$. The uniqueness of the minimum point is an immediate consequence of the strict convexity of J.

Now, if

$$\|y_1\|_{L^2(\Omega)} \leq \epsilon,$$

then

$$J(\varphi_0) \geq 0, \qquad \text{for each } \varphi_0 \in L_0^2(\Omega),$$

and the minimum is achieved in $\hat{\varphi}_0 = 0$. On the other hand, if $\hat{\varphi}_0 = 0$, then for each $\varphi_0 \in L_0^2(\Omega)$,

$$\lim_{t \to 0^+} J(t\varphi_0)/t \geq 0, \tag{41}$$

and taking

$$\varphi_0 = y_1/\|y_1\|_{L^2(\Omega)},$$

we have that

$$\|y_1\|_{L^2(\Omega)} \leq \epsilon.$$

This completes the proof of Lemma 4.1. □

Analogously, we have the following result for the functional $\hat{J}$ defined in (35)–(36).

Lemma 4.2. Let Ω be an open bounded domain with boundary of class C^2, and let $\Gamma_0 \subset \partial\Omega$ be such that the spectral unique continuation property (7) holds. Then, for each $\epsilon > 0$ and $y_1 \in L_0^2(\Omega)$, the functional $\hat{J}$ defined by (35)–(36) is continuous, strictly convex in $L_0^2(\Omega)$ and satisfies

$$\liminf \hat{J}(\varphi_0)/\|\varphi_0\|_{L^2(\Omega)} \geq \epsilon, \tag{42a}$$

$$\|\varphi_0\|_{L^2(\Omega)} \to \infty, \tag{42b}$$

$$\varphi_0 \in L_0^2(\Omega). \tag{42c}$$

Moreover, $\hat{J}$ achieves its minimum at some $\hat{\varphi}_0 \in L_0^2(\Omega)$ and

$$\hat{\varphi}_0 = 0 \Leftrightarrow \|y_1\|_{L^2(\Omega)} \leq \epsilon.$$

In Section 6, we shall show in detail why the minimizers of J and $\hat{J}$ provide the controls that we are looking for.

4.3. Enhanced Unique Continuation. The developments of Section 4.2 above show that the functionals J and $\hat{J}$ are coercive provided the following unique continuation property holds for the zero average solutions of the adjoint system:

$$\varphi_t + \Delta\varphi = 0, \qquad \text{in } \Omega \times (0, T), \tag{43a}$$

$$\partial\varphi/\partial n = 0, \qquad \text{on } \partial\Omega \times (0, T) \tag{43b}$$

$$\varphi = (1/|\Gamma_0|) \int_{\Gamma_0} \varphi \, d\Gamma, \qquad \text{on } \Gamma_0 \times (0, T) \tag{43c}$$

$$\varphi(T) = \varphi_0 \in L_0^2(\Omega), \tag{43d}$$

implying $\varphi \equiv 0$.

Note that the constant solution $\varphi \equiv 1$ for which obviously (43) fails is ruled out by the fact we are dealing only with solutions with data of zero average in $L_0^2(\Omega)$.

This enhanced uniqueness problem for the heat equation may be reduced to another one on the spectrum of the Neumann Laplacian. Indeed, let $\{\lambda_k\}_{k \geq 2}$ be the nonzero eigenvalues of the Laplacian with Neumann boundary conditions with multiplicity m_k. Let $\{w_j^k\}_{j=1}^{m_k}$ be the eigenfunctions associated to the eigenvalue λ_k constituting an orthonormal basis of $L_0^2(\Omega)$. Then, the Fourier expansion of φ is given by

$$\varphi(x, t) = \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} a_j^k e^{-\lambda_k(T-t)} w_j^k(x), \tag{44}$$

where

$$a_j^k = \int_\Omega y_0(x) w_j^k(x)\, dx. \tag{45}$$

Assume that the solution φ of (9) is such that

$$\varphi = (1/|\Gamma_0|) \int_{\Gamma_0} \varphi(x, t)\, d\Gamma = c(t), \qquad \text{on } \Gamma_0 \times (0, T). \tag{46}$$

Due to the analyticity of the semigroup generated by the heat equation and the fact that Ω is of class C^2, which allows to use elliptic regularity results, we know that $\varphi(x, t)$ is an analytic function of $t \in (-\infty, T)$ for every $x \in \bar{\Omega}$. Then, as a consequence of (46), we deduce that the same identity holds for all $t \in (-\infty, T)$; i.e.,

$$\varphi = (1/|\Gamma_0|) \int_{\Gamma_0} \varphi(x, t)\, d\Gamma = c(t), \qquad \text{on } \Gamma_0 \times (-\infty, T). \tag{47}$$

On the other hand,

$$\varphi(x, t) = \sum_{k=2}^{\infty} e^{-\lambda_k(T-t)} z^k(x), \qquad \text{in } \Omega \times (-\infty, T), \tag{48}$$

with

$$z_k = \sum_{j=1}^{m_k} a_j^k w_j^k.$$

As a consequence of (47),

$$\nabla_{\partial\Omega} = 0, \qquad \text{on } \Gamma_0 \times (-\infty, T). \tag{49}$$

Multiplying (47) and (48) by $e^{\lambda_2(T-t)}$, we have that

$$\nabla_{\partial\Omega} z_2 + \sum_{k=3}^{\infty} e^{(\lambda_2 - \lambda_k)(T-t)} \nabla_{\partial\Omega} z^k(x), \qquad \text{in } \Gamma_0 \times (-\infty, \mathrm{T}), \tag{50}$$

and taking limit $t \to -\infty$, we conclude that

$$\nabla_{\partial\Omega} z_2 = 0, \qquad \text{on } \Gamma_0 \times (-\infty, T). \tag{51}$$

Analogously, we prove that, for all $k \geq 2$,

$$\nabla_{\partial\Omega} z_k = 0, \qquad \text{on } \Gamma_0 \times (-\infty, T). \tag{52}$$

Hence, we obtain that, for all $k \geq 2$, z_k verifies

$$-\Delta z_k = \lambda_k z_k, \qquad \text{in } \Omega, \tag{53a}$$

$$\partial z_k / \partial n = 0, \qquad \text{on } \partial\Omega, \tag{53b}$$

$$z_k = (1/|\Gamma_0|) \int_{\Gamma_0} z_k \, d\Gamma, \qquad \text{on } \Gamma_0. \tag{53c}$$

If the spectral uniqueness property of Theorem 1.1 holds in the domain Ω for the subset Γ_0 of the boundary, we deduce that

$$z_k \equiv 0, \qquad \text{for all } k \geq 2.$$

Then, $\varphi \equiv 0$ and the enhanced uniqueness property for the parabolic equation (5) holds as well.

5. Proof of Theorem 1.1

In this section, we prove the generic, spectral unique continuation property of Theorem 1.1. To prove this result we show that set of deformations $u \in W^{3,\infty}(\Omega, \mathbb{R}^d)$, such that the unique continuation property (7) holds for all nonzero eigenvalues in $\Omega + u$ simultaneously, is a countable intersection of dense open subsets of $W^{3,\infty}(\Omega, \mathbb{R}^d)$.

We define $A_0 = W_0$ and, for each $n \in \mathbb{N}$, we consider the set

$$A_n = \{u \in W_0 \colon (7) \text{ holds for the branches of eigenvalues } \lambda_2(u), \dots, \lambda_n(u) \text{ in } \Omega + u\}.$$

We will prove that, for each $n \geq 0$, A_n is an open subset of W_0 and A_{n+1} is dense in A_n. Thus from the Baire lemma (see Lemma 2.1 and Lemma 2.2), we complete the proof.

(a) A_n Is Open. Let $u \in A_n$; that is, u is such that the unique continuation property (7) holds for the first n branches of nonzero eigenvalues in $\Omega + u$.

If A_n is not an open set, there exists a sequence of deformations $\{u_k\}_k$ such that

$$u_k \to u, \text{ in } W^{3,\infty}(\Omega, \mathbb{R}^d), \qquad \text{and} \qquad u_k \notin A_n, \text{ for all } k \in \mathbb{N}.$$

Therefore, for each k, there exists an eigenvalue $\lambda(u_k)$ and an associated eigenfunction $z(u_k)$ verifying

$$-\Delta z(u_k) = \lambda(u_k) z(u_k), \qquad \text{in } \Omega + u_k, \tag{54a}$$

$$\partial z(u_k)/\partial n(u_k) = 0, \qquad \text{on } \partial\Omega + u_k, \tag{54b}$$

$$z(u_k) = [1/|\Gamma_0 + u_k|] \int_{\Gamma_0 + u_k} z(u_k), \qquad \text{on } \Gamma_0 + u_k, \tag{54c}$$

and

$$\|z(u_k)\|_{L^2(\Omega + u_k)} = 1, \tag{55}$$

$u \to \lambda(u)$ being one of the branches $\lambda_2, \ldots, \lambda_n$ of positive eigenvalues.

Since $u \to \lambda(u)$ and $u \to z(u) \circ (I + u)$ are continuous functions from a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ with values in $\mathbb{R}$ and $H^2(\Omega)$, respectively, we deduce that, for a subsequence,

$$\lambda(u_k) \to \lambda(u) \text{ and } z(u_k) \circ (I + u_k) \to z(u) \circ (I + u), \qquad \text{in } H^2(\Omega),$$

where λ and z are the eigenvalues and eigenfunctions in the limit domain $\Omega + u$ belonging to one of the first n branches.

One can also pass to the limit in the constraint that the eigenfunctions $z(u_k)$ satisfy on Γ_0. By compactness, we also have that

$$\|z(u)\|_{L^2(\Omega + u)} = 1.$$

We deduce that

$$-\Delta z(u) = \lambda(u) z(u), \qquad \text{in } \Omega + u, \tag{56a}$$

$$\partial z(u)/\partial n(u) = 0, \qquad \text{on } \partial\Omega + u, \tag{56b}$$

$$z(u) = c_u = [1/|\Gamma_0 + u|] \int_{\Gamma_0 + u} z(u), \qquad \text{on } \Gamma_0 + u, \tag{56c}$$

and

$$\|z(u)\|_{L^2(\Omega + u)} = 1. \tag{57}$$

Since the unique continuation property (7) holds in the domain $\Omega + u$, as a consequence of (56) we have that $z(u) \equiv 0$, which is in contradiction with (57). This proves that A_n is open.

(b) A_{n+1} Is Dense in A_n. Let $u \in A_n \backslash A_{n+1}$. Without loss of generality, we may suppose that $u = 0$. In what follows, to simplify the notation, we will write

$$\lambda(u) = \lambda_{n+1}(u), \qquad z(u) = z_{n+1}(u), \tag{58}$$

and

$$\lambda = \lambda_{n+1}(0), \qquad z = z_{n+1}(0). \tag{59}$$

We proceed by contradiction. Assume that there exists a neighborhood U of $u = 0$ such that $U \subset A_n \backslash A_{n+1}$; that is, for all $u \in U$, there exists a nonzero function $z(u)$ which satisfies (56)–(57), with $\lambda(u) = \lambda_{n+1}(u)$. Since $u \to z(u)$ is a C^1 map from a neighborhood of $u = 0$ in $W^{3,\infty}(\Omega, \mathbb{R}^d)$ with values in $H^2(\Omega + u)$, the function

$$u \to C_u = [1/|\Gamma_0 + u|] \int_{\Gamma_0 + u} z(u),$$

is of class C^1 as well.

Notice that, if $c_u = 0$, from the classical uniqueness theorem of Holmgrem, we have that $z(u) = 0$. Therefore, for each $u \in U$, we have that $c_u \neq 0$.

Let $y(u) = z(u)/c_u$. Then, $y(u)$ satisfies

$$-\Delta y(u) = \lambda(u) y(u), \qquad \text{in } \Omega + u, \tag{60a}$$

$$\partial y(u)/\partial n(u) = 0, \qquad \text{on } \partial\Omega + u, \tag{60b}$$

$$y(u) = 1, \qquad \text{on } \Gamma_0 + u. \tag{60c}$$

From Theorem 2.2, we have that the local variations of $(\lambda(u), y(u))$ verify

$$-\Delta y'(u) = \lambda'(u) y + \lambda y'(u), \qquad \text{in } \Omega,$$

$$\partial y'(u)/\partial n = -(u \cdot n)\, \partial^2 y/\partial n^2 + \nabla y \cdot \nabla_{\partial\Omega}(u \cdot n), \qquad \text{on } \partial\Omega,$$

$$y'(u) = -(u \cdot n)\, \partial y/\partial n = 0, \qquad \text{on } \Gamma_0.$$

Hence, if we consider deformations u such that $u = 0$ on $\partial\Omega \backslash \Gamma_0$, we have that

$$-\Delta y'(u) = \lambda'(u) y + \lambda y'(u), \qquad \text{in } \Omega, \tag{61a}$$

$$\partial y'(u)/\partial n = -(u \cdot n)\, \partial^2 y/\partial n^2, \qquad \text{on } \Gamma_0, \tag{61b}$$

$$\partial y'(u)/\partial n = 0, \qquad \text{on } \partial\Omega \backslash \Gamma_0 \tag{61c}$$

$$y'(u) = 0, \qquad \text{on } \Gamma_0. \tag{61d}$$

On the other hand, multiplying (61) by $\varphi \in H^2(\Omega)$, we obtain that

$$\begin{aligned} &\lambda'(u) \int_\Omega y\varphi + \lambda \int_\Omega y'(u)\varphi \\ &= \int_\Omega \nabla y'(u) \cdot \nabla \varphi - \int_{\partial\Omega} [\partial y'(u)/\partial n]\varphi \\ &= -\int_\Omega y'(u)\Delta\varphi + \int_{\partial\Omega} (\partial\varphi/\partial n) y'(u) - \int_{\partial\Omega} [\partial y'(u)/\partial n]\varphi. \end{aligned} \tag{62}$$

Thus, with $\varphi = z$, we have that

$$\lambda'(u)\int_\Omega y^2 + \lambda\int_\Omega y'(u)y$$

$$= -\int_\Omega y'(u)\Delta y + \int_{\partial\Omega}(\partial y/\partial n)y'(u) - \int_{\partial\Omega}[\partial y'(u)/\partial n]y.$$

That is,

$$\lambda'(u)\int_\Omega y^2 = -\int_{\partial\Omega}[\partial y'(u)/\partial n]y$$

$$= \int_{\Gamma_0}(u\cdot n)(\partial^2 y/\partial n^2)y$$

$$= \int_{\Gamma_0}(u\cdot n)(-\lambda y)$$

$$= -\lambda\int_{\Gamma_0}(u\cdot n). \tag{63}$$

Taking into account that y is an eigenfunction of the Neumann problem (60), it follows that

$$\lambda\int_\Omega y'(u)y = -\int_\Omega y'(u)\Delta y + \int_\Omega (\partial y/\partial n)y'(u)y.$$

Therefore, if we consider deformations u such that

$$\int_{\Gamma_0}(u\cdot n) = 0, \tag{64}$$

we have that $\lambda'(u) = 0$ and $y'(u)$ satisfies

$$-\Delta y'(u) = \lambda y'(u), \qquad \text{in } \Omega, \tag{65a}$$

$$\partial y'(u)/\partial n = 0, \qquad \text{on } \partial\Omega\backslash\Gamma_0, \tag{65b}$$

$$y'(u) = 0, \qquad \text{on } \Gamma_0. \tag{65c}$$

Moreover, taking into account of the fact that, as a consequence of (60),

$$\nabla y(u) = 0, \qquad \text{on } \Gamma_0 + u,$$

we have

$$\partial y'(u)/\partial n = -(u\cdot n)(\partial^2 y/\partial n^2), \qquad \text{on } \Gamma_0. \tag{66}$$

Note that (65) is a new eigenvalue problem for the Laplacian with mixed Dirichlet-Neumann boundary conditions. Every eigenvalue λ of (65) has finite multiplicity. According to (66), the space $\{(u \cdot n)(\partial^2 y/\partial n^2)|_{\Gamma_0}\}$ is finite dimensional when u runs over the space of functions such that $u = 0$ on $\partial\Omega \backslash \Gamma_0$ and (64) holds. Taking into account that the latter is infinite-dimensional, we deduce that necessarily

$$\partial^2 y/\partial n^2 = 0, \qquad \text{on } \Gamma_0,$$

but that

$$\lambda y = -\Delta y = -\partial^2 y/\partial n^2 = 0 , \qquad \text{on } \Gamma_0.$$

That is, y is a solution of the problem

$$-\Delta y = \lambda y, \qquad \text{in } \Omega, \tag{67a}$$

$$\partial y/\partial n = 0, \qquad \text{on } \partial\Omega, \tag{67b}$$

$$y = 0, \qquad \text{on } \Gamma_0. \tag{67c}$$

Hence, by the Holmgrem uniqueness theorem, we have that $y \equiv 0$ in Ω, which is in contradiction with the fact that $y = 1$ on Γ_0. Hence, there exist arbitrarily small deformations $u \in A_{n+1}$. This completes the proof of the density of A_{n+1} in A_n. Now, applying the Baire lemma to the sequence of sets $\{A_n\}n$, the proof of Theorem 1.1 is complete. □

Remark 5.1. Note that, in the above proof, we cannot consider deformations u such that $u = 0$ on Γ_0. Indeed, if we consider deformations u such that the set Γ_0 is kept fixed, the local derivative of z, namely $z'(u)$, is not an eigenfunction of (65) associated to the eigenvalue λ.

As described in Section 4, as an immediate consequence of Theorem 1.1, Corollary 1.1 holds as well and the argument above does not apply.

6. Proof of Theorem 1.2

First of all, we prove the approximate controllability of the system (1)–(4). We shall then discuss the existence of bang–bang controls. According to the developments of Section 4 and Corollary 1.1, it is sufficient to check that the minimizer of the functional J introduced in (33)–(34) provides the control that we are looking for.

It is straightforward that the following proposition holds.

Proposition 6.1. Let $\varphi_0, \psi_0 \in L_0^2(\Omega)$, $\varphi_0 \not\equiv 0$. Then, the functional J is Gâteaux differentiable at φ_0 in the direction ψ_0. Moreover, the Gâteaux derivative at φ_0 in the direction ψ_0, denoted by $(\partial J/\partial \psi_0)(\varphi_0)$, is

$$(\partial J/\partial \psi_0)(\varphi_0) = \int_0^T \int_{\Gamma_0} L(\varphi_0) L(\psi_0) + [\epsilon/\|\varphi_0\|_{L^2(\Omega)}] \int_\Omega \varphi_0 \psi_0 - \int_\Omega y_1 \psi_0, \tag{68}$$

where the map L is defined by (38).

Since the unique continuation property (7) holds in the domain Ω, from Lemma 4.1, for each $\epsilon > 0$, there exists a unique minimum point $\hat{\varphi}_0$ for the functional J. Assume that $\hat{\varphi}_0 \neq 0$. Otherwise, $\|y_1\|_{L^2(\Omega)} \geq \epsilon$ and the control $v = 0$ suffices. The Gâteaux derivative of J in $\hat{\varphi}_0$ in the direction ψ_0 vanishes (see Proposition 6.1), that is, according to Proposition 6.1,

$$\int_0^T \int_{\Gamma_0} L(\hat{\varphi}_0) L(\psi_0) + [\epsilon/\|\hat{\varphi}_0\|_{L^2(\Omega)}] \int_\Omega \varphi_0 \psi_0 - \int_\Omega y_1 \psi_0 = 0, \quad \forall \psi_0 \in L_0^2(\Omega). \tag{69}$$

We define

$$v = \hat{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \hat{\varphi}, \qquad \text{on } \Gamma_0, \tag{70}$$

$\hat{\varphi}$ being the solution of (9) with the minimizer $\hat{\varphi}_0$ as the final datum.

We consider the control defined in (70). We multiply (1) by ψ, the solution of (9) with the final datum ψ_0. We have that

$$\int_\Omega y(T)\psi_0 = \int_0^T \int_{\Gamma_0} v\psi = \int_0^T \int_{\Gamma_0} v\left[\psi - (1/|\Gamma_0|) \int_{\Gamma_0} \psi\right] d\Gamma\, dt, \tag{71}$$

since v is of zero average. Thus, from (69) and (71), we obtain that

$$\int_\Omega (y_1 - y(T))\psi_0\, dx = \epsilon \left[\int_\Omega \hat{\varphi}_0 \psi_0\, dx\right] \Big/ \|\hat{\varphi}_0\|_{L^2(\Omega)} \leq \epsilon \|\psi_0\|_{L^2(\Omega)},$$

for all $\psi_0 \in L_0^2(\Omega)$. Therefore,

$$\|y_1 - y(T)\|_{L^2(\Omega)} \leq \epsilon,$$

since

$$\int_\Omega (y_1 - y(T))\psi_0 \, dx = 0.$$

This completes the proof of the constrained approximate controllability.

To prove the bang–bang controllability, we proceed as in Ref. 1. Firstly, from Lemma 4.2, we have that the functional $\hat{J}$ is strictly convex continuous in $L_0^2(\Omega)$; thus, it possesses a subdifferential at every $\varphi_0 \in L_0^2(\Omega)$. At its minimum $\hat{\varphi}_0$, we have that $0 \in \partial \hat{J}(\hat{\varphi}_0)$. Moreover, we have the following result.

Proposition 6.2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set of class C^2, and let Γ_0 be an open nonempty subset of the boundary. For every $\varphi_0, \psi_0 \in L_0^2(\Omega)$, $\varphi_0 \not\equiv 0$, we have that the subdifferential $\partial \hat{J}(\hat{\varphi}_0)$ verifies

$$\partial \hat{J}(\varphi_0) = \left\{ \xi \in L_0^2(\Omega) \colon \exists v \in \operatorname{sign}\left[\varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi \, d\Gamma \right] \right.$$

such that, for every $\psi_0 \in L_0^2(\Omega)$,

$$\begin{aligned}\int_\Omega \xi \psi_0 = & \left[\int_0^T \int_{\Gamma_0} \left| \varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi \right| d\Gamma \, dt \right] \\ & \times \left[\int_0^T \int_{\Gamma_0} \left(\psi - (1/|\Gamma_0|) \int_{\Gamma_0} \psi \right) v \, d\Gamma \, dt \right] \\ & + [\epsilon / \|\varphi_0\|_{L^2(\Omega)}] \int_\Omega \varphi_0(x) \psi_0(x) \, dx - \int_\Omega y_1(x) \psi_0(x) \, dx,\end{aligned}$$

ψ and φ being the solutions of (9) with the final datum ψ_0 and φ_0, respectively.

The proof of Proposition 6.2 is analogous to the proof of Proposition 2.3 in Ref. 1.

Now, let $\hat{\varphi}_0 \in L_0^2(\Omega)$ be the minimizer of the functional $\hat{J}$. Then, $0 \in \partial \hat{J}(\hat{\varphi}_0)$, that is, there exists a function

$$w \in \operatorname{sign}\left[\hat{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \hat{\varphi} \, d\Gamma \right]$$

such that, for every $\psi_0 \in L_0^2(\Omega)$,

$$0=\left[\int_0^T \int_{\Gamma_0} \left| \hat{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \hat{\varphi} \right| d\Gamma\, dt\right]$$

$$\times \left[\int_0^T \int_{\Gamma_0} \left(\psi - (1/|\Gamma_0|) \int_{\Gamma_0} \psi\right) w\, d\Gamma\, dt\right]$$

$$+ [\epsilon/\|\hat{\varphi}_0\|_{L^2(\Omega)}] \int_\Omega \hat{\varphi}_0(x)\psi_0(x) - \int_\Omega y_1 \psi_0$$

$$= \int_0^T \int_{\Gamma_0} \left[\psi - (1/|\Gamma_0|) \int_{\Gamma_0} \psi\right] v\, d\Gamma\, dt$$

$$+ [\epsilon/\|\hat{\varphi}_0\|_{L^2(\Omega)}] \int_\Omega \hat{\varphi}_0(x)\psi_0(x)\, dx - \int_\Omega y_1 \psi_0\, dx, \tag{72}$$

with the control function

$$v=\left[\int_0^T \int_{\Gamma_0} \left| \hat{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \hat{\varphi} \right| d\Gamma\, dt\right]\left[w - (1/|\Gamma_0|) \int_{\Gamma_0} w\, d\Gamma\right]. \tag{73}$$

On the other hand, the solution y of the problem (1) satisfies

$$\int_0^T \int_{\Gamma_0} \left[\psi - (1/|\Gamma_0|) \int_{\Gamma_0} \psi\right] v\, d\Gamma\, dt - \int_\Omega y(T)\psi_0\, dx = 0. \tag{74}$$

Hence, from (73)–(74) we have that

$$\int_\Omega (y_1 - y(T))\psi_0\, dx \leq \epsilon\left[\int_\Omega \hat{\varphi}_0 \psi_0\right] \Big/ \|\hat{\varphi}_0\|_{L^2(\Omega)} \leq \|\psi_0\|_{L^2(\Omega)}, \tag{75}$$

for all $\psi_0 \in L_0^2(\Omega)$. Therefore,

$$\|y_1 - y(T)\|_{L^2(\Omega)} \leq \epsilon.$$

This proves that the control can be chosen to be of quasi bang–bang form.

Note that the control that we have obtained in (73) is of the form

$$v = \mu[w - c(t)],$$

with μ a positive constant, $c = c(t)$, and

$$w \in \operatorname{sign}\left[\hat{\varphi} - (1/|\Gamma_0|) \int_{\Gamma_0} \hat{\varphi}\, d\Gamma\right].$$

Of course, if we are able to show that

$$\text{meas}\left\{\hat{\varphi} - (1/|\Gamma_0|)\int_{\Gamma_0} \hat{\varphi}\, d\Gamma = 0\right\} = 0,$$

we will have that $w = \pm 1$, a.e. on $\Gamma_0 \times (0, T)$, and therefore the control will be of bang–bang form.

When Ω is analytic, we can indeed obtain a bang–bang control. Indeed, if Ω is analytic, the zero set of the function

$$\varphi - (1/|\Gamma_0|)\int_{\Gamma_0} \varphi,$$

that is, the set

$$Z = \left\{(x, t) \in \Gamma_0 \times (0, T) \colon \varphi - (1/|\Gamma_0|)\int_{\Gamma_0} \varphi\right\}$$

has zero measure on $\Gamma_0 \times (0, T)$, when φ is a nontrivial solution of the adjoint system with datum in $L_0^2(\Omega)$.

Notice that, if Z does not have zero measure, from the analyticity of the solution $\varphi(x, t)$ of the adjoint problem (9) with final datum $\varphi_0 \not\equiv 0$ in $\bar{\Omega} \times (0, T)$, we have that the function φ is independent of x on $\partial\Omega \times (0, T)$. Then, by the enhanced uniqueness property, $\varphi \equiv 0$.

This proves that the control can be chosen to be of bang–bang form when Ω is analytic.

Remark 6.1. When the domain Ω is of class C^2, the argument above provides a control of quasi-bang–bang form. In this case, we do not know if the set

$$\left\{(x, t) \in \Gamma_0 \times (0, T) \colon \hat{\varphi}(x, t) = (1/|\Gamma_0|)\int_{\Gamma_0} \hat{\varphi}(s, t)\, ds\right\} \tag{76}$$

has zero measure.

7. Further Comments

In this section, we discuss some related control problems.

7.1. Approximate Dirichlet Control Problem for the Heat Equation. We consider the heat equation with Dirichlet boundary conditions

$$y_t - \Delta y = 0, \qquad \text{in } \Omega \times (0, T), \tag{77a}$$

$$y = v\chi_{\Gamma_0}, \qquad \text{on } \partial\Omega \times (0, T), \tag{77b}$$

$$y(0) = y_0, \qquad \text{in } \Omega, \tag{77c}$$

with $y_0 \in L^2(\Omega)$. Let $y_1 \in L^2(\Omega)$ such that

$$\int_\Omega y_0 \phi_1 = \int_\Omega y_1 \phi_1,$$

where ϕ_1 is the first eigenfunction of the Laplacian with Dirichlet boundary conditions.

The following problem is the Dirichlet analog of the problem discussed above: Given $\epsilon > 0$, is there a control function v such that

$$\|y(T) - y_1\|_{L^2(\Omega)} \leq \epsilon, \tag{78}$$

and

$$\int_\Omega y(x,t)\phi_1(x)\,dx = \int_\Omega y_0(x)\phi_1(x)\,dx, \qquad \forall t \in [0,T]? \tag{79}$$

Notice that this problem consists in obtaining an approximate control of the solution preserving (along the trajectory) the projection of the solution on the first eigenspace of the Direchlet Laplacian for all time $t > 0$.

This control problem can be reduced to the following unique continuation problem: If φ is the solution of the problem

$$-\varphi_t - \Delta\varphi = 0, \qquad \text{in } \Omega \times (0,T), \tag{80a}$$

$$\varphi = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{80b}$$

$$\varphi(T) = \varphi_0, \qquad \text{in } \Omega, \tag{80c}$$

satisfying

$$\partial\varphi/\partial n = \alpha(\partial\phi_1/\partial n), \qquad \text{on } \Gamma_0 \times (0,T), \tag{81}$$

where $\alpha = \alpha(t)$, then is $\varphi = \alpha(t)\phi_1$?

As above, we can reduce our unique continuation problem to a unique continuation property for the eigenfunctions of the Laplacian with Dirichlet boundary conditions. More precisely: Let ϕ be an eigenfunction of the Laplacian

$$-\Delta\phi = \lambda\phi, \qquad \text{in } \Omega,$$

$$\phi = 0, \qquad \text{on } \partial\Omega,$$

satisfying

$$\partial\phi/\partial n = \alpha(\partial\phi_1/\partial n), \qquad \text{on } \Gamma_0.$$

Can we guarantee that $\phi = c\phi_1$?

When Ω is a ball, this property fails. However, one may expect a generic positive answer.

This is a nonstandard unique continuation property. The methods that we have developed in the above sections seem to be insufficient to give an answer to this question. Therefore, this is an open problem.

7.2. Internal Control. Let $\omega \subset \Omega$ be an open nonempty set, and let χ_ω be the characteristic function of ω, $f \in L^2(\Omega \times (0, T))$, and $y_0 \in L^2(\Omega)$. We consider the problem

$$y_t - \Delta y = f\chi_\omega, \qquad \text{in } \Omega \times (0, T), \tag{82a}$$

$$y = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{82b}$$

$$y(0) = y_0, \qquad \text{in } \Omega. \tag{82c}$$

Let $y_1 \in L^2(\Omega)$ be such that

$$\int_\Omega y_0 \phi_1 = \int_\Omega y_1 \phi_1, \tag{83}$$

ϕ_1 being the first eigenfunction of the Laplacian with Dirichlet boundary conditions and $\epsilon > 0$.

We consider the following control problem: Can we find $f \in L^2(\omega \times (0, T))$ such that

$$\|y(T) - y_1\|_{L^2(\Omega)} \leq \epsilon \tag{84}$$

and

$$\int_\Omega y(x, t)\phi_1(x)\, dx = \int_\Omega y_0(x)\phi_1(x)\, dx, \qquad \forall t \in (0, T)? \tag{85}$$

This problem is similar to the one that we have presented in Section 7.1 above. But this time, the control acts on an open subset ω of Ω.

This problem has a positive answer for all domain $\Omega \subset \mathbb{R}^d$ and for every open nonempty set $\omega \subset \Omega$. To see this, first of all, we prove a simple unique continuation result.

Lemma 7.1. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2 and $T > 0$. Let $\omega \subset \Omega$ be a nonempty open set. Then, if φ is a solution of

$$-\varphi_t - \Delta\varphi = 0, \qquad \text{in } \Omega \times (0, T), \tag{86a}$$

$$\varphi = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{86b}$$

such that

$$\varphi(x, t) = \left[1 \Big/ \int_\omega \phi_1(x)\, dx\right] \int_\omega \varphi(x, t)\phi_1(x)\, dx, \quad \text{in } \omega \times (0, T), \tag{87}$$

necessarily $\varphi \equiv 0$.

Proof. Since

$$\varphi(x, t) = \rho(t), \qquad \text{in } \omega \times (0, T),$$

we have that

$$\Delta\varphi(x, t) = 0, \qquad \text{in } \omega \times (0, T).$$

Therefore,

$$\varphi_t(x, t) = 0, \qquad \text{in } \omega \times (0, T),$$

that is,

$$\varphi(x, t) = c, \qquad \text{in } \omega \times (0, T).$$

Thus,

$$\psi(x, t) = \varphi_t(x, t)$$

is a solution of the following problem:

$$-\psi_t - \Delta\psi = 0, \qquad \text{in } \Omega \times (0, T), \tag{88a}$$

$$\psi = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{88b}$$

$$\psi = 0, \qquad \text{in } \omega \times (0, T). \tag{88c}$$

Applying the Holmgrem uniqueness theorem in (88), we deduce that

$$\psi(x, t) = 0, \quad \text{in } \Omega \times (0, T) \Leftrightarrow \varphi(x, t) = c(x), \quad \text{in } \Omega \times (0, T). \tag{89}$$

Since φ satisfies (86), we deduce that

$$\Delta c = 0, \qquad \text{in } \Omega,$$

and

$$c = 0, \qquad \text{on } \partial\Omega,$$

which implies that

$$\varphi(x, t) = 0, \qquad \text{in } \Omega \times (0, T). \tag{90}$$

This completes the proof of Lemma 7.1. □

As a consequence of Lemma 7.1, we obtain the following result for every regular domain Ω and for every nonempty open set $\omega \subset \Omega$.

Theorem 7.1. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^2, and let $\omega \subset \Omega$ be a nonempty open set. Let $y_0, y_1 \in L^2(\Omega)$ be such that (83) holds. Then, for every $\epsilon > 0$ and $T > 0$, there exists $f \in L^2(\Omega \times (0, T))$ such that the solution of (82) verifies (84) and (85).

Proof. We consider the homogeneous adjoint problem of (82),

$$-\varphi_t - \Delta\varphi = 0, \qquad \text{in } \Omega \times (0, T), \tag{91a}$$

$$\varphi = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{91b}$$

$$\varphi(T) = \varphi^0, \qquad \text{in } \Omega. \tag{91c}$$

Then, we consider the functional $J: L^2(\Omega) \to \mathbb{R}$,

$$J(\varphi^0) = (1/2) \int_0^T \int_\omega \left| \varphi - \left(\int_\omega \varphi\phi_1 \bigg/ \int_\omega \phi_1 \right) \right|^2 dx\, dt + \epsilon \|\varphi^0\|_{L^2(\Omega)} - \int_\Omega y^1 \varphi^0\, dx, \tag{92}$$

or its bang–bang version

$$J(\varphi^0) = (1/2) \left[\int_0^T \int_\omega \left| \varphi - \left(\int_\omega \varphi\phi_1 \bigg/ \int_\omega \phi_1 \right) \right| dx\, dt \right]^2 + \epsilon \|\varphi^0\|_{L^2(\Omega)} - \int_\Omega y^1 \varphi^0\, dx. \tag{93}$$

Note that the coercivity of the functionals (92) and (93) is a direct consequence of Lemma 7.1. Moreover, if we consider the functional defined in (93), we can obtain a bang–bang control, due to the analyticity of solutions of (82) in $\Omega(0, T)$. □

7.3. Constrained Finite-Approximate Controllability. The approximate controllability condition (3) may be replaced by the following finite-approximate controllability, introduced in Ref. 10. Let $\Omega \subset R^d$ be an open bounded set of class C^2, and let $\Gamma_0 \subset \partial\Omega$ be an open nonempty subset of the boundary. Let $v \in L^2(\Gamma_0 \times (0, T))$, $y_0, y_1 \in L^2(\Omega)$, and $T > 0$ be such that

$$\int_\Omega y_0 = \int_\Omega y_1.$$

Let y be the solution of (1) with initial datum y_0. We consider the following problem: Is there a control v such that the system (1) is approximately controllable and simultaneously

$$\Pi_E(y(T)) = \Pi_E(y^1), \tag{94}$$

E being a finite dimensional subspace of $L^2(\Omega)$ and Π_E the corresponding orthogonal projection?

We have the following result.

Theorem 7.2. Let $\Omega \subset R^d$ be an open bounded set of class C^2, and let $\Gamma_0 \subset \partial\Omega$ be an open nonempty set such that the spectral unique continuation property (7) holds. Let E be a finite dimensional subspace of $L^2(\Omega)$, $y_0, y_1 \in L^2(\Omega)$, and $T > 0$ such that

$$\int_\Omega y_0 = \int_\Omega y_1.$$

Then, for each $\epsilon > 0$, there exists a control function $v \in L^2(\partial\Omega \times (0, T))$ such that the solution $y(x, t)$ of (1) verifies

$$\|y(T) - y_1\|_{L^2(\Omega)} \leq \epsilon, \qquad \Pi_E(y(T)) = \Pi_E(y_1),$$

and

$$\int_\Omega y(x, t)\, dx = \int_\Omega y_0\, dx, \qquad \forall t \in [0, T].$$

The proof is analogous to the proof of Theorem 1.2. It is sufficient to consider the functional J defined by

$$J(\varphi_0) = (1/2) \int_0^T \int_{\Gamma_0} \left[\varphi - (1/|\Gamma_0|) \int_{\Gamma_0} \varphi\right]^2 + \epsilon \|(I - \Pi_E)\varphi_0\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi_0. \tag{95}$$

We refer to Ref. 10 for the details of the proof in the unconstrained case.

7.4. An Open Problem. Notice that we have obtained a generic approximate controllability result with respect to the domain Ω. Moreover, in the case where Ω is a ball, we have shown that Theorem 1.1 is false.

The characterization of the domains Ω, in which these results hold, is an interesting problem.

We can see that the controllability result of Theorem 1.2 holds if and only if the following unique continuation property for the eigenfunctions of the Laplacian with Neumann boundary conditions holds:

$$-\Delta\varphi = \lambda\varphi, \qquad \text{in } \Omega, \tag{96a}$$

$$\partial\varphi/\partial n = 0, \qquad \text{on } \partial\Omega, \tag{96b}$$

$$\varphi = c, \qquad \text{on } \Gamma_0, \tag{96c}$$

implying $\varphi \equiv 0$.

As far as we know, the characterization of the domains Ω and the sets $\Gamma_0 \subset \partial\Omega$ such that (96) holds for all eigenfunctions simultaneously is an open problem.

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