The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential

by

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Abstract. We study the well-posedness and describe the asymptotic behavior of solutions of the heat equation with inverse-square potentials for the Cauchy-Dirichlet problem in a bounded domain and also for the Cauchy problem in $\mathbb{R}^N$. In the case of the bounded domain we use an improved form of the so called Hardy-Poincaré inequality and prove the exponential stabilization towards a solution in separated variables. In $\mathbb{R}^N$ we first establish a new weighted version of the Hardy-Poincaré inequality, and then show the stabilization towards a radially symmetric solution in self-similar variables with a polynomial decay rate.

This work complements and explains well-known work by Baras and Goldstein on the existence of global solutions and blow-up for these equations. In the present article the sign restriction on the data and solutions is removed, the functional framework for well-posedness is described and the asymptotic rates calculated. Examples of non-uniqueness are also given.

1 Introduction

The existence, uniqueness and behaviour of the solutions of the evolution equation

$$u_t = \Delta u + V(x)u, \tag{1.1}$$

and the elliptic version, $\Delta u + V(x)u + \mu u = 0$ (the associated spectral problem), is not very different from the corresponding properties of the heat kernel when the *potential* $V(x)$ is small enough, for instance for a bounded potential, or for potentials with moderate singularities. However, the situation changes dramatically for very singular potentials. The elliptic version (without u_t) with singular potentials arises in several contexts: one of them is the Schrödinger equation in quantum mechanics, cf. [22]. It also appears in the linearized analysis of standard combustion models which lead to blowup phenomena (see [2], [8], [11], [20], [24]). A particularly important case which appears in both contexts is the so-called *inverse-square potential*

$$V(x) = \frac{\lambda}{r^2}, \qquad r = |x|. \tag{1.2}$$

A strong change of character takes place for equation (1.1) and its elliptic version with such a potential at the critical value of the parameter λ, $\lambda_* = (N-2)^2/4$ if $N \geq 3$. We want to explain such a transition both for the stationary and the evolution problem. Our starting point is the following result by Baras and Goldstein, [1]:

The Cauchy-Dirichlet problem for equation (1.1) in an open set of $I\!R^N$ *with* $u(x,0) \geq 0$, $u(x,0) \not\equiv 0$, *and zero Dirichlet data has a global solution if* $\lambda \leq \lambda_*$ *and no solution, even locally in time, if* $\lambda > \lambda_*$. More precisely, in [1] the potential $V(x)$ is replaced by $V_n(x) = \max\{n, V(x)\}$ and the limit of the sequence of solutions $u_n(x,t)$ of the corresponding approximate problems is considered. It is then proved that: (a) when $\lambda \leq \lambda_*$ the sequence u_n converges monotonically to a solution u of the original problem; (b) when $\lambda > \lambda_*$ tends to infinity for all $(x,t) \in \Omega \times (0,\infty)$ (so-called *complete instantaneous blowup*).

A number of problems are naturally posed after this intriguing result. To quote, what is the real situation at the very unstable transition value $\lambda = \lambda_*$ that separates global existence from instantaneous blowup? can the sign restriction on the data be removed? can we characterize uniqueness in a proper functional class? do the solutions for $\lambda \leq \lambda_*$ decay, and if so at what rate? what is the difference between bounded and unbounded domains? are there any solutions for $\lambda > \lambda_*$? and some more.

In this paper we give answers to the foregoing questions. We consider here the properties of the initial-value problem

$$u_t = \Delta u + \frac{\lambda}{r^2}u, \tag{1.3}$$

$$u(x,0) = u_0(x), \tag{1.4}$$

posed in $\mathbb{R}^N$ or in a bounded subset Ω of $\mathbb{R}^N$ which contains the origin. In that case we add boundary data

$$u(x,t) = 0 \qquad \text{for } x \in \partial\Omega,\ t \geq 0. \tag{1.5}$$

We do not make any sign restriction on u_0 or u. We derive the properties of the evolution problems from a deeper analysis of the stationary operator. Indeed, we will show here that the behaviour of the solutions at the transition $\lambda = \lambda_*$ is intimately related to the classical Hardy inequality and its improved form (Brezis and Vazquez [8], Maz'ya [18]) that we shall refer to as the Hardy-Poincaré inequality. Indeed we will need improved versions of this inequality. These are one of the main contributions of the paper, cf. Theorems 2.2 and 9.1. The study is first performed in a bounded domain and then extended to the whole space. While the standard variational analysis applies to the case $\lambda < \lambda_*$, so that for every $u_0 \in L^2(\Omega)$ there exists a solution of (1.3)-(1.5) $u \in C([0,\infty); L^2(\Omega)) \cap L^2(0,\infty; H_0^1(\Omega))$, thus global in time, this is not true for $\lambda = \lambda_*$, where a suitable unique solution, the *good solution*, exists and is global in $L^2(\Omega)$, but blows up instantaneously in $H^1(\Omega)$. Even in the range $0 < \lambda < \lambda_*$, where the standard variational setting applies, there is lighter form of blowup which has already been remarked in [1] and marks a strong difference with the heat equation case $\lambda = 0$: the solutions $u \geq 0$ of the evolution problem, even with good initial data, are singular at the origin with a rate $u \geq C|x|^{-\alpha_1}$, for some $\alpha_1(N,\lambda) > 0$ which appears again below, hence they blow up in L^p for all large p and all $t > 0$. We make a precise description of the singular behaviour. While the main outline is similar when working in the whole space, the functional setting needs the introduction of weighted spaces.

The second objective of the paper is to establish the asymptotic behavior of solutions as time goes to infinity. It is proved that the solutions decay to zero as $t \to \infty$, but naturally the rate is different in the case where Ω is a bounded domain and when Ω is the whole space $\mathbb{R}^N$. In the first case we have exponential decay in $L^2(\Omega)$. In particular, when Ω is a ball the asymptotic behaviour of the solution of the Cauchy Problem for $\lambda \leq \lambda_*$ and the stabilization to a precise separated-variables solution with explicit exponential decay rate is proved. More precisely, we show that for every $u_0 \in L^2(\Omega)$ there exists a unique solution in the correct space and moreover this solution decays in time like

$$u(x,t) = O(e^{-\mu t}), \tag{1.6}$$

for a certain $\mu > 0$ that is exactly calculated when the domain is a ball of radius a as $\mu = z_{m,n}^2/a^2$, $z_{m,n}$ is the n-th positive zero of the Bessel function J_m, and $m^2 = \lambda_* - \lambda$, $m > 0$. The slowest and generic decay happens for $n = 1$.

An important additional property of the range under study, $0 < \lambda \leq \lambda_*$, appears in the form of non-uniqueness of distributional solutions, even when we assume that they are integrable and nonnegative. More precisely, the function

$$u = |x|^{-(N-2)/2} \log(1/|x|) \tag{1.7}$$

is a stationary solution of the evolution problem with critical coefficient λ_* which lies in $L^2(B_1(0))$ for all t and does not decay in time, while the *good solutions* must obey (1.6).

This solution does not live in $H_0^1(\Omega)$, but the 'error' amounts only to a logarithmic factor. A similar construction can be done for $0 < \lambda \le \lambda_*$ but then the bad solutions live in spaces which are farther away from the variational setting. This is worked out in Section 7, where we also remark that our non-uniqueness examples are closely related to Serrin's famous "pathological solutions" for elliptic equations in divergence form with bounded coefficients, [23].

We also discuss the extension question: indeed, the semigroup $S_t : u_0 \mapsto u(\cdot, t)$ can be extended to a larger class of initial data

$$L^1_{\alpha_1}(\Omega) = \int_\Omega |u_0(x)|\,|x|^{-\alpha_1}\,dx < \infty, \tag{1.8}$$

where α_1 is the smallest root of $\alpha(N-2-\alpha) = \lambda$. As already shown in [1] this choice is optimal in the sense that the exponent α_1 cannot be improved for positive data. But it can be extended for instance to the corresponding weighted space of measures $\mathcal{M}_{\alpha_1}(\Omega)$, see definition in (6.8). The regularity for $t > 0$ and the asymptotic properties of the L^2 semigroup are preserved. The extension is tied to the contractive character of the semigroup in a family of weighted L^p-spaces that we describe in detail in the final Appendix. In another direction, the analysis of Section 8 shows that for $\lambda > \lambda_*$ there are classes of *initial data of oscillating type* for which the solution exists globally in time. In other words, there is still a semigroup, defined in a restricted domain, for λ beyond the critical value.

Let us turn our attention to the case $\Omega = \mathbb{R}^N$. Then the standard Hardy inequality is sharp and therefore one cannot deduce from it any decay of the evolution solutions in the critical case $\lambda = \lambda_*$. We establish a new Hardy-Poincaré Inequality in the natural weighted space and establish asymptotic convergence to a self-similar profile with polynomial decay rate

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^N)} = O(t^{-1/2}). \tag{1.9}$$

More precisely, we show in Sections 9, 10 that the asymptotic behaviour of the good solution of the critical problem in the whole space with nonnegative initial data u_0 in a weighted space $L^2(K)$, $K = \mathtt{exp}(x^2/4t)$, is given in first approximation by a multiple of the explicit solution

$$u(x,t) = \frac{1}{|x|^{(N-2)/2}t}\,\mathtt{exp}\,(-\frac{x^2}{4t}). \tag{1.10}$$

This nontrivial solution exemplifies the non-uniqueness of general weak solutions in this setting, since it has a trivial initial trace, $u(\cdot,t) \to 0$ not only as a distribution but also in $L^1(\mathbb{R}^N)$ (though not in L^2) as $t \to 0$. A similar non-uniqueness result happens for all $0 < \lambda < \lambda_*$, see the discussion in §10.4.

In conclusion, we observe that there is a continuous transition from the slightly non-standard situation for $0 < \lambda < \lambda_*$ to the transition situation for $\lambda = \lambda_*$. In all these cases a unique global good solution exists for data in $L^2(\Omega)$ (or a larger space, as mentioned above). There is a sharp contrast with the range $\lambda > \lambda_*$, where all nonnegative solutions blowup instantaneously, i.e., the limit of natural approximations to the problem with non-negative initial data is $+\infty$ for all $t > 0$ and $|x| \neq 0$. This problem is a particularly clear example of the known fact that the attractors of a dynamical system, or the asymptotic exponents, need not depend continuously on the parameters.

Observe that when $N = 2$ the critical value of λ is $\lambda_* = 0$. Thus there is no $\lambda > 0$ for which the problem with homogeneous Dirichlet boundary conditions and nonnegative initial data has global in time solutions.

We end this introduction with some comments on related work. The literature on Hardy inequalities is extensive, see e.g. [18]. Equation (1.1) appears as the linearization of the the exponential reaction-diffusion equation $u_t = \Delta u + \lambda\, e^u$ around its stationary singular solution, and Hardy inequalities play a role in the stabilization of solutions. This is studied e.g. in [20, 8, 11]. The latter reference analyzes the case where $\lambda = 2(N-2)$, which for $N > 10$ implies $\lambda < \lambda_*$ and the results agree with ours. It discusses an alternative approach to Hardy inequalities in classes of radial solutions, and contains also second-order inequalities of the form $\int u^2/|x|^4\, dx \leq C \int (\Delta u)^2\, dx$. An interesting direction, different from the present one, happens when the singular potential blows up at the boundary, see [7] and its references. After this paper was complete the authors learned of the recent work [9], which deals with the same equation in a bounded domain. The authors of [9] give a new proof of Baras and Goldstein's result. They stress the importance of the Hardy inequality by showing that global solutions of the evolution problem exist roughly if and only if a Hardy-like inequality holds. This is also in agreement with our results below.

2 The Hardy and the Hardy-Poincaré inequalities

The classical form of the Hardy Inequality (shortly, HI) asserts that for every $u \in H^1(\mathbb{R}^N)$, $N \geq 3$, we have $u/r \in L^2(\mathbb{R}^N)$ and moreover

$$(2.1) \qquad \lambda_* \int_{\mathbb{R}^N} \frac{u^2}{r^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\, dx,$$

where the constant $\lambda_* = (N-2)^2/4$ is optimal and not attained in $H^1(\mathbb{R}^N)$. The same result applies for $u \in H_0^1(\Omega)$, if Ω is an open subset of $\mathbb{R}^N$, $N \geq 3$, with integrals in Ω. It is well-known that the difference of the integrals arising in (2.1) allows to get upper bounds of the L^2-norm of $r^{-\alpha}u$ for any $\alpha < 1$ (see [18], Section 1.2.6). In [8] the following sharp estimate was given when $\alpha = 0$:

Theorem 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Then there exists a second constant $C(\Omega) > 0$ such that for every $u \in H_0^1(\Omega)$*

$$(2.2) \qquad \int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx \geq C(\Omega) \int u^2\, dx.$$

The optimal value of the constant is given in a ball $B_a(0)$ by

$$C(\Omega) = z_0^2/a^2, \tag{2.3}$$

where z_0 is the first zero of the Bessel function $J_0(r)$, $z_0^2 = 0.57832...$ For a general bounded Ω we may use for C the constant corresponding to the ball of the same volume.

This result reduces for $N = 2$ to the standard Poincaré Inequality, while for $N \geq 3$ it represents a combination of the Hardy and Poincaré inequalities. We will call it the Hardy-Poincaré Inequality, HPI. The main purpose of this section is to improve the result to show that the left-hand side of (2.2) dominates the $W^{1,q}$-norm of u for all $1 \leq q < 2$.

Theorem 2.2 (Improved Hardy-Poincaré inequality) *Let Ω be a bounded open subset of $I\!R^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists a constant $C(q, \Omega) > 0$ such that*

$$\int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx \geq C(q, \Omega) \|u\|^2_{W^{1,q}(\Omega)} \tag{2.4}$$

holds for all $u \in H_0^1(\Omega)$.

Note the result is still (trivially) true for $N = 2$ with $\lambda_* = 0$ and $q = 2$. However, $q = 2$ is excluded for $N \geq 3$ since the constant λ_* in the Hardy inequality is optimal, cf. [8] and the spectral analysis below. As a consequence of Theorem 2.2 and Sobolev's imbeddings we have a control of the norm of u in the Sobolev spaces $H^s(\Omega) = W^{s,2}(\Omega)$ and more generally $W^{s,r}(\Omega)$ as follows.

Corollary 2.3 (Improved Hardy-Poincaré inequality. Second form) *Let Ω be a bounded open subset of $I\!R^N$, $N \geq 3$. Then for any $0 \leq s < 1$ and every $1 \leq r < r_* = 2n/(n - 2(1 - s))$ there exists a constant $C(s, r, \Omega) > 0$ such that*

$$C(s, r, \Omega) \|u\|^2_{W^{s,r}(\Omega)} \leq \int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx \tag{2.5}$$

holds for every $u \in H_0^1(\Omega)$.

For $N = 2$ the result holds, but then $\lambda_* = 0$ and there is no novelty. For $N \geq 3$ and $s = 0$ we recover the Hardy-Poincaré inequality.

The proof of Theorem 2.2 is divided into several steps. We present first the main calculation, which happens for radial functions in a ball.

Case 1. Ω IS A BALL CENTERED AT THE ORIGIN AND u IS RADIAL. By scaling we may assume that Ω is the unit ball $B = B_1(0)$. Let ω_N be the Lebesgue measure of B and let $S = S^{N-1} = \partial B$ be the unit sphere with $(N-1)$-dimensional Hausdorff measure $N\,\omega_N$.

Under the assumption that $u = u(r)$, $r = |x|$, we have to prove that for all $1 \le q < 2$ there exists a constant $C = C(q) > 0$ such that

$$C\left(\int_0^1 |u'|^q\, r^{N-1} dr\right)^{2/q} \le \int_0^1 \left[|u'|^2 - \lambda_* \frac{u^2}{r^2}\right] r^{N-1} dr \tag{2.6}$$

holds for every smooth function $u(r)$ defined for $0 \le r \le 1$ and such that $u(1) = 0$. The result for radial functions in $H_0^1(B)$ follows then by density. As in [8] we proceed via the change of variables

$$v(r) = r^{(N-2)/2}\, u(r), \tag{2.7}$$

a basic tool in what follows. Then

$$\int_0^1 \left[|u'|^2 - \lambda_* \frac{u^2}{r^2}\right] r^{N-1} dr = \int_0^1 |v'|^2 r dr, \tag{2.8}$$

cf. [8, pag. 454]. On the other hand,

$$\int_0^1 |u'|^q\, r^{N-1} dr = \int_0^1 |r^{-(N-2)/2} v'(r) - \frac{N-2}{2} r^{-N/2} v(r)|^q\, r^{N-1} dr \le$$

$$C_q \int_0^1 |v'|^q r^{N-1-(N-2)q/2} dr + C_{q,N} \int_0^1 |v|^q r^{N-1-Nq/2} dr = I_1 + I_2.$$

We bound the first integral as follows

$$\int_0^1 |v'|^q r^{N-1-(N-2)q/2} dr \le \left(\int_0^1 |v'|^2 r dr\right)^{q/2} \left(\int_0^1 r^\beta dr\right)^{(2-q)/2}$$

with $\beta = N - 1$. Obviously the last integral converges, therefore we have for $1 \le q \le 2$

$$I_1 \le C(\int_0^1 |v'|^2 r dr)^{q/2},$$

which has to be compared with (2.8). As for the other integral we have for every $p > q$

$$I_2 \le \left(\int_0^1 |v|^p r dr\right)^{q/p} \left(\int_0^1 r^\alpha dr\right)^{(p-q)/p}, \qquad \alpha = (N - 1 - \frac{Nq}{2} - \frac{q}{p})\frac{p}{p-q}.$$

The last integral converges iff $\alpha > -1$, i.e., if $q < 2$ and p is large enough, precisely for $p > 4q/N(2-q)$. Besides, using the standard imbedding of $H_0^1(B_2)$ into $L^p(B_2)$ in the two-dimensional ball, which is valid for any finite p, we have

$$\int_0^1 |v|^p\, r dr \le C_p \left(\int_0^1 |v'|^2 r dr\right)^{p/2}.$$

Thus, I_2 is also bounded above by a multiple of the right-hand side of (2.6). In this way the result is proved for radial functions in a ball.

Case 2. NONRADIAL FUNCTIONS IN A BALL. Again we may assume that Ω is the unit ball B. Using spherical coordinates $x = (r, \sigma)$ in B, we decompose u into spherical harmonics to get

$$u = \sum_{k=0}^{\infty} u_k(r) f_k(\sigma),$$

where the f_k constitute an orthonormal basis of $L^2(S^{N-1})$ consisting of eigenfunctions of the Laplace-Beltrami operator, which has eigenvalues

$$c_k = k(N + k - 2), \quad k \geq 0,$$

cf. [6, page 161]. In particular $f_0(\sigma) = 1$ and $u_0(r)$ is the projection of $u \in H_0^1(B)$ onto the space of radially symmetric functions. We now observe that

$$\int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx = N\omega_N \sum_{k=0}^{\infty} \int_0^1 \left[|u_k'|^2 - \lambda_* \frac{u_k^2}{r^2} + c_k \frac{u_k^2}{r^2} \right] r^{N-1} dr.$$

We now separate the sum of the terms corresponding to oscillating harmonics

$$I_1 = \sum_{k=1}^{\infty} \int_0^1 \left[|u_k'|^2 - (\lambda_* - c_k) \frac{u_k^2}{r^2} \right] r^{N-1} dr$$

from the radial term

$$I_0 = \int_0^1 \left[|u_0'|^2 - \lambda_* \frac{u_0^2}{r^2} \right] r^{N-1} dr.$$

This latter term has been estimated in the first step as follows:

$$I_0 \geq C \|u_0\|^2_{W^{1,q}(B)}.$$

The first one is quite easy to estimate once we have the radial result. Even with the result of Theorem 2.1 we get

$$\int_0^1 \left[|u_k'|^2 - \lambda_* \frac{u_k^2}{r^2} + c_k \frac{u_k^2}{r^2} \right] r^{N-1} dr \geq \frac{c_k}{\lambda_*} \int_0^1 |u_k'|^2 \, r^{N-1} dr.$$

Using the fact that $c_k \geq N - 1 > 0$ for $k \geq 1$, the sum over $k = 1, \cdots$ is bounded below by $C\|u - u_0\|^2_{H_0^1(B)}$. Joining this result to the conclusion of the previous step, Theorem 2.2 follows in a ball.

Case 3. GENERAL DOMAIN. Let Ω be a bounded subset of $\mathbb{R}^N$ and assume that $0 \in \Omega$ and $B_a(0) \subset \Omega$. We are going to prove the following

Claim: *there exist constants $C_1, C_2 > 0$ such that for every $u \in H_0^1(\Omega)$*

$$\int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{|x|^2} \right] dx \geq C_1 \|u\|^2_{W^{1,q}(\Omega)} - C_2 \|u\|^2_{L^2(\Omega)},$$

because this and Theorem 2.1 together imply the conclusion of our theorem, formula (2.4). To prove the Claim we first introduce a smooth cutoff function ϕ such that $0 \le \phi(x) \le 1$, with $\phi(x) = 1$ for all $x \in B_{a/2}(0)$ and $\phi(x) = 0$ when $|x| \ge a$. Setting $w_1 = u\phi$ and $w_2 = u(1-\phi)$, $u = w_1 + w_2$, we have

$$\begin{aligned} \int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx = \int_\Omega \left[|\nabla w_1|^2 - \lambda_* \frac{w_1^2}{r^2} \right] dx + \\ \int_\Omega \left[|\nabla w_2|^2 - \lambda_* \frac{w_2^2}{r^2} \right] dx + 2 \int_\Omega \left[\nabla w_1 \cdot \nabla w_2 - \lambda_* \frac{w_1 w_2}{r^2} \right] dx. \end{aligned} \tag{2.9}$$

We estimate the different terms in this decomposition. Since the support of w_2 is disjoint with the origin we have

$$\int_\Omega \frac{w_2^2}{|x|^2} dx + \int_\Omega \frac{w_1 w_2}{|x|^2} dx \le C \int_\Omega u^2 \, dx.$$

On the other hand,

$$\int_\Omega \nabla w_1 \cdot \nabla w_2 \, dx = \int_\Omega \phi(1-\phi)|\nabla u|^2 \, dx - \int_\Omega |\nabla \phi|^2 u^2 \, dx + \int_\Omega u \nabla u \cdot ((1-2\phi)\nabla\phi) \, dx.$$

Besides,

$$\int_\Omega u\nabla u \cdot [(1-2\phi)\nabla \phi] \; dx = -\frac{1}{2} \int_{B_a \setminus B_{a/2}} u^2 \mathtt{div}((1-2\phi)\nabla\phi) \, dx.$$

In this integration by parts the boundary terms vanish since $(1-2\phi)\nabla\phi = 0$ on $\partial(B_a \setminus B_{a/2})$. Combining the last two formulas we get

$$\int_\Omega \nabla w_1 \cdot \nabla w_2 \, dx \ge -C \int_\Omega u^2 \, dx.$$

This and (2.9) give

$$\int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx \ge \int_\Omega \left[|\nabla w_1|^2 - \lambda_* \frac{w_1^2}{r^2} \right] dx + \int_\Omega |\nabla w_2|^2 \, dx - C \int_\Omega u^2 \, dx.$$

Applying the result already proved in a ball to $w_1 \in H_0^1(B_a)$ we also have

$$\int_\Omega \left[|\nabla w_1|^2 - \lambda_* \frac{w_1^2}{r^2} \right] dx \ge C_1 \|w_1\|^2_{W^{1,q}(\Omega)}.$$

This concludes the proof. ∎

Remark The difficulty of the result of Theorem 2.2 is due to the singular behaviour of the weight $|x|^{-2}$ at the origin. Outside of the origin the situation falls into the standard estimates. Indeed, the above analysis shows that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $H_0^1(\Omega)$

$$\int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2} \right] dx \ge C_\varepsilon \|u\|^2_{H^1(\Omega \setminus B_\varepsilon(0))}. \tag{2.10}$$

Extension 1 In order to derive a sharper result about the behaviour of the functions, including the behaviour at the origin, we recall now the Trudinger Inequality, cf. [15, Theorem 7.15] or [16, page 33]: *there exist constants c_1 and c_2 depending only on N such that for every $u \in W_0^{1,N}(\Omega)$ the following holds:*

$$\frac{1}{\text{meas}\,(\Omega)} \int_\Omega \exp\left(\frac{u(x)}{c_1\,\|\nabla u\|_{L^N(\Omega)}}\right)^{N/(N-1)} dx \le c_2. \tag{2.11}$$

We may use the proof of the IHPI to derive a variant of the Trudinger Inequality that controls the function u also at the origin in terms of the bound

$$A = \int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{r^2}\right] dx. \tag{2.12}$$

Proposition 2.4 *There exists constants $c_1(N), c_2(\Omega) > 0$ such that for every $u \in H_0^1(\Omega)$*

$$\int_\Omega \mathtt{exp}\left(\frac{|x|^{(N-2)/2}u(x)}{c_1\,A^{1/2}}\right)^2 \frac{dx}{|x|^{N-2}} \le c_2 \tag{2.13}$$

with A given by (2.12).

Proof. In the radial case in a ball we have the control of v' in $L^2(B_2)$ by a multiple of $A^{1/2}$, where B_2 is the two-dimensional ball of the same radius. By the Trudinger inequality in $N = 2$,

$$\frac{1}{\pi\,a^2} \int_0^a \mathtt{exp}\left[\left(\frac{v(x)}{c_1'\,A^{1/2}}\right)^2\right] r dr \le c_2.$$

This is the formula that gives rise to (2.13). The non-radial components are easier to control, and so is the case of a general domain. ∎

Extension 2 The Hardy-Poincaré Inequality has a best constant $\lambda_* = (N-2)^2/4$, but its improved form has shown that we may still obtain a positive result

$$\int_\Omega \left[|\nabla u|^2 - V(r)\,u^2\right] dx \ge 0$$

if we put $V(r) = \lambda_* |x|^{-2} + c$ for some $c > 0$ which is precisely determined in Theorem 2.1. In order to push further the limit we examine next what happens with the potentials of the form

$$V(x) = \frac{\lambda_*}{|x|^2} + \frac{k}{|x|^p} \tag{2.14}$$

with $0 < p < 2$. There is a version of the HPI that gives a convenient bound also in that case. More precisely, *for any* $k > 0$ *and* $0 < p < 2$ *there exists a constant* $C(\Omega, k, p) > 0$ *such that for every* $u \in H_0^1(\Omega)$ *with* $\int u^2\,dx = 1$

$$\int_\Omega |\nabla u|^2\,dx \geq \int_\Omega \frac{\lambda_*}{r^2}u^2\,dx + \int_\Omega \frac{k}{r^p}u^2\,dx - C. \tag{2.15}$$

This result is proved in [18], Section 2.1.6. We can use the proof of Theorem 2.2. This is how to modify the analysis of the radial part in a ball. Under the stated conditions, working in the ball $B = B_1(0)$ we have with $p = 2 - \varepsilon$

$$\int_B |\nabla u|^2\,dx - \int_B \frac{\lambda_*}{r^2}u^2\,dx - \int_B \frac{k}{r^p}u^2\,dx = C\int_0^1 (v')^2 r\,dr - Ck\int_0^1 v^2\,r^{\varepsilon-1}\,dr.$$

where $C = N\omega_N$. We now observe that

$$\int_0^1 r^{\varepsilon-1}v^2\,dr \leq \left(\int_0^1 rv^{2q}\,dr\right)^{1/q}\left(\int_0^1 r^\beta\,dr\right)^{1-(1/q)}$$

with $\beta+1 = (\varepsilon q - 2)q/(q-1)$, which is positive for large q, so that the last integral is finite and bounded by $C_1\|v\|^2_{L^q(B')}$, where B' is the unit ball in $\mathbb{R}^2$. Interpolating

$$\|v\|^2_{L^q(B')} \leq \varepsilon\|v\|^2_{H_0^1(B')} + C_\varepsilon\|v\|^2_{L^2(B')},$$

we get the result in the radial case. The rest is easy.

3 Elliptic and Evolution Problems with subcritical parameter

As in the previous section we consider a bounded domain Ω in $\mathbb{R}^N$ with $N \geq 3$. We want to discuss the solvability of the parabolic problem with homogeneous boundary conditions of Dirichlet type:

$$u_t = \Delta u + V(x)\,u, \qquad \text{in } \Omega \times (0,\infty), \tag{3.1}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega, \tag{3.2}$$

$$u(x,t) = 0 \qquad \text{for } x \in \partial\Omega,\ t \geq 0 \tag{3.3}$$

where V is a locally integrable function defined in Ω. We are interested in potentials V with a singular behaviour at one point, say $0 \in \Omega$. The special example of interest is

$$V(x) = \frac{\lambda}{|x|^2}, \qquad \lambda \geq 0. \tag{3.4}$$

The study of this type of potential with critical value of the parameter, $\lambda_* = (N-2)^2/4$, is a main goal of this paper. More generally, concerning the well-posedness of the problem we will consider three cases:

• Subcritical Case : $V(x) = \lambda/|x|^2$ with $\lambda < \lambda_*$. The main results are generalized to potentials of the form $-C \le V(x) \le \lambda/|x|^2$ with $\lambda < \lambda_*$.

• Critical case: $V(x) = \lambda/|x|^2$ with $\lambda = \lambda_*$. Again, to be generalized to potentials with similar growth.

• Super-critical case: $V(x) \ge \lambda/|x|^2$ with $\lambda > \lambda_*$.

According to the results of [1] this last case corresponds to non-existence of solutions $u \ge 0$ due to instantaneous blow-up, hence we can concentrate on the first two.

3.1 General analysis of the sub-critical case

Assume that $-C \le V(x) \le \lambda/|x|^2$ with $\lambda < \lambda_*$. According to the Hardy-Poincaré Inequality we know that for $\lambda < \lambda_*$

$$\int_\Omega \left[|\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \right] dx = (1 - \frac{\lambda}{\lambda_*}) \int_\Omega |\nabla u|^2 \, dx +$$
$$\frac{\lambda}{\lambda_*} \int_\Omega \left[|\nabla u|^2 - \lambda_* \frac{u^2}{|x|^2} \right] dx \ge (1 - \frac{\lambda}{\lambda_*}) \int_\Omega |\nabla u|^2 \, dx + \frac{\lambda C(\Omega)}{\lambda_*} \int_\Omega u^2 \, dx.$$

Thus, $(\int_\Omega (|\nabla u|^2 - V(x) u^2 \, dx)^{1/2}$ is equivalent to the standard norm of $H_0^1(\Omega)$, and the operator $L = L(V)$ given by

$$L(V) = -\Delta - V(x)\, I$$

defines an isomorphism from $H_0^1(\Omega)$ into its dual

$$L : H_0^1(\Omega) \to H^{-1}(\Omega).$$

Combining the compact imbedding $H_0^1(\Omega) \to L^2(\Omega)$ and the dual imbedding $L^2(\Omega) \to H^{-1}(\Omega)$ we conclude that L defines by restriction an unbounded self-adjoint operator in $L^2(\Omega)$ with compact inverse.

Theorem 3.1 *There exists an orthonormal basis $\{e_k\}_{k\ge1}$ of $L^2(\Omega)$ constituted by eigenvectors of L with eigenvalue sequence*

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \ldots \to \infty,$$

so that

$$-\Delta e_k - V(x)\, e_k = \mu_k\, e_k \quad \text{in } \Omega, \qquad e_k = 0 \quad \text{on } \partial\Omega.$$

We recall that both sequences $\{e_k\}$ and $\{\mu_k\}$ depend on V. In what concerns the evolution problem, standard semigroup theory implies that there exists a unique solution in this functional framework. More precisely, we have

Theorem 3.2 *For any $u_0 \in L^2(\Omega)$ there exists a unique*

$$u \in C([0,\infty) : L^2(\Omega)) \cap L^2(0,\infty : H_0^1(\Omega))$$

which is a weak solution of the evolution problem. The solution can be developed with respect to the basis $\{e_k\}$ as follows:

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-\mu_k t}\, e_k(x), \tag{3.5}$$

where the a_k's are the Fourier coefficients of the initial data,

$$u_0 = \sum_{k=1}^{\infty} a_k e_k. \tag{3.6}$$

Finally, let us note that $L(V)$ generates an analytic semigroup of contractions in the pivot space $L^2(\Omega)$.

Asymptotic behaviour. As $t \to \infty$ we have exponential decay whose rate is given by the first eigenvalue. More precisely,

$$\|u(t)\|_{L^2(\Omega)} \le e^{-\mu_1 t}\|u_0\|_{L^2(\Omega)}, \tag{3.7}$$

$$\|u(t) - \sum_{k=1}^{K} a_k e^{-\mu_k t} e_k\|_{L^2(\Omega)} = O(e^{-\mu_{K+1} t})\|u_0\|_{L^2(\Omega)}. \tag{3.8}$$

3.2 The subcritical case in a ball

Assume now that the domain Ω is the ball $B = B_a(0)$ of radius $a > 0$ of $I\!R^N$ with $N \ge 3$ and let us take $V(x) = \lambda/|x|^2$, λ subcritical. We consider the same parabolic problem:

$$u_t = \Delta u + \frac{\lambda}{r^2}\, u, \qquad \text{in } \ B \times (0,\infty), \tag{3.9}$$

$$u(x,0) = u_0(x) \qquad \text{for } \ x \in B, \tag{3.10}$$

$$u(x,t) = 0 \qquad \text{for } |x| = a, t \ge 0. \tag{3.11}$$

In this case the spectrum can be explicitly computed using spherical coordinates, $x = (r,\sigma)$, $r > 0$, $\sigma \in S^{N-1}$, and this gives detailed information about the singularities and decay rates of the evolution problem. We denote by $f_j(\sigma)$ the eigenfunctions of the Laplace-Beltrami operator (shortly, LB), which constitute an orthonormal basis of $L^2(S^{N-1})$. The eigenvalues, that we call here c_j to distinguish from those of $L(V)$, are just $c_j = j(j+N-2)$, $j = 0, 1, 2, \ldots$ Then we look for eigenfunctions of $L(V)$ of the form

$$e(r,\sigma) = \phi(r) f_j(\sigma). \tag{3.12}$$

Then ϕ has to satisfy the eigenvalue equation

$$-\phi'' - \frac{N-1}{r}\phi' - \frac{\lambda}{r^2}\phi + \frac{c_j}{r^2}\phi = \mu\phi, \tag{3.13}$$

with condition at the border $\phi(a) = 0$. At $r = 0$ we would like to impose the condition of non-singularity, $\phi'(0) = 0$ but this is too restrictive. According to the variational theory, we only ask that $\phi' \in L^2((0,1); r^{N-1}dr)$. The equation can be written as

$$\phi'' + \frac{N-1}{r}\phi' + \left(\frac{\lambda - c_j}{r^2} + \mu\right)\phi = 0. \tag{3.14}$$

Using again one of the main ideas employed in the proof of the Hardy Inequality we now perform the change of variables

$$\phi(r) = \frac{\psi(r)}{r^{(N-2)/2}} \tag{3.15}$$

and then ψ solves the Bessel equation

$$\psi'' + \frac{1}{r}\psi' + \left(\mu - \frac{\lambda_* + c_j - \lambda}{r^2}\right)\psi = 0. \tag{3.16}$$

Now the boundary conditions are: $\psi'(0) = 0$, $\psi(a) = 0$. It follows that ψ has the form

$$\psi(r) = J_m(\sqrt{\mu}r), \tag{3.17}$$

where J_m is the m-th Bessel function, with $m = m(j, \lambda)$ given by

$$m^2 = \lambda_* + c_j - \lambda, \qquad m \geq 0, \tag{3.18}$$

so that m is always larger than 0 when $\lambda < \lambda_*$. Then J_m vanishes to order m at $r = 0$, $J_m(r) = cr^m + O(r^{m+1})$. Finally, the boundary condition $\Psi(a) = 0$ forces $\sqrt{\mu}\,a$ to be a zero of this Bessel function, $\mu = z_{m,n}^2/a^2$. We thus get

Theorem 3.3 *There exists a two-parameter family of eigenfunctions:*

$$e_{j,n}(r,\sigma) = r^{-(N-2)/2} J_m(\frac{z_{m,n}}{a}r) f_j(\sigma), \tag{3.19}$$

with free parameters $j \geq 0$, $n \geq 1$; the index $m = m(j) > 0$ is related to j by (3.18) and $z_{m,n}$ is the n-th zero of the Bessel function J_m. The corresponding eigenvalues are:

$$\mu_{j,n} = \frac{z_{m,n}^2}{a^2}. \tag{3.20}$$

The family $\{e_{j,n}\}$ is a complete orthogonal basis of $L^2(B)$ and $H_0^1(B)$.

SINGULAR BEHAVIOUR. SPLITTING. All the basis functions are C^∞ functions for $x \neq 0$. In studying the regularity of this basis at the origin we are interested in separating the radial component which is the most singular. We recall that $L^2(B)$ is the direct sum of the spaces

$$X_1 = L^2_r(B) = \{f \in L^2(B) : \ f = f(r)\}, \qquad X_2 = L^2_{za}(B) = \{f \in L^2(B) : \ \overline{f}(r) = 0\},$$

where $\overline{f}$ is the spherical average of a function in $L^2(B)$

$$\overline{f}(r) = \frac{1}{N\omega_N}\int_{|x|=r} f(r,\sigma)\,d\sigma. \tag{3.21}$$

We thus split any function $f \in L^2(B)$ into its radial and non-radial (or spherical zero-average) components, $f(r,\sigma) = f_1(r) + f_2(r,\sigma)$, by defining $f_1 = \overline{f}$.

With this decomposition we observe that the maximal singularity in the eigenfunction family corresponds to the sub-family of radial eigenfunctions. i.e., $j = 0$ (hence, $m(0,\lambda)^2 = \lambda_* - \lambda$), which represent the complete basis for the subspace X_1. We notice that for $0 < \lambda < \lambda_*$ all of them behave at $r = 0$ like

$$e_{0,n} = O(r^{m-(N-2)/2}), \tag{3.22}$$

therefore they are singular, since we have precisely $m(0,\lambda)^2 < \lambda_* = (N-2)^2/4$. As λ grows other separated-variables solutions with $j > 0$, corresponding to the zero-mean basis, develop singularities in their turn, precisely when $m(j,\lambda) < \sqrt{\lambda_*} = (N-2)/2$, which is equivalent to $\lambda > c_j$. Since $c_1 = N-1$ and $\lambda < \lambda_*$, this happens only in higher dimensions (see the discussion of the critical case below). In any case, all the singularities are compatible with the variational sense of the equation: thus, since $m > 0$, we have

$$|\nabla e|^2, \ \frac{1}{r^2}e^2 \in L^p(B) \quad \text{for some } p > 1.$$

Evolution in the ball. We can now construct the basis of separated-variables solutions

$$U_{j,n}(x,t) = e_{j,n}(x)\, e^{-\mu_{j,n}t}, \tag{3.23}$$

which allow us to solve the evolution problem in the form given in Theorem 3.2 and apply formulas (3.7), (3.8). In order to better visualize the results it is instructive to split the evolution into its radial and non-radial (zero-mean) parts. This is done as in the stationary case; time enters the splitting as a parameter. Thus, if u is a solution of the evolution problem, and we put

$$\overline{u}(r,t) = \frac{1}{N\omega_N}\int_{|x|=r} u(x,t)\,d\sigma, \tag{3.24}$$

then $\overline{u}$ satisfies the radial version of the problem, which reads

$$\overline{u}_t = \overline{u}_{rr} + \frac{N-1}{r}\,\overline{u}_r + \frac{\lambda}{r^2}\,\overline{u}, \tag{3.25}$$

with obvious initial and boundary conditions. On the other hand, the non-radial part $\tilde{u} = u - \overline{u}$ is a solution of the original problem with $\tilde{u}(t) \in X_2$ for all t.

RADIAL SOLUTIONS. Assume that $u = \overline{u}$ is a radially symmetric solution of (1.3) with potential $V(r)$. Then when we define

$$v = u\, r^{(N-2)/2}, \tag{3.26}$$

it is clear that

$$\int_\Omega u^2(r,\sigma)\, dx = N\,\omega_N \int_0^1 v^2(r,t) r dr.$$

We get the equation for v:

$$v_t = v_{rr} + \frac{1}{r} v_r + \left(V(r) - \frac{(N-2)^2}{4r^2} \right) v. \tag{3.27}$$

In the case $V = \lambda/r^2$ with $\lambda < \lambda_*$ we get the equation

$$v_t = v_{rr} + \frac{1}{r} v_r + \frac{(\lambda - \lambda_*)}{r^2}\, v, \tag{3.28}$$

and separation of variables leads to a radial mode

$$v(r,t) = e^{-\mu_{j,n}\, t} J_m(z_{m,n} r/a). \tag{3.29}$$

Going back to u we obtain the Fourier series. In particular, the separated-variable function with smallest time-decay is

$$U_1(r,t) = r^{-(N-2)/2}\, J_m(z_{m,1}\, r/a)\, e^{-\mu_1 t}, \tag{3.30}$$

which corresponds to $n = 1$ and $j = 0$, so that $m = m(0, \lambda)$ is given by $m^2 = \lambda_* - \lambda$, $m > 0$, $z_{m,1}$ the first zero of J_m, and $\mu_1 = \mu(0, 1) = z_{m,1}^2/a^2$. This quantity is larger than the $z_0^2/a^2 = z_{0,1}^2/a^2$ which appears in Theorem 2.1. We notice that for any $\lambda > 0$ this solution has a *standing singularity* at the origin of the form

$$U_1(r,t) \sim c(t)\, r^{m(0,\lambda)-(N-2)/2}.$$

The same singularity happens for the rest of the radial modes, hence for the general solution.

NON-RADIAL SOLUTIONS. Performing the change of variables $v = ur^{(N-2)/2}$ we obtain for a general solution

$$v_t = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} B(\sigma)\, v + \frac{(\lambda - \lambda_*)}{r^2}\, v, \tag{3.31}$$

where B is the Laplace-Beltrami operator in the sphere S^{N-1}. Take now as u the non-radial part $\tilde{u}$ and define accordingly $\tilde{v}(r, \sigma)$, which satisfies (3.31). The function can be expanded with respect to the zero-average eigenfunctions; it follows from that the higher decay rate of non-radial solutions. The corresponding separated-variable solutions have milder singularities than the radial ones. As we have said, all the singularities are compatible with the variational sense of the equation: since $m > 0$, we have

$$|\nabla u|^2, \frac{1}{r^2} u^2(\cdot, t) \in L^p(\Omega) \quad \text{for some } p > 1, \text{ uniformly in } t \geq 0.$$

Asymptotic behaviour. According to this analysis, the solution $u(x,t)$ with initial data u_0 can be approximated for large t by a multiple of the first separated-variable function $U_1(r,t)$.

Theorem 3.4 *As $t \to \infty$ we have*

$$\lim_{t\to\infty} e^{\mu_1 t} \|u(r) - a_1 U_1(r,t)\|_{L^2(\Omega)} \to 0, \tag{3.32}$$

and

$$a_1 = \int_\Omega u_0(x)\, U_1(r,0) dx / ||U_1(r,0)||_{L^2(\Omega)}. \tag{3.33}$$

This theorem shows that the solution $u = u(t)$ stabilizes as $t \to \infty$ towards the first mode of the radial component, unless this component is absent, because the other components have a faster exponential decay. We also recall that u, as well as U_1, belongs to $H_0^1(\Omega)$ for all $t > 0$. As we shall see, this is in contrast with the case $\lambda = \lambda_*$.

4 Analysis of the critical case

4.1 General analysis

We now proceed with the analysis of the most interesting case. In this subsection we assume that the potential satisfies

$$-C \leq V(x) \leq \frac{\lambda_*}{r^2}, \tag{4.1}$$

but $V(x) \leq \lambda |x|^{-2}$ is false for every $\lambda < \lambda_*$. The functional framework is now more delicate because the Hardy-Poincaré Inequality fails to provide the coercivity of the differential operator L in $H_0^1(\Omega)$. However, according to the Improved Hardy-Poincaré Inequality we know that

$$\int_\Omega \left\{ |\nabla u|^2 - \lambda_* \frac{u^2}{|x|^2} \right\} dx \geq C_q \|u\|^2_{W_0^{1,q}(\Omega)}$$

holds for every $1 \le q < 2$. This suggests that the evolution problem (3.1)-(3.3) should be well-posed in a suitable Hilbert space which is constructed as follows (cf. the theory of *Dirichlet forms*, [10]):

Definition. We denote by H the Hilbert space obtained as the completion of $\mathcal{D}(\Omega)$, or $H_0^1(\Omega)$, with respect to the norm

$$\|u\|_H = \left(\int_\Omega \left\{|\nabla u|^2 - V(r)\, u^2\right\} dx\right)^{1/2} \tag{4.2}$$

associated to the bilinear form

$$a(u,v) = \int_\Omega \{\nabla u \cdot \nabla v - V(r)\, uv\}\, dx.$$

This is the *energetic norm*, as described in [27, Chapter 5], or cf. [17]. By construction $L = -\Delta - V$ is the Riesz isomorphism from H into its dual H' associated with this bilinear form. We have the continuous imbeddings

$$H \hookrightarrow W_0^{1,q}(\Omega), \qquad H \hookrightarrow H_0^s(\Omega)$$

if $1 \le q < 2$ and $0 \le s < 1$. The second is also compact due to the fact that $W_0^{1,q}(\Omega)$ is compactly imbedded in $H_0^s(\Omega)$ for suitable $q = q(s)$ close enough to 2. Since $H_0^s(\Omega)$ is also compactly imbedded into $L^2(\Omega)$ we can then define a compact imbedding

$$H \to L^2(\Omega) \to H', \tag{4.3}$$

with $L^2(\Omega)$ as pivot space. We will see below that when $V = \lambda_*/r^2$ then H is larger than $H_0^1(\Omega)$, and smaller than $\cap_{q<2} W^{1,q}(\Omega)$.

We can now proceed with the study. The previous isomorphism can be viewed as a one-to-one map $H' \to H$, which implies that the corresponding operator L is an unbounded operator in H' with domain H, whose inverse is a compact and globally defined map from H' into itself. By restriction to $L^2(\Omega)$ we can define the surjective operator $L_* : D(L_*) \subset L^2(\Omega) \to L^2(\Omega)$ with domain

$$D_* = \{f \in H : -\Delta f - V(x)\, f \in L^2(\Omega)\}. \tag{4.4}$$

In the sequel we simply write L instead of L_* without fear of confusion. It is easy to see that L is self-adjoint with compact inverse. Therefore, it has an orthonormal basis of eigenfunctions in H, which we denote again by $\{e_k\}$, with eigenvalue sequence

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \ldots \to \infty. \tag{4.5}$$

Consequently, L generates an analytic semigroup of contractions on L^2. Then, conclusions similar to Theorems 3.1 and 3.2 hold in this context. In particular we have, cf. [17]

Theorem 4.1 *For any $u_0 \in L^2(\Omega)$ there exists a unique*

$$u \in C([0,\infty) : L^2(\Omega)) \cap L^2(0,\infty : H), \qquad u_t \in L^2(0,\infty : H'),$$

which is a weak solution of the evolution problem.

The solution can be developed with respect to the basis $\{e_k\}$ as before:

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-\mu_k t}\, e_k(x),$$

where the a_k's are the Fourier coefficients of the initial data. From the Fourier series we also have the decay expressions

$$\|u(t)\|_{L^2(\Omega)} \le e^{-\mu_1 t}\|u_0\|_{L^2(\Omega)} \tag{4.6}$$

$$\|u(t) - \sum_{k=1}^{K} a_k e^{-\mu_k t} e_k\|_{L^2(\Omega)} \le e^{-\mu_{K+1} t}\|u_0\|_{L^2(\Omega)}. \tag{4.7}$$

Indeed, we can argue directly: multiplying the equation by u and integrating in Ω we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \le -\int_\Omega |\nabla u|^2 dx + \int_\Omega V(x)\, u^2\, dx,$$

hence

$$\frac{1}{2}\frac{d}{dt}\int u^2\, dx \le -\mu_1 \int u^2\, dx, \tag{4.8}$$

where μ_1 is the first eigenvalue. And so on. Besides,

$$\langle L(e_k), e_j\rangle_{H'\times H} = \mu_k \int_\Omega e_k e_j\, dx = \mu_k \delta_{kj}, \tag{4.9}$$

i.e., they are orthogonal. Of course, the scalar product in H, which we denote by $(\cdot,\cdot)_H$, is related to this duality by

$$(e_k, e_j)_H = \langle L(e_k), e_j\rangle_{H'\times H}.$$

If $\|\cdot\|_H$ denotes the corresponding norm in H we have as well

$$\|u(t)\|_H \le e^{-\mu_1 t}\|u_0\|_H, \tag{4.10}$$

$$\|u(t) - \sum_{k=1}^{K} a_k e^{-\mu_k t} e_k\|_H \le e^{-\mu_{K+1} t}\|u_0\|_H. \tag{4.11}$$

We also have the standard regularizing effect from $L^2(\Omega)$ into H valid for semigroups generated by self-adjoint operators.

4.2 The critical case in a ball

Again the spectrum can be explicitly computed when $V(x) = \lambda_*/r^2$. Taking spherical coordinates and looking for eigenfunctions of $L(\lambda)$ of the form

$$e(r,\sigma) = \phi(r) f_j(\sigma),$$

we obtain equation (3.14) for ϕ with $\lambda = \lambda_*$. Putting also $\phi(r) = \psi(r)\, r^{-(N-2)/2}$ we get for ψ the equation

$$\psi'' + \frac{1}{r}\psi' + \left(\mu - \frac{c_j}{r^2}\right)\psi = 0, \tag{4.12}$$

where $c_j = j(j+N-2)$, $j \geq 0$, with boundary conditions $\psi'(0) = 0$, $\psi(a) = 0$. We get a complete family of solutions

Theorem 4.2 *There exists a two-parameter family of eigenfunctions*

$$e_{j,n}(r,\sigma) = r^{-(N-2)/2} J_m(\frac{z_{m,n}}{a}\, r) f_j(\sigma), \tag{4.13}$$

with m given by $m^2 = c_j$, $m \geq 0$. The corresponding eigenvalues are

$$\mu_{j,n} = \frac{z_{m,n}^2}{a^2}. \tag{4.14}$$

The family $\{e_{j,n}\}$ is a complete orthogonal basis of $L^2(B)$.

We recall that $z_{m,n}$ is the n-th zero of the Bessel function J_m. We also note that all the J_m vanish at $r=0$ but J_0 which has a finite positive value normalized to $J_0(0) = 1$.

Radial sub-basis. As pointed out in the subcritical case the eigenfunctions are smooth for $x \neq 0$ and may be singular at 0. The maximal singularity corresponds to the sub-family of eigenfunctions with $j = 0$ (hence, $m = 0$),

$$e_{0,n} = O(r^{-(N-2)/2}). \tag{4.15}$$

These functions represent the complete sub-basis for the subspace X_1 of radial functions in $L^2(B)$. They do not belong to the energy space $H_0^1(\Omega)$, but they are in H, as the theory predicts and is checked from the formulas, hence in $W^{1,q}(\Omega)$ for every $q < 2$. Indeed, the limit regularity of the basis functions is expressed by the fact that the gradients belong to the Marcinkiewicz space $M^2(\Omega)$. We recall that for $1 < p < \infty$ the Marcinkiewicz or weak-L^p space $M^p(\Omega) = L^{p,\infty}(\Omega)$ is defined as

$$M^p(\Omega) = \{f \text{ measurable in } \Omega,\ \text{meas}\{x : |f(x)| \geq \lambda\} \leq C\lambda^{-p}\}, \tag{4.16}$$

so that $|x|^{-\alpha}$ belongs to $M^p(B)$, $1 < p < \infty$, if and only if $p\,\alpha \leq N$. Cf. [5].

Zero-mean sub-basis. On the other hand, for $j \geq 1$ we have the sub-basis representing the space of functions with zero spherical mean, X_2. Then $m > 0$ and all of these functions belong to $H_0^1(\Omega)$. More precisely, since $m^2 = c_j \geq c_1 = N - 1$ we have

$$e_{j,n}(r, \sigma) = O(r^{m-(N-2)/2})$$

and this is never singular for any $m(j)$, $j > 0$ if $N \leq 6$. As the dimension $N \geq 7$ grows, more and more of these solutions become singular, but the singularity is always acceptable in the variational sense, i.e., the functions belong to $H^1(\Omega)$.

Evolution in a ball. The previous eigenfunctions produce evolution solutions of the form

$$U_{j,n}(x, t) = e_{j,n}(x)\, e^{-\mu_{j,n} t}, \tag{4.17}$$

which allow to write the Fourier form of a general solution. It is interesting as before to separate the evolution first into its radial and non-radial components.

Radial evolution. For radial functions we take $m = 0$ and we get maximum singularity. We also get minimum decay if $n = 1$

$$U_1(r, t) = r^{-(N-2)/2}\, J_0(z_{0,1}\, r/a)\, e^{-\mu_1 t}, \tag{4.18}$$

with decay exponent $\mu_1 = z_{0,1}^2/a^2$. We notice that for any $\lambda > 0$ this solution has a *standing singularity* at the origin of the form

$$U_1(r, t) \sim c(t)\, r^{-(N-2)/2}.$$

All the other radial separated-variables solutions develop the same type of singularity, but have larger decay. We can study the evolution of the whole radial component for critical potential by passing to the variable $v = r^{(N-2)/2}u$, which satisfies in general the equation

$$v_t = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} B(\sigma)\, v, \tag{4.19}$$

so that when $u = \overline{u}$ is radially symmetric then v evolves like a solution of the heat equation in two dimensions, $v_t = v_{rr} + (1/r)v_r$, and therefore, it becomes smooth for all $t > 0$. More precisely, we have as $t \to \infty$

$$\|v - C\, e^{-\mu_1 t} J_0(z_{0,1}\, r/a)\|_2 = O(e^{-\mu_2 t})$$

with $\mu_1 = z_{0,1}^2/a^2$, and norms in the 2-dimensional ball, so that the asymptotic result follows:

$$\lim_{t\to\infty} e^{\mu_1 t} \|u(r) - U_1(r, t)\|_{L^2(\Omega)} \to 0. \tag{4.20}$$

Remark. The rate $\mu_1 = z_{0,1}^2/a^2$ is precisely the quantity which appears in the HPI version of Theorem 2.1 (with the notation $z_0 = z_{0,1}$) which is proved in this way to be sharp, and is thus related to an important aspect of the theory, the critical time decay.

The asymptotic result of Theorem 3.4 holds, hence the solution becomes asymptotically radially symmetric in first approximation unless the radial component of the initial data is zero.

NON-RADIAL COMPONENT. There is better regularity for the non-radial component of the solution. We recall that in the proof of the Improved Hardy-Poincaré Inequality we have used the following partial result: for every $u \in H_0^1(\Omega)$ there exists $C(\Omega) > 0$ such that

$$\|u - \overline{u}\|_H^2 \geq C\|u - \overline{u}\|_{H_0^1(\Omega)}^2.$$

This allows to prove that the non-radial component of an evolution solution has better regularity, indeed the standard H^1 bound, hence

$$u(t) - \overline{u}(t) \in H_0^1(\Omega)$$

for every $t > 0$. In dimensions $N \leq 6$ this solution is even bounded for all $t > 0$. Let us take a more quantitative look at the separate evolution. With the notations of Section 3 the equation for

$$\tilde{v}(r, \sigma) = r^{(N-2)/2}(u(r, \sigma) - \overline{u}(r)) \tag{4.21}$$

is again (4.19). Moreover, for every $r > 0$ $\tilde{v}(r, \sigma)$ is a function of zero mean on S^{N-1}, hence

$$-\int_{S^{N-1}} \tilde{v}(r, \sigma)(B\tilde{v}(r, \sigma))\, d\sigma \geq (N-1) \int_{S^{N-1}} \tilde{v}(r, \sigma)^2\, d\sigma, \tag{4.22}$$

hence,

$$\frac{d}{dt}\int \tilde{v}^2 r dr \leq -c_1 \int \frac{\tilde{v}^2}{r^2}\, dx, \tag{4.23}$$

with integrals in the unit ball of $I\!R^2$. We conclude that $\tilde{v}$ decays exponentially fast in $L^2(I\!R^N)$. Hence, the influence of the non-radial part is negligible at the asymptotic level.

LIMIT. The solutions of the stationary and evolution problem in the critical case can be obtained as limit when $\lambda \to \lambda_*^+$ of the solutions for subcritical $\lambda < \lambda_*$. This is a particular case of very general results about convergence of semigroups of contractions.

5 The space H

We want to better understand the range H of the Dirichlet form for the elliptic problem in the critical case $V = \lambda_*/r^2$. The previous examples of solutions in separated variables show functions with a singularity of the form $f \sim |x|^{-(N-2)/2}$. This shows that H is larger than $H_0^1(\Omega)$. On the other hand, in view of the IHPI H must be included in $\cap_{q<2} W^{1,q}(\Omega)$. The same examples suggest that the gradient of the solutions lies in a Marcinkiewicz space, $|\nabla u| \in M^2(\Omega)$ (see definition (4.16)) but it is not true that H coincides with the space

$$\mathcal{V} = \{f \in L^2(\Omega) : |\nabla f| \in M^2(\Omega),\ f = 0 \text{ on } \partial\Omega\}. \tag{5.1}$$

In order to see this we examine a representative example, namely, the functions defined for $0 < r < r_0 < 1$ as

$$u(r) = r^{-(N-2)/2}(\log(1/r))^{\alpha}, \tag{5.2}$$

and continued smoothly up to the boundary of the ball $B_1(0)$, where $u = 0$. It is easy to see that u belongs to H if and only if $\alpha < 1/2$. This is easily checked by observing that for the dense set of functions on which we define the norm (4.2) and under the assumption of radial symmetry we have

$$\int_{B_1(0)} (|\nabla u|^2 - \frac{\lambda_*}{r^2}\, u^2)\, dx = C \int_0^1 (v')^2\, r dr$$

with $v = u\, r^{(N-2)/2}$. For $\alpha > 0$ the gradient of the solution (5.2) is not in M^2. The result shows that H is larger than $H_0^1(B)$ but it is smaller than $\cap_{q<2} W^{1,q}(B)$. Moreover, H is not contained in $\mathcal{V}$. The same happens for any bounded domain Ω containing the origin, since the problem under discussion depends only on the special integrability difficulties at the origin.

6 Optimal regularity and optimal initial data

We pass now to the question of optimal regularity of the solutions and at the same time the optimality of the class of functions to be taken as data. We recall that when V is a subcritical potential, like $V(x) = \lambda/r^2$ with $\lambda < \lambda_*$, we have constructed a unique solution of the evolution problem for all data $u_0 \in L^2(\Omega)$ in the classical energy spaces, $u \in C([0,\infty) : L^2(\Omega)) \cap L^2(0,\infty : H_0^1(\Omega))$. In particular, the maps

$$S_t : u_0 \mapsto u(\cdot, t) \tag{6.1}$$

form a semigroup of contractions in $L^2(\Omega)$. On the contrary, when V is the critical potential, $V(x) = \lambda_*/r^2$, we get out of the classical energy spaces and then the space H replaces $H_0^1(\Omega)$. We still get a semigroup of contractions in $L^2(\Omega)$.

6.I. As it is usual in evolution problems, the solutions belong for $t > 0$ to a better space than the natural space of the Dirichlet form. Thus, when $V = 0$ the solutions of the heat equation are C^∞-smooth, while the energy space is $H_0^1(\Omega)$. Contrary to that classical situation, in the critical case dealt with in this paper the corresponding spaces are not so different, indeed they are quite close when $\lambda = \lambda_*$. In general we have

Theorem 6.1 (Optimal Regularity) *Given $u_0 \in L^2(\Omega)$ the solution of the evolution problem with $0 < \lambda \le \lambda_*$ admits the estimates*

$$|u(x,t)| \le \frac{C}{|x|^{\alpha_1}}\, e^{-\mu_{0,1} t}, \qquad |\nabla u| \le \frac{C}{|x|^{\alpha_1+1}}\, e^{-\mu_{0,1} t}, \tag{6.2}$$

where $\alpha_1 = (N-2)/2 - m$, $m = \sqrt{\lambda_* - \lambda}$. and $C = C(u_0)$. In other terms, we may say that $u(t)$ is in $M^{p_*}(\Omega)$ and has a spatial gradient in $M^{q_*}(\Omega)$ for every $t > 0$, where the limiting exponents are given by

$$p_*(\lambda) = \frac{N}{\alpha_1}, \quad q_*(\lambda) = \frac{N}{\alpha_1 + 1} \tag{6.3}$$

These spaces are optimal for initial data with non-zero radial part, independently of the smoothness of u_0.

Proof. We start with radial functions in a ball $\Omega = B_a(0)$ and deal for simplicity with the case $\lambda = \lambda_*$. We write the Fourier decomposition

$$u(x,t) = \sum_n a_n C_n e^{-\mu_{0,n} t} e_{0,n}(x), \tag{6.4}$$

with basis functions $e_{j,n}$ defined in (4.13) and $C_n = 1/\|e_{0,n}\|_{L^2}$, so that $\sum a_n^2 < \infty$. Therefore,

$$\nabla u(x,t) = -\frac{N-2}{2|x|^{N/2}} \sum_n a_n C_n e^{-\mu_{0,n} t} J_0(\frac{z_{0,n}}{a} r) + \frac{1}{a|x|^{(N-2)/2}} \sum_n z_{0,n} a_n C_n e^{-\mu_{0,n} t} J_0'(\frac{z_{0,n}}{a} r).$$

Since the Bessel function and its derivative are bounded, $\mu_{0,n}$ is given by formula (4.14) and $z_{0,n}$ and C_n grow at a polynomial rate n, we conclude that

$$|x|^{N/2} |\nabla u(x,t)| \le C \sum_n a_n n^q e^{-\mu_{0,n} t} \tag{6.5}$$

which converges for $t > 0$, hence $|\nabla u| \in M^2(B)$. The nonradial part is smoother and belongs to $H_0^1(B)$. In this way the estimate for the gradient is obtained. The proof for the function $u(t)$ is similar. Optimality comes from inspection of the separate-variable solutions with $j = 0$, $n = 1$.

The proof for $\lambda < \lambda_*$ is the same using the results of Section 3 instead of Section 4.

In the case of a general domain, we argue much as in the Hardy analysis of Section 2. If Ω contains a ball $B = B_a(0)$ then the solution is C^∞ smooth outside of B. In order to study the situation around the origin we define $\tilde{u} = u\,\phi$ with a cutoff function ϕ as in Section 2. It satisfies in B an evolution equation of the form

$$\tilde{u}_t = \Delta \tilde{u} + V(x)\tilde{u} + f$$

with zero boundary data and f smooth and identically zero near $x = 0$. The variation of constants formula allows to complete the proof with the same type of estimates done above. ∎

Remarks. **(i)** The lower bound of the form $u \ge C(t)|x|^{-\alpha_1}$ is already established in [1] for nonnegative solutions. Estimate (6.2) shows that this is the type of bound that holds from above, and it also gives the value of $C(t)$. The explicit separate-variable solutions show that such rates are not reached for solutions which change sign.

(ii) It is interesting to note that the Marcinkiewicz spaces M^2 and $M^{2N/(N-2)}$ of the critical case (hence, the worst case) are the natural spaces in the L^1 theory for semilinear elliptic equations, like the equation $-\Delta u + \beta(u) = f$ with β a monotone function and $f \in L^1(I\!R^N)$ studied in [5].

The Theorem implies the following instantaneous blowup result

Corollary 6.2 *Let $u_0 \ge 0$, $u_0 \in L^2(\Omega)$, u_0 nontrivial. Then $u(\cdot,t)$ does not belong to $L^p_{loc}(\Omega)$ with $p \ge p_*$ for any $t > 0$, even if u_0 belongs to that space.*

6.II. We discuss next the extension of the solutions to a class of data as large as possible. We recall the results of [1] which show the existence of solutions $u \ge 0$ for problem (1.3)-(1.4) with $0 < \lambda \le \lambda_*$ under the assumption

$$\int_\Omega u_0(x)\,|x|^{-\alpha_1}\,dx < \infty, \tag{6.6}$$

where the exponent α_1 given in (6.2) is also proved to be optimal. Indeed, our analysis in Theorem 6.1 produces the estimates (6.2) under the condition that the Fourier coefficients

$$a_{j,n} = \int u_0(x)\, e_{j,n}(r,\sigma)\,dx$$

are well defined and form a bounded sequence. Now, $|x|^{-\alpha_1}$ is the worst regularity at the origin in the family $\{e_{j,n}\}$. A careful analysis of the proof of Theorem 6.1 allows to show that

Theorem 6.3 *The semigroup S_t generated in $L^2(\Omega)$ by Problem (1.3)-(1.5) with $0 < \lambda \le \lambda_*$ can be extended into a continuous semigroup in*

$$L_{\alpha_1}(\Omega) = \{f \text{ is measure in } \Omega, \quad \int_\Omega |x|^{-\alpha}|f|\,dx < \infty\}, \tag{6.7}$$

The extension is such that for every $t > 0$ S_t applies $L^1_{\alpha_1}(\Omega)$ into $L^2(\Omega)$, so that for $t \ge \tau > 0$ it falls into the L^2-semigroup studied in previous sections.

In particular, the extended semigroup enjoys the regularity properties (6.2). On the other hand, the set of initial data can be extended to the measure space

$$\mathcal{M}_{\alpha_1}(\Omega) = \{f \text{ is a locally finite measure in } \Omega, \quad \int_\Omega |x|^{-\alpha} d|f| < \infty\}. \tag{6.8}$$

and the solutions belong to $L^2(\Omega)$ for all $t > 0$. Since we find interesting to develop in detail the functional properties of the semigroup we will devote the final Appendix to treat that question aat length with a different technique, related to the contractive character of the semigroup in certain L^p-weighted spaces.

7 Uniqueness and non-uniqueness

We have constructed a unique solution of the evolution problem for λ critical or subcritical. There are a number of ways of characterizing the unique good solutions of these evolution problems, so-called *semigroup* or *mild solutions*, obtained from the positive operator resulting from the elliptic analysis, cf. [17, 19]. In all cases $\lambda > 0$ the singularity of the potential has a reflection in the singularity of the solutions, even with good initial data, and this entails a number of pathologies with respect to the standard properties of the classical heat equation, i.e., the case $\lambda = 0$. An important aspect that we want to underline here refers to the uniqueness of solutions, which is much more restrictive than in the heat equation case. We recall that for the heat equation, and for a number of related parabolic equations with good coefficients, distributional solutions are uniquely defined by the initial data under the sole restriction $u \in L^\infty(0, T : L^p\Omega))$ for some $p \geq 1$ (no matter which p). No such result is true for singular potentials of the form $V(x) = \lambda/r^2$ for any $\lambda > 0$, $\lambda \leq \lambda_*$.

SUBCRITICAL CASE. Let us consider first potentials of the form $V(x) = \lambda/r^2$ with $0 < \lambda < \lambda_*$ and discuss for the sake of simplicity the existence of stationary and radially symmetric solutions in a ball $\Omega = B_R$.

Theorem 7.1 *When $0 < \lambda < \lambda_*$ there exists a radially symmetric function $u(x)$ which solves the equation $\Delta u + V(x)u = 0$ in distribution sense in $B = B_R$, is smooth away from the origin, vanishes on the boundary and does not belong to $H^1_0(B)$. Moreover, $u \in L^p(B)$ for every $p < p(\lambda)$, where*

$$p(\lambda) = \frac{N}{\dfrac{N-2}{2} + m}, \qquad m = \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}. \tag{7.1}$$

Moreover, $|\nabla u| \in L^q(B)$ for $q < q(\lambda) = N/((N/2) + m)$.

Proof. Looking for radially symmetric and stationary solutions $u(r)$ of equation (1.1) and performing the change of variables $v(r) = u(r)\, r^{(N-2)/2}$ we arrive at the equation

$$v_{rr} + \frac{1}{r}\, v_r + \frac{\lambda_* - \lambda}{r^2}\, v = 0, \tag{7.2}$$

which admits a regular solution of the form $v_1(r) = C\, r^m$ with $m^2 = \lambda_* - \lambda$, $m > 0$ and a singular solution of the form $v_2(r) = C\, r^{-m}$. Undoing the change of variables we find u_1 and u_2,

$$u_1(r) = r^{-(N-2)/2+m}, \quad u_2(r) = C\, r^{-(N-2)/2-m}. \tag{7.3}$$

While u_1 belongs to $H_1(B)$, u_2 does not for any $\lambda > 0$. By combining them we may obtain a solution $u(r)$ which satisfies the boundary condition $u(R) = 0$ and inherits the regularity of u_2. Now, it is quite easy to check that u_2 is a distributional solution in the whole ball if $m < (N-2)/2$, which happens for $\lambda > 0$. The integrability conditions are obvious. ∎

We remark that $L^{p(\lambda)}(\Omega) \subset L^1_{\alpha_1}(\Omega)$ for all $\lambda \in (0, \lambda_*)$ (cf. the inclusion relations in the Appendix). In view of the semigroup construction of the previous section, since the present solution does not decay in time *it cannot be the solution that we obtain from the semigroup construction,* it is a *bad solution* in that sense. Observe that the bad solution is increasingly regular as λ increases, i.e., the non-uniqueness result becomes stronger. In the limit when $\lambda \to \lambda_*$ we have $q(\lambda) \to 2$, $p(\lambda) \to 2N/(N-2)$ (i.e., it approaches the variational regularity), while in the other limit $\lambda \to 0$ we have $q(\lambda) \to N/(N-1)$ and $p(\lambda) \to N/(N-2)$ (this is the standard regularity for elliptic equations with good coefficients and L^1 data). Let us also remark that in the limit $\lambda = 0$ the singular solution u_2 becomes the fundamental solution of the Laplace operator which fails to be a distributional solution of the equation $\Delta u = 0$ at the origin (because of the presence of a Dirac delta).

Concentrating our attention on L^2 data, we observe that $p(\lambda) > 2$ for $\lambda > \lambda_* - 1$. Our example implies the non-uniqueness of distributional solutions with initial data in $L^2(B)$ in the range $\lambda_* - 1 < \lambda < \lambda_*$. It leaves open the problem of same uniqueness question for $0 < \lambda \le \lambda_* - 1$, which is non-empty for $N > 4$. This is in any case a question related to a particular interest in $L^2(\Omega)$ that the Theorem 6.3 of the previous section must have dispelled.

Critical case. In the case $\lambda = \lambda_*$ a similar construction can be done. The corresponding singular function is not exactly the limit of the subcritical case but

$$u(x) = r^{-(N-2)/2}\log(r), \quad r = |x|. \tag{7.4}$$

This is a stationary solution of the problem in the unit ball which belongs to $L^2(\Omega)$, the equation is satisfied in the sense of distributions, even at the origin, but u is not the good solution because it does decay in time. Hence, it cannot be in H (note that only a $\log(r)$ factor separates it from H).

Serrin's example. These results have a very strong connection with the famous example of non-uniqueness described in [23] for the elliptic equation

$$\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = 0 \tag{7.5}$$

posed in a ball, with bounded coefficients: $a_{ij} = \delta_{ij} + (a-1)x_i x_j\, r^{-2}$. Here the existence of nontrivial solutions with zero boundary condition depends on the parameter $a > 1$. See also [21] for an improved version. Conditions of uniqueness based on the integrability of the solutions have been recently studied in [4]. Inspired in their analysis we conjecture that a suitable integrability condition on u (of the type $u \in L^\infty(0, T : L^p(B))$ with $p \ge p(\lambda)$) would imply uniqueness of the distributional solutions of the Cauchy problem.

Let us comment that, the non-uniqueness phenomenon being similar, our equation looks simpler than (7.5).

FINAL COMMENTS. It is to be noted that we could also address the failure of the H^1-setting by means of the concepts of *entropy solution*, [3], or *renormalized solution*. Both equivalent concepts, much used recently in the investigation of nonlinear evolution equations of the so-called p-Laplacian type, provide alternative functional settings where the solution can be uniquely characterized. These subjects deserve future attention. Let us also mention that the semigroup with critical potential can be obtained in the limit of the H_0^1-semigroups constructed in the subcritical case, $\lambda < \lambda_*$, hence we may say that it is a *limit solution*. Cf. [1] for other limit constructions, which also apply in this case.

8 Oscillating solutions for larger λ

There are explicit oscillating solutions for $\lambda > \lambda_*$, even for all λ there will be some solution in increasingly smaller spaces. In fact, the change of variables used above can be applied to the set of functions $X = \{\phi \in H_0^1(B_a(0)) : \overline{\phi} = 0\}$ to obtain an Improved Hardy Inequality

$$\int |\nabla u|^2 dx \geq \lambda_\# \int \frac{u^2}{r^2} dx, \qquad \phi \in X \tag{8.1}$$

with new optimal constant

$$\lambda_\# = \lambda_* + c_1 = \frac{N^2}{4}. \tag{8.2}$$

Let us recall that for radial functions the constant λ_* is un-improvable as a consequence of the blowup result of [1]. Writing this inequality for a general function of $H_0^1(B)$ gives

$$\int |\nabla u|^2 dx \geq \lambda_* \int \frac{\overline{u}^2}{r^2}\, dx + \lambda_\# \int \frac{\tilde{u}^2}{r^2} dx, \qquad \phi \in X, \tag{8.3}$$

where we have split $u(x) = \overline{u}(r) + \tilde{u}(r, \sigma)$. The basis of non-radial solutions of the separated-variables form $u = \phi(r) f_j(\sigma)$ used above can still be constructed for $\lambda > \lambda_*$ as long as

$$\lambda_* + c_j \geq \lambda. \tag{8.4}$$

9 The problem in $I\!R^N$. Fundamentals

We consider from now on the evolution problem (1.3)-(1.4) posed in the whole space $x \in I\!R^N$. For $\lambda \leq \lambda_*$ we can think of constructing a solution as limit of the solutions of the Cauchy-Dirichlet problem in balls $B_R(0)$ when $R \to \infty$. The classical Hardy inequality implies that the L^2-norm is nonincreasing in time for all bounded domains, so that the property holds in the limit and allows to construct a certain solution of the Cauchy problem in the whole space. But the proper characterization of the solution needs a more detailed study, specially in the case of interest $\lambda = \lambda_*$. Unfortunately, the Improved Hardy Inequality does not have a counterpart in the limit as the radius $R \to \infty$, so that we are not able to perform this task and also we are not able to calculate from it the decay rate of the solutions.

In this section we perform the detailed analysis using as leading idea the relevance of self-similar solutions which leads to the use of the weighted spaces naturally associated with the upcoming elliptic operator.

9.1. Similarity variables and functional preliminaries. We repeat the method used in the case of a ball, after replacing separate-variable solutions by self-similar solutions, as is natural to whole-space problems. We introduce similarity variables by means of the change of variables

$$w(y,s) = t^{N/4}u(t^{1/2}y,t), \qquad s = \log(t+1), \tag{9.1}$$

in other words $w(y,s) = e^{sN/4}u(e^{s/2}y, e^s - 1)$, and w satisfies

$$w_s = \Delta w + \frac{1}{2}y\cdot\nabla w + \frac{N}{4}w + \frac{\lambda}{r^2}w. \tag{9.2}$$

When the evolution is posed in that context y plays the role of space variable and s is the new time. We have the relation of norms

$$\int u^2(x,t)\,dx = \int w^2(y,s)\,dy, \tag{9.3}$$

which means that we can analyze the asymptotic behaviour of $u(x,t)$ by studying $w(y,s)$ for large s. Unless mention to the contrary, integrals are extended to $I\!R^N$ in the whole Section. The natural space to study the evolution of equation (9.2) is the weighted space $L^2(K)$ with weight $K = \exp\,(|y|^2/4)$

$$L^2(K) = \{f \in L^2(I\!R^N) : \int |f|^2\,K dy < \infty\}, \tag{9.4}$$

studied by Escobedo and Kavian in [12]. Indeed, multiplication of equation (9.2) by $w\,K$ and integration by parts gives

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 = \frac{N}{4}\|w\|^2 + \lambda\|\frac{w}{r}\|^2, \tag{9.5}$$

with norms $\|\cdot\|$ in $L^2(K)$. We have to analyze the coercivity of the quadratic functional

$$J(w) = \int |\nabla w|^2\,K(y)dy - \lambda_*\int \frac{w^2}{|y|^2}\,K dy, \tag{9.6}$$

whose Euler-Lagrange equation is $Lu = 0$ with operator

$$L = -\Delta - \frac{y}{2}\nabla - \frac{\lambda_*}{|y|^2} = -\frac{1}{K}\texttt{div}\ (K\nabla) - \frac{\lambda_*}{|y|^2}.$$

We will also use the weighted Sobolev space

$$H^1(K) = \{f \in L^2(K) : \ |\nabla f| \in L^2(K)\}, \tag{9.7}$$

endowed with the canonical norm

$$\|f\|_{H^1(K)} = \left[\int (f^2 + |\nabla f|^2)\, K dy\right]^{1/2}. \tag{9.8}$$

We need the following result from [12]: the imbedding $H^1(K) \to L^2(K)$ is compact, and besides

$$\int |y|^2 f^2\, K dy \leq 16 \int |\nabla f|^2\, K dy.$$

Moreover, the operator $L_0 = -\Delta - (y/2)\cdot\nabla$ is an isomorphism from $H^1(K)$ into its dual. Its restriction to $L^2(K)$ defines an unbounded self-adjoint operator in $L^2(K)$ with domain $H^2(K)$. This operator has a compact inverse. Its eigenvalues can be explicitly computed as

$$\mu_j = \frac{j + N - 1}{2}, \ j \geq 1, \tag{9.9}$$

and the corresponding null-space is given by

$$\text{Ker}\,(-\Delta - \frac{y}{2}\cdot\nabla - \mu_j I) = \text{Span}\{D^\alpha \phi_1 : \ |\alpha| = j - 1\},$$

where $\phi_1 = 1/K$ is the eigenfunction associated with the first eigenvalue $\mu_1 = N/2$. In particular it follows that

$$\int |\nabla f|^2\, K dy \geq \frac{N}{2}\int f^2\, K dy. \tag{9.10}$$

for every $f \in H^1(K)$. We write this orthonormal basis as $\{e_{j,l}\}$, where $j \geq 1$ and $l = 1, \cdots, l(j)$, with $l(j)$ the multiplicity of μ_j.

9.2. Hardy-Poincaré Inequality in the weighted space. The quadratic functional J introduced in (9.6) is well-defined in $H^1(K)$ when $N \geq 3$. We recall that when $N = 2$ we have $\lambda_* = 0$, therefore J is the quadratic form associated to L_0 and the evolution equation under consideration reduces to the constant-coefficient heat equation. The analysis of the evolution with critical λ will essentially use the following weighted version of the Hardy-Poincaré Inequality valid in $\mathbb{R}^N$:

Theorem 9.1 *For every $f \in H^1(K)$ we have*

$$J(f) = \int |\nabla f|^2\, K dy - \frac{(N-2)^2}{4}\int \frac{f^2}{|y|^2}\, K dy \geq \frac{N+2}{4}\int f^2\, K dy. \tag{9.11}$$

Both constants are optimal.

Proof of the Theorem. (i) Notice first that the inequality makes sense since

$$\lambda_* \int \frac{f^2}{|y|^2} K dy \le \int |\nabla f|^2 K dy,$$

i.e., the classical Hardy inequality holds in the weighted spaces. Indeed, for smooth functions with compact support,

$$f(y) = -\int_1^\infty \frac{d}{dt}(f(ty))\, dt = -y \cdot \nabla \int_1^\infty f(ty)\, dt,$$

so that

$$\|\frac{f(y)}{|y|}\|_{L^2(K)} \le \int_1^\infty \|\nabla f(ty)\|_{L^2(K)}\, dt \le$$

$$\|\nabla f\|_{L^2(K)} \int_1^\infty t^{-N/2}\, dt = \frac{2}{N-2}\|\nabla f\|_{L^2(K)}.$$

(ii) We proceed as in [8]. First of all, we use symmetrization to reduce ourselves to radial functions. We then perform the change of variables

$$g(y) = |y|^{(N-2)/2} f(y). \tag{9.12}$$

After some calculations, cf. Theorem 2.2, we get

$$J(f) = N\omega_N\{\int_0^\infty |g'|^2 r e^{r^2/4} dr - (N-2)\int_0^\infty g g' e^{r^2/4} dr\}. \tag{9.13}$$

Integrating by parts the last term we get

$$\int_0^\infty g g' e^{r^2/4} dr = \frac{1}{2}\int_0^\infty (g^2)_r e^{r^2/4} dr = -\frac{1}{4}\int_0^\infty g^2 e^{r^2/4} dr.$$

In order to justify this integration we reduce ourselves to consider smooth functions with compact support, which is allowed by density. We thus arrive at

$$J(f) = N\omega_N\{\int_0^\infty |g'|^2 r e^{r^2/4} dr + \frac{N-2}{4}\int g^2 r e^{r^2/4} dr\}. \tag{9.14}$$

This is a functional in two dimensions which up to a constant factor is the radial version of the functional

$$H(g) = \int_{I\!R^2} |\nabla g|^2 K dy + \frac{N-2}{4}\int_{I\!R^2} g^2 K dy,$$

In view of (9.10) for $N=2$ we have

$$H(g) \ge \frac{N+2}{4}\int_{I\!R^2} g^2\, K dy.$$

Coming back to $J(f)$ we get the desired result. The optimal constants are achieved for the function

$$f(y) = |y|^{-(N-2)/2}\mathtt{exp}(-|y|^2/4),$$

obtained from the first eigenfunction of the operator L in $L^2(K)$. □

FUNCTIONAL SPACE. In view of this estimate it is natural to introduce the Hilbert space H, completion of $H^1(K)$ with respect to the norm $\|f\|_H = (J(f))^{1/2}$. We have the continuous imbeddings

$$H^1(K) \to H \to L^2(K). \tag{9.15}$$

As in the case of the bounded domain we have

Proposition 9.2 *The imbedding $H \to L^2(K)$ is compact.*

Proof. The proof relies on the separate control of the radial and nonradial parts. The latter is the more regular and we have the following result

Lemma 9.3 *There exists a positive constant $C > 0$ such that for every $f \in H^1(K)$*

$$\|f - \overline{f}\|_H \geq C\|f - \overline{f}\|_{H^1(K)}, \tag{9.16}$$

where $\overline{f}$ denotes the radially symmetric component of f obtained by spherical averaging.

Assuming this result and taking into account the compact imbedding $H^1(K) \to L^2(K)$ it is sufficient to analyze the subspace of radially symmetric functions to conclude the proof. We argue by contradiction. Assuming that the imbedding is not compact, then there exists a sequence f_j, $j \geq 1$, of radial functions in H such that

(i) $f_j \to 0$ weakly in H, but

(ii) $\|f_j\|_{L^2(K)} = 1$.

We set $g_j(r) = r^{(N-2)/2} f_j(r)$. Then

$$\int_0^\infty |g_j'|^2 r e^{r^2/4} dr + \frac{N-2}{4} \int_0^\infty |g_j|^2 r e^{r^2/4} dr \leq C \tag{9.17}$$

and

$$0 < C_1 \leq \int_0^\infty |g_j'|^2 r e^{r^2/4} dr \leq C_2.$$

In view of (9.17) we deduce that the sequence of functions $g_j : \mathbb{R}^2 \to \mathbb{R}$ are bounded in $H^1(K)$, the two-dimensional version of this space. Therefore, g_j is relatively compact in $L^2(K)$ in 2D. There exists a radially symmetric function $g \in H^1(K)$ such that

$$\int_0^\infty |g_j - g|^2 r e^{r^2/4} dr \to 0$$

as $j \to \infty$. This shows that the corresponding function $f = r^{-(N-2)/2} g$ is such that

$$f_j \to f \quad \text{weakly in } L^2(K).$$

Moreover, g_j converges weakly in $H^1(K)$ to g (in 2D), hence

$$f_j \to f \quad \text{weakly in} \quad H^1(K).$$

By the assumption (i) we must have $f = 0$, hence $g = 0$, but this contradicts assumption (ii). This concludes the proof for radial functions. $\square$

Proof of the Lemma. We develop an arbitrary function $h \in H^1(K)$ in spherical harmonics with the notation of Sections 3, 4

$$h = \sum_{j \geq 0} h_j(r) f_j(\sigma).$$

We have $\overline{h} = h_0(r)$. Therefore,

$$h - \overline{h} = \sum_{j \geq 1} h_j(r) f_j(\sigma).$$

Moreover,

$$J(h - \overline{h}) = N\omega_N \sum_{j \geq 1} \int_0^\infty \left[|h_j'|^2 - \lambda_* \frac{h_j^2}{r^2} + \mu_j \frac{h_j^2}{r^2} \right] r^{N-1} e^{r^2/4} dr.$$

We have $\mu_j \geq \mu_1 = N$ if $j \geq 1$. Thus, according to Theorem 9.1 there exists a constant $C > 0$ such that

$$J(h - \overline{h}) \geq C \sum_{j \geq 1} \int_0^\infty \left[|h_j'|^2 + \mu_j \frac{h_j^2}{r^2} \right] r^{N-1} e^{r^2/4} dr \sim \|h - \overline{h}\|^2_{H^1(K)}.$$

9.3. Spectral decomposition. The elliptic operator

$$L_* = -\Delta - \frac{y}{2} \cdot \nabla - \lambda_* \frac{I}{|y|^2} \tag{9.18}$$

will play a key role in the analysis of the Cauchy problem below. Note that

$$L_* f = -\frac{1}{K} \texttt{div}\ (K \nabla f) - \lambda_* \frac{f}{|y|^2}.$$

Therefore, we formally have

$$(L_* f, f)_{L^2(K)} = \int |\nabla f|^2 K dy - \lambda_* \int \frac{f^2}{|y|^2} K dy.$$

It follows that L_* is the Riesz isomorphism from H onto its dual H'. By restriction we define the unbounded operator $L_\#$ in $L^2(K)$ with domain

$$D(L_\#) = \{ f \in H : \ L_* f \in L^2(K) \}. \tag{9.19}$$

It follows from Proposition 9.2 that $L_\#$ is self-adjoint with compact inverse. Therefore, it admits a basis of eigenfunctions $\{e_j\}$ with eigenvalues

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \ldots \to \infty, \tag{9.20}$$

Moreover, Theorem 9.1 says that

$$\mu_1 \geq \frac{N+2}{4}. \tag{9.21}$$

The e_j's are an orthonormal basis in $L^2(K)$. We also have

$$\|e_j\|_H = \sqrt{\mu_j}, \quad (e_k, e_j)_H = 0 \ \text{ if } \ i \neq j.$$

COMPUTATION OF THE SPECTRUM. Let $N \geq 3$ and $\lambda \leq \lambda_*$. We consider the eigenvalue problem

$$-\Delta e - \frac{y}{2} \cdot \nabla e - \frac{\lambda}{|y|^2} e = \mu\, e \quad \text{in } \mathbb{R}^N, \qquad e \in H^1(K).$$

We write $e(r, \sigma) = \phi(r) f_j(\sigma)$ where f_j is the j-th eigenvalue of the Laplace-Beltrami operator with eigenvalue c_j. The equation for ϕ is then

$$\phi_{rr} + (\frac{N-1}{r} + \frac{r}{2})\phi_r + (\frac{\lambda - c_j}{r^2} + \mu)\phi = 0 \tag{9.22}$$

under the condition that

$$\int_0^\infty (|\phi|^2 + |\phi_r|^2) r^{N-1} e^{r^2/4} dr < \infty. \tag{9.23}$$

The change of variables $\phi(r) = r^{-(N-2)/2}\psi(r)$ gives

$$\psi'' + (\frac{1}{r} + \frac{r}{2})\psi' + \left(\mu - \frac{N-2}{4} - \frac{\lambda_* + c_j - \lambda}{r^2}\right)\psi = 0. \tag{9.24}$$

The indicial equation of the Fröbenius series for (9.24) is as in the bounded case

$$m^2 = \lambda_* - \lambda + c_j. \tag{9.25}$$

This implies a behaviour at the origin of the regular solutions of the form $\psi(r) \sim r^m$, with m the nonnegative root of (9.25), which agrees with the bounded case as it should, since the effect of the singular potential on the singularity of the eigenfunctions at zero is a local effect. The first eigenfunction is precisely

$$\psi_1(r) = r^m e^{-r^2/4}, \qquad \phi_1(r) = r^{m-(N-2)/2} e^{-r^2/4}, \tag{9.26}$$

with eigenvalue

$$\mu_1(\lambda) = \frac{N+2+2m}{4}, \quad m = \sqrt{\lambda_* - \lambda}. \tag{9.27}$$

For $\lambda = 0$ we have $m = (N-2)/2$ and we recover $\mu_1(0) = N/2$, while in the limit case $\lambda = \lambda_*$ we have $m = 0$ and thus $\mu_1(\lambda_*) = (N+2)/4$. This latter result confirms that the HPI in the whole space, formula (9.11), has sharp constants.

10 The Cauchy problem in $\mathbb{R}^N$. Well-posedness and asymptotics

By introducing the similarity variables (9.1) we arrive at the equivalent evolution equation (9.2) for $w(y,s)$:

$$w_s = \Delta w + \frac{1}{2} y \cdot \nabla w + \frac{N}{4} w + \frac{\lambda}{r^2} w,$$

with initial data $w(y,0) = u(x,0)$.

10.1 Review of the heat equation. Before considering the Cauchy problem for $\lambda > 0$ let us briefly review the situation for $\lambda = 0$, the classical heat equation. Then the equation is

$$w_s = \Delta w + \frac{1}{2} y \cdot \nabla w + \frac{N}{4} w,$$

with initial data $w(y,0) = u(y,0)$ in $\mathbb{R}^N$. As a consequence of the results of [12] the following holds: For any $u_0 \in L^2(K)$ this initial-value problem admits a unique solution $w \in C([0,\infty) : L^2(K)) \cap L^2(0,\infty : H^1(K))$. Moreover,

$$w(y,s) = \sum_{j \geq 1} e^{-(\mu_j - N/4)s} \left[\sum_{l=1}^{l(j)} a_{j,l} e_{j,l} \right].$$

Taking into account that for $j \geq 1$ $\mu_j \geq \mu_1 \geq N/2$ we have

$$\|w(s)\|_{L^2(K)} \leq e^{-Ns/4} \|u_0\|_{L^2(K)} \tag{10.1}$$

Returning to the original variables it follows that

$$\int u^2(z, e^s)\, dz \leq \int u^2(z, e^s - 1)\, dz \leq \int u^2(z, e^s - 1) \texttt{exp}\ (\frac{z^2}{4e^s})\, dz =$$

$$= e^{Ns/2} \int u^2(e^{s/2} y, e^s - 1) \exp\ (\frac{|y|^2}{4})\, dy = \|w(s)\|_{L^2(K)} \leq e^{-Ns/4} \|u_0\|_{L^2(K)}.$$

In particular,

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \leq t^{-N/4} \|u_0\|_{L^2(K)}. \tag{10.2}$$

Therefore, the use of the similarity variables allows us to establish the well-known decay rate in the L^2 norm of order $t^{-N/4}$ for the solution of the classical heat equation. Note however that (10.1) is known to hold for solutions with initial data in $L^2(\mathbb{R}^N)$ and we can replace the $L^2(K)$ by the $L^2(\mathbb{R}^N)$ norm in the right-hand term of the estimate. We thus obtain the classical result in a weaker form.

10.2 The subcritical case. We now study the Cauchy problem in $\mathbb{R}^N$, $N \geq 3$, when $0 < \lambda < \lambda_*$. According to Theorem 9.1 the operator

$$A(\lambda) = -\Delta - \frac{y}{2} \cdot \nabla - \frac{\lambda}{|y|^2} I$$

is an isomorphism from $H^1(K)$ into $H^{-1}(K)$. When restricted to $L^2(K)$ it becomes a self-adjoint operator with compact inverse. Thus, $L^2(K)$ admits an orthonormal basis of eigenfunctions of $A(\lambda)$ with eigenvalues $\mu_j(\lambda)$, $j \geq 1$,

$$-\Delta e_j - \frac{y}{2} \cdot \nabla e_j - \frac{\lambda}{|y|^2} e_j = \mu_j(\lambda) e_j$$

in $I\!R^N$. We also have

$$\int |\nabla e_j|^2 \, K dy - \lambda \int \frac{|e_j|^2}{|y|^2} K dy = \mu_j(\lambda)$$

and

$$\int \nabla e_j \cdot \nabla e_k K dy - \lambda \int \frac{e_j e_k}{|y|^2} K dy = 0$$

if $j \neq k$. In particular, $e_j \in H^1(K)$ for all $j \geq 1$. This will be the main difference with the case $\lambda = \lambda_*$ where, as indicated before, the eigenfunctions lie in a larger space H. We also have for all $f \in H^1(K)$ the inequality

(10.3) $$\int |\nabla f|^2 K dy - \lambda \int \frac{|f|^2}{|y|^2} K dy \geq \mu_1(\lambda) \int f^2 \, K dy,$$

with $\mu_1(\lambda)$ given by formula (9.27). The following result holds

Theorem 10.1 *Assume that $N \geq 3$ and that $\lambda < \lambda_*$. Then, for any $u_0 \in L^2(K)$ the Cauchy problem admits a unique solution $u \in C([0, \infty) : L^2(K)) \cap L^2(0, \infty : H^1(K))$.*

Let us state some properties of the solution. If the initial data is developed as

(10.4) $$u_0 = \sum_{j \geq 1} a_j e_j,$$

then,

(10.5) $$w = \sum_{j \geq 1} a_j e^{-\nu_j s} e_j, \qquad \nu_j = \mu_j - \frac{N}{4} = \frac{1}{2}(1 + m(\lambda, j)).$$

We also have for all $s > 0$ and $u_0 \in L^2(K)$

(10.6) $$\|w(s)\|_{L^2(K)} \leq e^{-\nu_1(\lambda)s} \|u_0\|_{L^2(K)}, \quad \nu_1(\lambda) = \frac{1}{2} + \frac{1}{2}\sqrt{\lambda_* - \lambda}.$$

All these results are a direct consequence of the preceding considerations. The decay rate (10.6) can be easily obtained by classical energy estimates. Indeed, multiplying the w-equation by wK and integrating by parts

$$\frac{d}{ds} \int w^2 \, K dy + \int |\nabla w|^2 \, K dy - \frac{N}{4} \int w^2 \, K dy - \lambda \int \frac{w^2}{|y|^2} K dy = 0.$$

According to (10.3) we deduce that

$$\frac{d}{ds}\int w^2\,K dy + \Big(\mu_1(\lambda) - \frac{N}{4}\Big)\int w^2\,K dy \le 0,$$

which implies (10.6). Note that $\nu_1(\lambda) = N/4$ for $\lambda = 0$, and we recover the decay rate of the heat equation. On the other hand, for $\lambda = \lambda_*$ we get $\nu_* = 1/2$, which, as expected, is a slower decay rate (since $N \ge 3$).

Using formula (9.3) and the fact that $K \ge 1$, we have in the original variables

Corollary 10.2 *Assume that $N \ge 3$ and that $0 < \lambda < \lambda_*$. Then, for every $t > 0$ and all $u_0 \in L^2(K)$*

(10.7) $$\|u(t)\|_{L^2(I\!R^N)} \le t^{-\nu_1(\lambda)}\|u_0\|_{L^2(K)}$$

This estimate is sharp since we have an explicit solution with smallest decay corresponding to the first eigenvalue of operator $L_\#$ calculated above, which has precisely this rate.

10.3 The critical case. According to this analysis and the developments of Subsection 9.3 in similarity variables, in particular to (10.6) when $\lambda = \lambda_*$, the estimate is

(10.8) $$\|w(s)\|_{L^2(K)} \le e^{-s/2}\|u_0\|_{L^2(K)}$$

for every $u_0 \in L^2(K)$. The separated-variables function with smallest decay corresponds to the first eigenvalue of operator $L_\#$ and reads

(10.9) $$w(y,s) = |y|^{-(N-2)/2}e^{-|y|^2/4}e^{-s/2},$$

i.e., in the original variables

(10.10) $$U_1(x,t) = \frac{1}{|x|^{(N-2)/2}t}\,\mathtt{exp}\,(-\frac{x^2}{4t}).$$

Theorem 10.3 *Assume that $N \ge 3$ and $\lambda = \lambda_*$. Then, for any $u_0 \in L^2(K)$ the Cauchy problem admits a unique solution $u \in C([0,\infty) : L^2(K)) \cap L^2(0,\infty : H)$. Moreover, for every $u_0 \in L^2(K)$ and every $t > 0$*

(10.11) $$\|u(t)\|_{L^2(I\!R^N)} \le t^{-1/2}\|u_0\|_{L^2(K)}$$

and

(10.12) $$\lim_{t\to\infty} t^{1/2}\|u(x,t) - a_1 U_1(r,t)\|_{L^2(I\!R^N)} \to 0,$$

where

(10.13) $$a_1 = \int_{I\!R}^N u_0(x)\,U_1(r,0)\,Kdx/||U_1(r,0)||_{L^2(K)}.$$

Note that obtaining this result requires the use of similarity variables and the functional framework of subsection 9.3. Otherwise, the classical Hardy inequality does not provide any information about the coercivity of the operator $-\Delta - \lambda_* |x|^{-2} I$, hence no time decay like (10.11).

Remark The exact result that we get for u in the Cauchy Problem is

$$\int u^2(x,t)\, e^{\frac{x^2}{4(t+1)}}\, dx \le \frac{1}{1+t} \int u_0^2(x)\, e^{\frac{x^2}{4t}}\, dx. \tag{10.14}$$

10.4 Initial behaviour, non-uniqueness and dependence on λ. Observe that, as expected, $U_1(t)$ is not in $H^1_{loc}(I\!R^N)$. Moreover, this is a nontrivial solution with interesting initial behaviour, since

$$U_1(x,t) \to 0 \qquad \text{as } t \to 0 \text{ for every } x \ne 0,$$

while it diverges at $x = 0$ for all $t > 0$ (isolated standing singularity). Besides, $U_1(t) \in L^p(I\!R^N)$ for every $p < 2N/(N-2)$ with

$$\|U_1(t)\|_p = C\, t^{-\alpha} \qquad \text{with } \alpha = \frac{N+2}{4} - \frac{N}{2p},$$

so that $\alpha = 1/2$ for $p = 2$ and $\alpha \to 1$ as $p \to 2N/(N-2)$. This means that U_1 is a semigroup solution for all $t \ge \tau > 0$ which takes on trivial initial data not only in distribution sense but also in $L^p(B)$ for all $p < 2N/(N+2)$ and every ball B that contains the origin. In particular, for $p = 1$ we have $\alpha = (2-N)/4 < 0$, which corresponds exactly to a decay $\|w(s)\|_1 \sim e^{-s/2}$, as expected. On the other hand, the L^p-norm is constant for $p = 2N/(N+2)$ and increases as $t \to 0$ for $p > N/(N+2)$, e.g., for $p = 2$, when $\alpha = 1/2$.

By studying the behaviour for $0 < \lambda < \lambda_*$ we find a way of connecting this solution to the fundamental solution of the heat equation, and 'explain' how the non-standard singularity at $(x,t) = (0,0)$ arises. Indeed, when studying the asymptotic behaviour for this range of parameters the first solution is also explicit

$$U_1(x,t;\lambda) = |x|^{m - \frac{N-2}{2}} t^{-(1+m)} \exp\left(-\frac{x^2}{4t}\right) \tag{10.15}$$

with $m = (\lambda_* - \lambda)^{1/2}$, cf. (9.25). For $\lambda > 0$ this function has a singularity at $x = 0$ for all $t > 0$ and $u(x,0) = 0$ for $x \ne 0$. Contrary to the critical case, $U_1(t) \in H^1(I\!R^N)$ for $\lambda < \lambda_*$ and $t > 0$. A result like Theorem 10.3 holds and we see that the decay exponent $-1/2$ in (10.11) is the limit as $\lambda \to \lambda_*$ of the decay exponent of the subcritical case. Again, the nontrivial solution $U_1(x,t;\lambda)$ takes on trivial initial data in distribution sense and in $L^1(I\!R^N)$, even in $L^p(K)$ for some $p > 1$ and close to 1. Finally, in the limit $\lambda \to 0$ we obtain the fundamental solution of the heat equation and the initial data are no more trivial, but a Dirac mass instead.

11 Appendix. Contractivity properties and the extended semigroup

We devote some space to the question, already addressed in Section 6, about the optimal domain of definition of the semigroup generated by equation (1.3). For definiteness we work in a bounded domain Ω. We prove three kinds of results: extension, contractivity and regularizing effect. The two former are closely connected. The latter uses the techniques developed in the process.

We begin by proving that the L^2-semigroup can be extended into a uniquely defined semigroup in certain spaces $L^p_\alpha(\Omega)$ in which it enjoys the contraction property. These spaces are defined as

$$L^p_\alpha(\Omega) = \{f \text{ measurable in } \Omega, \quad \int_\Omega |x|^{-\alpha}|f|^p\,dx < \infty\}, \tag{11.1}$$

for $1 \le p < \infty$, $\alpha \in I\!R$. Then, $L^p_0(\Omega)$ is the usual Lebesgue space $L^p(\Omega)$. Clearly, $\alpha < \alpha'$ implies that $L^p_{\alpha'}(\Omega) \subset L^p_\alpha(\Omega)$ for all p and $L^p_\beta(\Omega) \subset L^1_\alpha(\Omega)$ if $\beta + n(p-1) > \alpha p$. The following result gives the exact range of the contraction property.

Proposition 11.1 *The semigroup S_t generated by Problem (1.3)-(1.5) with $0 < \lambda \le \lambda_*$ can be extended into a continuous semigroup in $L^1_{\alpha_1}(\Omega)$. The semigroup is contractive in $L^p_\alpha(\Omega)$ with finite $p > 1$ and $\alpha \in I\!R$ if*

$$(\alpha - \frac{1}{2}(N-2)(2-p))^2 \le p^2(\lambda_* - \lambda). \tag{11.2}$$

In particular, for $\lambda = \lambda_*$ this condition is satisfied exactly by

$$\alpha_*(p) = (1/2)(N-2)(2-p), \tag{11.3}$$

while for $0 < \lambda < \lambda_*$ it is satisfied in an α interval $[\alpha_1(\lambda,p), \alpha_2(\lambda,p)]$ of length $2p\sqrt{\lambda_* - \lambda}$ around that value. The maximal interval happens for $\lambda = 0$ and it grows with p.

Proof. (1) Let us establish first the contraction property in the simpler case $p = 1$. A calculation already done in [1] shows that the semigroup is contractive in the L^1_α-norm as long as

$$c(\alpha) = \alpha(N-2-\alpha) \ge \lambda, \tag{11.4}$$

hence $\alpha_1(\lambda) \le \alpha \le \alpha_2(\lambda)$. The important exponent $\alpha_1 = \alpha_1(1,\lambda)$ appears in [1] through the weight function $\phi_\alpha(x) = |x|^{-\alpha}$, which satisfies

$$-\Delta\phi_\alpha = \frac{c(\alpha)}{|x|^2}\,\phi_\alpha, \qquad c(\alpha) = \alpha(N-2-\alpha), \tag{11.5}$$

so that $0 \le c(\alpha) \le \lambda_*$ for $0 \le \alpha \le N-2$. Then, $\alpha_1 = \alpha_1(\lambda)$ is the smaller solution of the equation $c(\alpha) = \lambda$. For $p = 1$ we have $\alpha_1(\lambda) = \alpha_2(\lambda) = (N-2)/2$ for $c = \lambda_*$ and $\lambda < c(\alpha) \le \lambda_*$ for $\alpha_1(\lambda) < \alpha < \alpha_2(\lambda)$ if $0 < \lambda < \lambda_*$. Indeed, we have for solutions $u \ge 0$

$$\frac{d}{dt}\int_\Omega u\phi_\alpha\, dx \le -\int_\Omega u\,\Delta\phi_\alpha\, dx + \int_\Omega \lambda\frac{u}{|x|^2}\phi_\alpha\, dx \le 0, \tag{11.6}$$

since $\Delta\phi_\alpha = -c(\alpha)\phi_\alpha/|x|^2$. This formal calculation is true for the nonnegative L^2 solutions. Solutions with changing sign are split into the positive and negative parts, $u_0 = u_0^+ - u_0^-$, so that $|u_0| = u_0^+ + u_0^-$, and use the previous fact in the form

$$(S_t(u_0))^\pm \le S_t(u_0^\pm).$$

By standard extension we construct a contraction semigroup in the spaces $L^1_\alpha(\Omega))$ with the above α, hence a solution $u \in C([0,\infty) : L^1_\alpha(\Omega))$ for data in $L^1_\alpha(\Omega)$.

(2) For the full calculation we take ϕ_α as above and $1 < p < \infty$ we proceed as usual in order to find the acceptable range of parameters α and p: we multiply the equation formally by $|u|^{p-2}u\,\phi_\alpha$ and integrate by parts. Without loss of generality we may also assume $u \ge 0$ and then we obtain the relation

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,\phi_\alpha\, dx &= -\int_\Omega \nabla(u^{p-1}\phi_\alpha)\cdot\nabla u + \int_\Omega \lambda\frac{u^p}{|x|^2}\,\phi_\alpha\, dx\\ &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla(u^{p/2})|^2\phi_\alpha dx - \int_\Omega u^{p-1}(\nabla u\cdot\nabla\phi_\alpha)\, dx + \int_\Omega \lambda\frac{u^p}{|x|^2}\,\phi_\alpha\, dx.\end{aligned} \tag{11.7}$$

It is then easily checked that there exists a contraction semigroup in $L^p_\alpha(\Omega)$ iff the second member is nonpositive (the elliptic operator is then called L^p_α-accretive). To proceed further, we write $v = u^{p/2}$, $\phi_\alpha = \psi^2$ and use

$$\int |\nabla(v\psi)|^2\, dx = \int |\nabla v|^2\psi^2\, dx + \int v^2|\nabla\psi|^2\, dx + 2\int v\psi(\nabla v\cdot\nabla\psi)\, dx,$$

so that

$$-\int |\nabla(u^{p/2})|^2\phi_\alpha dx = -\int |\nabla(u^{p/2}\psi)|^2 dx + \frac{1}{2}\int \nabla(u^p)\cdot\nabla\phi_\alpha\, dx + \int u^p\,|\nabla\psi|^2\, dx.$$

After using the HI on the first term on the right and using the fact that $|\nabla\psi| = (|\alpha|/2)\psi/|x|$ we get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,\phi_\alpha\, dx \le \{\lambda - \frac{c(\alpha)}{p} - \frac{(p-1)}{p^2}((N-2)^2 - 2\,c(\alpha) - \alpha^2\}\int_\Omega \frac{u^p}{|x|^2}\,\phi_\alpha\, dx. \tag{11.8}$$

This means that contraction holds in $L^p_\alpha(\Omega)$ if

$$\lambda + \frac{p-1}{p^2}(2\,c(\alpha) + \alpha^2) \le \frac{c(\alpha)}{p} + \frac{4(p-1)\lambda_*}{p^2}, \tag{11.9}$$

which is equivalent to condition (11.2).

(3) Once the contraction property is established, the extension of the semigroup is immediate.

REMARKS. The following comments tend to shed some light on the contraction range given by formula (11.2). Thus, the critical value λ_* appears as the maximal value of λ for which (11.2) can be solved; then necessarily $\alpha = \alpha_*$. For $p = 2$ this value is 0 while for $p = 1$ we obtain $\alpha_* = (N-2)/2$. Given λ and p, the right-hand end of the admissible α-interval is located at

$$\alpha_2(\lambda, p) = \frac{1}{2}(N-2)(2-p) + p\sqrt{\lambda_* - \lambda} \le N-2,$$

which is the value reached for $\lambda = 0$ (independent of p). For $p = 2$ the interval becomes

$$\alpha^2 \le 4(\lambda_* - \lambda).$$

When $\alpha = 0$ (i.e., the L^p theory without weights) we get the conclusion that Problem (1.3)-(1.4) generates a semigroup of contractions in $L^p(\Omega)$, $1 < p < \infty$, if and only if

$$\frac{\lambda}{(N-2)^2} \le \frac{1}{p} - \frac{1}{p^2}. \tag{11.10}$$

The maximum value of λ happens for $p = 2$ and is precisely $\lambda = \lambda_*$. Hence, on one hand the $L^2(\Omega)$-theory covers the widest range of λ, and on the other hand the $L^2(\Omega)$ setting is the only acceptable L^p-setting for $\lambda = \lambda_*$. When $1 < \lambda < \lambda_*$ we have $\lambda \to 0$ as p tends to either 1 or infinity, and the range of acceptable p, according to (11.2), is the closed interval limited by the values $p_1(\lambda)$ and $p_2(\lambda)$ given by

$$\frac{1}{p} = \frac{1}{2} \pm \frac{\sqrt{\lambda_* - \lambda}}{N-2},$$

hence $p_1 < 2 < p_2$ for $\lambda < \lambda_*$. Actually, the calculation is much simplified in this case and we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p dx = -\frac{4(p-1)}{p^2}\int_\Omega |\nabla(|u|^{p/2})|^2 dx + \int_\Omega \lambda \frac{|u|^p}{|x|^2}\, dx. \tag{11.11}$$

Finally, let us note that in the use the Hardy inequality we have neglected to recall the remainder calculated in the IHPI, which implies that there is a further positive term in the left-hand side of (11.8) of the form

$$C\|u^{p/2}\phi^{1/2}\|^2_{W^{1,s}(\Omega)}, \quad s < 2$$

that also becomes controlled. We will use this control later. □

Our last purpose is to prove that the extended semigroup consists for $t > 0$ of the same class of solutions constructed in Sections 3 and 4, whose regularity was explained in Section 6.

Proposition 11.2 *The semigroup S_t can be extended into a continuous semigroup in $L^1_{\alpha_1}(\Omega)$ such that for every $t>0$ S_t applies $L^1_{\alpha_1}(\Omega)$ into $L^2(\Omega)$.*

Proof. We divide the proof into several steps, consisting in two kinds of regularity improvements: better regularity with respect to the weight and better L^p regularity. Without loss of generality we work with nonnegative u.

(I) Take first $0<\lambda<\lambda_*$ and $\alpha_1(\lambda)<\alpha<\alpha_2(\lambda)$ and let us improve the *integrability with respect to a weight.* We prove that whenever $u_0\in L^1_\alpha(\Omega)$ then $u(\tau)\in L^1_{\alpha'}(\Omega)$ for a.e. $\tau>0$ with $\alpha'\le\min\{\alpha+2,\alpha_2\}$. By iteration and using the contraction property we can then conclude that

$$u(t)\in L^1_\alpha(\Omega)\quad \forall t>0, \alpha\le\alpha_2.$$

This is done by repeating the proof of Step 1 of previous Proposition and observing by our choice of α we have $c(\alpha)>\lambda$, so that

$$\frac{d}{dt}\int_\Omega u\phi_\alpha\,dx+(c(\alpha)-\lambda)\int_\Omega\frac{u}{|x|^2}\phi_\alpha\,dx\le 0. \tag{11.12}$$

It follows that

$$\int_\Omega u(\tau)\phi_\alpha\,dx+(c-\lambda)\int_0^\tau\int\frac{u}{|x|^2}\phi_\alpha\,dxdt\le\int_\Omega u_0\phi_\alpha\,dx \tag{11.13}$$

which means that the integrals in the first member are finite. The contraction property allows then to derive the estimate for $u(t)\phi_\alpha|x|^{-2}$ in $L^1(\Omega)$ uniformly in time $t\ge\tau/2$ as long as $\alpha+2\le\alpha_2$. We thus get $u(\tau)\in L^1_\alpha(\Omega)$ for all $\alpha\le\alpha_2$. We note in passing that it holds for $\alpha=(N-2)/2$, which is the exponent corresponding to $\lambda=\lambda_*$ and will be used below.

(II) *Case* $\alpha=\alpha_1$, $\lambda<\lambda_*$. This case is a bit more complicated. We can deal as follows: we multiply the equation by $h(u)\phi$, where h is the typical smooth approximation of the Heaviside function with $0\le h\le 1$, $h'\ge 0$, $h(0)=0$, and ϕ is now a perturbation of ϕ_{α_1} of the form

$$\phi=\phi_{\alpha_1}-b\phi_\alpha,\qquad \alpha=\alpha_1-\varepsilon,$$

with $b>0$ small so that $\phi>0$ in Ω. Calling $j(r)$ the primitive of h with $j(0)=0$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega j(u)\,\phi\,dx&=\int_\Omega h(u)\Delta u\phi\,dx+\lambda\int_\Omega\frac{h(u)u}{|x|^2}\,\phi\,dx\\ &=-\int_\Omega h'(u)|\nabla u|^2\phi\,dx+\int j(u)\Delta\phi\,dx+\lambda\int_\Omega\frac{uh(u)}{|x|^2}\,\phi\,dx.\end{aligned}$$

We now take $h(u)\sim 1-u^{-\delta}$ for all $u\gg 1$ so that $j(u)\sim u-u^{1-\delta}/(1-\delta)$ and $uh(u)-j(u)=\delta u^{1-\delta}/(1-\delta)$, and use the fact that

$$\Delta\phi=-c(\alpha_1)|x|^{-(2+\alpha_1)}+bc(\alpha)|x|^{-(2+\alpha)}$$

to get $(*)$

$$
\begin{aligned}
&c\int_\Omega u(t)\phi\,dx + \frac{4\delta}{(1-\delta)^2}\iint |\nabla(u^{(1-\delta)/2})|^2\,\phi\,dxdt + (\lambda - c(\alpha))b\iint \frac{u}{|x|^{2+\alpha}}\,dxdt\\
&\le \int_\Omega u_0\phi\,dx + \frac{\delta\lambda}{1-\delta}\iint \frac{u^{1-\delta}}{|x|^{2+\alpha_1}}\,dxdt + C,
\end{aligned}
$$

where $\lambda - c(\alpha) > 0$ is small of order ε. Using the calculus inequality $ab \le (1/p)a^p + (1/q)b^q$, valid for $a, b > 0$ and $1 < p < \infty$, $q = p/(p-1)$ we get (with $p = 1/(1-\delta)$, $q = 1/\delta$, $a = u^{1-\delta}|x|^{-\alpha(1-\delta)}$, $b = |x|^{\alpha(1-\delta)-\alpha_1}$)

$$
\frac{u^{1-\delta}}{|x|^{2+\alpha_1}} \le \varepsilon\frac{u}{|x|^{2+\alpha}} + C(\varepsilon)\frac{1}{|x|^{2+\gamma}}
$$

where $\gamma = (\alpha_1 - \alpha)\delta + \alpha$. The last term is integrable if $2 + \gamma < N$. Since $\alpha < \alpha_1 \le (N-2)/2$ it holds if

$$
(\alpha_1 - \alpha)\delta \le (N-2)/2.
$$

This is a smallness condition to be imposed on $\varepsilon\,\delta$. Substituting all these estimates into $(*)$ we get bounds for the quantities

$$
\iint \frac{u}{|x|^{2+\alpha}}\,dxdt, \qquad \iint |\nabla(u^{(1-\delta)/2})|^2\,|x|^{-\alpha_1}\,dxdt. \tag{11.14}
$$

The first one places us in the conditions of Step 1.

(III) Under the same conditions on u_0 for $\lambda < \lambda_*$, we obtain *better L^p regularity*. Indeed, by Step 2 we know that for a.e. every time $\int_\Omega |\nabla(u(t)^{(1-\delta)/2})|^2\phi < \infty$, so that (cf. [18], page 97) $u(t)^{(1-\delta)/2}\phi^{1/2} \in L^{2N/(N-2)}(\Omega)$, hence $u(t) \in L^p_\beta(\Omega)$ with $p = (1-\delta)N/(N-2)$ and $\beta = \alpha_1 N/(N-2) > 0$. Observe that for $N = 3$ we have already arrived at the L^2 regularity. For $N \ge 4$ we still have to continue further. Starting from $u(t_0) \in L^p_\alpha(\Omega)$ with α strictly inside interval determined by condition (11.2), we multiply the equation by $u^{p-1}\phi_\alpha$ and argue as in Step 2 of Proposition 11.1 to obtain not only the contraction property, but also a small gap in the algebraic inequality, that allows for the boundedness of the integral

$$
\int\int |\nabla(u^{p/2}\phi^{1/2})|^2\,dx
$$

which as before allows to obtain a regularity $L^q_\beta(\Omega)$ with $q = pN/(N-2)$ and suitable β. In this way we arrive at $q = 2$ by iteration if necessary.

(IV) *Case* $\lambda = \lambda_*$, $\alpha = \alpha_1(\lambda_*) = (N-2)/2$. When we do the previous Step 2 the improvement of integrability with respect to α is of no use since the interval $[\alpha_1, \alpha_2]$ is just a point. The gradient estimate in (11.14) implies that

$$
\iint u^p\,|x|^{-\beta}\,dxdt < \infty,
$$

for $p < N/(N-2)$ and $\beta = \alpha(N-2)/2 = N/2$, as calculated as above. Again, for $N = 3 >$ the proof is complete, since $L^1_\beta \subset L^1$. particular we can choose them inside the range of (11.2)., Otherwise, the calculation of Step 3 can now be applied to improve the p. Since now the algebraic condition in the contraction calculation of Proposition 11.1 is exact we have to use the remainder of the IHPI. As we have remarked there is a further positive term under control of the form

$$\|u^{p/2}\phi_\alpha^{1/2}\|_{W^{1,s}(\Omega)}, \quad s < 2.$$

By Sobolev's embedding this means that the integrability exponent is improved in p with the correct weight $\alpha = \alpha_*(p)$. By iteration we get to $p = 2$. ∎

Remark. The extension of the previous bounds to cover measures as initial data, $u_0 \in \mathcal{M}_{\alpha_1}$, is immediate in view of the estimates.

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