NUMERICAL HYPOCOERCIVITY FOR THE KOLMOGOROV EQUATION

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Abstract. We prove that a finite-difference centered approximation for the Kolmogorov equation in the whole space preserves the decay properties of continuous solutions as $t \to \infty$, independently of the mesh-size parameters. This is a manifestation of the property of numerical hypo-coercivity and it holds both for semi-discrete and fully discrete approximations. The method of proof is based on the energy methods developed by Herau and Villani, employing well-balanced Lyapunov functionals mixing different energies, suitably weighted and equilibrated by multiplicative powers in time. The decreasing character of this Lyapunov functional leads to the optimal decay of the L^2-norms of solutions and partial derivatives, which are of different order because of the anisotropy of the model.

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1. Introduction and main results

The celebrated Kolmogorov equation ([11])

$$
\begin{cases} \partial_t f - \partial_{xx} f - x\partial_y f = 0, & (x,y) \in \mathbb{R}^2\,, t > 0 \\ f(x,y,0) = f_0(x,y), & (x,y) \in \mathbb{R}^2 \end{cases} \tag{1.1}
$$

is one of the most paradigmatic examples of degenerate advection-diffusion equations which have the property of hypo-ellipticity, ensuring the C^∞ regularity of solutions for $t > 0$ ([7]). In the present case, the generator of the semigroup is constituted by the superposition of operators ∂_{xx} and $x\partial_y$. Despite the presence of a first order term, that could lead to transport phenomena and, consequently, to the lack of smoothing, the regularizing effect is ensured by the fact that the commutator of these two operators is non-trivial, allowing to gain regularity in the variable y. A full characterization of hypo-ellipticity can be found in [7]).

Solutions of (1.1) experience also decay properties as $t \to \infty$. This is also a manifestation of hypo-coercivity (in the sense developed by Villani [15], [16]) as a byproduct of the hidden interaction of the two operators entering in the generator of the semigroup.

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In this particular case, using the Fourier transform, the fundamental solution of (1.1) (starting from an initial Dirac mass $\delta_{(x_0,y_0)}$) can be computed explicitly getting the following anisotropic Gaussian kernel

$$K_{(x_0,y_0)}(x,y,t) = \tag{1.2}$$

$$\frac{1}{3\pi^2 t^2} \exp\left[-\frac{1}{\pi^2}\left(\frac{3|y-(y_0+tx_0)|^2}{t^3} + \frac{3(y-(y_0+tx_0))(x-x_0)}{t^2} + \frac{|x-x_0|^2}{t}\right)\right]$$

which exhibits different diffusivity and decay scales in the variables x and y.

In view of the structure of the fundamental solution, one can deduce the following decay rates:

$$\|f(t)\|_{L^2} + \sqrt{t}\,\|\partial_x f(t)\|_{L^2} + t^{\frac{3}{2}}\|\partial_y f(t)\|_{L^2} \le C||f_0||_{L^2} \tag{1.3}$$

for solutions with initial data f_0 in L^2. Similar decay properties can be predicted by scaling arguments, due to the invariance properties of the equation in (1.1).

These decay properties are of anisotropic nature and of a different rate in the x and y-directions. Indeed, in the x-direction, as in the classical heat equation, we observe a decay rate of the order of $t^{-1/2}$, while, in the y-variable, the decay is of order $t^{-3/2}$.

The obtention of these decay properties by energy methods has been a challenging topic of particular interest when dealing with more general convection-diffusion models that do not allow the explicit computation of the kernel. In this effort, the asymptotic behavior of Kolmogorov equation and several other relevant kinetic models was investigated intensively through the concept and techniques of hypo-coercivity, which allow to make explicit the hidden diffusivity and dissipativity of the involved operators (see [15], [16] and the previous references therein).

The literature on the asymptotic behaviour of models related with Kolmogorov equation is huge. We refer for instance to [9], [10], [3] for earlier works, and to [5], [6] for more recent approaches. Roughly speaking, it is by now well known that, constructing well-adapted Lyapunov functionals through variations of the natural energy of the system, one can make the dissipativity properties of the semigroup emerge and then obtain the sharp decay rates. These techniques have been developed also in other contexts such as partially dissipative hyperbolic systems (see [1]).

The goal of this paper is to introduce a numerical scheme that preserves this hypo-coercivity property at the numerical level, uniformly on the mesh-size parameters. The issue is relevant from a computational point of view since, as it has been observed in a number of contexts (wave propagation, dispersivity of Schrödinger equations, conservation laws, etc. [17], [8]), the convergence property in the classical sense of numerical analysis (a property that concerns finite-time horizons) is not sufficient to ensure the asymptotic behavior of the PDE solutions to be captured correctly. The fact that the numerical approximation schemes preserve the decay properties of continuous solutions can be considered as a manifestation of the property of *numerical hypo-coercivity.*

In this paper this issue is analyzed in the context of finite-difference schemes in uniform grids exploiting, at the discrete level, the approach developed by

Herau [5] (see also Villani [15, Appendix A.21]). As we shall see this approach is very efficient on obtaining the sharp decay rates for the numerical solutions, provided the numerical schemes are properly chosen.

We refer to [4] for a different attempt in this direction, using similarity variables, and to [12] where time-splitting techniques, distinguishing diffusive and convective terms, are shown to preserve the decay properties.

Similarly as in [5], [6], the following abstract setting is the natural one for addressing the long time asymptotic behaviour. This framework is extensively developed in [15], [16]. We briefly summarize below a byproduct of this approach which will serve to our purposes.[1]

Given two operators A, B, we will denote by $[A, B]$ the standard commutator, i.e. $[A, B] = AB - BA$.

Proposition 1.1. *([15]) Let X be a Hilbert space and $A, B : X \to X$ be linear operators. Assume that B is anti-symmetric, i.e.*

$$B^* = -B$$

and let $f \in L^2(0, T; X)$ be a solution to

$$\partial_t f + A^*Af + Bf = 0\,. \tag{1.4}$$

Assume that

$$A, A^*, B \quad \text{commute with} \quad [A, B]\,, \tag{1.5}$$

and that $[A, A^*]$ is bounded relatively to I and A, which means that there exists $\beta > 0$ such that

$$\|[A, A^*]x\|_X \le \beta\,(\|x\|_X + \|Ax\|_X) \qquad \forall x \in X\,. \tag{1.6}$$

Then we have, for $t_0 > 0$,

$$\|f(t)\|_X^2 + t\,\|Af(t)\|_X^2 + t^3\|[A, B]f(t)\|_X^2 \le C\|f(0)\|_X^2 \qquad \forall t < t_0\,, \tag{1.7}$$

for some $C > 0$ only depending on $\beta\,t_0$.

In particular, if $[A, A^*] = 0$, the constant C does not depend on t_0 and estimate (1.7) holds true uniformly for all times $t \in (0, \infty)$.

Note that, in the case of the continuous Kolmogorov equation, we have $Af = \partial_x f$, $Bf = x\partial_y f$ and $[A, B]f = \partial_y f$. Therefore, setting $X = L^2$, Proposition 1.1 provides an estimate of the H^1-norm of f; namely, we deduce (1.3). Moreover, since A and A^* commute, this estimate holds for every time $t > 0$.

Note also that, as mentioned above, these decay estimates are sharp according to the explicit form of the fundamental solution.

For the reader's convenience, we provide a proof of Proposition 1.1, on the lines of the arguments given by [15, Appendix A.21], at the end of the paper.

[1] We refer to [15] for more details on the domains of the unbounded operators involved in this abstract result (roughly, the domains must share some common topological vector space where computations are allowed) as well as on the meaning of the abstract evolution equation. Here we put the emphasis on the functional structure of the model under consideration, that will be the key to obtain uniform (with respect to the mesh-size parameter) results when dealing with numerical approximations. In that case the computations involved in the proof can be justified easily manner working in the discrete setting of functions defined on $[h\mathbb{Z}]^2$.

Actually, as we shall see, the same techniques allow to go further in the above argument recovering estimates for higher order terms. In practice, when dealing with the Kolmogorov model, this leads to $L^2 - H^2$ estimates that, by Sobolev embedding, yield $L^2 - L^\infty$ estimates. Duality and scaling arguments allow then to prove the $L^1 - L^\infty$ decay estimate of the order of t^{-2} of the fundamental solution, which is sharp.

More precisely, the following holds:

Proposition 1.2. *In addition to the assumptions of Proposition 1.1, assume also that $[A^*A, B]$ is bounded relatively to $A[A, B]$, namely that for some $\beta > 0$:*

$$\|[A^*A, B]x\|_X \le \beta \|A[A, B]\|_X \qquad \forall x \in X\,. \tag{1.8}$$

Then, for some $C > 0$ we also have

$$t^2\|A^*Af(t)\|_X^2 + t^4\|A[A, B]f(t)\|_X^2 + t^6\|[A, B]^2f(t)\|_X^2 \le C\|f(0)\|_X^2 \quad \forall t < t_0\,, \tag{1.9}$$

and for every $t > 0$ if A, A^* commute.

The proof of this Proposition will be given in Section 3 at the end of the paper. Notice that condition (1.8) is always satisfied whenever $A^* = -A$ since we already assumed that A commutes [2] with $[A, B]$. In fact, some variants (possibly weaker conditions) than (1.9) could be alternatively assumed, but this is enough for our purposes here.

We now describe the main results of this paper obtained in the context of finite difference schemes. We shall adapt the previous abstract results to the discrete setting to obtain uniform (with respect to the mesh-size parameter) decay rates. Note however that in the discrete setting scaling arguments cannot be employed. The obtention of the discrete analog of the $L^1 - L^\infty$ decay will therefore require of extra arguments, based on a new form of the Gagliardo-Nirenberg's inequality that we present and proof.

1.1. **Finite difference scheme.**

At the numerical level we propose to employ a finite difference scheme.

Let us consider a uniform grid on $\mathbb{R}^2$ with mesh step h, and let $P_{i,j} = (ih, jh)$ denote a generic point in $\mathbb{R}^2_h$, with $i, j \in \mathbb{Z}$. The values of a function f at $P_{i,j}$ are denoted by $f_{i,j}$ and we assume that $(f_{i,j}) \in \ell^2_h$, namely that $h^2 \sum_{i,j\in\mathbb{Z}^2} f_{i,j}^2 < \infty$. Setting $f := (f_{i,j})$, we denote

$$\|f\|_{\ell^2_h} = \left(h^2 \sum_{i,j} f_{i,j}^2 \right)^{\frac{1}{2}}\,.$$

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[2] Indeed

$$|[A^*A, B]x|^2 = ((A^*AB - BA^*A)x, [A^*A, B]x) = (A^*[A, B]x + [A^*, B]Ax, [A^*A, B]x)$$

hence, since $A^* = -A$ and A commutes with $[A, B]$,

$$|[A^*A, B]x|^2 = -2(A[A, B]x, [A^*A, B]x) \le 2\|A[A, B]x\||[A^*A, B]x|$$

Henceforth, in the notation we will omit to explicitly recall the space ℓ^2_h, since all norms will be computed in this space. Similarly, for any two grid functions $f = (f_{i,j})$ and $g = (g_{i,j})$, we will denote without further specification their scalar product in ℓ^2_h as

$$(f,g) = h^2 \sum_{i,j} f_{i,j}\, g_{i,j}.$$

Let us also introduce the elementary finite difference operators A_h, B_h acting on grid functions f as follows:

$$A_h f = \frac{f_{i,j} - f_{i-1,j}}{h} \tag{1.10}$$

and

$$B_h f = x_i \, \frac{f_{i,j+1} - f_{i,j-1}}{2h}\,. \tag{1.11}$$

The reader can easily check that B_h is an anti-symmetric operator thanks to the choice of discretizing ∂_y with the centered difference at j.

Moreover, we have

$$A_h^* f = -\,\frac{f_{i+1,j} - f_{i,j}}{h}$$

and so $A_h^* A_h$ is the classical three-point discretization of the second derivative ∂_x^2:

$$A_h^* A_h f = -\,\frac{1}{h^2}(f_{i+1,j} + f_{i-1,j} - 2 f_{i,j}).$$

Notice that the commutator operator $[A_h, B_h]$ turns out to be

$$[A_h, B_h] f = \frac{f_{i-1,j+1} - f_{i-1,j-1}}{2h}$$

which should be read as an approximation of ∂_y.

One can readily check that A_h, A_h^* and B_h commute with $[A_h, B_h]$, and in addition $[A_h, A_h^*] = 0$. Therefore, the assumptions used for the estimates of Section 2 are satisfied in this discrete version.

1.2. **Semi-discrete scheme.**

A first application of the previously defined discrete operators can be given for time continuous, discrete space approximations. Namely, one can introduce the following semi-discrete finite difference centered scheme as an approximation of the Kolmogorov equation:

$$f'_{i,j}(t) - \frac{1}{h^2}(f_{i+1,j}(t) + f_{i-1,j}(t) - 2 f_{i,j}(t)) - x_i \, \frac{f_{i,j+1}(t) - f_{i,j-1}(t)}{2h} = 0, \tag{1.12}$$

for all $i, j \in \mathbb{Z}^2$.

System (1.12) can be recast in abstract form as

$$f' + A_h^* A_h f + B_h f = 0$$

to be interpreted as an evolution equation in ℓ^2_h.

Whenever $h > 0$ is fixed, the semi-discrete model (1.12) is a system of infinitely many coupled ODEs. At this level, let us notice that the operator B_h is unbounded in ℓ^2_h and, therefore, the generation of the semigroup property requires some care. However, truncating the variable x_i entering in its definition, the operator B_h can be approximated by a bounded operator for which the generation of the semigroup is straightforward. Passing to the limit in the truncated operator and using uniform energy bounds allows us to show the existence of solutions for the full semi-discrete equation. Moreover, solutions can be shown to be unique by energy estimates as well. Thus, solutions of the semi-discrete system exist, they are unique, and they depend smoothly on time and on the initial data. Of course the scheme being also consistent and stable in the ℓ^2_h-sense (it is actually dissipative in ℓ^2_h) it is convergent in the classical sense of numerical analysis, in finite intervals $[0, T]$ and as $h \to 0$.

But here we are rather interested on the asymptotic correctness of the scheme as t tends to ∞. In other words, we wish to show that solutions of the semi-discrete scheme reproduce the decay properties of continuous solutions as $t \to \infty$. Note that this issue is particularly relevant when facing applications in long-time horizons since, in practice, numerical simulations have to be performed with a fixed (possibly very small but fixed) mesh-size $h > 0$.

The properties of A_h and B_h mentioned above imply that the structure of the continuous problem is preserved in the discrete version. In particular, as a direct consequence of the abstract result in Proposition 1.1 we obtain the following uniform (with respect to the mesh-size parameters) discrete estimates.

Theorem 1.3. *There exists $C > 0$ independent of the mesh-size parameter $h > 0$ so that every solution of (1.12) satisfies*

$$\|f(t)\|^2_{\ell^2_h} + t\|D_x f(t)\|^2_{\ell^2_h} + t^3\|D_y f(t)\|^2_{\ell^2_h} \le C\|f_0\|^2_{\ell^2_h}, \qquad \forall t > 0,$$

whatever the initial datum f_0 is given in ℓ^2_h.

Here and in the sequel we have used and we shall use the notation:

$$D_x f := \frac{f_{i,j} - f_{i-1,j}}{h}, \qquad D_y f := \frac{f_{i,j+1} - f_{i,j-1}}{2h}, \tag{1.13}$$

for the first order discrete derivatives in x and y respectively. Note that they are defined differently, adapted to the numerical scheme under consideration.

1.3. **Fully discrete scheme.** We now consider a fully discrete (time-implicit scheme), by discretizing the time variable as well. Let $\Delta t > 0$ be a positive mesh-size parameter. We also denote by $t_n = n\Delta t$, $n = 0, \ldots, \infty$ the discrete time.

We approximate (1.1) with the following *implicit Euler scheme*

$$\frac{f^{n+1}_{i,j} - f^n_{i,j}}{\Delta t} + A^*_h A_h f^{n+1}_{i,j} + B_h f^{n+1}_{i,j} = 0 \tag{1.14}$$

for all points (ih, jh) in the grid $\mathbb{R}^2_h$ and all $n = 0, \ldots, \infty$, where all the discrete operators have been introduced before.

System (1.14) generates a stable dynamics in ℓ^2_h. In fact, it is easy to see that, given any initial condition $f^0_{i,j} = f_0(P_{i,j})$ for all (i, j), such that $f^0 \in \ell^2_h$, there exists a unique ℓ^2_h solution $\{f^n\}_{0<n<\infty} = \{(f^n_{i,j})\}$ to (1.14). Notice that uniqueness holds in ℓ^2_h, despite the fact that B_h is an unbounded operator. Existence can be easily proved by truncating the drift x_i and passing to the limit: indeed, if we truncate x_i then B_h becomes a bounded operator and in that case the time-discrete dynamics is well-defined as a direct consequence of

Lax-Milgram's Lemma. Since the estimates are uniform in ℓ_h^2, when we let the truncation parameter tend to infinity we obtain a solution in ℓ_h^2, and actually the unique in this class.

Being consistent and stable in ℓ_h^2 the fully-discrete implicit scheme (1.14) is also unconditionally convergent.

We are now interested on the problem of asymptotic behavior as $t_n \to \infty$, and, more precisely, on whether the same decay properties of the Kolmogorov equation hold uniformly with respect to the mesh-size parameters.

We develop a discrete version of the continuous approach described above which leads to the following fully-discrete uniform hypo-coercivity estimates, under a suitable CFL-type condition.

Theorem 1.4. *Let $\{f^n\}_{0<n<\infty}$ be a solution to (1.14). Assume that*

$$|\Delta t|^3 \leq Ch^2 \tag{1.15}$$

for some $C > 0$.

Then we have

$$\|f^n\|_{\ell_h^2}^2 + t_n\|D_x f^n\|_{\ell_h^2}^2 + t_n^3\|D_y f^n\|_{\ell_h^2}^2 \leq C\|f^0\|_{\ell_h^2}^2, \quad \forall n \geq 0, \tag{1.16}$$

for some constant C independent of n and h, where D_x, D_y are as in (1.13).

Remark 1.5. The estimatewe are going to prove is as follows

$$\|f^n\|_{\ell_h^2}^2 + t_n\|D_x f^n\|_{\ell_h^2}^2 + t_n^3\|D_y f^n\|_{\ell_h^2}^2 \leq C\|f^0\|_{\ell_h^2}^2 + C|\Delta t|^3\|D_y f^0\|_{\ell_h^2}^2, \tag{1.17}$$

which holds for any choice of Δt and h. If we wish an estimate only depending on $\|f^0\|$, then it is enough to take a time step Δt fulfilling the CFL condition (1.15).

These estimates show that, despite the initial datum is only assumed to be in ℓ_h^2, the solutions, at time $t > 0$, belong to higher order H^1-discrete Sobolev spaces, and that this occurs uniformly on the mesh-size parameters. This constitutes an evidence of the fact that hypo-coercivity is preserved at the numerical level. In addition, we will also show that the scheme yields uniform second order estimates and, eventually, the $L^1 - L^\infty$ decay estimate as well, again with the optimal time rate.

The rest of this paper is organized as follows. In the next section we give the proof of Theorem 1.4 and further decay estimates for second order derivatives and the sup-norm. In Section 3, for the sake of completeness, we give a proof of the continuous hypo-coercivity estimates by Herau and Villani, following Appendix A.21 in [15], based on the construction of well-balanced Lyapunov functionals, and derive some higher order extensions. These results are of direct application both for the PDE models and for its semi-discrete approximations, thus leading, directly, to the proof of Theorem 1.3. We conclude with a section devoted to discuss some other closely related issues.

2. Fully discrete decay estimates

Here we give a proof of the fully discrete decay estimates.

We will perform all computations assuming that $\sum_{i,j} |x_i||f^n_{i,j}|2 < \infty$ for every n. There is actually no loss of generality in doing that; indeed, since, by assuming that f^0 satisfies this condition, one can easily prove that the condition is preserved for any $n \geq 1$. Moreover, the final estimates (1.16) obtained are stable in ℓ^2_h, since they only depend on the ℓ^2_h-norm. Therefore, by using an approximation argument on f^0 and the ℓ^2_h-stability estimates, these estimates are proved to hold for any $f^0 \in \ell^2_h$ and for the (unique) ℓ^2_h solution of (1.14).

Proof of Theorem 1.4.

We proceed in several steps. To simplify the presentation we denote simply by $\|\cdot\|$ the norm in ℓ^2_h. Moreover, we fix some integer N as a generic possible stopping time of our scheme.

Step 1: Estimate of $\|f^N\|^2$. First of all, we multiply (1.14) by $f^{n+1}_{i,j}$, sum over i, j and over n (which is the discrete analog of a time-integration), obtaining

$$h^2 \Delta t \sum_{n=0}^{N-1} \sum_{i,j} \frac{f^{n+1}_{i,j} - f^n_{i,j}}{\Delta t} f^{n+1}_{i,j} = -h^2 \Delta t \sum_{n=0}^{N-1} \sum_{i,j} |A_h f^{n+1}_{i,j}|^2$$
$$= -\Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2.$$

Since we have

$$h^2 \sum_{i,j} |f^N_{i,j}|^2 - |f^0_{i,j}|^2 = h^2 \sum_{n=0}^{N-1} \sum_{i,j} |f^{n+1}_{i,j}|^2 - |f^n_{i,j}|^2$$
$$= 2h^2 \Delta t \sum_{n=0}^{N-1} \sum_{i,j} \frac{f^{n+1}_{i,j} - f^n_{i,j}}{\Delta t} f^{n+1}_{i,j} - h^2 \sum_{n=0}^{N-1} \sum_{i,j} (f^{n+1}_{i,j} - f^n_{i,j})^2$$

we deduce

$$\|f^N\|^2 \leq \|f^0\|^2 - \Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2 . \tag{2.1}$$

Step 2: Estimate of $\|A_h f^N\|^2$. Secondly, we compute

$$
\begin{aligned}
h^2 \sum_{i,j} t_N |A_h f_{i,j}^N|^2 &= h^2 \sum_{n=0}^{N-1} \sum_{i,j} t_{n+1} |A_h f_{i,j}^{n+1}|^2 - t_n |A_h f_{i,j}^n|^2 \\
&= h^2 \sum_{n=0}^{N-1} \sum_{i,j} (t_{n+1} - t_n) |A_h f_{i,j}^{n+1}|^2 + h^2 \sum_{n=0}^{N-1} t_n \sum_{i,j} |A_h f_{i,j}^{n+1}|^2 - |A_h f_{i,j}^n|^2 \\
&= h^2 \Delta t \sum_{n=0}^{N-1} \sum_{i,j} |A_h f_{i,j}^{n+1}|^2 + 2h^2 \Delta t \sum_{n=0}^{N-1} t_n \sum_{i,j} \frac{A_h (f_{i,j}^{n+1} - f_{i,j}^n)}{\Delta t} A_h f_{i,j}^{n+1} \\
&\quad - h^2 \sum_{n=0}^{N-1} t_n \sum_{i,j} |A_h (f_{i,j}^{n+1} - f_{i,j}^n)|^2
\end{aligned} \tag{2.2}
$$

where we used that $t_{n+1} - t_n = \Delta t$. Notice that the two latter sums do not see the term at $n = 0$ because of the penalisation term t_n. Thus we will need a separate estimate for f^1.

Using equation (1.14) and $B_h^* = -B_h$ we have, for $n = 0, \ldots, N-1$:

$$
\begin{aligned}
&h^2 \Delta t \sum_{i,j} \frac{A_h (f_{i,j}^{n+1} - f_{i,j}^n)}{\Delta t} A_h f_{i,j}^{n+1} = \\
&= -\Delta t (A_h A_h^* A_h f^{n+1}, A_h f^{n+1}) - \Delta t (A_h B_h f^{n+1}, A_h f^{n+1}) \\
&= -\Delta t \|A_h^* A_h f^{n+1}\|^2 - \Delta t ([A_h, B_h] f^{n+1}, A_h f^{n+1}).
\end{aligned} \tag{2.3}
$$

In the special case $n = 0$ the above equality reads as

$$
\|A_h f^1\|^2 + \Delta t \|A_h^* A_h f^1\|^2 = (A_h f^0, A_h f^1) - \Delta t \, ([A_h, B_h] f^1, A_h f^1)
$$

which implies

$$
\|A_h f^1\|^2 + \frac{1}{2} \Delta t \, \|A_h^* A_h f^1\|^2 \le \frac{1}{2\Delta t} \|f^0\|^2 - \Delta t \, ([A_h, B_h] f^1, A_h f^1) \, . \tag{2.4}
$$

Recall that $t_1 = \Delta t$. On account of (2.3) and (2.4) we deduce from (2.2)

$$
\begin{aligned}
&t_N \|A_h f^N\|^2 \le \Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2 \\
&- 2\Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2 - \frac{1}{2} \Delta t \, t_1 \, \|A_h^* A_h f^1\|^2 \\
&- 2\Delta t \sum_{n=0}^{N-1} t_n ([A_h, B_h] f^{n+1}, A_h f^{n+1}) - \Delta t \, t_1 \, ([A_h, B_h] f^1, A_h f^1) \\
&\qquad + \frac{1}{2} \|f^0\|^2 - h^2 \sum_{n=0}^{N-1} t_n \sum_{i,j} |A_h (f_{i,j}^{n+1} - f_{i,j}^n)|^2.
\end{aligned} \tag{2.5}
$$

Step 3: Estimate of the crossed term. We go on computing

$$t_N^2([A_h,B_h]f^N,A_hf^N) = \sum_{n=0}^{N-1}\left\{t_{n+1}^2([A_h,B_h]f^{n+1},A_hf^{n+1}) - t_n^2([A_h,B_h]f^n,A_hf^n)\right\}$$

which we split, similarly as before:

(2.6)

$$\begin{aligned} t_N^2([A_h,B_h]f^N,A_hf^N) &= \sum_{n=0}^{N-1}(t_{n+1}^2-t_n^2)([A_h,B_h]f^{n+1},A_hf^{n+1}) \\ &+ \sum_{n=0}^{N-1} t_n^2\left\{([A_h,B_h](f^{n+1}-f^n),A_hf^{n+1}) + ([A_h,B_h]f^{n+1},A_h(f^{n+1}-f^n))\right\} \\ &- \sum_{n=0}^{N-1} t_n^2([A_h,B_h](f^{n+1}-f^n),A_h(f^{n+1}-f^n)) \,. \end{aligned}$$

The second line is estimated through the equation (1.14)

$$\begin{aligned} &([A_h,B_h](f^{n+1}-f^n),A_hf^{n+1}) + ([A_h,B_h]f^{n+1},A_h(f^{n+1}-f^n)) \\ &\quad = -\Delta t([A_h,B_h](A_h^*A_h+B_h)f^{n+1},A_hf^{n+1}) \\ &\quad - \Delta t([A_h,B_h]f^{n+1},A_h(A_h^*A_h+B_h)f^{n+1}) \,. \end{aligned}$$

Using the properties of operators A_h and B_h as in Proposition 1.1 we obtain

$$\begin{aligned} &- ([A_h,B_h](A_h^*A_h+B_h)f^{n+1},A_hf^{n+1}) \\ &\qquad - ([A_h,B_h]f^{n+1},A_h(A_h^*A_h+B_h)f^{n+1}) \\ &= -\|[A_h,B_h]f^{n+1}\|^2 \\ &\qquad - ((A_h^*A_h+A_hA_h^*)[A_h,B_h]f^{n+1},A_hf^{n+1}) \end{aligned} \tag{2.7}$$

If we use in addition that A_h and A_h^* commute we have

$$((A_h^*A_h+A_hA_h^*)[A_h,B_h]f^{n+1},A_hf^{n+1}) = 2(A_h[A_h,B_h]f^{n+1},A_h^2f^{n+1})$$

and

$$\|A_h^2f\|^2 = \|A_h^*A_hf\|^2$$

so that

(2.8)

$$\left|((A_h^*A_h+A_hA_h^*)[A_h,B_h]f^{n+1},A_hf^{n+1})\right| \le 2\|A_h[A_h,B_h]f^{n+1}\|\,\|A_h^*A_hf^{n+1}\| \,.$$

Using now (2.7) and (2.8) we conclude the estimate

$$\begin{aligned} &([A_h,B_h](f^{n+1}-f^n),A_hf^{n+1}) + ([A_h,B_h]f^{n+1},A_h(f^{n+1}-f^n)) \\ &\quad \le -\Delta t\|[A_h,B_h]f^{n+1}\|^2 + 2\Delta t\|A_h[A_h,B_h]f^{n+1}\|\,\|A_h^*A_hf^{n+1}\| \,. \end{aligned} \tag{2.9}$$

We can specialize this estimate for $n=0$, which reads as

$$\begin{aligned} &\left\{([A_h,B_h](f^1-f^0),A_hf^1) + ([A_h,B_h]f^1,A_h(f^1-f^0))\right\} \\ &\quad \le -\Delta t\|[A_h,B_h]f^1\|^2 + \Delta t\|A_h[A_h,B_h]f^1\|\,\|A_h^*A_hf^1\| \end{aligned}$$

hence

(2.10)

$$
\begin{aligned}
&2([A_h,B_h]f^1,A_hf^1) \le -\Delta t\|[A_h,B_h]f^1\|^2 + \Delta t\|A_h[A_h,B_h]f^1\|\ \|A_h^*A_hf^1\| \\
&+([A_h,B_h]f^1,A_hf^0) + ([A_h,B_h]f^0,A_hf^1) \\
&\le -\Delta t\|[A_h,B_h]f^1\|^2 + \Delta t\|A_h[A_h,B_h]f^1\|\ \|A_h^*A_hf^1\| + 2\|f^0\|\,\|A_h[A_h,B_h]f^1\|
\end{aligned}
$$

where we used that $\|C\| = \|C^*\|$ and, since A_h commutes with A_h^*, we have $\|A_h^*[A_h,B_h]f^1\| = \|A_h[A_h,B_h]f^1\|$.

We use (2.10) to estimate the value at $n=0$ of the first right-hand side term of (2.6). Together with (2.9), we obtain from (2.6) that

(2.11)

$$
\begin{aligned}
&t_N^2([A_h,B_h]f^N,A_hf^N) \le \sum_{n=1}^{N-1}(t_{n+1}^2-t_n^2)([A_h,B_h]f^{n+1},A_hf^{n+1}) \\
&-\frac{1}{2}t_1^2\,\Delta t\,\|[A_h,B_h]f^1\|^2 + \frac{1}{2}t_1^2\,\Delta t\,\|A_h[A_h,B_h]f^1\|\ \|A_h^*A_hf^1\| \\
&+t_1^2\,\|f^0\|\,\|A_h[A_h,B_h]f^1\| \\
&\quad -\Delta t\sum_{n=0}^{N-1}t_n^2\|[A_h,B_h]f^{n+1}\|^2 + 2\Delta t\sum_{n=0}^{N-1}t_n^2\|A_h[A_h,B_h]f^{n+1}\|\ \|A_h^*A_hf^{n+1}\| \\
&\quad -\sum_{n=0}^{N-1}t_n^2([A_h,B_h](f^{n+1}-f^n),A_h(f^{n+1}-f^n))\,.
\end{aligned}
$$

Step 4: Estimate of $\|[A_h,B_h]f^N\|^2$. Finally, we estimate

$$
t_N^3\|[A_h,B_h]f^N\|^2 = \sum_{n=0}^{N-1}\left\{t_{n+1}^3\|[A_h,B_h]f^{n+1}\|^2 - t_n^3\|[A_h,B_h]f^n\|^2\right\}
$$

and again we read this term as follows

$$
\begin{aligned}
t_N^3\|[A_h,B_h]f^N\|^2 &= \sum_{n=0}^{N-1}(t_{n+1}^3-t_n^3)\|[A_h,B_h]f^{n+1}\|^2 \\
&+2\sum_{n=0}^{N-1}t_n^3([A_h,B_h](f^{n+1}-f^n),[A_h,B_h]f^{n+1}) \\
&-\sum_{n=0}^{N-1}t_n^3\|[A_h,B_h](f^{n+1}-f^n)\|^2\,.
\end{aligned}
$$

From the equation (1.14) we have, since A_h, B_h commute with $[A_h,B_h]$,

$$
([A_h,B_h](f^{n+1}-f^n),[A_h,B_h]f^{n+1}) = -\Delta t\|A_h[A_h,B_h]f^{n+1}\|^2
$$

which we also use for $n = 0$ to estimate the first step f^1. Then we get

$$\begin{aligned} t_N^3 \|[A_h, B_h] f^N\|^2 &= \sum_{n=1}^{N-1} (t_{n+1}^3 - t_n^3) |[A_h, B_h] f^{n+1}|^2 \\ &- t_1^3 \, \Delta t \|A_h [A_h, B_h] f^1\|^2 + t_1^3 ([A_h, B_h] f^0, [A_h, B_h] f^1) \\ &- 2\Delta t \sum_{n=0}^{N-1} t_n^3 \|A_h [A_h, B_h] f^{n+1}\|^2 - \sum_{n=0}^{N-1} t_n^3 \|[A_h, B_h](f^{n+1} - f^n)\|^2 \,. \end{aligned} \tag{2.12}$$

Step 5: Conclusion. Take now positive numbers $\lambda, a, b, c > 0$. Putting together (2.1), (2.5), (2.11) and (2.12) we obtain

(2.13)

$$\begin{aligned} &\lambda \|f^N\|^2 + a t_N \|A_h f^N\|^2 + b t_N^2 ([A_h, B_h] f^N, A_h f^N) + c t_N^3 \|[A_h, B_h] f^N\|^2 \\ &\le (\lambda + \frac{a}{2}) \|f^0\|^2 + (a - \lambda) \Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2 \\ &\quad - \frac{a}{2} \Delta t \, t_1 \, \|A_h^* A_h f^1\|^2 - a \Delta t \, t_1 \, ([A_h, B_h] f^1, A_h f^1) \\ &\quad - 2a \Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2 - 2a \Delta t \sum_{n=0}^{N-1} t_n ([A_h, B_h] f^{n+1}, A_h f^{n+1}) \\ &\quad - a \sum_{n=0}^{N-1} t_n \|A_h (f^{n+1} - f^n)\|^2 \\ &\quad + b \sum_{n=1}^{N-1} (t_{n+1}^2 - t_n^2) ([A_h, B_h] f^{n+1}, A_h f^{n+1}) - \frac{1}{2} b t_1^2 \, \Delta t \, \|[A_h, B_h] f^1\|^2 \\ &\quad + \frac{1}{2} b t_1^2 \, \Delta t \, \|A_h [A_h, B_h] f^1\| \; \|A_h^* A_h f^1\| + b \, t_1^2 \, \|f^0\| \, \|A_h [A_h, B_h] f^1\| \\ &\quad - b \Delta t \sum_{n=0}^{N-1} t_n^2 \|[A_h, B_h] f^{n+1}\|^2 + 2b \Delta t \sum_{n=0}^{N-1} t_n^2 \|A_h [A_h, B_h] f^{n+1}\| \; \|A_h^* A_h f^{n+1}\| \\ &\quad - b \sum_{n=0}^{N-1} t_n^2 ([A_h, B_h](f^{n+1} - f^n), A_h (f^{n+1} - f^n)) \\ &\quad + c \sum_{n=1}^{N-1} (t_{n+1}^3 - t_n^3) \|[A_h, B_h] f^{n+1}\|^2 \\ &\quad - c \, t_1^3 \, \Delta t \|A_h [A_h, B_h] f^1\|^2 + c \, t_1^3 ([A_h, B_h] f^0, [A_h, B_h] f^1) \\ &\quad - 2c \Delta t \sum_{n=0}^{N-1} t_n^3 \|A_h [A_h, B_h] f^{n+1}\|^2 - c \sum_{n=0}^{N-1} t_n^3 \|[A_h, B_h](f^{n+1} - f^n)\|^2 \,. \end{aligned}$$

Notice that, by choosing $b^2 - 4ac < 0$, so that the quadratic form $ax^2 + bxy + cy^2$ is positive definite, we have

$$
\begin{aligned}
&- a \sum_{n=0}^{N-1} t_n \|A_h(f_{i,j}^{n+1} - f_{i,j}^n)\|^2 - b \sum_{n=0}^{N-1} t_n^2([A_h, B_h](f^{n+1} - f^n), A_h(f^{n+1} - f^n)) \\
&- c \sum_{n=0}^{N-1} t_n^3 \|[A_h, B_h](f^{n+1} - f^n)\|^2 \leq 0\,.
\end{aligned}
$$

Hence we can drop the sum of those three terms in the right-hand side of (2.13). Moreover, we estimate

$$
\begin{aligned}
2a\Delta t \sum_{n=0}^{N-1} t_n \left|([A_h, B_h]f^{n+1}, A_h f^{n+1})\right| \leq{}& \frac{b}{2}\Delta t \sum_{n=0}^{N-1} t_n^2 \|[A_h, B_h]f^{n+1}\|^2 \\
&+ \frac{2a^2}{b}\Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
2b\Delta t \sum_{n=0}^{N-1} t_n^2 \|A_h[A_h, B_h]f^{n+1}\| \; \|A_h^* A_h f^{n+1}\| \leq{}& c\Delta t \sum_{n=0}^{N-1} t_n^3 \|A_h[A_h, B_h]f^{n+1}\|^2 \\
&+ \frac{b^2}{c}\Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2\,.
\end{aligned}
$$

Therefore, we deduce from (2.13)

(2.14)

$$
\begin{aligned}
&\lambda\|f^N\|^2 + at_N\|A_h f^N\|^2 + bt_N^2([A_h, B_h]f^N, A_h f^N) + ct_N^3\|[A_h, B_h]f^N\|^2 \\
&\leq (\lambda + \frac{a}{2})\|f^0\|^2 + (a + \frac{2a^2}{b} - \lambda)\Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2 - 2a\Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2 \\
&\quad - \frac{b}{2}\Delta t \sum_{n=0}^{N-1} t_n^2 \|[A_h, B_h]f^{n+1}\|^2 - c\Delta t \sum_{n=0}^{N-1} t_n^3 \|A_h[A_h, B_h]f^{n+1}\|^2 \\
&\quad - \frac{a}{2}\Delta t\, t_1 \,\|A_h^* A_h f^1\|^2 - a\Delta t\, t_1\, ([A_h, B_h]f^1, A_h f^1) - \frac{1}{2}\, bt_1^2\, \Delta t\, \|[A_h, B_h]f^1\|^2 \\
&\quad + \frac{1}{2}\, bt_1^2\, \Delta t\, \|A_h[A_h, B_h]f^1\| \; \|A_h^* A_h f^1\| + b\, t_1^2\, \|f^0\|\, \|A_h[A_h, B_h]f^1\| \\
&\quad + b \sum_{n=1}^{N-1} (t_{n+1}^2 - t_n^2)([A_h, B_h]f^{n+1}, A_h f^{n+1}) + c \sum_{n=1}^{N-1} (t_{n+1}^3 - t_n^3)\|[A_h, B_h]f^{n+1}\|^2 \\
&\quad - c\, t_1^3\, \Delta t \|A_h[A_h, B_h]f^1\|^2 + c\, t_1^3([A_h, B_h]f^0, [A_h, B_h]f^1)\,.
\end{aligned}
$$

We are left with the estimate of the last four lines. Since $t_{n+1} - t_n = \Delta t$ and $t_{n+1} \leq 2t_n$ for $n \geq 1$, we have

$$b \sum_{n=1}^{N-1} (t_{n+1}^2 - t_n^2)([A_h, B_h]f^{n+1}, A_h f^{n+1}) \leq \frac{b}{4} \Delta t \sum_{n=1}^{N-1} t_n^2 \|[A_h, B_h]f^{n+1}\|^2 + 9\, b \Delta t \sum_{n=1}^{N-1} \|A_h f^{n+1}\|^2$$

and similarly we estimate

$$c \sum_{n=1}^{N-1} (t_{n+1}^3 - t_n^3)\|[A_h, B_h]f^{n+1}\|^2 \leq 12\, c\, \Delta t \sum_{n=1}^{N-1} t_n^2 \|[A_h, B_h]f^{n+1}\|^2 .$$

Moreover, we estimate

$$-a\Delta t\, t_1 \,([A_h, B_h]f^1, A_h f^1) \leq \frac{b}{4} t_1^2 \Delta t \|[A_h, B_h]f^1\|^2 + \frac{a^2}{b} \,\Delta t \|A_h f^1\|^2$$

and

$$\begin{aligned} &\frac{1}{2}\, b t_1^2 \,\Delta t \,\|A_h[A_h, B_h]f^1\| \; \|A_h^* A_h f^1\| + b\, t_1^2 \,\|f^0\| \,\|A_h[A_h, B_h]f^1\| \\ &\leq \frac{c}{2}\, t_1^3 \,\Delta t \|A_h[A_h, B_h]f^1\|^2 + \frac{b^2}{4c} \; t_1 \Delta t \|A_h^* A_h f^1\|^2 + \frac{b^2}{c} \|f^0\|^2 \end{aligned}$$

where we used that $t_1 = \Delta t$. Therefore, we obtain

(2.15)

$$\begin{aligned} &\lambda \|f^N\|^2 + a t_N \|A_h f^N\|^2 + b t_N^2 ([A_h, B_h]f^N, A_h f^N) + c t_N^3 \|[A_h, B_h]f^N\|^2 \\ &\leq (\lambda + \frac{a}{2} + \frac{b^2}{c}) \|f^0\|^2 \\ &\quad + (a + 3\frac{a^2}{b} + 9b - \lambda)\Delta t \sum_{n=0}^{N-1} \|A_h f^{n+1}\|^2 - 2a\Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2 \\ &\quad + (12c - \frac{b}{4})\Delta t \sum_{n=1}^{N-1} t_n^2 \|[A_h, B_h]f^{n+1}\|^2 - c\Delta t \sum_{n=0}^{N-1} t_n^3 \|A_h[A_h, B_h]f^{n+1}\|^2 \\ &\quad + (\frac{b^2}{4c} - \frac{a}{2})\Delta t\, t_1 \,\|A_h^* A_h f^1\|^2 - \frac{b}{4} t_1^2 \,\Delta t \,\|[A_h, B_h]f^1\|^2 \\ &\quad - \frac{c}{2}\, t_1^3 \,\Delta t \|A_h[A_h, B_h]f^1\|^2 + c\, t_1^3 ([A_h, B_h]f^0, [A_h, B_h]f^1) . \end{aligned}$$

Recall that we choose $\lambda >> a >> b >> c$. Young's inequality in the last term allows us to conclude that

$$\begin{aligned} &\lambda \|f^N\|^2 + a t_N \|A_h f^N\|^2 + b t_N^2 ([A_h, B_h]f^N, A_h f^N) + c t_N^3 \|[A_h, B_h]f^N\|^2 \\ &\leq K \,\|f^0\|^2 + K\, t_1^3 \|[A_h, B_h]f^0\|^2 . \end{aligned}$$

In particular, whenever $|\Delta t|^3 \leq C\, h^2$, we end up with

$$\|f^N\|^2 + t_N \|A_h f^N\|^2 + t_N^2 ([A_h, B_h]f^N, A_h f^N) + t_N^3 \|[A_h, B_h]f^N\|^2 \leq K \,\|f^0\|^2$$

for some K independent of h and T. □

2.1. **Second order estimates and L^∞ decay.** We wish now to show other forms of the regularizing effect and the L^∞ decay. We start with second order estimates which are the discrete equivalent of Proposition 1.2.

Proposition 2.1. *Let $\{f^n\}_{0<n<\infty}$ be a solution to (1.14) and assume that the condition (1.15) holds. Then we have*

$$t_n^2\|D_xD_xf^n\|_{\ell_h^2}^2+t_n^4\|D_xD_yf^n\|_{\ell_h^2}^2+t_n^6\|D_yD_yf^n\|_{\ell_h^2}^2\le C\|f^0\|_{\ell_h^2}^2,\quad \forall n\ge 0, \tag{2.16}$$

for some constant C independent of n and h.

Proof. Since A_h, A_h^*, B_h all commute with $[A_h,B_h]$, we deduce that $[A_h,B_h]f^{n+1}$ satisfies the same implicit equation (1.14). Hence the conclusion of Theorem 1.4 applies to $[A_h,B_h]f^{n+1}$ and we get

$$t_{n-k}\|D_x[A_h,B_h]f^n\|_{\ell_h^2}^2+t_{n-k}^3\|D_y[A_h,B_h]f^n\|_{\ell_h^2}^2\le C\|[A_h,B_h]f^k\|_{\ell_h^2}^2,$$

for any $k<n$. Since $\|[A_h,B_h]f^k\|=\|D_yf^k\|$, combining this estimate with Theorem 1.4 we get

$$t_{n-k}\|D_x[A_h,B_h]f^n\|_{\ell_h^2}^2+t_{n-k}^3\|D_y[A_h,B_h]f^n\|_{\ell_h^2}^2\le Ct_k^{-3}\|f^0\|_{\ell_h^2}^2\,.$$

If $n\ge 2$, we can choose $k=[\frac{n}{2}]$ so that $t_{n-k}\ge\frac{t_n}{2}$, $t_k\ge\frac{t_n}{4}$ and we deduce

$$t_n^4\|D_x[A_h,B_h]f^n\|_{\ell_h^2}^2+t_n^6\|D_y[A_h,B_h]f^n\|_{\ell_h^2}^2\le C\|f^0\|_{\ell_h^2}^2\,. \tag{2.17}$$

Moreover, the estimate (2.17) also holds if we have $n=1$ since $\|[A_h,B_h]f^0\|_{\ell_h^2}^2\le\frac{1}{h^2}\|f^0\|_{\ell_h^2}^2\le Ct_1^{-3}\|f^0\|_{\ell_h^2}^2$ due to the condition (1.15).

Notice that

$$\|D_xg\|_{\ell_h^2}^2=h^2\sum_{i,j}\frac{|g_{i,j}-g_{i-1,j}|^2}{h^2}=h^2\sum_{i,j}\frac{|g_{i+1,j}-g_{i,j}|^2}{h^2}$$

so $\|D_x[A_h,B_h]f^n\|^2$ is the same as $\|D_xD_yf^n\|^2$, with our notations of D_x, D_y. The same is true for $\|D_y[A_h,B_h]f^n\|$, so through (2.17) we get the second and third term in estimate (2.16).

Now we estimate the second x-derivative. We have, for any stopping time N:

(2.18)

$$\begin{aligned}
h^2\sum_{i,j}t_N^2|A_h^*A_hf_{i,j}^N|^2&=h^2\sum_{n=0}^{N-1}\sum_{i,j}t_{n+1}^2|A_h^*A_hf_{i,j}^{n+1}|^2-t_n^2|A_h^*A_hf_{i,j}^n|^2\\
&=h^2\sum_{n=0}^{N-1}\sum_{i,j}(t_{n+1}^2-t_n^2)|A_h^*A_hf_{i,j}^{n+1}|^2+h^2\sum_{n=0}^{N-1}t_n^2\sum_{i,j}|A_h^*A_hf_{i,j}^{n+1}|^2-|A_h^*A_hf_{i,j}^n|^2\\
&\le h^2\Delta t\sum_{n=0}^{N-1}\sum_{i,j}(t_{n+1}+t_n)|A_h^*A_hf_{i,j}^{n+1}|^2\\
&\quad+2h^2\Delta t\sum_{n=0}^{N-1}t_n^2\sum_{i,j}\frac{A_h^*A_h(f_{i,j}^{n+1}-f_{i,j}^n)}{\Delta t}A_h^*A_hf_{i,j}^{n+1}\,.
\end{aligned}$$

On the other hand, again from (1.14) we get

$$\begin{aligned}
&\sum_{i,j} \frac{A_h^* A_h (f_{i,j}^{n+1} - f_{i,j}^n)}{\Delta t} A_h^* A_h f_{i,j}^{n+1} \\
&= -\|A_h A_h^* A_h f^{n+1}\|^2 - ([A_h^* A_h, B_h] f^{n+1}, A_h^* A_h f^{n+1}) \\
&\leq -\|A_h A_h^* A_h f^{n+1}\|^2 + \frac{1}{2t_n} \|A_h^* A_h f^{n+1}\|^2 + \frac{t_n}{2} \|[A_h^* A_h, B_h] f^{n+1}\|^2 .
\end{aligned}$$

Since we have

$$[A_h^* A_h, B_h] = A_h^* [A_h, B_h] + [A_h^*, B_h] A_h$$

and both can be estimated in terms of $A_h[A_h, B_h] f^{n+1}$, we have

$$\|[A_h^* A_h, B_h] f^{n+1}\| \leq c \|A_h [A_h, B_h] f^{n+1}\|$$

for some $c > 0$. Therefore we conclude from (2.18)

$$\begin{aligned}
h^2 \sum_{i,j} t_N^2 |A_h^* A_h f_{i,j}^N|^2 &\leq c\, h^2 \Delta t \sum_{n=0}^{N-1} t_n \|A_h^* A_h f^{n+1}\|^2 + h^2\, t_1\, \Delta t\, \|A_h^* A_h f^1\|^2 \\
&\quad + c\, h^2\, \Delta t \sum_{n=0}^{N-1} t_n^3\, \|A_h [A_h, B_h] f^{n+1}\| .
\end{aligned} \tag{2.19}$$

Putting this together with (2.15), we conclude that

$$t_N^2 \|A_h^* A_h f^N\|^2 \leq c\, \|f^0\|^2$$

for some $c > 0$. This gives the desired estimate on $D_x D_x f^n$ at any index n, completing (2.16). □

We now use the following discrete Gagliardo-Nirenberg type estimates. For any $p > 1$, we will use the discrete ℓ_h^p space and we denote its norm as

$$\|f\|_p := \left(h^2 \sum_{i,j} |f_{i,j}|^p \right)^{\frac{1}{p}}$$

Let us recall that $\ell_h^p \subset \ell_h^q$ whenever $p \leq q$.

The first result we use is a standard H^1 embedding estimate. Namely, there exists a constant C, independent of h, such that we have

$$\|f\|_4^4 \leq C \|D_x f\|_2 \|D_y f\|_2\, \|f\|_2^2 \tag{2.20}$$

for every $f \in \ell_h^2$.

Inequality (2.20) can be proved as a byproduct of more general discrete Sobolev type inequalities (in particular of the type given below).

The second result that we prove is, to our knowledge, less standard and is a precised version of the L^∞ Sobolev embedding estimate.

Theorem 2.2. *For any $p > 2$ and $m \geq 1$, there exists a constant C, only depending on p, m such that*

$$\|f\|_\infty \leq C\left(\|D_x f\|_p \|D_y f\|_p\right)^{\frac{p}{m(p-2)+2p}} \|f\|_m^{\frac{m(p-2)}{m(p-2)+2p}} \tag{2.21}$$

for every $f \in \ell_h^r(\mathbb{R}^2)$ with $r = \min(2, m)$.

The proof of Theorem 2.2 can be given using the same method given in [14] for the corresponding continuous version, which provides an alternative proof of the usual Gagliardo-Nirenberg inequality without using the Morrey embedding. For the reader's convenience, we sketch a purely discrete version of this proof, which could be equally applied to other kinds of discrete schemes, like finite volumes. In fact, in the present case of finite differences on regular grids, one could provide an alternative proof applying the continuous result to a suitable continuous interpolation of the discrete values $f_{i,j}$.

Proof. With no loss of generality one can assume that f has compact support. We start by writing

$$\sum_{i,j} \frac{1}{2}(|f_{i,j}|^2 + |f_{i,j-1}|^2) \leq \sum_{i,j} |f_{i,j}| \frac{|f_{i,j}| + |f_{i,j-1}|}{2} + \sum_{i,j} |f_{i,j-1}| \frac{|f_{i,j}| + |f_{i,j-1}|}{2}. \tag{2.22}$$

Notice that

$$\frac{|f_{i,j}| + |f_{i,j-1}|}{2} = \sum_{l=-\infty}^{j-1} \frac{|f_{i,l+1}| - |f_{i,l-1}|}{2} \leq \sum_{l=-\infty}^{j-1} \frac{|f_{i,l+1} - f_{i,l-1}|}{2}$$

hence

$$\begin{aligned}\sum_{i,j} |f_{i,j}| \frac{|f_{i,j}| + |f_{i,j-1}|}{2} &\leq \sum_{i,j} \sum_{k=-\infty}^{i} |f_{k,j} - f_{k-1,j}| \sum_{l=-\infty}^{j-1} \frac{|f_{i,l+1} - f_{i,l-1}|}{2} \\ &\leq \left(\sum_{k,j} |f_{k,j} - f_{k-1,j}|\right)\left(\sum_{i,l} \frac{|f_{i,l+1} - f_{i,l-1}|}{2}\right).\end{aligned}$$

Similarly we reason for the second term in the right-hand side of (2.22), which now implies

$$\sum_{i,j} \frac{1}{2}(|f_{i,j}|^2 + |f_{i,j-1}|^2) \leq 2\left(\sum_{k,j} |f_{k,j} - f_{k-1,j}|\right)\left(\sum_{i,l} \frac{|f_{i,l+1} - f_{i,l-1}|}{2}\right).$$

In terms of ℓ_h^p norms, this gives

$$\|f\|_2^2 \leq 2\|D_x f\|_1 \|D_y f\|_1 \tag{2.23}$$

where, as above, D_x and D_y are respectively the left and centered finite difference.

Now we apply this inequality to the function f^γ, with $\gamma > 1$. Since

$$|D_x f^\gamma| \leq \gamma(|f_{i,j}|^{\gamma-1} + |f_{i-1,j}|^{\gamma-1})|D_x f|\,; \quad |D_y f^\gamma| \leq \gamma(|f_{i,j-1}|^{\gamma-1} + |f_{i,j+1}|^{\gamma-1})|D_y f|$$

using Hölder inequality we get
(2.24)

$$h^2\sum_{i,j}|f_{i,j}|^{2\gamma}\leq c\,\gamma^2\left(h^2\sum_{i,j}|D_xf|^p\right)^{\frac1p}\left(h^2\sum_{i,j}|D_yf|^p\right)^{\frac1p}\left(h^2\sum_{i,j}|f_{i,j}|^{(\gamma-1)p'}\right)^{\frac{2}{p'}}.$$

We use now interpolation, assuming $\gamma\geq p$ so that $\gamma\leq(\gamma-1)p'<2\gamma$; we get

$$\left(h^2\sum_{i,j}|f_{i,j}|^{(\gamma-1)p'}\right)^{\frac{2}{p'}}\leq\left(h^2\sum_{i,j}|f_{i,j}|^{2\gamma}\right)^{\frac{2\theta}{p'}}\left(h^2\sum_{i,j}|f_{i,j}|^{\gamma}\right)^{\frac{2(1-\theta)}{p'}}$$

with $\theta=\frac{(\gamma-1)p'-\gamma}{\gamma}$. Plugging this inequality into (2.24) we get

$$\left(h^2\sum_{i,j}|f_{i,j}|^{2\gamma}\right)^{1-\frac{2\theta}{p'}}\leq c\,\gamma^2\|D_xf\|_p\,\|D_yf\|_p\left(h^2\sum_{i,j}|f_{i,j}|^{\gamma}\right)^{\frac{2(1-\theta)}{p'}}$$

which implies

$$\|f\|_{2\gamma}\leq\left[c\,\gamma^2\|D_xf\|_p\,\|D_yf\|_p\right]^{\frac{p'}{2\gamma(p'-2\theta)}}\|f\|_\gamma^{\frac{1-\theta}{p'-2\theta}}. \tag{2.25}$$

Inequality (2.25) can be iterated choosing $\gamma=m\,2^n$ (with $m\geq p$) and letting n tend to infinity. The iteration can be proved to converge (see [14]) and therefore there exists two positive constants α,β, and some $C>0$ such that

$$\|f\|_\infty\leq C\,\left[\|D_xf\|_p\,\|D_yf\|_p\right]^\alpha\|f\|_m^\beta\,.$$

The consistency of the scheme now implies that, letting $h\to0$, the same inequality must be true for smooth functions $f\in C^1_c(\mathbb{R}^2)$, so that the constants α,β are fixed in a unique way as in the continuous case ([14]) and inequality (2.21) is proved for $m\geq p$.

Finally, (2.21) extends to m smaller than p by using (2.24) (eventually iterated a finite number of times to go from ℓ^m_h to ℓ^p_h) and it is true, by density, for all functions $f\in\ell^r_h(\mathbb{R}^2)$ with $r=\min(2,m)$. □

Let us see how we deduce the L^∞ decay from the above estimates.

Theorem 2.3. *Let $\{f^n\}_{0<n<\infty}$ be a solution to (1.14) and assume that condition (1.15) holds. Then we have*

$$t_n\|f^n\|_\infty\leq C\|f^0\|_{\ell^2_h},\quad\forall n\geq0, \tag{2.26}$$

for some constant C independent of n and h.

Proof. Using (2.20) for D_xf^n and D_yf^n, respectively, together with Theorem 1.4 and Proposition 2.1, we get

$$\|D_xf^n\|_4\leq Ct_n^{-1}\|f^0\|_2\,,\qquad\|D_yf^n\|_4\leq Ct_n^{-2}\|f^0\|_2\,.$$

Applying now Theorem 2.2 to f^n with $m=2$ and $p=4$ we find

$$\begin{aligned}\|f^n\|_\infty&\leq C\,\left(\|D_xf^n\|_4\|D_yf^n\|_4\right)^{\frac13}\|f^n\|_2^{\frac13}\\&\leq Ct_n^{-1}\|f^0\|_2\,.\end{aligned}$$

Hence we conclude. □

Finally, we observe now that the $L^1 - L^2$ decay can be deduced by duality from the above result, by using the solution of the backward dual problem

$$-\frac{g_{i,j}^{n+1} - g_{i,j}^n}{\Delta t} + A_h^* A_h g_{i,j}^n - B_h g_{i,j}^n = 0$$

for all points (ih, jh) in the grid $\mathbb{R}_h^2$ and all $n = 0, \dots, N$. Multiplying by $f_{i,j}^{n+1}$ and summing we get

$$-\sum_{n=0}^{N-1} \sum_{i,j} [\frac{g_{i,j}^{n+1} - g_{i,j}^n}{\Delta t}] f_{i,j}^{n+1} = \sum_{n=0}^{N-1} \sum_{i,j} [\frac{f_{i,j}^{n+1} - f_{i,j}^n}{\Delta t}] g_{i,j}^n \,,$$

hence

$$(g^N, f^N) = (g^0, f^0)\,.$$

Applying to g the result of Theorem 2.3 we have $\|g^0\|_\infty \leq C\, \|g^N\|_2 / t_N$ and since g^N is arbitrary we conclude

$$\|f^N\|_2 \leq C t_N^{-1} \|f^0\|_1\,.$$

By combining the two regularizing effects we obtain

$$\|f^n\|_\infty \leq C t_{n-k}^{-1} \|f^k\|_2 \leq C t_{n-k}^{-1} t_k^{-1} \|f^0\|_1$$

so that, by choosing $k = [\frac{n}{2}]$, we end up with the further estimate

$$t_n^2 \|f^n\|_\infty \leq C \|f^0\|_1 \qquad \forall n \geq 2\,.$$

3. Time-continuous hypo-coercivity

For the sake of completeness we recall the main steps of the proof of Proposition 1.1 in Villani in [15], [16].

We proceed formally assuming that solutions are sufficiently smooth for the computations to be justified. In practice this requires suitable density arguments concerning the domains of the operators. Note however that these arguments are obvious in the semi-discrete case, which is where we will apply this result. All norms and scalar products considered below are in the space X, so we omit to specify that in the following.

3.1. **Proof of Proposition 1.1.** We start by the classical decay of the X-norm, which is given by the anti-symmetric character of B:

$$\frac{d}{dt}\|f(t)\|^2 = -2(f(t), A^* A f(t)) = -2\|Af(t)\|^2 \tag{3.1}$$

Next we compute

$$\begin{aligned}\frac{d}{dt}\left\{t\|Af(t)\|^2\right\} &= \|Af(t)\|^2 - 2t(Af(t), A\partial_t f) \\ &= \|Af(t)\|^2 - 2t(A^*Af(t), A^*Af(t)) - 2t(Af(t), ABf(t))\end{aligned}$$

and since $(Ax, ABx) = (Ax, [A, B]x)$ because B is antisymmetric, we obtain

$$\frac{d}{dt}\left\{t\|Af(t)\|^2\right\} = \|Af(t)\|^2 - 2t\|A^*Af(t)\|^2 - 2t(Af(t), [A, B]f(t)) \tag{3.2}$$

We proceed by computing the evolution of the mixed term

$$\begin{aligned}\frac{d}{dt}\left\{t^2(Af(t),[A,B]f(t))\right\} &= 2t(Af(t),[A,B]f(t)) \\ &\quad + t^2(A\partial_t f,[A,B]f(t)) + t^2(Af(t),[A,B]\partial_t f) \\ &= 2t(Af(t),[A,B]f(t)) - t^2(AA^*Af,[A,B]f(t)) \\ &\quad - t^2(ABf,[A,B]f(t)) - t^2(Af(t),[A,B](A^*A+B)f(t))\end{aligned}$$

In the last term we use that all three operators A, A^* and B commute with $[A,B]$, so

$$\begin{aligned}(Af(t),[A,B](A^*A+B)f(t)) &= (Af(t),A^*A[A,B]f(t)) + (Af(t),B[A,B]f(t)) \\ &= (Af(t),A^*A[A,B]f(t)) - (BAf(t),[A,B]f(t))\end{aligned}$$

since B is antisymmetric. Therefore we obtain

$$\begin{aligned}\frac{d}{dt}\left\{t^2(Af(t),[A,B]f(t))\right\} &= 2t(Af(t),[A,B]f(t)) - t^2(AA^*Af(t),[A,B]f(t)) \\ &\quad - t^2([A,B]f(t),[A,B]f(t)) - t^2(Af(t),A^*A[A,B]f(t))\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}\left\{t^2(Af(t),[A,B]f(t))\right\} &= 2t(Af(t),[A,B]f(t)) \\ &\quad - t^2\|[A,B]f(t)\|^2 - t^2(Af(t),(A^*A+AA^*)[A,B]f(t)).\end{aligned} \tag{3.3}$$

Lastly we compute the time derivative of the commutator:

$$\begin{aligned}\frac{d}{dt}\left\{t^3\|[A,B]f(t)\|^2\right\} &= 3t^2\|[A,B]f(t)\|^2 + 2t^3([A,B]\partial_t f,[A,B]f(t)) \\ &= 3t^2\|[A,B]f(t)\|^2 - 2t^3([A,B](A^*A+B)f(t),[A,B]f(t))\end{aligned}$$

and since A, A^* and B commute with $[A,B]$, using the anti-symmetric property of B we get

$$\frac{d}{dt}\left\{t^3\|[A,B]f(t)\|^2\right\} = 3t^2\|[A,B]f(t)\|^2 - 2t^3\|A[A,B]f(t)\|^2. \tag{3.4}$$

We collect now (3.1)–(3.4) to construct a suitable Lyapunov functional. Observe that, for $\lambda > 0$ and suitably chosen positive numbers a, b, c, the function

$$Vf(t) := \lambda\|f(t)\|^2 + at\|Af(t)\|^2 + bt^2(Af(t),[A,B]f(t)) + ct^3\|[A,B]f(t)\|^2$$

is coercive and we have

$$Vf(t) \ge \gamma\left\{\|f(t)\|^2 + t\|Af(t)\|^2 + t^3\|[A,B]f(t)\|^2\right\}$$

for some $\gamma > 0$. In particular, this is true whenever $b^2 < 4ac$; so, in the following we will require this condition to be satisfied. From the above computations, we have

$$\begin{aligned}\frac{d}{dt}(Vf(t)) &= (a-2\lambda)\|Af(t)\|^2 - 2a\,t\|A^*Af(t)\|^2 + 2(b-a)t(Af(t),[A,B]f(t)) \\ &\quad - (b-3c)t^2\|[A,B]f(t)\|^2 - bt^2(Af(t),(A^*A+AA^*)[A,B]f(t)) \\ &\quad - 2c\,t^3\|A[A,B]f(t)\|^2\,.\end{aligned} \tag{3.5}$$

We will therefore take $b > 3c$ and estimate

(3.6)
$$|2(b-a)t(Af(t),[A,B]f(t))| \leq \frac{(b-3c)}{2}t^2\|[A,B]f(t)\|^2 + \frac{2(b-a)^2}{(b-3c)}\|Af(t)\|^2.$$

We are left with the estimate of the "bad term" $(Af,(A^*A+AA^*)[A,B]f)$. First we notice that

$$(Af, A^*A+AA^*)[A,B]f) = (Af,[A,A^*][A,B]f) + (A^2f, A[A,B]f) \tag{3.7}$$

and then we estimate the above two terms. We use now assumption (1.6) to get

$$|(Af,[A,A^*][A,B]f)| \leq \beta\|Af\|\,(\|[A,B]f\| + \|A[A,B]f\|) \tag{3.8}$$

Moreover, from the identity

$$\|A^2f\|^2 = (Af,[A^*,A]Af) + \|A^*Af\|^2$$

we estimate, again using (1.6),

$$\|A^2f\|^2 \leq \|Af\|\beta(\|Af\| + \|A^2f\|) + \|A^*Af\|^2$$

which implies

$$\frac{1}{2}\|A^2f\|^2 \leq (\beta + \frac{\beta^2}{2})\|Af\|^2 + \|A^*Af\|^2\,.$$

Therefore, for the second term in (3.7) we have

$$\big|(A^2f, A[A,B]f)\big| \leq (\sqrt{2\beta+\beta^2}\|Af\| + \sqrt{2}\|A^*Af\|)\|A[A,B]f\|\,. \tag{3.9}$$

From (3.8) and (3.9) we derive the estimate

$$\begin{aligned} b\,t^2\,|(Af,A^*A+AA^*)[A,B]f)| &\leq b\,t^2\beta\|Af\|\,\|[A,B]f\| \\ &+ b\,t^2(\sqrt{2\beta+\beta^2}\|Af\| + \sqrt{2}\|A^*Af\|)\|A[A,B]f\| \\ &\leq \frac{(b-3c)}{4}\,t^2\|[A,B]f\|^2 + c\,t^3\|A[A,B]f\|^2 + \frac{b^2}{c}\,t\,\|A^*Af\|^2 \\ &+ \left\{\frac{b^2}{(b-3c)}\beta^2\,t^2 + \frac{b^2}{2c}(2\beta+\beta^2)\,t\right\}\|Af\|^2. \end{aligned} \tag{3.10}$$

Using (3.6) and (3.10) in (3.5) we obtain

(3.11)
$$\begin{aligned} \frac{d}{dt}(Vf(t)) &= (K(a,b,c)\left[1+\beta^2t^2+(\beta+\beta^2)\,t\right] - 2\lambda)\|Af(t)\|^2 \\ &- (2a - \frac{b^2}{c})\,t\|A^*Af(t)\|^2 - \frac{(b-3c)}{4}t^2\|[A,B]f(t)\|^2 - c\,t^3\|A[A,B]f(t)\|^2\,, \end{aligned}$$

for some constant K depending on a,b,c. We can choose $b > 3c$, and then a sufficiently large, in particular such that $b^2 - 4ac < 0$, which implies that the Lyapunov function V is coercive. Finally, choosing λ large enough we obtain that

$$\frac{d}{dt}(Vf(t)) < 0\,.$$

Notice that λ depends on t_0 whenever $t \leq t_0$. However, if $\beta = 0$ (i.e. A, A^* commute), then λ is independent of t and the estimate holds for every $t > 0$. The decreasing character of $V(f(t))$ implies (1.7). $\square$

3.2. **Higher order estimates.** We now prove Proposition 1.2.

Proof of Proposition 1.2. We notice that

$$\begin{aligned}\frac{d}{dt}\left\{t^2\|A^*Af(t)\|^2\right\} &= 2t\|A^*Af(t)\|^2 - 2t^2\|AA^*Af(t)\|^2\\ &\quad - 2t^2([A^*A, B]f(t), A^*Af(t)).\end{aligned}$$

Hence from assumption (1.8) we obtain

$$\begin{aligned}\frac{d}{dt}\left\{t^2\|A^*Af(t)\|^2\right\} &\leq 2t\|A^*Af(t)\|^2 - 2t^2\|AA^*Af(t)\|^2\\ &\quad + \beta^2\, t^3\|A[A, B]f(t)\|^2 + t\|A^*Af(t)\|^2.\end{aligned}$$

Recalling (3.11), we deduce that

$$\frac{d}{dt}\left\{V(f(t)) + \varepsilon\, t^2\|A^*Af(t)\|^2\right\} < 0$$

provided ε is small. Therefore, we gain the further estimate

$$t^2\|A^*Af(t)\|^2 \leq C\|f(0)\|^2\,.$$

Finally, observe that $[A, B]f$ satisfies the same abstract equation as f, since A^*, A and B commute with $[A, B]$. Therefore we can apply the conclusion of Proposition 1.1 to $[A, B]f$, obtaining

$$\begin{aligned}t\,\|A[A, B]f(t)\|_X^2 + t^3\|[A, B]^2f(t)\|_X^2 &\leq C\|[A, B]f(t/2)\|_X^2\\ &\leq \frac{C}{t^3}\|f(0)\|_X^2\,.\end{aligned}$$

This completes (1.9). $\square$

3.3. $L^1 \to L^\infty$ **decay for the Kolmogorov equation.** Let us again specialize to the Kolmogorov equation (1.1). In this case Proposition 1.2 provides estimates for the second derivatives with the right scaling:

$$t\|\partial_{xx}f(t)\|_{L^2} + t^2\|\partial_{xy}f(t)\|_{L^2} + t^3\|\partial_{yy}f(t)\|_{L^2} \leq C\|f_0\|_{L^2}\,.$$

In particular, by using the embedding of H^2 into L^∞, or the precised Gagliardo-Nirenberg estimate (2.21) in the continuous version, this inequality implies the $L^2 \to L^\infty$ regularizing effect. On account of the scaling, this reads as

$$\|f(t)\|_\infty \leq \frac{C}{t}\|f_0\|_{L^2}\,.$$

By duality, this gives the $L^1 \to L^2$ regularizing estimate, and, finally, by composition the $L^1 \to L^\infty$ estimate

$$\|f(t)\|_\infty \leq \frac{C}{t^2}\|f_0\|_{L^1}\,.$$

4. Further comments and open problems.

As mentioned above, the content of this paper is a first contribution in the context of numerical hypo-coercivity for Kolmogorov-type equations. We planned to keep the exposition at the simplest level and this is why we concentrated on the very precise Kolmogorov equation (1.1) to serve as a reference model. Of course, more general classes of equations could now be investigated with similar tools and many interesting questions remain open. We mention here some of them together with some further remarks.

- **Higher dimensional problems.** The same results apply for the Kolmogorov equation in higher space dimensions, i. e. when both x and y belong to $\mathbb{R}^d$ and this for all $d \geq 1$. This is so provided the equations are discretized in each variable by the same centered schemes in cartesian grids using finite-differences.
- **The CFL condition.** As stated in Theorem 1.4 a CFL condition (1.15) was imposed to ensure the numerical hypo-coercivity of the fully discrete scheme. This was so to ensure uniform estimates using only the ℓ_h^2-norm of the discrete initial data (1.16). In the absence of that condition we were only able to prove the weaker version (1.17). It would be interesting to analyze whether this CFL condition, that arises naturally when developing the Lyapunov approach at the time-discrete level, is sharp.
- **Non-uniform meshes.** Whether the same results hold for more finite-difference general schemes, associated to variable grids, for instance, is an interesting open problem. In the context of wave equations it is well known that the propagation properties of solutions are sensitive to the geometry of the numerical mesh. Whether numerical hypo-coercivity experiences the same phenomena is an interesting open problem.
- **Other numerical approximation methods.** It would be interesting to analyze whether the same results hold for other approximation schemes, such as the finite element method. Note however that, in particular, the fact that the scheme applied to approximate the B operator $x\partial_y$ leads to an antisymmetric operator B_h plays a key role in the proof. In fact the proof of numerical hypo-coercivity above does not apply even in the finite-difference context if, rather than employing the centered discretization above, one uses a forward or backward finite-difference. It would also be interesting to exhibit examples showing that the numerical hypo-coercivity property actually fails, in the absence of this anti-symmetry property.

  Finally, we have analyzed the time-discrete implicit Euler scheme. Similar issues arise and would be worth considering for more general time-discretization schemes.

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