then $P(A_n \text{ i.o.}) \geq \alpha$. The case $\alpha = 1$ contains (6.6).

1.7. Strong Law Of Large Numbers

We are now ready to give Etemadi's proof of

(7.1) **Strong law of large numbers.** Let $X_1, X_2, \ldots$ be pairwise independent identically distributed random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \ldots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.

Proof As in the proof of weak law of large numbers, we begin by truncating.

(a) **Lemma.** Let $Y_k = X_k 1_{(|X_k| \leq k)}$ and $T_n = Y_1 + \cdots + Y_n$. It is sufficient to prove that $T_n/n \to \mu$ a.s.

Proof $\sum_{k=1}^{\infty} P(|X_k| > k) \leq \int_0^\infty P(|X_1| > t)\, dt = E|X_1| < \infty$ so $P(X_k \neq Y_k \text{ i.o.}) = 0$. This shows that $|S_n(\omega) - T_n(\omega)| \leq R(\omega) < \infty$ a.s. for all n, from which the desired result follows. □

The second step is not so intuitive but it is an important part of this proof and the one given in Section 1.8.

(b) **Lemma.** $\sum_{k=1}^{\infty} \text{var}(Y_k)/k^2 \leq 4E|X_1| < \infty$.

Proof To bound the sum, we observe

$$\text{var}(Y_k) \leq E(Y_k^2) = \int_0^\infty 2yP(|Y_k| > y)\, dy \leq \int_0^k 2yP(|X_1| > y)\, dy$$

so using Fubini's theorem (since everything is ≥ 0 and the sum is just an integral with respect to counting measure on $\{1, 2, \ldots\}$)

$$\begin{aligned}
\sum_{k=1}^{\infty} E(Y_k^2)/k^2 &\leq \sum_{k=1}^{\infty} k^{-2} \int_0^\infty 1_{(y<k)}\, 2y\, P(|X_1| > y)\, dy \\
&= \int_0^\infty \left\{ \sum_{k=1}^{\infty} k^{-2} 1_{(y<k)} \right\} 2yP(|X_1| > y)\, dy
\end{aligned}$$

Since $E|X_1| = \int_0^\infty P(|X_1| > y)\, dy$, we can complete the proof by showing

(c) **Lemma.** If $y \geq 0$ then $2y \sum_{k>y} k^{-2} \leq 4$.

Proof We being with the observation that if $m \geq 2$ then

$$\sum_{k \geq m} k^{-2} \leq \int_{m-1}^{\infty} x^{-2} dx = (m-1)^{-1}$$

When $y \geq 1$ the sum starts with $k = [y] + 1 \geq 2$ so

$$2y \sum_{k>y} k^{-2} \leq 2y/[y] \leq 4$$

since $y/[y] \leq 2$ for $y \geq 1$ (the worst case being y close to 2). To cover $0 \leq y < 1$ we note that in this case

$$2y \sum_{k>y} k^{-2} \leq 2\left(1 + \sum_{k=2}^{\infty} k^{-2}\right) \leq 4 \qquad \square$$

The first two steps, (a) and (b) above, are standard. Etemadi's inspiration was that since X_n^+, $n \geq 1$, and X_n^-, $n \geq 1$, satisfy the assumptions of the theorem and $X_n = X_n^+ - X_n^-$, we can without loss of generality suppose $X_n \geq 0$. As in the proof of (6.8) we will prove the result first for a subsequence and then use monotonicity to control the values in between. This time however, we let $\alpha > 1$, and $k(n) = [\alpha^n]$. Chebyshev's inequality implies that if $\epsilon > 0$

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) &\leq \epsilon^{-2} \sum_{n=1}^{\infty} \text{var}(T_{k(n)})/k(n)^2 \\
&= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \text{var}(Y_m) \\
&= \epsilon^{-2} \sum_{m=1}^{\infty} \text{var}(Y_m) \sum_{n:k(n) \geq m} k(n)^{-2}
\end{aligned}$$

where we have used Fubini's theorem to interchange the two summations (everything is ≥ 0). Now $k(n) = [\alpha^n]$ and $[\alpha^n] \geq \alpha^n/2$ for $n \geq 1$, so summing the geometric series and noting that the first term is $\leq m^{-2}$

$$\sum_{n:\alpha^n \geq m} [\alpha^n]^{-2} \leq 4 \sum_{n:\alpha^n \geq m} \alpha^{-2n} \leq 4(1-\alpha^{-2})^{-1} m^{-2}$$

Combining our computations shows

$$\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) \leq 4(1-\alpha^{-2})^{-1} \epsilon^{-2} \sum_{m=1}^{\infty} E(Y_m^2) m^{-2} < \infty$$

by (b). Since ϵ is arbitrary $(T_{k(n)} - ET_{k(n)})/k(n) \to 0$. The dominated convergence theorem implies $EY_k \to EX_1$ as $k \to \infty$, so $ET_{k(n)}/k(n) \to EX_1$ and we have shown $T_{k(n)}/k(n) \to EX_1$ a.s. To handle the intermediate values, we observe that if $k(n) \le m < k(n+1)$

$$\frac{T_{k(n)}}{k(n+1)} \le \frac{T_m}{m} \le \frac{T_{k(n+1)}}{k(n)}$$

(here we use $Y_i \ge 0$), so recalling $k(n) = [\alpha^n]$ we have $k(n+1)/k(n) \to \alpha$ and

$$\frac{1}{\alpha} EX_1 \le \liminf_{n\to\infty} T_m/m \le \limsup_{m\to\infty} T_m/m \le \alpha EX_1$$

Since $\alpha > 1$ is arbitrary the proof is complete. □

The next result shows that the strong law holds whenever EX_i exists.

(7.2) **Theorem.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. If $S_n = X_1 + \cdots + X_n$ then $S_n/n \to \infty$ a.s.

Proof Let $M > 0$ and $X_i^M = X_i \wedge M$. The X_i^M are i.i.d. with $E|X_i^M| < \infty$ so if $S_i^M = X_1^M + \cdots + X_n^M$ then (7.1) implies $S_n^M/n \to EX_i^M$. Since $X_i \ge X_i^M$ it follows that

$$\liminf_{n\to\infty} S_n/n \ge \lim_{n\to\infty} S_n^M/n = EX_i^M$$

The monotone convergence theorem implies $E(X_i^M)^+ \uparrow EX_i^+ = \infty$ as $M \uparrow \infty$, so $EX_i^M = E(X_i^M)^+ - E(X_i^M)^- \uparrow \infty$ and we have $\liminf_{n\to\infty} S_n/n \ge \infty$ which implies the desired result. □

The rest of this section is devoted to applications of the strong law of large numbers.

Example 7.1. Renewal theory. Let $X_1, X_2, \ldots$ be i.i.d. with $0 < X_i < \infty$. Let $T_n = X_1 + \ldots + X_n$ and think of T_n as the time of nth occurrence of some event. For a concrete situation consider a diligent janitor who replaces a light bulb the instant it burns out. Suppose the first bulb is put in at time 0 and let X_i be the lifetime of the ith lightbulb. In this interpretation T_n is the time the nth light bulb burns out and $N_t = \sup\{n : T_n \le t\}$ is the number of light bulbs that have burnt out by time t.

(7.3) **Theorem.** If $EX_1 = \mu \le \infty$ then as $t \to \infty$, $N_t/t \to 1/\mu$ a.s. $(1/\infty = 0)$