groups is known as group theory. Among the various branches of this science the theory of substitution groups is the oldest, having been founded by A. L. Cauchy, about 1845, and first embodied in the form of a separate treatise by C. Jordan, *Traité des substitutions et équations algébriques*, 1870.

The theory of substitutions contains two large branches. The older of these is sometimes called the theory of permutation groups, and is based on the possible interchanges of letters, while the other branch is involved in the theory of linear transformations, and is commonly known as the theory of linear substitution groups.

The theory of finite abstract groups is intimately connected with the two theories of substitution groups just noted and was first embodied in the form of a separate treatise by W. Burnside, *Theory of groups of finite order*, 1897; second edition with greater emphasis on linear groups, 1911. These three theories are sometimes referred to as *algebraic group theory*. There is, however, no clear line of distinction between algebraic group theory and the group theories of analysis and geometry.

The group theory of analysis may also be divided into three large branches, viz., theory of finite continuous groups, theory of infinite continuous groups, and theory of groups of automorphic functions. The first of these theories was first developed in a systematic manner by S. Lie, *Theorie der Transformationsgruppen*, three large volumes, while the last was treated in volume 1 of *Automorphe Functionen* by R. Fricke and F. Klein, 1897. No systematic treatise on the general theory of infinite continuous groups has yet been published.

Geometric group theory is based on the group theories of algebra and analysis. In geometry the group concept has entered more widely into the various developments than in algebra or in analysis. Among the treatises devoting considerable space to geometric groups we may mention Klein's *Einleitung in die höhere Geometrie*, II, 1893, and Lie's *Geometrie der Berührungstransformationen*, 1896.

C. Alasia prepared a general bibliography on group theory, which was published in volumes 18–22 of *Rivista di fisica matematica e scienze naturali*, Pavia. A bibliography relating to finite groups together with many historical data may be found in the *Constructive development of group theory* by B. S. Easton, 1902. Among the treatises on the theory of groups which were not noted above are the following: E. Netto, *Substitutionentheorie*, 1882; translated into Italian by G. Battaglini, 1885, and into English by F. N. Cole, 1892; S. Lie and G. Scheffers, *Vorlesungen über Differentialgleichungen*, 1891, and *Vorlesungen über kontinuierliche Gruppen*, 1893; G. Vivanti, *Teoria dei gruppi di transformazioni*, 1898; translated into French by A. Boulanger, 1904; L. Bianchi, *Lezioni sulla theoria dei gruppi di sostituzioni*, 1900; L. E. Dickson, *Linear Groups*, 1901; J. E. Campbell, *Theory of Continuous Groups*, 1903; J. A. de Séguier, *Groupes Abstraits*, 1904; G. Fubini, *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908; H. Hilton, *Finite Groups*, 1908; E. Netto *Gruppen- und Substitutionentheorie*, 1908; J. A. de Séguier, *Groupes de Substitutions*, 1912; Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, 1916; H. F. Blichfeldt, *Finite Collineation Groups*, 1917.

FUNDAMENTALS IN THE MATHEMATICS OF INVESTMENT.

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§ 1. Introduction.

In most books on the mathematics of investment there is a wealth of formula somewhat forbidding to the casual reader, however necessary it may be to the accountant or actuary. It is the object of this paper to present in rather compact form some of the fundamentals of the subject, with a few general formulas of wide application.

§ 2. Interest and Discount.

The mathematics of investment deals with the *increment* of *value*. If P units of value—say P dollars—at one moment of time are worth or conceived to be worth S units at a later moment, the increment $S - P$ is called the *interest*[1]

[1] For the general theory, it is immaterial whether the change of value is brought about by a loan or by a series of commercial transactions, indeed, whether the increment is positive or negative.

(on P), and this same difference is called the *discount* (on S) for the period of time determined by the given moments. The ratios

$$r = \frac{S - P}{P}, \qquad u = \frac{S - P}{S} \tag{1}$$

are respectively the *rate* of *interest* and the *rate* of *discount* for the period.

Discount is also called *interest in advance*; S is called the *amount* of P, and P the *present value* or *present worth* of S; P is often called the *principal* or *capital*.

If each dollar of P increases by r, the total increase is $S - P$, the given increase. The interest-rate for a period, then, may just as well be defined as the increase of *one* (dollar) during the period. In place of a dollar, any unit of money or value may be used.

Likewise, if from each dollar of S, the value at the end of the interval, the same deduction u is made, the total deduction is $S - P$, the difference between the ultimate and initial value of the money in question. Thus u may be defined as the discount on *one* (dollar) for the period.

The sum of money $1/(1 + r)$ at the beginning of the period becomes *one* at the end of the period. The former is, then, the *present value* or *present worth* of the latter, and may be designated by w. Then

$$\frac{1}{1 + r} = w = 1 - u. \tag{2}$$

Illustration.—A man borrows \$100 for one year at a discount of 4%—or at 4% interest in advance. In this case, he actually receives \$96 and must pay \$100 at the end of the year. Thus, \$1 due in one year is worth \$.96 now; $w = .96$. The discount on \$1 is \$.04; $u = .04$. The interest is \$4 on \$96, the interest rate is about $4\frac{1}{6}\%$; $r = .0416 +$.

§ 3. Derived Interest Rates.

Let r be the interest rate for each of n consecutive periods[1] of time. Then an initial P (dollars) becomes $P(1 + r)$ at the end of the first period, $P(1 + r)^2$ at the end of the second period, $\cdots$, $P(1 + r)^n$ at the end of the nth period. For the sake of simplicity, the P may be dropped. The amount of *one* for the entire period or *term*—formed by joining the n consecutive periods—is

$$1 + r_n = (1 + r)^n,$$

where, by the definition of § 2, r_n is the *interest* on *one* for the entire period or term.

In conformity with this, the amount of *one* for t periods,[2] where t is any

[1] The periods need not be of equal length. The interest charged for February may be the same as for March. So far as the theory goes, there is no reason why the periods should be even approximately equal.

[2] Even this does not require the periods to be of equal length. Moments of time are merely to be in one-to-one correspondence with real numbers, the later of two moments to be associated with the number algebraically greater.

positive real number, is defined to be

$$1 + r_t = (1 + r)^t. \tag{3}$$

The interest on *one* for the entire period is then

$$r_t = (1 + r)^t - 1. \tag{4}$$

If w_t and u_t refer to this new period, it follows from (2) and (3) that

$$\frac{1}{1 + r_t} = 1 - u_t = w_t = w^t = (1 - u)^t. \tag{5}$$

Illustration.—Many banks pay a "*nominal* 4%" *convertible semi-annually.* This means that they pay 2% for 6 months. Then \$100 becomes \$102 at the end of 6 months, and this \$102 becomes \$104.04 at the end of the next 6 months. This makes \$4.04 the interest on \$100 for one year, and the interest rate for the year is $4\frac{1}{25}\%$. This is in conformity with (4) where the original period is 6 mo., $r = .02$, $t = 2$, $r_t = .0404$.

§ 4. Compound Interest vs. Simple Interest.

If in Equation (4), the Binomial Theorem is used and only two terms retained, the result is simple interest on *one* (dollar). The error of the approximation, using simple interest for compound, is the sum of the terms after the second in the binomial expansion.

If P at one moment is worth Q at another moment, then P and Q will be said to be *equivalent* to each other. Thus, equivalence involves the notion of time as well as of value.

A fundamental property of *compound interest* is the following:

Two sums of money each equivalent to a third sum of money are equivalent to each other.

Thus P at one time is equivalent to $P(1 + r)^t$ after the lapse of the time t. This in turn is equivalent to $P(1 + r)^{t+t'}$ after the further lapse of time t'. But the latter is also equivalent to the original principal P after the lapse of the time $t + t'$.

Thus the initial and the ultimate value are each equivalent to the middle value, and they are equivalent to each other.

But, if *simple interest* is used, two sums of money each equivalent to a third are not equivalent to each other.

For major financial computations, simple interest would be absurd—although it often gives a permissible approximation for a fraction of a year.

Thus, the amount must be an exponential function of the time; it can not be a linear function of the time.

The amount is, indeed, a linear function of the principal. Thus often we may ignore the principal at first, and merely use it as a multiplier as the concluding step in a problem.

Illustration.—If at 5% *simple* interest, \$100 is loaned for 4 years, it becomes \$120; and if this is reloaned for 6 years, it becomes \$156. Whereas, if the \$100 were loaned for 10 years straight the amount would be only \$150.

But if \$100 is kept continuously at strict 5% *compound* interest for 10 years, the amount will be exactly \$100 $(1.05)^{10}$, even if the money changes hands a dozen times.

§ 5. Perpetuities.

A *perpetuity*[1] is an infinite series of values, associated with moments of time which extend indefinitely into the future. These moments are usually thought of as the end moments of the periods into which they divide time. When it is desirable to associate the values—usually called *payments*—with the initial moments of the periods, the perpetuity is called a *perpetuity-due*, or, often, an *immediate perpetuity*. Unless otherwise sta ed, the payments of the perpetuity are to be taken as all equal; indeed, frequently it is understood that each payment is a payment of *one* (dollar).

If r is the interest rate for each period, the *present value of a perpetuity* of *one* per period, payable at the end of each period forever is

$$b_\infty = \frac{1}{r}; \tag{6}$$

and the *present value of a perpetuity-due* of *one* per period, payable at the beginning of each period forever is

$$B_\infty = \frac{1}{u}. \tag{7}$$

For $1/r$ will yield as interest *one* at the end of each period forever; and $1/u$ will yield as interest in advance *one* at the beginning of each period forever. The interest is to be withdrawn as soon as it falls due.

Illustration.—A building must be reconstructed at the end of every 25 years, at an expense of \$10,000. What sum of money put out at 4% compound interest will pay for the renewals forever? Let r_{25} be the interest rate for the period of 25 years. Then from (6) the required endowment is

$$10{,}000\left(\frac{1}{r_{25}}\right) = \frac{10{,}000}{r_1}\left(\frac{r_1}{r_{25}}\right) = 250{,}000\ (.0240120), = \$6003,$$

as found by using a monetary table.

A perpetuity deferred t periods is a series of perpetual payments, the first payment to be made after the lapse of $1 + t$ periods,—thus the first payment is made t periods later than it would normally be made. This t may be any positive real number,—indeed t may be negative, if the *forborne perpetuity*, to be considered presently, is counted as a special case of a deferred annuity. It follows from (5) and (6) that

[1] If \$4 is to be collected as interest on \$100 at the end of each year forever, this is called a perpetuity.

$$_{t|}\, b_\infty = w^t \left(\frac{1}{r}\right) \tag{8}$$

is the present value of *one* per period forever, the first payment to be made after the lapse of $1 + t$ periods.

Likewise, for the *perpetuity-due deferred t periods,*

$$_{t|}\, B_\infty = w^t \left(\frac{1}{u}\right) \tag{9}$$

is the present value of *one* per period forever, the first payment to be made after the lapse of t periods.

Now $(1 + r)^t b_\infty$ at any given moment is equivalent to b_∞ for that moment earlier by t periods, and hence is the "present value" of a perpetuity whose first payment was made $t - 1$ periods earlier. Thus, for this *forborne perpetuity,*

$$_{-t|}\, b_\infty = (1 + r)^t \left(\frac{1}{r}\right) \tag{10}$$

is the sum of the "amounts" of the earlier payments of *one* per period, begun $t - 1$ periods earlier, and the "present values" of the later payments of *one* per period ad infinitum.

Likewise, for the *perpetuity-due forborne t periods,*

$$_{-t|}\, B_\infty = (1 + r)^t \left(\frac{1}{u}\right) \tag{11}$$

is the sum of the amounts of the earlier payments of *one* per period, begun t periods earlier, and the present values of the later payments of *one* per period in regular continuation forever.

§ 6. Annuities.

An *annuity* is a series of periodic payments. The payments are usually equal and limited in number.

A perpetuity may be considered as an annuity with an infinite number of payments. *And an annuity may be considered as the difference between two perpetuities starting at different times.*

The value of an annuity may be required at any time. But usually its value is required (1) at the time of the first payment, or (2) one period before this payment is made, or (3) at the time of the last payment, or (4) one period later than the last payment. Taking n as the number of payments, these four values, in order, are

$$B_n = \frac{1}{u} - w^n \left(\frac{1}{u}\right) = \frac{u_n}{u}, \tag{12}$$

$$b_n = \frac{1}{r} - w^n \left(\frac{1}{r}\right) = \frac{u_n}{r}, \tag{13}$$

$$c_n = (1 + r)^n \left(\frac{1}{r}\right) - \frac{1}{r} = \frac{r_n}{r}, \tag{14}$$

$$C_n = (1 + r)^n \left(\frac{1}{u}\right) - \frac{1}{u} = \frac{r_n}{u}, \tag{15}$$

where each payment is *one*, as is seen by referring to Equations (3)–(11).

These four formulas can be proven without reference to perpetuities. To prove (14), note that an initial *one* (dollar) is worth r per period, payable at the end of each period for n periods, together with *one* at the end of the nth period—this terminal *one* is principal returned. But an initial *one* is also worth $(1 + r)^n$ at the end of the nth period. Hence r per period for n periods is worth $(1 + r)^n - 1$ at the end of the nth period. Thus *one* per period for n periods is worth $\{(1 + r)^n - 1\}/r = r_n/r$, at the end of the nth period.

The most common proof of these formulas involves the summing of a geometric progression.

Illustration.—If a man deposits, in a bank that pays 4%, \$100 at the end of each year for 25 years, the accumulation to his credit at the end of the 25 years will be \$4,164.59, as found from a table based upon (14).

The terms *deferred* and *forborne* are applied to annuities in the same way as to perpetuities. Thus, the present value of an annuity *deferred* t periods is $w^t b_n$.

B_n and C_n are respectively the "present value" and "accumulation" of an *annuity-due,*—here a payment is made at the *beginning* of each interval.

In (12)–(15), the r and u are respectively the interest-rate and discount-rate for the *period between payments.* If an interest rate is given for some other period—say for a period of length t—then by (3) and (2) these formulas (12)–(15) may be transformed to involve r_t explicitly.

The following interesting and useful relations may be easily proved algebraically or arithmetically:

$$b_n = w^n c_n, \qquad B_n = w^n C_n,$$

$$\frac{1}{b_n} = \frac{1}{c_n} + r, \qquad \frac{1}{B_n} = \frac{1}{C_n} + u.$$

§ 7. Varying Annuities and Perpetuities.

It has been found that $1/u$ will furnish *one* at the beginning of each period forever. Required the capital that will furnish *one* at the beginning of the first period, *two* at the beginning of the second period, and so on, increasing forever. These payments will be furnished if a perpetuity-due is started at the beginning of each period. The capital required for this perpetuity-due of perpetuities-due is

$$(IB)_\infty = \frac{1}{u^2}. \tag{17}$$

The formula for a perpetuity-due of perpetuities is

$$(Ib)_\infty = \frac{1}{u} \cdot \frac{1}{r} = \frac{1}{r^2} + \frac{1}{r}. \tag{18}$$

This yields *one* at the *end* of the first period, *two* at the *end* of the second period, and so on, increasing forever.

An *increasing* annuity or perpetuity is one in which the successive payments are in order, *one, two, three,* etc., as in counting. Thus, an increasing annuity of n payments is an increasing perpetuity minus an increasing perpetuity started n periods later, minus also n times a perpetuity of *one* started likewise at the latter moment.

A *varying* annuity or perpetuity may have payments forming an arithmetic progression of second or higher order. A reader interested in such annuities is referred to "The Institute of Actuaries' Text Book, Part I," pages 40–48.

§ 8. Complete Annuities.

Suppose that the interest r on *one* (dollar) is collected at the end of each of m consecutive periods. If the loan is continued for the fraction t of a period, and the "*face*" of this loan—viz. *one*—is then collected, interest for that fraction t of a period is also due, to the extent of r_t, as per (4). Now the interest payments of r each form an annuity, and the single payment of r_t is said to *complete* this annuity. Likewise, if $1/r$ is loaned, an annuity of *one* per period results, which is *completed* by the payment of r_t/r.

Let the whole term of the loan be n periods. Then $n = m + t$. Now an initial $1/r$ has at the end of n periods the value $(1 + r)^n/r$. Thus, reasoning as in § 6, *one* per period for m periods, followed by the completing payment of r_t/r, is worth at the end of n periods,

$$\frac{1}{r}(1 + r)^n - \frac{1}{r} = \frac{r_n}{r},$$

as in (14). Formula (14) has thus been made valid for all positive real values of n provided that the completing payment is r_t/r when n is not integral. This completing payment approaches one when t approaches one, as obviously it should.

Indeed, formulas (12)–(15) are valid for all positive values of n if the completing payment of r_t/r is made at the end of the n periods.

Likewise they hold if the set of m regular payments of *one* each is preceded by an initial completing payment of u_t/u, made earlier than the first regular payment by the fraction t of a period.

Indeed, these formulas will remain valid if any set of k consecutive regular payments is replaced by a single payment of u_k/u at the beginning of the term of the special set, or by a single payment of r_k/r at the end of the term of the special set.

If the t above is a fraction of a small period of time, such as a year, a fair approximation for r_t/r, or indeed for u_t/u, will be t itself, in most cases—as easily seen by using the Binomial Theorem.

Illustration.—On a loan of \$2,500 at 4%, interest payments of \$100 have been collected regularly for a certain number of years. The lender wishes the loan of \$2,500 paid 6 months after the last interest payment has been made. In practice \$50, the simple interest on the \$2,500 for 6 months would be collected. But this is not the equitable interest. \$100 payable at the end of a year is equivalent to \$49.50 payable at the end of 6 months and \$49.50 at the end of the year. The interest on \$49.50 for 6 months makes up the extra dollar. An annuity of \$100 per year is completed (exactly) by \$49.50 at a moment 6 months later than a regular payment. This illustrates the fact that *for a fraction of one period, simple interest is greater than compound interest,* so that a money lender can well afford to substitute simple interest for compound interest for a fraction of the specified interest period.

§ 9. Common Notation.

To gain generality, certain commonly accepted symbols have been avoided in this paper thus far.

When the year is taken as the unit of time, the usual symbols are as follows:

$$r = i, \qquad u = d, \qquad w = v,$$

$$b_n = a_{\overline{n}|}, \qquad c_n = s_{\overline{n}|}, \qquad B_n = \mathbf{a}_{\overline{n}|}, \qquad C_n = \mathbf{s}_{\overline{n}|}.$$

Thus

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \qquad s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

The corresponding rate of interest for one one-mth of a year is j_m/m; and the corresponding rate of discount is f_m/m. Here j_m is called the "*nominal rate of interest,*" and f_m the "*nominal rate of discount.*"

The equations

$$1 + i = \left(1 + \frac{j_m}{m}\right)^m = v^{-1} = (1 - d)^{-1} = \left(1 - \frac{f_m}{m}\right)^{-m} = e^{\delta}$$

connect the most fundamental of these quantities where the force of interest

$$\delta = \log_e (1 + i) = \lim_{m=\infty} j_m = \lim_{m=\infty} f_m,$$

with i constant.

§ 10. Conclusion.

No attempt has been made to display all the formulas most frequently used, or to introduce the reader to the many fascinating applications in connection with premiums on bonds, amortization schedules, wearing values of machinery, sinking funds, etc.

Few students of mathematics have any conception of the beauty or difficulty of certain problems arising in actual business transactions.

But an intense arithmetic appreciation of the important relations underlying annuities and perpetuities will go far toward equipping a student to solve problems of this kind.

BOOK REVIEW.

SEND ALL COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

Introduction to Mathematical Statistics. By CARL J. WEST, Ph.D., Assistant Professor of Mathematics, Ohio State University. R. G. Adams and Co., Columbus, 1918.

In spite of the wide divergence between the original sources and purposes of mathematical statistics, the present development of the subject seems to be along two main lines—that is, either it presents statistical information consisting usually of numerical measurements in a form easily and quickly interpreted by the eye, or it derives and applies formulas for the purpose of measuring various phenomena presented by the measurements.

The first line of development calls for very elementary mathematical knowledge and has been well treated by several authors, especially Brinton. The second line of development has been treated in its various phases in scientific journals and a few English and German books; it involves mathematical principles ranging from the most elementary to the most abstruse but was never presented by an American statistician in approximately complete form until Dr. West's *Introduction to Mathematical Statistics* appeared. The usefulness of the German books in this country is impaired by the language while all the other foreign books on the subject are too voluminous to be used as textbooks in American colleges.

As may be expected, Dr. West's book gives a treatment of both lines of development and, considering the great amount of literature of the second kind in various scientific journals, shows good judgment in the selection of important principles. No doubt, Dr. West's book will be widely used—especially as a textbook in colleges—with few rivals for many years to come, although any author of a statistical textbook would merit considerable praise even though he did little more than encourage wider study and help to standardize methods.

The reviewer agrees with Dr. West in most of his introductory statements but insists that a knowledge of at least the calculus is indispensable to a full understanding of the principles involved in the second line of development mentioned above, especially the work of Pearson and his followers, some of which is treated by Dr. West.

The printers show a lack of experience with scientific books or else the proof was not carefully read; the alignment is poor in places—for example, letters are out of line in eighteen different places on page 18—and typographical errors are fairly frequent.