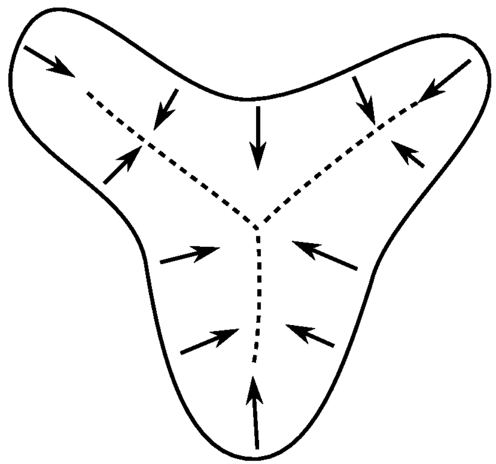
Singular sets of Hamilton-Jacobi equations and cut loci

by Pablo Angulo Ardoy and Luis Guijarro

Universidad Autónoma de Madrid

$$\begin{aligned} H(p, du(p)) &= 1 & \quad p &\in \Omega \\ u(p) &= g(p) & \quad p &\in \partial\Omega \end{aligned}$$



Hamilton-Jacobi equations

Find $H : \Omega \to \mathbb{R}$ *such that:*

$$\begin{aligned} H(p, du(p)) &= 1 \qquad && p \in \Omega \\ u(p) &= g(p) && p \in \partial\Omega \end{aligned} \tag{1}$$

- $H^{-1}(1) \cap T_p^*\Omega$ is **convex** for every p, and contains 0.
- Ω is a smooth and **compact manifold with boundary**, H and g are smooth.
- The boundary data satisfies a **compatibility condition** (more about it later)

$$|g(y) - g(z)| < d(y, z) \tag{2}$$

A geometrical interpretation

1. We can assume H is a norm in each vector space $T_p^*\Omega$ (if necessary, replace H with $\tilde{H}(p,w)=t$, for the only $t>0$ such that $H(p,\frac{1}{t}w)=1$)

2. Define *the dual norm* φ in $T_p\Omega$

$$\varphi_p(v)=\sup\{\langle v,\alpha\rangle_p\colon \alpha\in T_p^*\Omega,\ H(p,\alpha)=1\}$$

3. This is a **Finsler** metric, which induces a distance d in Ω

4. The metric is Riemannian iff H is quadratic in its second argument.

Classical solution by characteristic curves

The HJ equations are first order PDEs, and thus *there is a solution using characteristic curves*, defined *only in a neighborhood of* $\partial\Omega$.

If $x = \gamma(t)$ for a characteristic γ with $\gamma(0) = y$, then $u(x) = t + g(y)$

The **characteristic curves are geodesics** of φ, whose initial condition at $y \in \partial\Omega$ is the vector V_y satisfying:

$$\varphi_y(V_y) = 1 \qquad \widehat{V_y}|_{T_y(\partial\Omega)} = dg \qquad V_y \text{ points inwards}$$

In particular, if g is constant and φ Riemannian, V is perpendicular to $\partial\Omega$.

For a vector V in a Finsler space, $w = \hat{V}$ is its **dual one-form**, given by:

$$w_j = \frac{\partial\varphi}{\partial V^j}(p, V)$$

This is the usual definition of dual form if φ is a riemannian metric.

Viscosity solution

A *viscosity solution* is a solution in a weak sense, **defined in all Ω**.

- The *inspiration* is too *add a viscosity term* to the HJ equations and make $\varepsilon \to 0$:

$$H(p, du(p)) + \varepsilon \triangle u(p) = 1$$

- The *definition* of viscosity solutions relies on test functions that *touch u from above (and below)*.

- There are other equivalent definitions (e.g., with *semiconcave* functions).

The solution obtained with characteristic curves coincides with the viscosity solution where both are defined.

Lax-Oleinik formula

Ths viscosity solution is given by a formula involving the Finsler distance:

$$u(p) = \inf_{q \in \partial\Omega} \{d(p, q) + g(q)\}$$

Comments

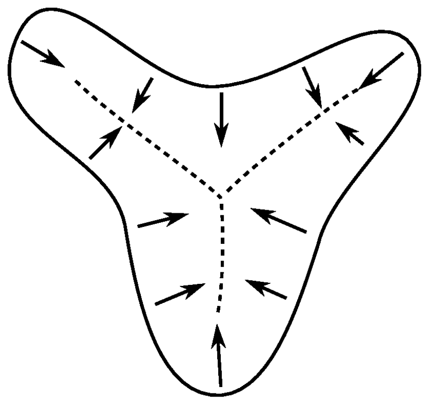
- The compatibility condition $|g(y) - g(z)| < d(y, z)$ is necessary and sufficient for solutions to exist.
- If $g = 0$, then u is the *distance to the boundary*.
- The solution is not C^1 in all of Ω.

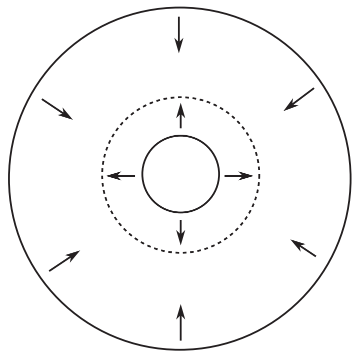
The singular set

Characteristic curves from $\partial\Omega$ intersect each other if continued indefinitely.

The extra information required to *get the viscosity solution from the classical one* is a criterion to decide which characteristic curve is used to compute the value of u at a given point.

This extra information is the singular set of the solution u:



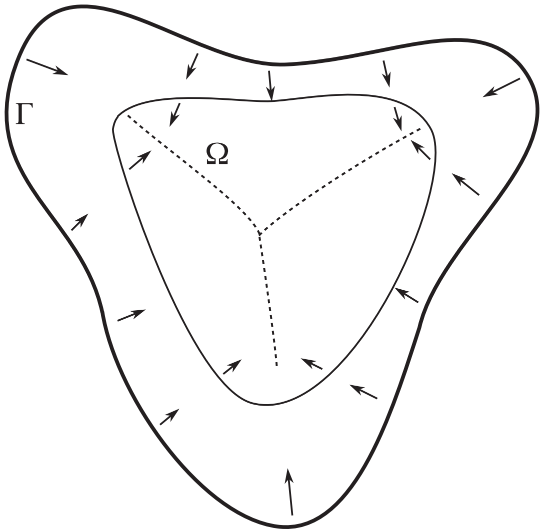


Let Sing be the closure of the singular set of u

What do we know about the singular set of the viscosity solution?

If $g=0$: u is the distance to the boundary, **Sing** is the **cut locus**.

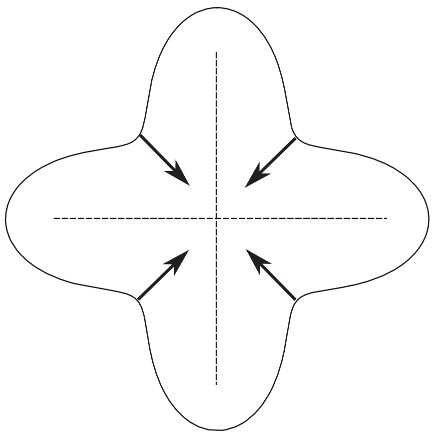
And indeed, a solution with $g \neq 0$ in Ω is the *restriction* of the solution with $g=0$ in a bigger set $\Gamma \supset \Omega$:



$$\begin{aligned} H(p, dv(p)) &= 1 \qquad && p \in \Gamma \\ v(p) &= 0 && p \in \partial\Gamma \end{aligned}$$

$$u = v|_\Omega$$
$$\mathrm{Sing}(u) = \mathrm{Sing}(v)$$

Some special cases



If H *depends only* on du,
the characteristics are straight lines.
(this includes the eikonal equation, $|\nabla u| = 1$*)*

If Ω is a *simply connected plane* region,
Sing is a *tree*.

If furthermore, Ω, H and g are all *analytic*,
then **Sing** is a *finite* tree.

If Ω is not planar, but Ω, H and g are *analytic*
then **Sing** is a stratified smooth manifold.

Structure of the singular set

Singular sets (cut loci) are studied by PDE and geometry people

- The singular set is a deformation retract of Ω (obvious).

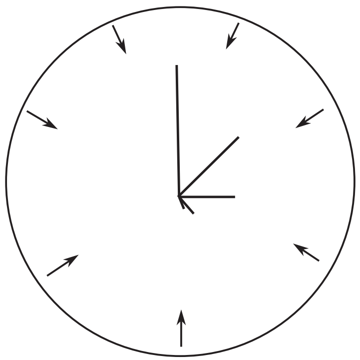
- It is the union of a $(n-1)$-dimensional smooth manifold consisting of points with two minimizing geodesics and a set of Hausdorff dimension at most $n-2$ (Hebda87, Itoh-Tanaka98, Barden-Le97, Mantegazza-Menucci03 for the riemannian case).

- The singular set is stratified by the dimension of the subdifferential ∂u (Alberti-Ambrosio-Cannarsa-Etcetera92-94).

- It has finite Hausdorff measure $\mathcal{H}^{n-1}$ (Itoh-Tanaka00 for the riemannian case, Li-Nirenberg05 for general case).

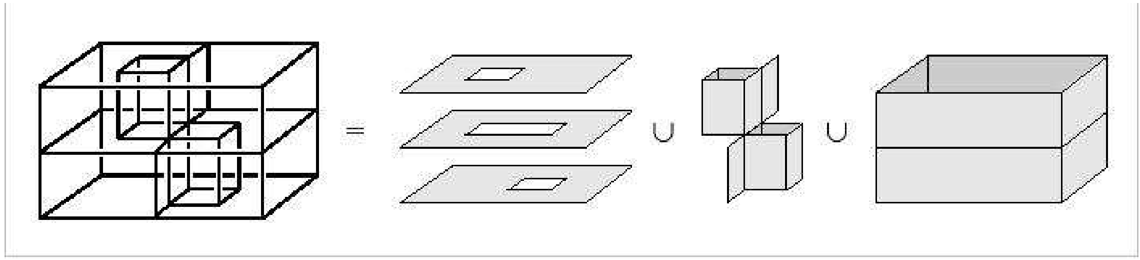
- If we add a generic perturbation to H or Ω, **Sing** becomes a stratified smooth manifold (Buchner78).

However, a cut locus can be pretty bad:



In Gluck-Singer78, the authors show there are plenty of *non-triangulable* cut loci.

Sing has the homotopy of Ω, but its topology may be non-trivial. The cut locus of a ball in $\mathbb{R}^3$ could be *the house with two rooms*:



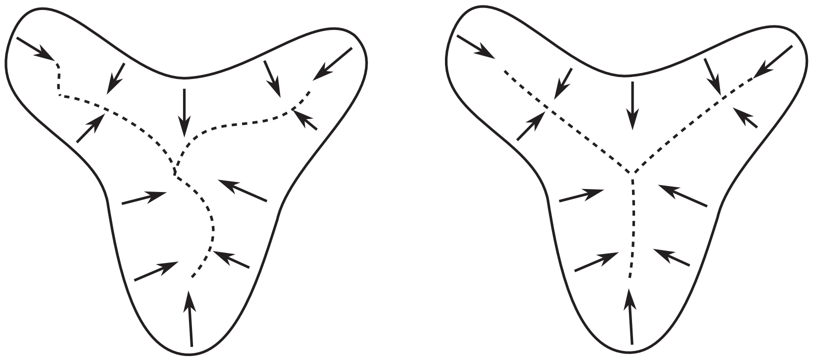
This figure was taken from the book *Algebraic Topology* by *Allen Hatcher*

Balanced split locus

Definition 1. *We say* $\boldsymbol{S \subset \Omega}$ ***splits*** $\boldsymbol{\Omega}$ *iff every point* $p \in \Omega \setminus S$ *belongs to a unique characteristic from* $\partial\Omega$ *contained in* $\Omega \setminus S$.

If S splits Ω, and $p \in \Omega \setminus S$, let $\boldsymbol{R_p}$ be the speed of the characteristic from $\partial\Omega$ to p in $\Omega \setminus S$. If $p \in S$, let R_p be the limit set of vectors R_q when $q \to p$.

Definition 2. *S is a split locus iff* $S = \overline{\{p \in S : \# R_p \geqslant 2\}}$



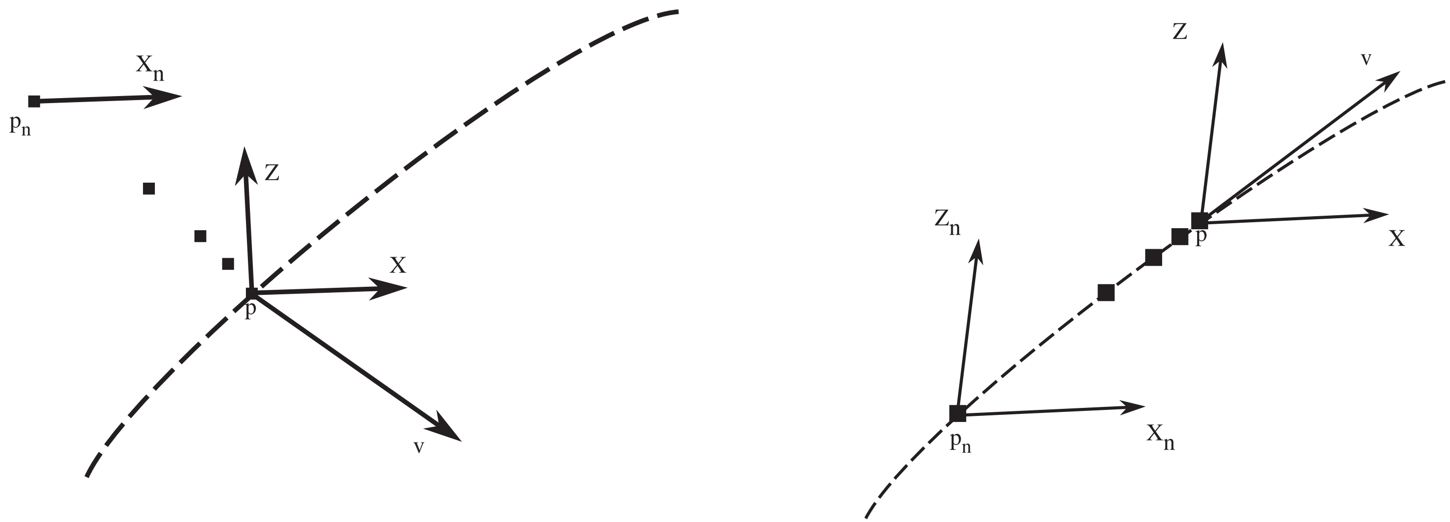
An arbitrary split locus and the singular set of u

Equivalently, S is a **split locus** iff S is closed, it splits Ω, and no closed $S' \subsetneq S$ splits Ω.

Definition 3. *A split locus S is balanced iff the following holds:*

Let p_n be a sequence of points and $X_n \in R_{p_n}$ be a sequence of vectors. If $p_n \to p$, $X_n \to X$, and the vector from p_n to p converges to v, *then:*

$$\hat{X}(v) \geqslant \hat{Z}(v) \quad \forall Z \in R_p$$



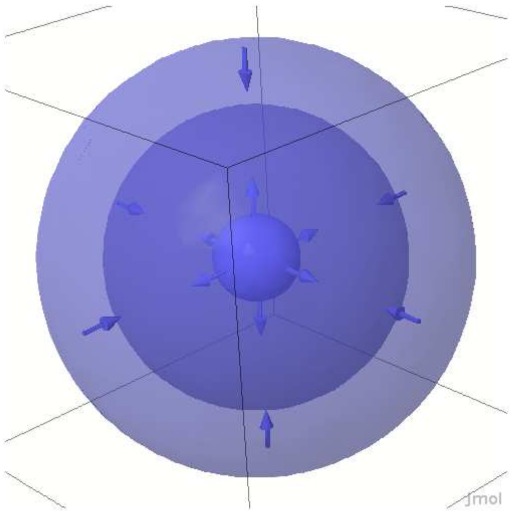
In riemannian geometry, $\hat{X}(v) = <X, v> = |v||X|\cos(\angle(X, v))$, so the balanced property means that *the angle of the incoming vector with the limit vector X is smaller* than the angle it makes with any other vector of R_p.

Main result

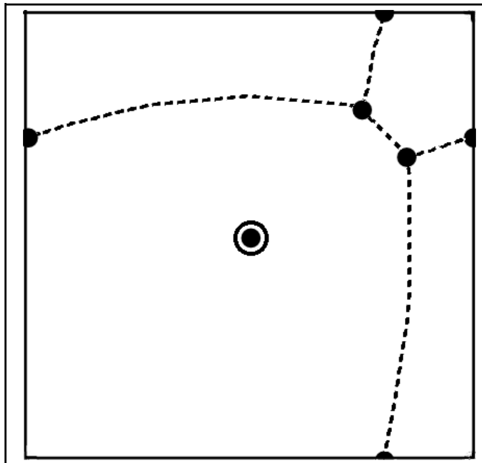
Is the singular set of the viscosity solution the unique balanced split locus?

Ω is simply connected *and* $\partial\Omega$ connected	$\rightarrow$	The singular set is the unique balanced split locus
Ω is simply connected, $\partial\Omega$ is **not** connected	$\rightarrow$	We can add a different constant to g at each component of $\partial\Omega$ and get different balanced split loci
General case	$\rightarrow$	Balanced split loci are parametrized by a neighborhood of 0 in $H_{n-1}(\Omega, \mathbb{R})$

Examples



In the **ring** between two concentric spheres ($g = 0$ and euclidean metric) there is a one dimensional family of balanced split loci.



In a square with opposite sides identified (a **flat torus**), there is a 2 dimensional family. ($\partial\mathbb{T}$ is a tiny circle around the central point) We confirm $\dim H_1(\mathbb{T}) = 2$

Motivation: Cleave points

Let Φ be the *geodesic flow* in $T\Omega$, with domain $D(\Phi)$. Define

$$V = \{(t,z)\colon z \in \partial\Omega, t \in [0,\infty), (t, V_z) \in D(\Phi)\}$$

$$F\colon V \to \Omega \qquad F(t,z) = \pi(\Phi(t, V_z))$$

(V_z is the initial speed of the characteristic starting at z)

- $x = (t,z) \in V$ is **regular** if F is a local diffeomorphism at x
- $x = (t,z) \in V$ is **conjugate of order k** if the rank of $d_x F$ is $n-k$

A point $p \in S$ is a **cleave point** iff $R_p = \{X_1, X_2\}$, with $X_i = d_{x_i} F(\frac{\partial}{\partial t})$, and both x_1 and x_2 *regular points* of F.

> *At a cleave point p, the balanced condition implies:*
>
> $$T_p S = \ker(\widehat{X_1} - \widehat{X_2})$$
>
> *Unique local solution to this differential equation through any point*

Proof of main theorem: more structure results

To prove our theorem we first had to adapt the existing structure results to Finsler geometry and/or to balanced split locus.

Theorem 4. *A balanced split locus S consists of cleave points (a smooth manifold of dimension $n-1$), and a set of Hausdorff dimension at most $n-2$.*

Proof. We extended previous results to Finsler manifolds. The proof is similar to the existing one, using Morse-Sard-Federer. □

Theorem 5. *A balanced split locus is stratified by* $\dim\left(\mathrm{span}(\widehat{R_p})\right)$.

Proof. Similar to the proofs for semiconcave functions by Albano, Alberti, Ambrosio, Cannarsa, Soner... □

Let $\lambda_k(z) > 0$ be the value of t where the geodesic $\Phi(t,z)$ has its k-th order conjugate point.

Let $\rho(z)$ be the minimum t such that $F(t,z) \in S$.

Theorem 6. *All* $\lambda_k\colon \partial\Omega \to \mathbb{R}$ *are Lipschitz functions.*

Proof. This result is new for Finsler manifolds. Our proof is different from the one in Itoh-Tanaka00, and uses the Malgrange preparation theorem. □

Theorem 7. $\rho\colon \partial\Omega \to \mathbb{R}$ *is a Lipschitz function.*

Proof. This was known for Finsler manifolds (Li-Nirenberg05), but we had to repeat it for balanced split loci. Our proof is unrelated to theirs, and has more in common with Itoh-Tanaka00. □

Corollary 8. $\mathcal{H}^{n-1}(S) < \infty$ *for a balanced split locus* S.

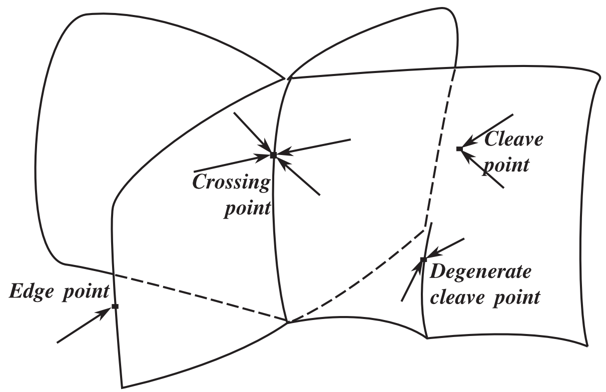
We also proved the following:

Theorem 9. *The set of points $p \in \Omega$ such that R_p contains a conjugate geodesic of order $\geqslant 2$ has Hausdorff dimension $\leqslant n-3$.*

Proof. The set of conjugate points of order 2 is the union of two sets: Q_2^1 and Q_2^2. The image of Q_2^2 has Hausdorff dimension $\leqslant n-3$ (uses Morse-Sard-Federer), and vectors in Q_2^1 do not map to vectors in the sets R_p. $\square$

Remark 10. In more standard terminology, this can be rephrased as *"the set of points that can be joined to $\partial\Omega$ with a minimizing geodesic conjugate of order 2 has Hausdorff dimension $\leqslant n-3$"*.

The restriction to *minimizing* geodesics is essential: the Hausdorff dimension of $F(Q_2^1)$ may well be $n-2$.



Corollary 11. *A balanced split locus S consists of:*

- ***Cleave points*** *($R_p = \{X_1, X_2\}$, each X_i is regular)(a smooth non-connected hypersurface)*
- ***Edge points*** *(R_p consists of one conjugate point of order 1) (Hausdorff dimension $n-2$)*
- ***Degenerate cleave points*** *($R_p = \{X_1, X_2\}$, X_i may be conjugate of order 1) (Hausdorff dimension $n-2$)*
- ***Crossing points*** *($\widehat{R_p} = \{\hat{X} : X \in R_p\}$ is contained in an affine 2D plane, R_p has regular and conjugate points of order 1) (*rectifiable *set of dimension $n-2$)*
- ***Remainder*** *(Hausdorff dimension $n-3$)*

Comment: this is interesting to study brownian motion on manifolds.

Proof of main theorem: a current

Each characteristic curve carries a value for u. A point in $\Omega \setminus S$ gets only one value, but a point in S gets a possible value from each geodesic from $\partial\Omega$ contained in $\Omega \setminus S$.

Let $\mathcal{C}_j$ be the *connected components of* the set of *cleave points*. Each cleave point gets one candidate value for u from either side: u_l and u_r

We define a current T of dimension $n-1$:

$$T(\phi) = \sum_j \left(\int_{\mathcal{C}_{j,l}} \phi u_l + \int_{\mathcal{C}_{j,r}} \phi u_r \right) \tag{3}$$

here $\mathcal{C}_{j,i}$ means $\mathcal{C}_j$ with the orientation induced by a fixed orientation in Ω, and the vector tangent to the geodesic coming from side $i = l, r$.

If $T = 0$, then u can be defined unambiguously, and it's continuous.

The main step of the proof is to show $\boldsymbol{\partial T = 0}$

Once we have this, it is not hard to show that if two currents T_1 and T_2 obtained from two balanced split loci S_1 and S_2 represent the same homology class in $H_{n-1}(\Omega)$, then $T_1 = T_2$.

For example, if Ω is simply connected and $\partial\Omega$ connected, and T is closed, then $T = dP$, where $P(\phi) = \int \phi f$ for a density $f \in L^n$. But $d\,P|_{\Omega\setminus S} = T|_{\Omega\setminus S} = 0$ implies f is locally constant outside S. Under our hypothesis, f is constant and $T = 0$.

For ϕ with support in a neighborhood of a cleave point:

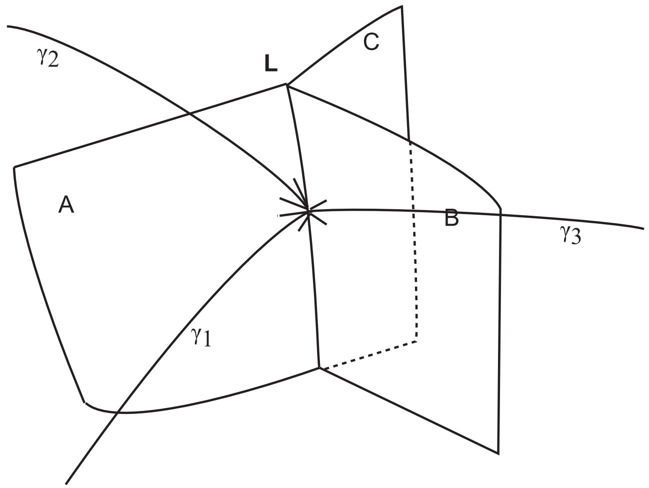
$$\partial T(\phi) = T(d\phi) = \int_{\mathcal{C}_{j,r}} d\phi(u_r - u_l) = \int_{\mathcal{C}_{j,r}} \phi d(u_r - u_l)$$

But $du_i = \widehat{X_i}$ for the incoming vector X_i ($i = l, r$).
By the balanced condition, $T\mathcal{C}_j \subset \ker(\widehat{X_r} - \widehat{X_l})$, so the integral is 0.

For ϕ with support in a neighborhood of a (generic) edge point:

Near a generic edge point q, S is a smooth *hypersurface with boundary*, with q a boundary point. $u_r - u_l$ is contant, and converges to zero as we approach the boundary.

For ϕ in a neighborhood of a (generic) crossing point:



$$
\begin{aligned}
\partial T(\phi) &= T(d\phi) \\
&= \int_{A_1} d\phi u_1 + \int_{A_2} d\phi u_2 + \int_{B_1} d\phi u_1 + \\
&\quad + \int_{B_3} d\phi u_3 + \int_{C_2} d\phi u_2 + \int_{C_3} d\phi u_3 \\
&= \int_{A_1} \phi d(u_1 - u_2) + \int_{B_3} \phi d(u_3 - u_1) \\
&\quad + \int_{C_2} \phi d(u_2 - u_3) \\
&\quad + \int_{L} \phi(u_1 - u_1 + u_2 - u_2 + u_3 - u_3) \\
&= 0
\end{aligned}
$$

Proof for general points:

Non-generic edge and crossing points can be quite more complicated than that, with *a countable amount of components* $\mathcal{C}_j$ in any neighborhood.

Thanks to the structure results, **we only have to deal with non-conjugate geodesics and geodesics of order 1**.

Lemma 12. *Let $x \in V$ be non-conjugate or conjugate of order 1, and $p = F(x)$. There are neighborhoods O_x and $U_p = F(O_x)$ such that for any $q \in U$ and $(t_i, z_i) \in O_x$ $(i = 1, 2)$ such that $X_i = d_{(t_i, z_i)}F(\frac{\partial}{\partial t}) \in R_q$, we have:*

$$t_1 + g(z_1) = t_2 + g(z_2)$$

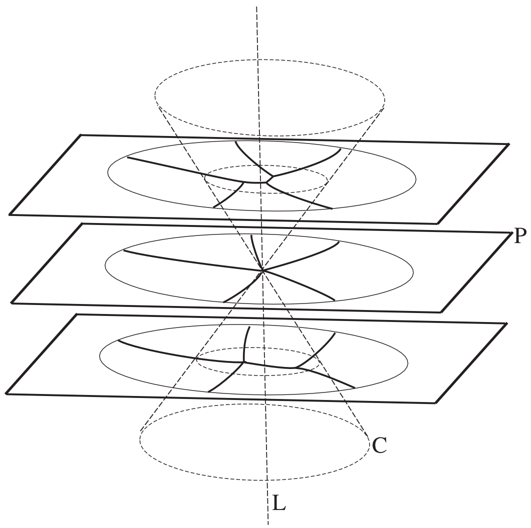
Thus, the value of u computed from all incoming directions in O_x is the same.

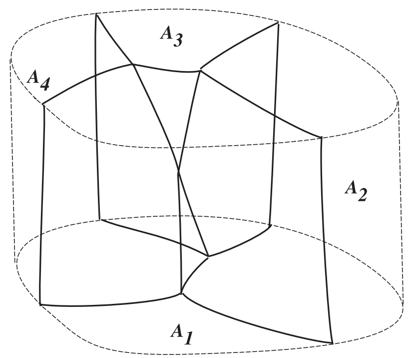
Lemma 13. *Let $p \in S$ be a degenerate cleave point, with $R_p = \{X_1, X_2\}$ with $X_i = d_{(t_i, z_i)}F(\frac{\partial}{\partial t})$.*

Let O_{x_i} be neighborhoods as in the above lemma. Let A_i be the set of q such that R_q contains a vector $d_x F(\frac{\partial}{\partial t})$ for a point $x \in O_{x_i}$. Then $A_1 \cap A_2$ is a Lipschitz hypersurface. We can apply the argument for cleave points to show that $\partial T = 0$ at degenerate cleave points.

Lemma 14. *Let $p \in S$ be a general crossing point. There is a finite amount of open sets O_i as in lemma 12 such that any $X \in R_p$ is of the form $X = d_{x_i}F(\frac{\partial}{\partial t})$ for some $x_i \in O_i$.*

- *All $A_i \cap A_j$ are Lipschitz hypersurfaces*
- *Let $\Sigma = \cup(A_i \cap A_j \cap A_k)$. In certain coordinates, the intersections of Σ with coordinate planes $\{x_1 = a_1, \ldots, x_{n-2} = a_{n-2}\}$ are Lipschitz trees*
- *At general crossing points, we also have $\partial T = 0$.*





Extensions

- The set of points in aFinsler manifold Ω that can be joined to $\partial\Omega$ with a minimizing geodesic conjugate of order k has Hausdorff dimension $\leqslant n-k-1$.

- Other first order PDEs

  - HJ-equations with dependence on u

  - Non-convex H

  - Sub-riemannian geometry?

References

Cut and singular loci up to codimension 3 http://arxiv.org/abs/0806.2229 (Annales de l'Institut Fourier)
Balanced split sets and Hamilton-Jacobi equations http://arxiv.org/abs/0807.2046 (Calculus of Variations and Partial Differential Equations, vol 40)