Singular sets of Hamilton-Jacobi equations and cut loci

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\[ H(p, du(p)) = 1 \quad p \in \Omega \]

\[ u(p) = g(p) \quad p \in \partial \Omega \]
Hamilton-Jacobi equations

Find $H: \Omega \rightarrow \mathbb{R}$ such that:

\begin{align*}
H(p, du(p)) &= 1 \quad p \in \Omega \\
u(p) &= g(p) \quad p \in \partial \Omega
\end{align*}

(1)

- $H^{-1}(1) \cap T^*_p \Omega$ is \textbf{convex} for every $p$, and contains 0.

- $\Omega$ is a smooth and \textbf{compact manifold with boundary}, $H$ and $g$ are smooth.

- The boundary data satisfies a \textbf{compatibility condition} (more about it later)

\begin{equation}
|g(y) - g(z)| < d(y, z)
\end{equation}

(2)
A geometrical interpretation

1. We can assume $H$ is a norm in each vector space $T_p^*\Omega$ (if necessary, replace $H$ with $\tilde{H}(p, w) = t$, for the only $t > 0$ such that $H(p, \frac{1}{t}w) = 1$).

2. Define the dual norm $\varphi$ in $T_p\Omega$

   $$\varphi_p(v) = \sup \{ \langle v, \alpha \rangle_p : \alpha \in T_p^*\Omega, H(p, \alpha) = 1 \}$$

3. This is a Finsler metric, which induces a distance $d$ in $\Omega$

4. The metric is Riemannian iff $H$ is quadratic in its second argument.
Classical solution by characteristic curves

The HJ equations are first order PDEs, and thus there is a solution using characteristic curves, defined only in a neighborhood of $\partial \Omega$.

If $x = \gamma(t)$ for a characteristic $\gamma$ with $\gamma(0) = y$, then $u(x) = t + g(y)$

The characteristic curves are geodesics of $\varphi$, whose initial condition at $y \in \partial \Omega$ is the vector $V_y$ satisfying:

$$\varphi_y(V_y) = 1 \quad \hat{V}_y|_{T_y(\partial \Omega)} = dg \quad V_y \text{ points inwards}$$

In particular, if $g$ is constant and $\varphi$ Riemannian, $V$ is perpendicular to $\partial \Omega$.

For a vector $V$ in a Finsler space, $w = \hat{V}$ is its dual one-form, given by:

$$w_j = \frac{\partial \varphi}{\partial V_j}(p, V)$$

This is the usual definition of dual form if $\varphi$ is a riemannian metric.
Viscosity solution

A viscosity solution is a solution in a weak sense, defined in all $\Omega$.

- The inspiration is too add a viscosity term to the HJ equations and make $\varepsilon \to 0$:

\[ H(p, du(p)) + \varepsilon \Delta u(p) = 1 \]

- The definition of viscosity solutions relies on test functions that touch $u$ from above (and below).

- There are other equivalent definitions (e.g., with semiconcave functions).

The solution obtained with characteristic curves coincides with the viscosity solution where both are defined.
Lax-Oleinik formula

The viscosity solution is given by a formula involving the Finsler distance:

\[ u(p) = \inf_{q \in \partial \Omega} \{ d(p, q) + g(q) \} \]

Comments

- The compatibility condition \(|g(y) - g(z)| < d(y, z)\) is necessary and sufficient for solutions to exist.

- If \( g = 0 \), then \( u \) is the distance to the boundary.

- The solution is not \( C^1 \) in all of \( \Omega \).
The singular set

Characteristic curves from $\partial \Omega$ intersect each other if continued indefinitely.

The extra information required to \textit{get the viscosity solution from the classical one} is a criterion to decide which characteristic curve is used to compute the value of $u$ at a given point.

\textit{This extra information is the singular set of the solution} $u$:

Let $\text{Sing}$ be the closure of the singular set of $u$. 
What do we know about the singular set of the viscosity solution?

If \( g = 0 \): \( u \) is the distance to the boundary, \( \text{Sing} \) is the cut locus.

And indeed, a solution with \( g \neq 0 \) in \( \Omega \) is the restriction of the solution with \( g = 0 \) in a bigger set \( \Gamma \supset \Omega \):

\[
\begin{align*}
H(p, dv(p)) &= 1 & p \in \Gamma \\
       v(p) &= 0 & p \in \partial \Gamma \\
   u = v|_\Omega & \\
\text{Sing}(u) &= \text{Sing}(v)
\end{align*}
\]
Some special cases

If $H$ depends only on $du$,
the characteristics are straight lines.
*(this includes the eikonal equation, $|\nabla u| = 1$)*

If $\Omega$ is a *simply connected plane* region,
$\text{Sing}$ is a *tree*.

If furthermore, $\Omega$, $H$ and $g$ are all *analytic*,
then $\text{Sing}$ is a *finite* tree.

If $\Omega$ is not planar, but $\Omega$, $H$ and $g$ are *analytic*,
then $\text{Sing}$ is a stratified smooth manifold.
Structure of the singular set

Singular sets (cut loci) are studied by PDE and geometry people

- The singular set is a deformation retract of $\Omega$ (obvious).

- It is the union of a $(n-1)$-dimensional smooth manifold consisting of points with two minimizing geodesics and a set of Hausdorff dimension at most $n-2$ (Hebda87, Itoh-Tanaka98, Barden-Le97, Mantegazza-Menucci03 for the riemannian case).

- The singular set is stratified by the dimension of the subdifferential $\partial u$ (Alberti-Ambrosio-Cannarsa-Etcetera92-94).

- It has finite Hausdorff measure $\mathcal{H}^{n-1}$ (Itoh-Tanaka00 for the riemannian case, Li-Nirenberg05 for general case).

- If we add a generic perturbation to $H$ or $\Omega$, $\text{Sing}$ becomes a stratified smooth manifold (Buchner78).
However, a cut locus can be pretty bad:

In Gluck-Singer78, the authors show there are plenty of non-triangulable cut loci.

Sing has the homotopy of $\Omega$, but its topology may be non-trivial. The cut locus of a ball in $\mathbb{R}^3$ could be the house with two rooms:

This figure was taken from the book *Algebraic Topology* by Allen Hatcher
Balanced split locus

Definition 1. We say $S \subset \Omega$ **splits** $\Omega$ iff every point $p \in \Omega \setminus S$ belongs to a unique characteristic from $\partial \Omega$ contained in $\Omega \setminus S$.

If $S$ splits $\Omega$, and $p \in \Omega \setminus S$, let $R_p$ be the speed of the characteristic from $\partial \Omega$ to $p$ in $\Omega \setminus S$. If $p \in S$, let $R_p$ be the limit set of vectors $R_q$ when $q \to p$.

Definition 2. $S$ is a split locus iff $S = \{ p \in S : \#R_p \geq 2 \}$

An arbitrary split locus and the singular set of $u$

Equivalently, $S$ is a **split locus** iff $S$ is closed, it splits $\Omega$, and no closed $S' \subset S$ splits $\Omega$. 

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Definition 3. A split locus $S$ is balanced iff the following holds:

Let $p_n$ be a sequence of points and $X_n \in R_{p_n}$ be a sequence of vectors. If $p_n \to p$, $X_n \to X$, and the vector from $p_n$ to $p$ converges to $v$, then:

$$\hat{X}(v) \geq \hat{Z}(v) \quad \forall Z \in R_p$$

In riemannian geometry, $\hat{X}(v) = \langle X, v \rangle = |v| |X| \cos(\angle(X, v))$, so the balanced property means that the angle of the incoming vector with the limit vector $X$ is smaller than the angle it makes with any other vector of $R_p$. 

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Main result

*Is the singular set of the viscosity solution the unique balanced split locus?*

\( \Omega \) is simply connected \( \rightarrow \) The singular set is the unique balanced split locus

and \( \partial \Omega \) connected

\( \Omega \) is simply connected, \( \partial \Omega \) is not connected \( \rightarrow \) We can add a different constant to \( g \) at each component of \( \partial \Omega \) and get different balanced split loci

General case \( \rightarrow \) Balanced split loci are parametrized by a neighborhood of 0 in \( H_{n-1}(\Omega, \mathbb{R}) \)
Examples

In the ring between two concentric spheres \((g = 0 \text{ and euclidean metric)}\) there is a one dimensional family of balanced split loci.

In a square with opposite sides identified (a flat torus), there is a 2 dimensional family. \((\partial \mathcal{T} \text{ is a tiny circle around the central point)}\)

We confirm \(\dim H_1(\mathcal{T}) = 2\)
Motivation: Cleave points

Let $\Phi$ be the geodesic flow in $T\Omega$, with domain $D(\Phi)$. Define

$$V = \{(t, z): z \in \partial\Omega, t \in [0, \infty), (t, V_z) \in D(\Phi)\}$$

$$F: V \rightarrow \Omega \quad F(t, z) = \pi(\Phi(t, V_z))$$

($V_z$ is the initial speed of the characteristic starting at $z$)

- $x = (t, z) \in V$ is \textbf{regular} if $F$ is a local diffeomorphism at $x$

- $x = (t, z) \in V$ is \textbf{conjugate of order $k$} if the rank of $d_x F$ is $n - k$

A point $p \in S$ is a **cleave point** iff $R_p = \{X_1, X_2\}$, with $X_i = d_x F(\frac{\partial}{\partial t})$, and both $x_1$ and $x_2$ \textit{regular points} of $F$.

At a cleave point $p$, the balanced condition implies:

$$T_p S = \ker (\hat{X}_1 - \hat{X}_2)$$

\textit{Unique local solution to this differential equation through any point}
Proof of main theorem: more structure results

To prove our theorem we first had to adapt the existing structure results to Finsler geometry and/or to balanced split locus.

**Theorem 4.** A balanced split locus $S$ consists of cleave points (a smooth manifold of dimension $n - 1$), and a set of Hausdorff dimension at most $n - 2$.

**Proof.** We extended previous results to Finsler manifolds. The proof is similar to the existing one, using Morse-Sard-Federer. □

**Theorem 5.** A balanced split locus is stratified by $\dim(\text{span}(\mathcal{R}_p))$.

**Proof.** Similar to the proofs for semiconcave functions by Albano, Alberti, Ambrosio, Cannarsa, Soner... □
Let $\lambda_k(z) > 0$ be the value of $t$ where the geodesic $\Phi(t, z)$ has its $k$-th order conjugate point.

Let $\rho(z)$ be the minimum $t$ such that $F(t, z) \in S$.

**Theorem 6.** All $\lambda_k: \partial \Omega \to \mathbb{R}$ are Lipschitz functions.

**Proof.** This result is new for Finsler manifolds. Our proof is different from the one in Itoh-Tanaka00, and uses the Malgrange preparation theorem. □

**Theorem 7.** $\rho: \partial \Omega \to \mathbb{R}$ is a Lipschitz function.

**Proof.** This was known for Finsler manifolds (Li-Nirenberg05), but we had to repeat it for balanced split loci. Our proof is unrelated to theirs, and has more in common with Itoh-Tanaka00. □

**Corollary 8.** $\mathcal{H}^{n-1}(S) < \infty$ for a balanced split locus $S$. 
We also proved the following:

**Theorem 9.** The set of points \( p \in \Omega \) such that \( R_p \) contains a conjugate geodesic of order \( \geq 2 \) has Hausdorff dimension \( \leq n - 3 \).

**Proof.** The set of conjugate points of order 2 is the union of two sets: \( Q_2^1 \) and \( Q_2^2 \). The image of \( Q_2^2 \) has Hausdorff dimension \( \leq n - 3 \) (uses Morse-Sard-Federer), and vectors in \( Q_2^1 \) do not map to vectors in the sets \( R_p \). \( \Box \)

**Remark 10.** In more standard terminology, this can be rephrased as “the set of points that can be joined to \( \partial \Omega \) with a minimizing geodesic conjugate of order 2 has Hausdorff dimension \( \leq n - 3 \)”.

The restriction to minimizing geodesics is essential: the Hausdorff dimension of \( F(Q_2^1) \) may well be \( n - 2 \).
Corollary 11. A balanced split locus $S$ consists of:

- **Cleave points** ($R_p = \{X_1, X_2\}$, each $X_i$ is regular) (a smooth non-connected hypersurface)

- **Edge points** ($R_p$ consists of one conjugate point of order 1) (Hausdorff dimension $n - 2$)

- **Degenerate cleave points** ($R_p = \{X_1, X_2\}$, $X_i$ may be conjugate of order 1) (Hausdorff dimension $n - 2$)

- **Crossing points** ($\hat{R}_p = \{\hat{X} : X \in R_p\}$ is contained in an affine 2D plane, $R_p$ has regular and conjugate points of order 1) (rectifiable set of dimension $n - 2$)

- **Remainder** (Hausdorff dimension $n - 3$)

Comment: this is interesting to study brownian motion on manifolds.
Proof of main theorem: a current

Each characteristic curve carries a value for $u$. A point in $\Omega \setminus S$ gets only one value, but a point in $S$ gets a possible value from each geodesic from $\partial \Omega$ contained in $\Omega \setminus S$.

Let $C_j$ be the connected components of the set of cleave points. Each cleave point gets one candidate value for $u$ from either side: $u_l$ and $u_r$.

We define a current $T$ of dimension $n - 1$:

$$T(\phi) = \sum_j \left( \int_{C_{j,l}} \phi u_l + \int_{C_{j,r}} \phi u_r \right)$$

(3)

here $C_{j,i}$ means $C_j$ with the orientation induced by a fixed orientation in $\Omega$, and the vector tangent to the geodesic coming from side $i = l, r$.

If $T = 0$, then $u$ can be defined unambiguously, and it’s continuous.
The main step of the proof is to show $\partial T = 0$

Once we have this, it is not hard to show that if two currents $T_1$ and $T_2$ obtained from two balanced split loci $S_1$ and $S_2$ represent the same homology class in $H_{n-1}(\Omega)$, then $T_1 = T_2$.

For example, if $\Omega$ is simply connected and $\partial \Omega$ connected, and $T$ is closed, then $T = dP$, where $P(\phi) = \int \phi f$ for a density $f \in L^n$. But $dP|_{\Omega \setminus S} = T|_{\Omega \setminus S} = 0$ implies $f$ is locally constant outside $S$. Under our hypothesis, $f$ is constant and $T = 0$.

For $\phi$ with support in a neighborhood of a cleave point:

$$\partial T(\phi) = T(d\phi) = \int_{C_{j,r}} d\phi(u_r - u_l) = \int_{C_{j,r}} \phi d(u_r - u_l)$$

But $du_i = X_i$ for the incoming vector $X_i$ ($i = l, r$).

By the balanced condition, $TC_j \subset \ker (\widehat{X_r} - \widehat{X_l})$, so the integral is $0$.

For $\phi$ with support in a neighborhood of a (generic) edge point:

Near a generic edge point $q$, $S$ is a smooth hypersurface with boundary, with $q$ a boundary point. $u_r - u_l$ is constant, and converges to zero as we approach the boundary.
For $\phi$ in a neighborhood of a (generic) crossing point:

$$\partial T(\phi) = T(d\phi)$$

$$= \int_{A_1} d\phi u_1 + \int_{A_2} d\phi u_2 + \int_{B_1} d\phi u_1 +$$
$$+ \int_{B_3} d\phi u_3 + \int_{C_2} d\phi u_2 + \int_{C_3} d\phi u_3$$

$$= \int_{A_1} \phi d(u_1 - u_2) + \int_{B_3} \phi d(u_3 - u_1)$$
$$+ \int_{C_2} \phi d(u_2 - u_3)$$
$$+ \int_{L} \phi(u_1 - u_1 + u_2 - u_2 + u_3 - u_3)$$

$$= 0$$

**Proof for general points:**

Non-generic edge and crossing points can be quite more complicated than that, with a countable amount of components $C_j$ in any neighborhood.
Thanks to the structure results, we only have to deal with non-conjugate geodesics and geodesics of order 1.

**Lemma 12.** Let \( x \in V \) be non-conjugate or conjugate of order 1, and \( p = F(x) \). There are neighborhoods \( O_x \) and \( U_p = F(O_x) \) such that for any \( q \in U \) and \((t_i, z_i) \in O_x \) (\( i = 1, 2 \)) such that \( X_i = d_{(t_i, z_i)} F(\frac{\partial}{\partial t}) \in R_q \), we have:

\[
t_1 + g(z_1) = t_2 + g(z_2)
\]

Thus, the value of \( u \) computed from all incoming directions in \( O_x \) is the same.

**Lemma 13.** Let \( p \in S \) be a degenerate cleave point, with \( R_p = \{X_1, X_2\} \) with \( X_i = d_{(t_i, z_i)} F(\frac{\partial}{\partial t}) \).

Let \( O_{x_i} \) be neighborhoods as in the above lemma. Let \( A_i \) be the set of \( q \) such that \( R_q \) contains a vector \( d_x F(\frac{\partial}{\partial t}) \) for a point \( x \in O_{x_i} \). Then \( A_1 \cap A_2 \) is a Lipschitz hypersurface. We can apply the argument for cleave points to show that \( \partial T = 0 \) at degenerate cleave points.
Lemma 14. Let $p \in S$ be a general crossing point. There is a finite amount of open sets $O_i$ as in lemma \red{12} such that any $X \in R_p$ is of the form $X = d_{x_i}F(\frac{\partial}{\partial t})$ for some $x_i \in O_i$.

- All $A_i \cap A_j$ are Lipschitz hypersurfaces
- Let $\Sigma = \cup (A_i \cap A_j \cap A_k)$. In certain coordinates, the intersections of $\Sigma$ with coordinate planes $\{x_1 = a_1, \ldots, x_{n-2} = a_{n-2}\}$ are Lipschitz trees
- At general crossing points, we also have $\partial T = 0$. 

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  \item Let $\Sigma = \cup (A_i \cap A_j \cap A_k)$. In certain coordinates, the intersections of $\Sigma$ with coordinate planes $\{x_1 = a_1, \ldots, x_{n-2} = a_{n-2}\}$ are Lipschitz trees
  \item At general crossing points, we also have $\partial T = 0$.
\end{itemize}
Extensions

• The set of points in a Finsler manifold $\Omega$ that can be joined to $\partial \Omega$ with a minimizing geodesic conjugate of order $k$ has Hausdorff dimension $\leq n - k - 1$.

• Other first order PDEs
  
  - HJ-equations with dependence on $u$
  
  - Non-convex $H$
  
  - Sub-riemannian geometry?

References

Cut and singular loci up to codimension 3 http://arxiv.org/abs/0806.2229 (Annales de l’Institut Fourier)