AN INTRODUCTION TO ISOPARAMETRIC FOLIATIONS

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Abstract. A hypersurface in a Riemannian manifold is called isoparametric if it and its nearby equidistant hypersurfaces have constant mean curvature. These geometric objects, as well as their important generalization to isoparametric submanifolds of codimension greater than one, appear in families called isoparametric foliations.

In these notes we present an introduction to isoparametric foliations, starting from the problem in Geometric Optics that motivated their study, and then explaining the main results known so far, with focus on some recent techniques.

Contents

1. Motivation: a problem in Geometric Optics

> *"According to the Huygens principle, one of the most simple models of what wave propagation in an isotropic media should be consists in a family of parallel surfaces that are intersected perpendicularly at every point by a set of straight lines. The sequence of parallel surfaces, each one of which can be considered as the envelope of a set of spheres of radius equal to the distance between the surface and one of the previous ones, constitutes the family of wavefronts."*

This comment, due to the italian mathematician Carlo Somigliana in 1919 in the paper [83], though elementary from the viewpoint of classical Geometric Optics, represented the starting point of a remarkable research line in the field of Geometry from the beginning

of the 20th century until nowadays: the study of the so-called isoparametric hypersurfaces and their generalizations. As we will see later, renowned mathematicians as Beniamino Segre, Élie Cartan and Tullio Levi-Civita, among others, studied these geometric objects at some point of their careers. A bit more recently, Yau even included the classification problem of isoparametric hypersurfaces in spheres in his influential list of open problems in Geometry [104].

Let us sketch here the problem of Geometric Optics that Somigliana addressed in [83]. We will present it in the more general setting of Riemannian manifolds, instead of the particular case of $\mathbb{R}^3$ studied by Somigliana.

We start by considering a wave in an ambient Riemannian manifold $\bar{M}$, that is, we have a smooth solution $\varphi\colon \bar{M}\times\mathbb{R}\to\mathbb{R}$, $(x,t)\mapsto\varphi(x,t)$, to the wave equation

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial t^2},$$

where Δ is the Laplace-Beltrami operator of $\bar{M}$, $x\in\bar{M}$ represents the spatial variables, and $t\in\mathbb{R}$ the time variable. Recall that the *wavefronts* of φ are the sets of points in $\bar{M}$ that share a common phase (i.e. the same oscillating state) at a fixed instant t_0. In other words, if we fix an instant $t_0\in\mathbb{R}$ and define a function $f\colon\bar{M}\to\mathbb{R}$ by means of $f(x)=\varphi(x,t_0)$, for each $x\in\bar{M}$, the wavefronts of φ at t_0 are the level sets of f. Now, we will impose two conditions on the wave.

Firstly, we will assume that the wave φ is stationary, that is, its wavefronts are time-independent. This implies that the wavefronts coincide with the level sets of f. Let then M be a wavefront and $x_0\in M$. We define the function $c\colon\mathbb{R}\to\mathbb{R}$, $t\mapsto c(t)=\varphi(x_0,t)$. Due to the imposed condition, c does not depend on $x_0\in M$, but only on the wavefront M. Then, we have

$$\Delta f(x)=\Delta\varphi(x,t_0)=\frac{\partial^2\varphi}{\partial t^2}(x,t_0)=c''(t_0)$$

for any $x\in M=f^{-1}(c(t_0))$. Thus, we get that the Laplacian of f is constant along the level sets of f.

Now we impose a second condition. We require the wavefronts of φ to be equidistant to each other. Due to the first assumption, this second one is equivalent to the condition that the level sets of f are equidistant to each other. In each singular level set we have that $\|\nabla f\|=0$. Let then $M_a=f^{-1}(a)$ and $M_b=f^{-1}(b)$ be two regular level sets such that the interval $[a,b]\subset\mathbb{R}$ does not have any critical value of f. Note that M_a and M_b are embedded hypersurfaces in $\bar{M}$. The distance between M_a and M_b is measured along geodesics orthogonal to both hypersurfaces, due to the first variation formula. Let $\gamma\colon[0,l]\to M$ be one of these unit-speed geodesics with $p=\gamma(0)\in M_a$ e $\gamma(l)\in M_b$. Since ∇f is orthogonal to the level sets of f, we have that the tangent vector to γ is $\dot\gamma=\nabla f/\|\nabla f\|$. Hence, the distance between M_a and M_b is

$$d(M_a,M_b)=l=\int_0^l\frac{1}{\|\nabla f\|}\langle\nabla f,\dot\gamma\rangle dt=\int_0^l\frac{1}{\|\nabla f\|}(f\circ\gamma)'dt=\int_a^b\frac{1}{\|\nabla f\|}ds, \tag{1.1}$$

and thus the mean value theorem guarantees that

$$\|\nabla f(p)\| = \lim_{b\to a} \frac{b-a}{d(M_a, M_b)}.$$

This value for $\|\nabla f(p)\|$ if independent of the point $p \in M_a$ chosen as starting point for γ. We conclude that the norm of the gradient of f is constant along each level set of f.

Thus, we see that a stationary wave with equidistant wavefronts determines a smooth function f such that $\|\nabla f\|$ and ∇f are constant along the level sets of f. This leads directly to the notion of isoparametric map, which was probably introduced by Levi-Civita [59] in 1937.

Definition 1.1. A nonconstant smooth real function $f\colon \bar{M} \to \mathbb{R}$ on a Riemannian manifold $\bar{M}$ is called an *isoparametric map* if there exist real functions Φ and Ψ of one real variable such that

$$\|\nabla f\| = \Phi \circ f \qquad \text{and} \qquad \Delta f = \Psi \circ f.$$

An *isoparametric family of hypersurfaces* is the collection $\{f^{-1}(c) : c \in \mathbb{R}\}$ of level sets of an isoparametric map f.

Sometimes it is convenient to require some regularity conditions on Φ and Ψ, but for our purposes it will not be necessary. See [99] for more details.

As we have seen, the constancy of the norm of the gradient along the level sets of an isoparametric map f means, roughly speaking, that the level sets are equidistant to each other. In Section 2 we will see that the analogous condition for the Laplacian of f also has a geometric interpretation: the regular level sets of f have constant mean curvature. Indeed, Cartan characterized the hypersurfaces that are regular level sets of isoparametric maps by the condition of defining locally a foliation of equidistant constant mean curvature hypersurfaces. We will prove this in Theorem 2.1 below. We use this result to provide the definition of isoparametric hypersurface.

Definition 1.2. An immersed hypersurface M of a Riemannian manifold $\bar{M}$ is called an *isoparametric hypersurface* if, for each $p \in M$, there exists an open neighbourhood U of p in M such that U and the nearby equidistant hypersurfaces to U have constant mean curvature.

Note that, given an immersed hypersurface M and a point $p \in M$, there always exists an open neighbourhood U of p in M such that U is an embedded hypersurface with unit normal vector field ξ, and the equidistant hypersurfaces $U^r = \{\exp_q(r\xi_q) : q \in U\}$, for r small enough, are embedded.

The study of isoparametric hypersurfaces and their generalizations enjoys nowadays a long history that evidenced many bridges with different areas of Mathematics. Apart from Riemannian Geometry, the theory of isoparametric hypersurfaces interacts with the theory of Lie groups, isometric actions, symmetric spaces, Algebraic Topology, Algebraic Geometry, Differential Equations, and submanifold theory in Hilbert spaces. We refer to the excellent survey [93] for references on this interplay. Moreover, although here we will focus on the more geometric aspects, isoparametric hypersurfaces have shown to be

important in modelling some physical phenomena. We refer, for example, to [78] and [81] for certain applications to problems in Fluid Mechanics, and to [65] for an application to the study of isothermic surfaces.

These introductory notes have been written as a support tool for a minicourse on isoparametric foliations at Universidade de São Paulo. They are based on my PhD thesis [34], the resulting articles [29], [30], [36], as well as in the book [6] by Berndt, Console and Olmos and the notes [41] by Ferus.

These notes are organized as follows. In Section 2 we present the basic properties of isoparametric hypersurfaces and homogeneous hypersurfaces, with focus on the spaces of constant curvature. In Section 3 we state the classification of isoparametric hypersurfaces in Euclidean and real hyperbolic spaces. The main points of the outstanding classification problem of isoparametric hypersurfaces in spheres are presented in Section 4. Then we move on to spaces of nonconstant curvature. In Section 5 we review what is known about isoparametric hypersurfaces in manifolds of nonconstant curvature and, in Section 6, we give an introduction to the algebraic structure of a symmetric space of noncompact type, and apply it to construct examples of isoparametric hypersurfaces. In Section 7 we introduce the notion of isoparametric submanifold of arbitrary codimension, and review the main known results about this generalization. Finally, in Section 8, we explain the main steps of the recent classification of isoparametric submanifolds in complex projective spaces.

We also include three appendices in these notes, where we review the basics of Jacobi field theory applied to the study of equidistant and focal submanifolds (Appendix A), of isometric actions (Appendix B), and of symmetric spaces (Appendix C).

2. Basic properties of isoparametric hypersurfaces

In this section we prove some basic results about isoparametric hypersurfaces, first for general Riemannian manifolds, and later focusing on the case of spaces of constant curvature. We also introduce the important subclass of homogeneous isoparametric hypersurfaces.

2.1. Isoparametric maps versus isoparametric hypersurfaces.

We start by proving the relation between isoparametric maps and isoparametric hypersurfaces announced above.

Theorem 2.1. *Let $\bar{M}$ be a Riemannian manifold. Let $f\colon \bar{M} \to \mathbb{R}$ be an isoparametric map, $c \in \mathbb{R}$ a regular value for f, and $M = f^{-1}(c)$ the corresponding level hypersurface. Then M is an isoparametric hypersurface.*

Conversely, if M is an isoparametric hypersurface in $\bar{M}$, then for each $p \in M$ there is an open neighbourhood U of p in M such that U is a regular level set of an isoparametric map $f\colon V \to \mathbb{R}$, for some open subset V of $\bar{M}$.

Proof. Let $\xi = \nabla f / \|\nabla f\|$ be a unit normal vector field to the hypersurface $M = f^{-1}(c)$. Then the shape operator $\mathcal{S}$ of M with respect to ξ is given by

$$\langle \mathcal{S}X, Y\rangle = -\langle \bar{\nabla}_X \xi, Y\rangle = -\frac{1}{\|\nabla f\|}\langle \bar{\nabla}_X \nabla f, Y\rangle = -\frac{1}{\|\nabla f\|}\operatorname{Hess}_f(X,Y).$$

Hence, if $\{E_1, \dots, E_{n-1}\}$ is an orthonormal frame on M, the mean curvature H of M is

$$H = \operatorname{tr}\mathcal{S} = \sum_{i=1}^{n-1}\langle \mathcal{S}E_i, E_i\rangle = -\frac{1}{\|\nabla f\|}\sum_{i=1}^{n-1}\operatorname{Hess}_f(E_i,E_i) = -\frac{1}{\|\nabla f\|}(\Delta f - \operatorname{Hess}_f(\xi,\xi)) \tag{2.1}$$
$$= -\frac{1}{\|\nabla f\|}(\Delta f - \frac{1}{\|\nabla f\|^2}\langle \bar\nabla_{\nabla f}\nabla f, \nabla f\rangle) = -\frac{1}{\|\nabla f\|}(\Delta f - \frac{1}{2\|\nabla f\|^2}\nabla f(\|\nabla f\|^2)),$$

which is constant along $M = f^{-1}(c)$, since f is isoparametric and, thus, $\|\nabla f\|$ and Δf are constant along the level sets of f.

Since c is an arbitrary regular value, the first part of the theorem will be proved when we show that nearby regular level sets of f are equidistant. To show this, it is enough to prove that the integral curves of the unit vector field ξ defined on the regular stratum of f are geodesics. Indeed, if we know that, a calculation similar to (1.1) allows to conclude that nearby regular level sets are equidistant. Thus, let us show that $\bar\nabla_\xi\xi = 0$. First, since ξ has unit length, we have $\langle\bar\nabla_\xi\xi,\xi\rangle = 0$. Second, let X be an arbitrary smooth vector field on (an open set) of $\bar M$ being tangent to the regular level sets of f. We know that $\xi(f)$ is constant along the regular level sets of f, so $X(\xi(f)) = 0$. Moreover, $X(f) = 0$ and thus $\xi(X(f)) = 0$. Then

$$0 = \frac{1}{\|\nabla f\|}[X,\xi](f) = \langle\xi,[X,\xi]\rangle = \langle\bar\nabla_X\xi,\xi\rangle + \langle\bar\nabla_\xi X,\xi\rangle = \langle\bar\nabla_\xi X,\xi\rangle = -\langle\bar\nabla_\xi\xi, X\rangle.$$

Altogether, we have that ξ is a geodesic vector field.

For the converse, let M be an isoparametric hypersurface in $\bar M$. Fix a point $p \in M$ and an open neighbourhood U of p in M such that U is embedded with unit normal vector field ξ. Then there is an $\varepsilon > 0$ such that the equidistant hypersurfaces $U^r = \{\exp_q(r\xi_q) : q \in U\}$, for $r \in (-\varepsilon,\varepsilon)$, are embedded and have constant mean curvature. Consider the open set $V = \cup_{r\in(-\varepsilon,\varepsilon)}U^r$ of $\bar M$ and define the map $f\colon V \to (-\varepsilon,\varepsilon)$ sending a point $q \in U^r$ to r. It is easy to show that $\|\nabla f\| = 1$. Using (2.1) and the fact that each $U^r = f^{-1}(r)$ has constant mean curvature, we also get that Δf is constant along each level set $f^{-1}(r)$. This shows that f is an isoparametric map and concludes the proof. □

Let us conclude this subsection by stating a general result about isoparametric families of hypersurfaces due to Wang [99]; see also [98], [42], [44] and [67].

Theorem 2.2. *Let $f\colon \bar M \to \mathbb{R}$ be an isoparametric map in a complete connected Riemannian manifold $\bar M$. We assume that the functions Φ and Ψ in Definition 1.1 are smooth, and continuous, respectively. Let $J = f(\bar M)$ and consider the level sets $M_+ = f(\max J)$ and $M_- = f(\min J)$, if they exist. Then:*

(a) *M_- and M_+, if they exist, are smooth submanifolds of $\bar M$.*
(b) *The interior of J only has regular values.*
(c) *Each regular level set of f is a tube around M_+ and M_-, if exist.*
(d) *M_- and M_+ are minimal submanifolds.*
(e) *If $\bar M$ is closed, there exists at least one regular level set of f which is a minimal hypersurface.*

The submanifolds M_- and M_+ are often called *focal varieties* or *focal submanifolds* of f. In principle, they can be disconnected, or of codimension one. To avoid this undesired behaviour, Ge and Tang introduced the notion of *proper isoparametric map*; see [42] for more details.

2.2. **Ambient spaces of constant curvature: isoparametric hypersurfaces versus hypersurfaces with constant principal curvatures.** When the ambient space has constant curvature, the isoparametricity of a hypersurface turns out to be equivalent to the constancy of the principal curvatures of the hypersurface. This is a crucial result due to Cartan [13].

We will denote by $\bar{M}(\kappa)$ a real space form of curvature κ, that is, a complete Riemannian manifold with constant sectional curvature κ. We start by making precise the definition of hypersurface with constant principal curvatures, and by stating a very useful result that allows to calculate the extrinsic geometry of equidistant hypersurfaces.

Definition 2.3. A hypersurface M has *constant principal curvatures* if for any open set U of M with unit normal vector field ξ on U, the eigenvalues of the shape operator of U with respect to ξ are constant on U.

Lemma 2.4. *Let M be an embedded hypersurface with global unit normal vector field ξ on a space form $\bar{M}(\kappa)$ defining embedded equidistant hypersurfaces $M^r = \{\exp_p(r\xi_p) : p \in M\}$, for $r \in (-\varepsilon, \varepsilon)$. Denote by $\mathcal{S}$ the shape operator of M with respect to ξ, and by $\mathcal{S}^r$ the shape operator of M^r with respect to η^r, where $\eta^r_{\exp_p(r\xi_p)} = (\exp_p)_{*r\xi_p}(r\xi_p)$, $p \in M$. Then, with respect to a basis of parallel translated vectors along the geodesics $\gamma_p(t) = \exp_p(t\xi_p)$, $t \in (-\varepsilon, \varepsilon)$, $p \in M$, we have that*

$$\mathcal{S}^r = -D'(r)D(r)^{-1},$$

where $D(r) = c_\kappa(r)I - s_\kappa(r)\mathcal{S}$, I is de identity transformation, and

$$c_\kappa(t) = \begin{cases} 1 & \text{if } \kappa = 0 \\ \cos(t\sqrt{\kappa}) & \text{if } \kappa > 0 \\ \cosh(t\sqrt{-\kappa}) & \text{if } \kappa < 0 \end{cases} \qquad s_\kappa(t) = \begin{cases} t & \text{if } \kappa = 0 \\ \sin(t\sqrt{\kappa}) & \text{if } \kappa > 0 \\ \sinh(t\sqrt{-\kappa}) & \text{if } \kappa < 0. \end{cases}$$

In particular, if λ is a principal curvature of M at p, then

$$\lambda(r) = \hat{\kappa}\frac{\operatorname{sign}(\kappa)\tan_\kappa(r) + \lambda}{1 - \lambda\tan_\kappa(r)}$$

is a principal curvature of M^r at $\exp_p(r\xi_p)$, where $\hat{\kappa} = 1$, $\sqrt{\kappa}$ or $\sqrt{-\kappa}$ according to $\kappa = 0$, > 0 or < 0, and where $\tan_\kappa = s_\kappa/c_\kappa$.

Proof. Use standard Jacobi field theory; see Appendix A. □

Theorem 2.5. *Let M be a hypersurface in a space form $\bar{M}(\kappa)$. Then M is isoparametric if and only if M has constant principal curvatures.*

Proof. First note that it is enough to work locally, and thus we can assume that M defines nearby equidistant embedded hypersurfaces.

Let us start with the sufficiency. Assume that M has g distinct constant principal curvatures $\lambda_1, \dots, \lambda_g$. Then Lemma 2.4 implies that the principal curvatures of a nearby equidistant hypersurface M^r at distance r are

$$\lambda_i(r) = \hat{\kappa}\frac{\operatorname{sign}(\kappa)\tan_\kappa(r) + \lambda_i}{1 - \lambda_i \tan_\kappa(r)}, \qquad i = 1, \dots, g,$$

which are constant functions on M^r. Hence, for r small enough, M^r has constant principal curvatures, and, thus, constant mean curvature. Hence each of these M^r, including M, are isoparametric hypersurfaces.

In order to prove the necessity, let M be an isoparametric hypersurface of dimension $n-1$ and let $\lambda_1, \dots, \lambda_{n-1}$ be the principal curvature functions of M. We have to show that these functions are constant. For simplicity, we will only do the proof for the flat case $\kappa = 0$. Then, the principal curvatures of M^r at $\exp_p(r\xi_p)$, with $p \in M$, are $\frac{\lambda_i(p)}{1-r\lambda_i(p)}$, $i = 1, \dots, n-1$. By assumption, the mean curvature of M^r at any point $\exp_p(r\xi_p) \in M^r$ is

$$H(r) = \sum_{i=1}^{n-1} \frac{\lambda_i(p)}{1 - r\lambda_i(p)}.$$

For a fixed $p \in M$, this defines an analytic function H on an open subset of $\mathbb{R}$. Since we have

$$H(0) = \sum_{i=1}^{n-1} \lambda_i(p), \quad H'(0) = \sum_{i=1}^{n-1} \lambda_i^2(p), \quad \dots, \quad H^{(n-2)}(0) = \sum_{i=1}^{n-1} \lambda_i^{n-1}(p),$$

we deduce that the λ_i do not depend on $p \in M$, and thus, M has constant principal curvatures. For $\kappa \neq 0$, one can see that the $\lambda_i(p)$ are determined by the poles of some function that does not depend on p, and thus conclude similarly. □

It is worthwhile to mention at this point that isoparametricity and constancy of the principal curvatures are not equivalent in general if the ambient space has nonconstant curvature. The first counterexamples were constructed by Wang [97] in complex projective spaces $\mathbb{C}P^n$. Many more examples have been found recently; see for example [29] and [30].

Proposition 2.6. *Let M be a hypersurface with constant principal curvatures in a space form $\bar{M}(\kappa)$. If λ is a principal curvature of M, then the corresponding principal curvature distribution T_λ is autoparallel, that is, $\nabla_{T_\lambda} T_\lambda \subset T_\lambda$.*

In particular, each principal curvature distribution is integrable, and any leaf of such a distribution is totally geodesic in M and totally umbilical in $\bar{M}$.

Proof. Let $\mu \neq \lambda$ be another principal curvature of M. Then the Codazzi equation applied to vector fields $X, Y \in \Gamma(T_\lambda)$ and $Z \in \Gamma(T_\mu)$ reads

$$\begin{aligned} 0 &= \langle (\nabla_X \mathcal{S})Z, Y\rangle - \langle (\nabla_Z \mathcal{S})X, Y\rangle \\ &= \langle \nabla_X \mathcal{S}Z, Y\rangle - \langle \nabla_X Z, \mathcal{S}Y\rangle - \langle \nabla_Z \mathcal{S}X, Y\rangle + \langle \nabla_Z X, \mathcal{S}Y\rangle \\ &= \mu\langle \nabla_X Z, Y\rangle - \lambda\langle \nabla_X Z, Y\rangle - \lambda\langle \nabla_Z X, Y\rangle + \lambda\langle \nabla_Z X, Y\rangle \\ &= (\mu-\lambda)\langle \nabla_X Z, Y\rangle = (\lambda-\mu)\langle Z, \nabla_X Y\rangle. \end{aligned}$$

Thus, we deduce that $\langle \nabla_{T_\lambda} T_\lambda, T_\mu\rangle = 0$. Since $\mu \neq \lambda$ is arbitrary, the result follows. □

We are now ready to prove a key formula, due to Cartan, for the study of isoparametric hypersurfaces in space forms. The proof below has been extracted from [6].

Theorem 2.7 (**Cartan's fundamental formula**). *Let M be an isoparametric hypersurface in a space form $\bar{M}(\kappa)$. Let $\lambda_1, \dots, \lambda_g$ be the distinct constant principal curvatures of M, with respective multiplicities $m_1, \dots, m_g$. Then, for all $i \in \{1, \dots, g\}$, we have*

$$\sum_{j=1,\, j\neq i}^{g} m_j \frac{\kappa + \lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0$$

Proof. We change the notation slightly and consider $\lambda_1, \dots, \lambda_{n-1}$ the constant principal curvatures of M, where $n-1$ is the dimension of M. Let $\{E_1, \dots, E_{n-1}\}$ be a local orthonormal frame tangent to M such that $\mathcal{S}E_i = \lambda_i E_i$, $i = 1, \dots, n-1$. The Gauss equation applied to fields E_i and E_j with $\lambda_i \neq \lambda_j$, using the fact that T_{λ_i} and T_{λ_j} are

integrable by Proposition 2.6, yields

$$\begin{aligned}
\kappa + \lambda_i\lambda_j &= R(E_i, E_j, E_j, E_i) = \langle \nabla_{E_i}\nabla_{E_j}E_j, E_i\rangle - \langle \nabla_{E_j}\nabla_{E_i}E_j, E_i\rangle - \langle \nabla_{[E_i,E_j]}E_j, E_i\rangle \\
&= E_i\langle \nabla_{E_j}E_j, E_i\rangle - \langle \nabla_{E_j}E_j, \nabla_{E_i}E_i\rangle - E_j\langle \nabla_{E_i}E_j, E_i\rangle \\
&\quad + \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \langle \nabla_{[E_i,E_j]}E_j, E_i\rangle \\
&= \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \langle \nabla_{[E_i,E_j]}E_j, E_i\rangle \\
&= \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \frac{1}{\lambda_j - \lambda_i}\langle (\nabla_{[E_i,E_j]}\mathcal{S})E_j, E_i\rangle \\
&= \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \frac{1}{\lambda_j - \lambda_i}\langle (\nabla_{E_i}\mathcal{S})E_j, [E_i, E_j]\rangle \\
&= \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \frac{1}{\lambda_j - \lambda_i}(\langle (\nabla_{E_i}\mathcal{S})E_j, \nabla_{E_i}E_j\rangle - \langle (\nabla_{E_i}\mathcal{S})E_j, \nabla_{E_j}E_i\rangle \\
&= \langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle - \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_i}\langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle \\
&= 2\langle \nabla_{E_i}E_j, \nabla_{E_j}E_i\rangle = 2\sum_{k=1}^{n-1}\langle \nabla_{E_i}E_j, E_k\rangle\langle E_k, \nabla_{E_j}E_i\rangle \\
&= 2\sum_{\substack{k=1\\ \lambda_k\neq\lambda_i,\lambda_j}}^{n-1}\frac{\langle (\nabla_{E_k}\mathcal{S})E_i, E_j\rangle^2}{(\lambda_j - \lambda_k)(\lambda_i - \lambda_k)},
\end{aligned}$$

where in the last equality we have used Codazzi equation and the fact that $\nabla_{E_k}\mathcal{S}$ is self-adjoint. Dividing by $\lambda_i - \lambda_j$ and adding on j we have

$$\sum_{\substack{j=1\\ \lambda_j\neq\lambda_i}}^{n-1}\frac{\kappa + \lambda_i\lambda_j}{\lambda_i - \lambda_j} = 2\sum_{\substack{j,k=1\\ \lambda_j\neq\lambda_i\neq\lambda_k\neq\lambda_j}}^{n-1}\frac{\langle (\nabla_{E_k}\mathcal{S})E_i, E_j\rangle^2}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_i - \lambda_k)} = -\sum_{\substack{k=1\\ \lambda_k\neq\lambda_i}}^{n-1}\frac{\kappa + \lambda_i\lambda_k}{\lambda_i - \lambda_k},$$

which proves the result. □

2.3. **Homogeneous hypersurfaces.** An important family of examples of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures is given by the codimension-one orbits of isometric actions on a given ambient Riemannian manifold. These are the so-called homogeneous hypersurfaces.

Definition 2.8. An *(extrinsically) homogeneous hypersurface* of a Riemannian manifold $\bar{M}$ is a codimension-one orbit of the action of a Lie subgroup H of the isometry group $\mathrm{Isom}(\bar{M})$ of $\bar{M}$.

Those isometric actions $H \times \bar{M} \to \bar{M}$ that admit a hypersurface as an orbit are called *cohomogeneity one actions*. Thus, homogeneous hypersurfaces are precisely the orbits of maximum dimension of cohomogeneity one actions.

Example 2.9. Easy examples of cohomogeneity one actions are the action of $SO(n)$ on $\mathbb{R}^n$ producing concentric spheres, the action of $SO(n-1)$ on S^n or $\mathbb{R}P^n$ yielding geodesic

spheres (and a cut locus $\mathbb{R}P^{n-1}$ in the projective case), the action of $\mathbb{R}^{n-1}$ on $\mathbb{R}^n$ via translations producing parallel hyperplanes, or the action of the Heisenberg group on the hyperbolic 3-space $\mathbb{R}H^n$ that gives rise to the horosphere foliation. We will see more examples throughout these notes.

It is easy to show that homogeneous hypersurfaces are isoparametric and with constant principal curvatures. Determining when the converse is true is, probably, the most important question in the area.

Proposition 2.10. *A homogeneous hypersurface is isoparametric and has constant principal curvatures.*

Proof. Let $M = H \cdot p$ be a codimension-one orbit of an isometric action $H \times \bar{M} \to \bar{M}$ through a point $p \in \bar{M}$. For any two points $q, x \in M$ there exists $h \in H$ such that $h(M) = M$ and $h(q) = x$. Then the shape operators of M at q and x are related by $\mathcal{S}_x = h_* \mathcal{S}_q h_*^{-1}$. Thus, they have the same eigenvalues. This shows that homogeneous hypersurfaces have constant principal curvatures and, thus, constant mean curvature.

Let γ be a geodesic normal to M at some point $p \in M$. The tangent space to any orbit of H is generated by Killing vector fields induced by H. If X is a Killing vector field induced by the action of H, then $\bar{\nabla} X$ is skew-symmetric, and thus $\langle \bar{\nabla}_{\dot{\gamma}} X, \dot{\gamma} \rangle = 0$, which proves that $\langle X, \dot{\gamma} \rangle$ is constant along γ. Since $\langle X, \dot{\gamma} \rangle$ vanishes at p and X is arbitrary, we have that γ is perpendicular to the other orbits it intersects. This shows that nearby orbits to M are equidistant to it, and hence, they all are isoparametric. □

3. The classification in Euclidean and hyperbolic spaces

The relevance of Cartan's fundamental formula stems from the fact that it can be applied to obtain the complete classification of isoparametric hypersurfaces in Euclidean and real hyperbolic spaces. Historically, the first one to obtain the complete classification in the Euclidean case was Segre [80], after Somigliana's [83] and Levi-Civita's [59] classifications for $\mathbb{R}^3$. The hyperbolic case was solved by Cartan [13].

Below we apply Cartan's formula to obtain the classification in Euclidean spaces.

Theorem 3.1 (**Classification of isoparametric hypersurfaces in $\mathbb{R}^n$**). *An isoparametric hypersurface in a Euclidean space $\mathbb{R}^n$ has $g \in \{1, 2\}$ principal curvatures and is an open part of one of the following hypersurfaces:*

(a) *an affine hyperplane $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$,*
(b) *a sphere S^{n-1} in $\mathbb{R}^n$,*
(c) *a generalized cylinder $S^k \times \mathbb{R}^{n-k-1}$, $k \in \{1, \dots, n-2\}$.*

Proof. Putting $\kappa = 0$ in Cartan's formula, we get $\sum_{j=1,\, j\neq i}^{g} m_j \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0$, for any principal curvature λ_i of an isoparametric hypersurface M in $\mathbb{R}^n$. Suppose that $g \geq 2$. By reversing the orientation of the normal vector if needed, we can take λ_i as the lowest positive principal curvature. Then all terms $m_j \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j}$ are negative, except if $\lambda_j = 0$. Then, Cartan's formula implies that at most λ_i and $\lambda_j = 0$ are principal curvatures. Thus we have shown that either $g = 1$ or $g = 2$ and, in the second case, one of the principal curvatures is zero.

If $g = 1$ and $\lambda_1 = 0$, we have an open part of a totally geodesic hyperplane, thus we have case (a). If $g = 1$ and $\lambda_1 \neq 0$, we have a totally umbilical non-totally geodesic hypersurface, so we are in case (b). Finally, if $g = 2$ standard Jacobi field theory implies that we have an open part of a distance tube around a totally geodesic subspace of codimension at least 2, that is, we are in case (c). □

Similar ideas as in the Euclidean case can be applied to obtain the following classification for real hyperbolic spaces.

Theorem 3.2 (**Classification of isoparametric hypersurfaces in** $\mathbb{R}H^n$)**.** *An isoparametric hypersurface in a real hyperbolic space* $\mathbb{R}H^n$ *has* $g \in \{1, 2\}$ *principal curvatures and is an open part of one of the following hypersurfaces:*

(a) *a totally geodesic real hyperbolic hyperspace* $\mathbb{R}H^{n-1}$ *in* $\mathbb{R}H^n$ *or one of its equidistant hypersurfaces,*
(b) *a distance tube around a totally geodesic real hyperbolic subspace* $\mathbb{R}H^k$ *in* $\mathbb{R}H^n$*,* $k \in \{1, \ldots, n-2\}$*,*
(c) *a geodesic sphere in* $\mathbb{R}H^n$*,*
(d) *a horosphere in* $\mathbb{R}H^n$*.*

An important consequence of these classifications is the fact that all isoparametric hypersurfaces in Euclidean and hyperbolic spaces are open parts of homogeneous hypersurfaces. In view of Proposition 2.10, this yields the classifications of homogeneous hypersurfaces in these spaces.

Another relevant observation is that every isoparametric hypersurface in $\mathbb{R}^n$ or $\mathbb{R}H^n$ is an open part of a complete isoparametric hypersurface which defines, via normal displacements, a decomposition $\mathcal{F}$ of the ambient space into equidistant submanifolds, which we can call leaves. These decomposition $\mathcal{F}$ is easily seen to be an isoparametric family of hypersurfaces, that is, its leaves are the level sets of an isoparametric map on the ambient manifold. Since, as mentioned above, all examples in these two settings are homogeneous, such an isoparametric family coincides with the set of orbits of an isometric action of cohomogeneity one. With the terminology that we will introduce in Section 7, the isoparametric family $\mathcal{F}$ above is also called an *isoparametric singular Riemannian foliation* of codimension one or, simply, and *isoparametric foliation* of codimension one. Thus, we can restate the observation by saying that every isoparametric hypersurface in $\mathbb{R}^n$ or $\mathbb{R}H^n$ is an open part of a regular leaf of an isoparametric foliation that filles the whole ambient space.

4. The classification problem in spheres

In this section we give a quick review of the outstanding problem of the classification of isoparametric hypersurfaces in spheres, which is still open nowadays.

4.1. **Cartan's results and the homogeneous examples.** Cartan also investigated isoparametric hypersurfaces in spheres in the articles [14], [15], [16]. In this setting, since $\kappa > 0$, the fundamental formula does not provide as much information as for $\kappa \leq 0$. In fact, the problem in spheres is much more involved and rich. Cartan was able to classify

isoparametric hypersurfaces in spheres S^n with $g \in \{1,2,3\}$ principal curvatures. The examples with $g = 1$ are just geodesic spheres, while those with $g = 2$ are tubes around totally geodesic submanifolds S^k of S^n with $1 \leq k \leq n-2$. For $g = 3$, Cartan showed that all three multiplicities m_i are equal, and one has $m = m_1 = m_2 = m_3 \in \{1,2,4,8\}$. He also proved that the corresponding isoparametric hypersurfaces are tubes around certain embedding of the projective plane $\mathbb{F}P^2$ in S^{3m+1}, where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, for $m = 1,2,4,8$, respectively. Moreover, Cartan found two examples of isoparametric hypersurfaces with four principal curvatures in S^5 and in S^9, but he could get neither a classification for $g \geq 4$, nor an upper bound on g (as for $\mathbb{R}^n$ and $\mathbb{R}H^n$).

Example 4.1. Let us describe briefly the isoparametric family of hypersurfaces with $g = 3$ principal curvatures in S^4 discovered by Cartan. We start by considering an orthogonal decomposition of the Lie algebra $\mathfrak{su}(3) = \mathfrak{k} \oplus \mathfrak{p}$ into the sum of the compact Lie subalgebra $\mathfrak{k} = \mathfrak{so}(3)$ and its 5-dimensional orthogonal complement $\mathfrak{p}$ with respect to the inner product given by $\langle X, Y\rangle = -\operatorname{tr}(XY)$. The action

$$\begin{array}{rcl} SO(3) \times \mathfrak{p} & \to & \mathfrak{p} \\ (k, X) & \mapsto & kXk^{-1} \end{array}$$

is then isometric with respect to the inner product on $\mathfrak{p}$ defined above. Thus, we have an orthogonal representation of the compact Lie group $SO(3)$ on the 5-dimensional vector space $\mathfrak{p}$. The restriction of this action to the unit sphere of $\mathfrak{p}$ is then an isometric action. Fix the following points in this unit sphere

$$p = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad q = \frac{1}{\sqrt{6}} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

Then, the Lie algebra of the isotropy group of p for the action of $K = SO(3)$ is trivial, since one can calculate that $[X, p] = 0$ implies that $X = 0$, for any $X \in \mathfrak{k}$. Hence, the dimension of the orbit of the above action through p equals the dimension of $SO(3)$, which is 3. Thus, the action above is a cohomogeneity one action on the sphere, and the orbit through p is a homogeneous isoparametric hypersurface in S^4. On the other hand, an elementary calculation shows that the isotropy group of q is isomorphic to $S(O(2) \times O(1))$. This implies that the orbit of $K = SO(3)$ through q is isometric to a real projective plane $\mathbb{R}P^2 = SO(3)/S(O(2) \times O(1))$. This is the so-called Veronese embedding of $\mathbb{R}P^2$ in S^4. By direct calculation or by the general theory of isoparametric hypersurfaces in spheres (see below), this immersion is minimal.

This example admits an extremely important generalization to the so-called isotropy representation of a symmetric space.

Example 4.2. Let $N = G/K$ be a symmetric space with no Euclidean factor, where G is the connected component of the identity in the isometry group $\operatorname{Isom}(N)$, and K is the isotropy group, that is, those elements in G that fix a base point $o \in N$. Then, K acts

infinitesimally on the tangent space T_oN by

$$\begin{array}{rcl} K \times T_oN & \to & T_oN \\ (k, v) & \mapsto & k_*v. \end{array}$$

This action is the *isotropy representation* of the symmetric space G/K. It is an isometric action with respect to the inner product of the symmetric space, and thus induces an isometric action on the unit sphere of T_oN.

Now let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition associated with the pair (G, K); that is, $\mathfrak{p}$ is the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G with respect to the Killing form of G. Consider the map $\phi\colon G \to N$, $g \mapsto \phi(g) = g(o)$. Its differential ϕ_{*e} at the identity $e \in G$ maps $\mathfrak{k}$ to 0, and $\mathfrak{p}$ isomorphically onto T_oN. Thus, it turns out that the isotropy representation above can be rewritten as the adjoint action

$$\begin{array}{rcl} K \times \mathfrak{p} & \to & \mathfrak{p} \\ (k, X) & \mapsto & \mathrm{Ad}(k)X. \end{array}$$

If the symmetric space $N = G/K$ has rank two, then the restriction of the isotropy representation to the unit sphere of T_oN (or of $\mathfrak{p}$) has cohomogeneity one. Note that we have an explicit list of symmetric spaces of rank two; see Table 1 below. Observe also that Example 4.1 above corresponds precisely to the isotropy representation of the rank-two symmetric space $SU(3)/SO(3)$.

Coming back to the development of the study of isoparametric hypersurfaces, Cartan also noticed that all examples known to him (those in spheres, but also those in $\mathbb{R}^n$ and $\mathbb{R}H^n$) were homogeneous. This observation led him to ask the following question: is every isoparametric hypersurface extrinsically homogeneous? A surprising negative answer would only come several decades later.

The study of isoparametric hypersurfaces was taken up again in the early seventies. Nomizu [73] shows that the focal manifolds of an isoparametric family of hypersurfaces in a sphere are minimal; the *focal manifolds* of an isoparametric family are those elements of the family with codimension greater than one. About that time Hsiang and Lawson [51] derived the classification of cohomogeneity one actions on spheres:

Theorem 4.3. [51] *Each cohomogeneity one action on a sphere S^n is orbit equivalent to the isotropy representation of a Riemannian symmetric space of rank 2. Every such action has exactly two singular orbits, while the other orbits are principal and tubes around each one of the singular ones.*

Based on the work of Hsiang and Lawson, Takagi and Takahashi [87] determined the principal curvatures of homogeneous (isoparametric) hypersurfaces in spheres. According to these results, every homogeneous hypersurface in a sphere is a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two. In Table 1 all symmetric spaces of rank 2 are shown, together with their dimensions, the number g of principal curvatures and the multiplicities of the corresponding homogeneous hypersurfaces.

g	Multiplicities	Symmetric space G/K	$\dim G/K$
1	$l-2$	$S^1 \times S^{l-1}$	l
2	$(k, l-k-2)$	$S^{k+1} \times S^{l-k-1}$	l
3	1	$SU(3)/SO(3)$	5
3	2	$SU(3)$	8
3	4	$SU(6)/Sp(3)$	14
3	8	E_6/F_4	26
4	$(2,2)$	$Sp(2)$	10
4	$(4,5)$	$SO(10)/U(5)$	20
4	$(1, k-2)$	$SO(k+2)/SO(2) \times SO(k)$	$2k$
4	$(2, 2k-3)$	$SU(k+2)/\mathrm{S}(U(2) \times U(k))$	$4k$
4	$(4, 4k-5)$	$Sp(k+2)/Sp(2) \times Sp(k)$	$8k$
4	$(9,6)$	$E_6/Spin(10) \cdot U(1)$	32
6	$(1,1)$	$G_2/SO(4)$	8
6	$(2,2)$	G_2	14

TABLE 1. Compact symmetric spaces of rank 2 corresponding to the homogeneous isoparametric families in spheres

4.2. **Münzner's structural results.** A consequence of Takagi and Takahashi's work is that the number of principal curvatures g of a homogeneous hypersurface in a sphere satisfies $g \in \{1, 2, 3, 4, 6\}$. In two remarkable articles [70], [71], Münzner was able to prove that the same restriction on g holds for every (not necessarily homogeneous) isoparametric hypersurface in a sphere. Münzner's papers contain a deep analysis of the structure of isoparametric families of hypersurfaces in spheres, using both geometric and topological methods. Apart from the restriction on g, we emphasize other two consequences of Münzner's work.

The first one is that, if $\lambda_1 < \cdots < \lambda_g$ are the principal curvatures of an isoparametric hypersurface in a sphere, they can be written as $\lambda_i = \cot \theta_i$, for $0 < \theta_1 < \cdots < \theta_g < \pi$, with

$$\theta_i = \theta_1 + \frac{i-1}{g}\pi, \qquad i = 1, \ldots, g.$$

Moreover, if the corresponding multiplicities are $m_1, \ldots, m_g$, they satisfy $m_i = m_{i+2}$ (indices modulo g); in particular, if g is odd, all the multiplicities coincide, and if g is even, there are at most two different multiplicities.

The second result is the algebraic character of isoparametric hypersurfaces in spheres. More precisely, a hypersurface M in S^n is isoparametric if and only if $M \subset F^{-1}(c) \cap S^n$, where F is a homogeneous polynomial of degree g on $\mathbb{R}^{n+1}$ satisfying the differential

equations

$$\|\nabla F(x)\|^2 = g^2\|x\|^{2g-2},$$
$$\Delta F(x) = \frac{1}{2}(m_2 - m_1)g^2\|x\|^{g-2}, \qquad x \in \mathbb{R}^{n+1}.$$

The intersection of S^n with the level sets of such an F form an isoparametric family of hypersurfaces in S^n. From this result, it also follows that every isoparametric hypersurface in S^n is an open part of a complete isoparametric hypersurface in S^n, which is in turn a leaf of an isoparametric family of hypersurfaces that fills the whole S^n (this happened also for $\mathbb{R}^n$ and $\mathbb{R}H^n$). A polynomial F like the one above is called a *Cartan-Münzner polynomial*. Notice that, according to this result, the classification problem of isoparametric hypersurfaces in spheres is reduced to a problem of Algebraic Geometry, but a very difficult one.

Each isoparametric family in S^n determined by a Cartan-Münzner polynomial F has exactly two focal submanifolds of codimensions $m_1 + 1$ and $m_2 + 1$, which correspond to the level sets $M_+ = F|_{S^n}^{-1}(1)$ and $M_- = F|_{S^n}^{-1}(-1)$, regardless of the number of principal curvatures. Each principal curvature $\lambda_i = \cot\theta_i$, $i = 1, \ldots, g$, of a hypersurface M in the family gives rise to two antipodal focal points that correspond to the instants $t = \theta_i$ and $t = \theta_i + \pi$ in any unit speed geodesic γ in S^n normal to the hypersurface M and with $\gamma(0) \in M$. The $2g$ focal points are equally spaced at intervals of length π/g along the normal geodesic γ, and they lie alternately on the two focal submanifolds M_+ and M_-.

4.3. **The inhomogeneous examples.** Since the restriction on g obtained by Münzner coincides with the one for homogeneous hypersurfaces, Cartan's question on the homogeneity of isoparametric hypersurfaces became even more attractive. However, in 1975 Ozeki and Takeuchi gave a negative answer to this question [75]. They constructed some Cartan-Münzner polynomials that give rise to isoparametric hypersurfaces with $g = 4$ that are not homogeneous, because their multiplicities do not coincide with the possible multiplicities of the homogeneous examples.

Some years later, Ferus, Karcher and Münzner [40] found a much larger family of inhomogeneous examples that included the ones given by Ozeki and Takeuchi. For each representation of a Clifford algebra they constructed a Cartan-Münzner polynomial that yields an isoparametric family of hypersurfaces with $g = 4$. We call these examples *hypersurfaces of FKM-type* or *of Clifford type*, and the corresponding isoparametric families are called *FKM-foliations*. We have preferred to postpone the construction of these examples to §4.5 below.

Most of the isoparametric hypersurfaces of FKM-type are inhomogeneous, and this inhomogeneity was proved in [40] in a direct way, without using the classification of homogeneous hypersurfaces. As a consequence of this result, one gets the existence of an infinite countable collection of noncongruent inhomogeneous isoparametric families in spheres. This made the study of isoparametric hypersurfaces in spheres a much more appealing and interesting topic of research.

4.4. **Towards the final classification.** Even today, all known isoparametric hypersurfaces in spheres are either homogeneous or of FKM-type; and all those hypersurfaces with $g = 4$ are of FKM-type, with the exception of two homogeneous families of hypersurfaces with multiplicities $(2, 2)$ and $(4, 5)$. A first step towards a classification would be to determine the possible triples (g, m_1, m_2) that an isoparametric hypersurface with $g = 4$ or $g = 6$ can take. Several authors have contributed to this question (we just mention some of them, and refer to the surveys [93] and [18] for further references). In [70] and [71], Münzner already found some restrictions, which were improved by Abresch [1]. In particular, Abresch showed that the only possible triples with $g = 6$ are $(6, 1, 1)$ and $(6, 2, 2)$; moreover, there exist homogeneous examples in both cases. The determination of all possible triples with $g = 4$ was established by Stolz in 1999 [84]. He proved that every isoparametric hypersurface with $g = 4$ constant principal curvatures in a sphere has the multiplicities of one of the known homogeneous or inhomogeneous examples; in other words, the possible triples $(4, m_1, m_2)$ are $(4, 2, 2)$, $(4, 4, 5)$ and the ones of FKM-type hypersurfaces (see Table 2 in §4.5).

As we mentioned before, isoparametric hypersurfaces in spheres with $g \in \{1, 2, 3\}$ had been classified by Cartan. In 1976, Takagi [86] showed that if $g = 4$ and one of the multiplicities is one, then the hypersurface is homogeneous and of FKM-type. Ozeki and Takeuchi [76] proved that those isoparametric hypersurfaces with $g = 4$ and one multiplicity equal to 2 are homogeneous and, except for the case of multiplicities $(2, 2)$ (which corresponds to the homogeneous example in S^9 obtained by Cartan), also of FKM-type. In 1985, Dorfmeister and Neher [37] proved the uniqueness of the hypersurface with triple $(6, 1, 1)$, which is hence homogeneous. Quite recently, in 2007-2008, Cecil, Chi and Jensen [19], and independently Immervoll [52], proved that, with a few possible exceptions, every isoparametric hypersurface with $g = 4$ is one of the known examples. More precisely, if the multiplicities (m_1, m_2) of an isoparametric hypersurface with $g = 4$ in a sphere satisfy $m_2 \geq 2m_1 - 1$, then such hypersurface must be of FKM-type. Together with other known results, this one gives a classification of the case $g = 4$ with the exception of the pairs of multiplicities $(3, 4)$, $(4, 5)$, $(6, 9)$ and $(7, 8)$. The methods used in both articles are different: while Cecil, Chi and Jensen make use of the theory of moving frames and commutative algebra, Immervoll uses the tool of isoparametric triple systems developed by Dorfmeister and Neher [37]. In the last years, on the one hand, Chi went on studying the exceptional cases with $g = 4$ in [20] and [21], leaving only open the case of multiplicities $(7, 8)$. On the other hand, Miyaoka has investigated the case $(g, m_1, m_2) = (6, 2, 2)$ in the article [66], where the uniqueness and homogeneity of such isoparametric family is claimed. However, in view of the recent errata [68] and the work in progress [82] by Siffert, it seems that this result has not been confirmed yet.

4.5. **The FKM examples.** In this subsection we present the construction and some of the main properties of the isoparametric families in spheres constructed by Ferus, Karcher and Münzner. The exposition here is taken from the paper [36]. For details missing here we refer to the original paper [40].

Let $V = \mathbb{R}^{2n+2}$ be a Euclidean space and $(P_0, \dots, P_m)$ an $(m+1)$-tuple of symmetric real matrices of order $2n+2$. Thus, we regard each P_i as a selfadjoint endomorphism of V. This $(m+1)$-tuple $(P_0, \dots, P_m)$ is called a *(symmetric) Clifford system* if the matrices satisfy $P_iP_j + P_jP_i = 2\delta_{ij}\,\mathrm{Id}$ for all $i, j \in \{0, \dots, m\}$, where δ_{ij} is the Kronecker delta. We also define $\mathcal{P} = \mathrm{span}\{P_0, \dots, P_m\}$ and endow this vector space with the inner product given by $\langle P, P'\rangle = (1/\dim V)\,\mathrm{tr}(PP')$, for $P, P' \in \mathcal{P}$.

Assume that $m_2 = n - m > 0$. Then the FKM-foliation $\mathcal{F}_{\mathcal{P}}$ associated with the Clifford system $(P_0, \dots, P_m)$ is defined by the level sets of $F|_{S(V)}$, where $S(V)$ is the unit sphere of V and $F\colon V \to \mathbb{R}$ is the Cartan-Münzner polynomial:

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2.$$

The corresponding isoparametric hypersurfaces have $g = 4$ principal curvatures with multiplicities $(m_1, m_2) = (m, n-m)$. This construction does not depend on the particular matrices $P_0, \dots, P_m$, but only on the unit sphere $S(\mathcal{P})$ of $\mathcal{P}$. $S(\mathcal{P})$ is called the Clifford sphere of the foliation. Moreover, two FKM-foliations are congruent if and only if their Clifford spheres are conjugate under an orthogonal transformation of V.

For each integer $m \geq 1$, we define $\delta(m)$ as the smallest natural number such that there exists a Clifford system $(P_0, \dots, P_m)$ on $V = \mathbb{R}^{2\delta(m)}$. Equivalently, $2\delta(m)$ is the dimension of any irreducible Clifford Cl^*_{m+1}-module; see [58, Chapter I] for more information on Clifford algebras and modules. In addition, if $(P_0, \dots, P_m)$ is a Clifford system on $V = \mathbb{R}^{2n+2}$, then there is a natural number k such that $n+1 = k\delta(m)$. Conversely, for a fixed $m \geq 1$, if k is a natural number such that $m_2 = n - m \geq 1$, with $n = k\delta(m) - 1$, then there exists a Clifford system $(P_0, \dots, P_m)$ on $V = \mathbb{R}^{2n+2} = \mathbb{R}^{2k\delta(m)}$ that gives rise to an FKM-foliation on S^{2n+1}.

The classification result of FKM-foliations given in [40] ensures that for $m \not\equiv 0 \,(\mathrm{mod}\, 4)$, there exists only one isoparametric FKM-foliation for each natural $k \geq (m+2)/\delta(m)$, up to congruence. However, if $m \equiv 0 \,(\mathrm{mod}\, 4)$, for each natural $k \geq (m+2)/\delta(m)$ there are exactly $[k/2] + 1$ FKM-foliations up to congruence. Here $[\cdot]$ denotes the integer part of a real number. This different behaviour, depending on whether m is multiple of 4 or not, is due to the fact that, if $m \equiv 0 \,(\mathrm{mod}\, 4)$, there exist exactly two irreducible representations $\mathfrak{d}_+$, $\mathfrak{d}_-$ of the Clifford algebra Cl^*_{m+1} up to equivalence, whereas there is only one, say $\mathfrak{d}$, if $m \not\equiv 0 \,(\mathrm{mod}\, 4)$. Thus, every representation of Cl^*_{m+1} on $V = \mathbb{R}^{2n+2}$ has the form $\oplus_{i=1}^{k}\mathfrak{d}$ if $m \equiv 0 \,(\mathrm{mod}\, 4)$, or the form $(\oplus_{i=1}^{k_+}\mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-}\mathfrak{d}_-)$ if $m \not\equiv 0 \,(\mathrm{mod}\, 4)$, for certain integers k_+, k_- such that $k = k_+ + k_-$.

In Table 2 we show the pairs of multiplicities $(m_1, m_2) = (m, n-m)$ of the principal curvatures of the hypersurfaces of FKM type, for low values of m and k. When a pair (m_1, m_2) is not underlined, we will understand that there is only one FKM-foliation with those multiplicities, up to congruence; the underlinings $\underline{(m_1, m_2)}$, $\underline{\underline{(m_1, m_2)}}$... point to the existence of two, three... FKM-foliations with multiplicities (m_1, m_2), respectively.

For a fixed m, the examples in the corresponding column of Table 2 are mutually non-congruent. Nevertheless, it can happen that examples in two different columns (i.e. with

m	1	2	3	4	5	6	7	8	9	10	...
$\delta(m)$	1	2	4	4	8	8	8	8	16	32	...
$k=1$	–	–	–	–	$(5,2)$	$(6,1)$	–	–	$(9,6)$	$(10,21)$	...
$k=2$	–	$(2,1)$	$(3,4)$	$\underline{(4,3)}$	$(5,10)$	$(6,9)$	$(7,8)$	$\underline{(8,7)}$	$(9,22)$	$(10,53)$	...
$k=3$	$(1,1)$	$(2,3)$	$(3,8)$	$\underline{(4,7)}$	$(5,18)$	$(6,17)$	$(7,16)$	$\underline{(8,15)}$	$(9,38)$	$(10,85)$	...
$k=4$	$(1,2)$	$(2,5)$	$(3,12)$	$\underline{\underline{(4,11)}}$	$(5,26)$	$(6,25)$	$(7,24)$	$\underline{\underline{(8,23)}}$	$(9,54)$	$(10,117)$	...
$k=5$	$(1,3)$	$(2,7)$	$(3,16)$	$\underline{\underline{(4,15)}}$	$(5,34)$	$(6,33)$	$(7,32)$	$\underline{\underline{(8,31)}}$	$(9,70)$	$(10,149)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 2. Small multiplicities (m_1, m_2) of the FKM-hypersurfaces

different m) are congruent to each other. In fact, this is the case for several pairs of examples: the families with pairs of multiplicities $(2,1)$, $(6,1)$, $(5,2)$ and one of the two families with pair $(4,3)$ are congruent, respectively, to those families with pairs $(1,2)$, $(1,6)$, $(2,5)$ e $(3,4)$. These exhaust all coincidences up to congruence. Moreover, the only FKM-foliations that are homogeneous correspond to those families with multiplicities $(1,k-2)$ for $k \geq 3$, $(2,2k-3)$ for $k \geq 2$, one of the examples with multiplicities $(4,4k-5)$ for each $k \geq 2$, and the foliations with multiplicities $(5,2)$, $(6,1)$ and $(9,6)$.

5. Isoparametric hypersurfaces in spaces of nonconstant curvature

As shown by Cartan, in a space of constant curvature, an isoparametric hypersurface is the same as a hypersurface with constant principal curvatures. However, this equivalence does not hold in general for ambient manifolds of nonconstant curvature, as will be comented below. Thus, two different properties of hypersurfaces in ambient manifolds of nonconstant curvature generalize in a natural way the property of being an isoparametric hypersurface in a real space form: the original notion of isoparametric hypersurface, but also the notion of hypersurface with constant principal curvatures. The study of hypersurfaces with constant principal curvatures, particularly in complex space forms, has been a fruitful area of research in the last decades. We refer the reader to the [28] and [33] for further information on this topic. However, in these notes we focus on the original notion of isoparametric hypersurface.

The most efficient method to construct isoparametric hypersurfaces in a space of nonconstant curvature is by means of cohomogeneity one actions: their principal orbits are then homogeneous isoparametric hypersurfaces. Cohomogeneity one actions have been studied in different settings. For example, they have shown to be useful in the construction of manifolds with nonnegative or positive sectional curvature [46], [100], [45], of Einstein, Einstein-Kähler and Einstein-Weyl structures [4], [12], in order to investigate Yang-Mills equations [96], and to construct hyper-Kähler Calabi metrics [23], special Lagrangian submanifolds [53] or Ricci solitons [26]. Moreover, the classification of cohomogeneity one actions has been studied for certain spaces, with especial attention to symmetric spaces;

see the paper [55] by Kollross for the compact case, and the paper [10] by Berndt and Tamaru for the noncompact case.

Apart from cohomogeneity one actions, another method of construction of isoparametric hypersurfaces is by means of warped products. It is known that the fibers $\{b\} \times F$ of a warped product $B \times_f F$ are totally umbilical submanifolds; see [74]. If we take the basis B to be of dimension one, F any Riemannian manifold, and $f\colon B \to \mathbb{R}$ any smooth positive function, the warped product $B \times_f F$ carries an isoparametric family of totally umbilical hypersurfaces given by the fibers $\{b\} \times F$, where $b \in B$. This allows to obtain a large family of examples of isoparametric families of hypersurfaces with exactly one constant principal curvature. If we take F to be an intrinsically inhomogeneous Riemannian manifold, then the corresponding isoparametric hypersurfaces cannot be extrinsically homogeneous.

However, these two families of examples are rather particular. We are interested in obtaining methods of construction of isoparametric hypersurfaces which, in principle, can be inhomogeneous and have extrinsic geometry more complicated than the fibers of a warped product, which are totally umbilical. The natural setting to address this purpose is to take ambient spaces with lots of symmetries. Symmetric spaces are then natural candidates.

The first examples of isoparametric hypersurfaces with nonconstant principal curvatures were found by Wang [97] in the complex projective space $\mathbb{C}P^n$, by projecting some of the inhomogeneous isoparametric hypersurfaces of FKM-type in odd-dimensional spheres S^{2n+1} to $\mathbb{C}P^n$ via the Hopf map. Other inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in complex projective spaces were constructed by Xiao [102] and by Ge, Tang and Yan [43]. These examples are again related to the isoparametric hypersurfaces in spheres. The idea of using the Hopf map to project isoparametric hypersurfaces in a sphere to a complex projective space has been systematically explored in [36]. Explaining this method is the aim of Section 8.

Another large set of examples is given by small geodesic spheres in the non-symmetric Damek-Ricci spaces. These are certain solvable Lie groups endowed with a left-invariant metric which are harmonic as Riemannian manifolds; they were constructed by Damek and Ricci [25]. One characterization of harmonicity is that sufficiently small geodesic spheres have constant mean curvature, and hence, are isoparametric. The family of Damek-Ricci spaces includes the Riemannian symmetric spaces of noncompact type and rank one as particular cases (these are precisely real, complex and quaternionic hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$ and $\mathbb{H}H^n$, and the Cayley hyperbolic plane $\mathbb{O}H^2$). However, for those non-symmetric Damek-Ricci spaces, the small geodesic spheres have nonconstant principal curvatures, in spite of being isoparametric. In §6.2 we will talk about Damek-Ricci spaces.

Recently, Díaz-Ramos and Domínguez-Vázquez have constructed many inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in the complex hyperbolic space [29] and, more generally, in Damek-Ricci spaces [30]. The aim of Section 6, and in particular of §6.3, is to explain the construction of these hypersurfaces.

6. Isoparametric hypersurfaces in Damek-Ricci spaces

In this section our purpose is to explain a method for the construction of isoparametric hypersurfaces, which are generically inhomogeneous, in the symmetric spaces of noncompact type, nonconstant curvature and rank one. This method can indeed be applied to the more general setting of Damek-Ricci spaces, which are certain noncompact Lie groups with left-invariant metric. We provide several powerful tools that are used in the construction of the examples and that can be used for many other purposes (see for example [8], [31], [88]). Thus, in §6.1 we explain how a symmetric space of noncompact type can be regarded as a Lie group endowed with a left-invariant metric: the so-called solvable model. In §6.2 we present the definition of Damek-Ricci spaces, which is motivated by the solvable model of a rank one noncompact symmetric space. Finally, the construction method of isoparametric hypersurfaces in Damek-Ricci spaces is presented in §6.3.

6.1. **The solvable model of a noncompact symmetric space.** Our aim in this section is to provide a model of any symmetric space $M = G/K$ of noncompact type as a solvable Lie group AN equipped with a left-invariant metric. The proof of this general fact is based on the Iwasawa decomposition of the noncompact symmetric space. We will content ourselves with presenting the construction without giving the proofs. The reader is referred to [32, Chapter 2] for a more detailed description and to [54, §6.4] for general information on the Iwasawa decomposition of semisimple Lie groups; see also [9].

Let M be a symmetric space of noncompact type (see Appendix C for a quick introduction to symmetric spaces). Then M admits the representation as a coset space G/K, where G is the identity connected component of the isometry group of M, and K is the isotropy group at some point $o \in M$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. It is known that $\mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$.

Let ad and Ad be the adjoint maps of $\mathfrak{g}$ and G, respectively. Let $\mathcal{B}$ be the Killing form of $\mathfrak{g}$, that is, $\mathcal{B}\colon (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto \mathcal{B}(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) \in \mathbb{R}$, which is a nondegenerate bilinear form by virtue of Cartan's criterion for semisimple Lie algebras. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the *Cartan decomposition* of $\mathfrak{g}$ with respect to $o \in M$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\mathcal{B}$. This means that we have the bracket relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $\mathcal{B}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

The *Cartan involution* θ corresponding to the Cartan decomposition above is the automorphism of the Lie algebra $\mathfrak{g}$ defined by $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. Moreover, it turns out that $\mathcal{B}_\theta(X,Y) = -\mathcal{B}(\theta X, Y)$ defines a positive definite inner product on $\mathfrak{g}$ satisfying the relation $\mathcal{B}_\theta(\operatorname{ad}(X)Y, Z) = -\mathcal{B}_\theta(Y, \operatorname{ad}(\theta X)Y)$ for all X, Y, $Z \in \mathfrak{g}$.

We take now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. It happens that the dimension of $\mathfrak{a}$ is precisely the rank of the symmetric space M. The set $\{\operatorname{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting self-adjoint (with respect to $\mathcal{B}_\theta$) endomorphisms of $\mathfrak{g}$, and hence simultaneously diagonalizable. By definition, their common eigenspaces are the *(restricted) root spaces* of the simple Lie algebra $\mathfrak{g}$, and their nonzero eigenvalues (which do depend on $H \in \mathfrak{a}$) are

the *(restricted) roots* of $\mathfrak{g}$. Denoting by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$, if we define for each $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\},$$

then the *(restricted) root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ has the form

$$\mathfrak{g} = \mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right),$$

where Σ is the set of all restricted roots. These mutually $\mathcal{B}_\theta$-orthogonal subspaces $\mathfrak{g}_\lambda$ are precisely the root spaces. Moreover, $\mathfrak{a} \subset \mathfrak{g}_0$, and for every $\lambda, \mu \in \mathfrak{a}^*$, we have that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Furthermore, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \cong \mathfrak{u}(n-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$. Note that each root space $\mathfrak{g}_\lambda$ is normalized by $\mathfrak{g}_0$.

Now we fix a criterion of positivity in the set of roots: we take any hyperplane in $\mathfrak{a}^*$ not containing any root, and we declare the roots on one side of the hyperplane as positive, and the roots on the other side as negative. We denote by Σ^+ the subset of positive roots. Define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ as the sum of the root spaces corresponding to all positive roots. Due to the properties of the root space decomposition, $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$ is nilpotent.

Example 6.1. It is convenient to have in mind a specific example. The simplest one is when M is the real hyperbolic plane $\mathbb{R}H^2$. We refer to [9] for a description of the elements mentioned in this subsection for the case $M = \mathbb{R}H^2$. However, with view on the construction of new isoparametric hypersurfaces, the first interesting space is $M = \mathbb{C}H^n$: the complex hyperbolic space. In this case we have $G = SU(1, n)$ and $K = S(U(1)U(n))$. Note that G is simple, and K compact with one-dimensional center, which means that $\mathbb{C}H^n$ is a Hermitian symmetric space. Indeed, $\mathbb{C}H^n$ is a complete simply connected homogeneous Kähler manifold of constant holomorphic negative sectional curvature. Since $\mathbb{C}H^n$ has rank one, then $\dim \mathfrak{a} = 1$ in this case. Moreover, one can show that the restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ in this case adopts the form

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$$

for a certain covector $\alpha \in \mathfrak{a}^*$. We have that the set of roots is $\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$. It is possible to calculate all this explicitely in terms of matrices; we refer to [32] for a detailed exposition. It turns out that $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = \dim \mathfrak{a} = 1$ and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 2n - 2$. Now, fixing a criterion of positivity is equivalent to declare α or $-\alpha$ as positive root. If we choose the first possibility, then we have $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, which is a nilpotent Lie subalgebra of $\mathfrak{g}$ with center $\mathfrak{g}_{2\alpha}$. In fact $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional generalized Heisenberg algebra, and then $\mathfrak{a} \oplus \mathfrak{n}$ is the Lie algebra of a Damek-Ricci space. See §6.2 for a description of generalized Heisenberg algebras and Damek-Ricci spaces.

We come back to our general description. The direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is called the *Iwasawa decomposition* of the semisimple Lie algebra $\mathfrak{g}$. It is important to mention that, even though $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$, the previous decomposition of $\mathfrak{g}$ is just a decomposition in a direct sum of vector subspaces, but neither an orthogonal decomposition, nor a direct sum of Lie algebras.

Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a}\oplus\mathfrak{n}$, respectively. The Iwasawa decomposition theorem at the Lie group level ensures that the product map $(k,a,n)\in K\times A\times N\mapsto kan\in G$ is a diffeomorphism. Again, we just mean that G and $K\times A\times N$ are diffeomorphic as manifolds, but not that G is isomorphic to the direct product of the groups K, A and N. It follows from the Iwasawa decomposition that the solvable group AN acts simply transitively on M; in other words, we can identify M with AN as we will show soon.

Consider now the differentiable map

$$\phi\colon h\in G\mapsto h(o)\in M.$$

Since AN acts simply transitively on M, the map $\phi|_{AN}\colon AN\to M$ is a diffeomorphism, and one can identify $\mathfrak{a}\oplus\mathfrak{n}$ with the tangent space T_oM. The metric g of the symmetric space M as a Riemannian manifold induces a metric ϕ^*g on AN. The Riemannian manifolds (AN,ϕ^*g) and (M,g) are then trivially isometric. Let us denote by L_h the left translation in G by the element $h\in G$. As the metric g on M is invariant under isometries (and then under elements of G), it follows that

$$L_h^*(\phi^*g)=L_h^*\phi^*(h^{-1})^*g=(h^{-1}\circ\phi\circ L_h)^*g=\phi^*g,\quad \text{for all } h\in G,$$

because $(h^{-1}\circ\phi\circ L_h)(h')=h^{-1}(hh'(o))=h'(o)=\phi(h')$ for all $h'\in G$. Therefore the metric ϕ^*g on AN is left-invariant. From now on, we will denote this metric by $\langle\cdot,\cdot\rangle_{AN}$. Thus, we have obtained that M can be seen as a solvable Lie group AN endowed with a left-invariant metric.

Example 6.2. Particularizing this theory for the case of the complex hyperbolic space, what we see is that we can regard $\mathbb{C}H^n$ as solvable Lie group AN endowed with a left-invariant metric, and where its Lie algebra $\mathfrak{a}\oplus\mathfrak{n}=\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ can be identified with the tangent space $T_o\mathbb{C}H^n$, with $\dim\mathfrak{a}=\dim\mathfrak{g}_{2\alpha}=1$ and $\dim\mathfrak{g}_\alpha=2n-2$. By means of $\phi|_{AN}$ we can also equip AN with the Kähler structure induced by the one in $\mathbb{C}H^n$, and we obtain the corresponding complex structure J on AN, and also on $\mathfrak{a}\oplus\mathfrak{n}$. Some calculations with matrices would show that the complex structure J on $\mathfrak{a}\oplus\mathfrak{n}$ leaves $\mathfrak{g}_\alpha$ invariant and $J\mathfrak{a}=\mathfrak{g}_{2\alpha}$. Thus we can see $\mathfrak{g}_\alpha$ as a complex vector space $\mathbb{C}^{n-1}$.

Let $B\in\mathfrak{a}$ be a vector such that $\langle B,B\rangle_{AN}=1$ and define $Z=JB\in\mathfrak{g}_{2\alpha}$. Then $\langle Z,Z\rangle_{AN}=1$. Let now a, b, x, y be real numbers and U, $V\in\mathfrak{g}_\alpha$. One can show that the Lie bracket of $\mathfrak{a}\oplus\mathfrak{n}$ is given by

$$\frac{1}{\sqrt{-c}}[aB+U+xZ,bB+V+yZ]=-\frac{b}{2}U+\frac{a}{2}V+\left(-bx+ay+\langle JU,V\rangle_{AN}\right)Z,$$

where c is the constant holomorphic sectional curvature of $\mathbb{C}H^n$. Furthermore, the Levi-Civita connection $\bar\nabla$ of $(AN,\langle\cdot,\cdot\rangle_{AN})$ can be calculated by the expression (cf. [7, §2]):

$$\begin{aligned}\frac{1}{\sqrt{-c}}\bar\nabla_{aB+U+xZ}(bB+V+yZ)=&\left(xy+\frac{1}{2}\langle U,V\rangle_{AN}\right)B-\frac{1}{2}\left(bU+yJU+xJV\right)\\&+\left(-bx+\frac{1}{2}\langle JU,V\rangle_{AN}\right)Z.\end{aligned}$$

These formulas agree with the corresponding formulas for $\mathbb{C}H^n$ seen as a Damek-Ricci space (cf. §6.2). Indeed, the construction described so far for $\mathbb{C}H^n$ can be carried out also for the other rank one symmetric spaces of noncompact type: the hyperbolic spaces over the reals $\mathbb{R}H^n$, over the quaternions $\mathbb{H}H^n$, and the hyperbolic plane over the octonions $\mathbb{O}H^2$ (the so-called Cayley hyperbolic plane). For these spaces, $\dim \mathfrak{a} = 1$, since all have rank one, and the corresponding root space decomposition of $\mathfrak{g}$ adopts the same form as for $\mathbb{C}H^n$, but now the dimensions vary: we have $\dim \mathfrak{g}_\alpha = 0, 1, 3, 7$ according to real, complex, quaternionic or octonionic cases, respectively. An orthonormal basis of $\mathfrak{g}_\alpha$ can be regarded as the set of natural complex structures that we have in each case. The spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ are precisely the only Damek-Ricci spaces that are symmetric spaces. Thus, for rank one noncompact symmetric spaces we can use both the theory of symmetric spaces and the theory of Damek-Ricci spaces.

6.2. **Damek-Ricci spaces.** We have mentioned that rank one symmetric spaces of noncompact type are Damek-Ricci spaces. These manifolds were constructed by Damek and Ricci [25] to provide counterexamples to the Lichnerowicz conjecture, stating that every harmonic manifold is locally isometric to a two-point homogeneous space (that is, a rank one symmetric space, or Euclidean space). In fact, the only Damek-Ricci spaces that are not counterexamples to this conjecture are precisely the noncompact symmetric spaces of rank one: the hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$. Our purpose here is to present a succint description of these manifolds, with the only aim of using this description for the construction that we will present below in §6.3.

Definition 6.3. Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces. Define the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and endow it with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular. Define a linear map $J\colon Z \in \mathfrak{z} \mapsto J_Z \in \operatorname{End}(\mathfrak{v})$ such that $\langle J_Z U, V \rangle_{\mathfrak{n}} = -\langle U, J_Z V \rangle_{\mathfrak{n}}$ for all $U, V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Consider the Lie algebra structure on $\mathfrak{n}$ given by

$$\langle [U, V], X \rangle_{\mathfrak{n}} = \langle J_X U, V \rangle_{\mathfrak{n}}, \quad [X, V] = [U, Y] = [X, Y] = 0, \quad \text{for all } U, V \in \mathfrak{v},\ X, Y \in \mathfrak{z}.$$

Then $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center $\mathfrak{z}$, and, if $J_Z^2 = -\langle Z, Z \rangle_{\mathfrak{n}} \operatorname{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$, $\mathfrak{n}$ is said to be a *generalized Heisenberg algebra* or an *H-type algebra*. The associated simply connected nilpotent Lie group N, endowed with the induced left-invariant Riemannian metric, is called a *generalized Heisenberg group* or an *H-type group*.

For U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$, we have the following properties of generalized Heisenberg algebras:

$$J_X J_Y + J_Y J_X = -2\langle X, Y \rangle_{\mathfrak{n}} \operatorname{id}_{\mathfrak{v}}, \qquad [J_X U, V] - [U, J_X V] = -2\langle U, V \rangle_{\mathfrak{n}} X,$$
$$\langle J_X U, J_X V \rangle_{\mathfrak{n}} = \langle X, X \rangle_{\mathfrak{n}} \langle U, V \rangle_{\mathfrak{n}}, \qquad \langle J_X U, J_Y U \rangle_{\mathfrak{n}} = \langle X, Y \rangle_{\mathfrak{n}} \langle U, U \rangle_{\mathfrak{n}}.$$

In particular, for any unit $Z \in \mathfrak{z}$, J_Z is an almost Hermitian structure on $\mathfrak{v}$.

The classification of generalized Heisenberg algebras is known. In fact, it follows from the classification of representations of Clifford algebras of vector spaces with negative definite quadratic forms. The ultimate reason for this is that the map $J\colon \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ can be extended to the Clifford algebra $Cl(\mathfrak{z}, q)$, where q is the quadratic form given by $q(Z) = -\langle Z, Z \rangle$, in such a way that $\mathfrak{v}$ becomes now a Clifford module over $Cl(\mathfrak{z}, q)$ (see [11,

Chapter 3]). In particular, for each $m \in \mathbb{N}$ there exist an infinite number of non-isomorphic generalized Heisenberg algebras with $\dim \mathfrak{z} = m$.

Now we can give the construction of Damek-Ricci spaces.

Definition 6.4. Let $\mathfrak{a}$ be a one-dimensional real vector space, B a non-zero vector in $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ a generalized Heisenberg algebra, where $\mathfrak{z}$ is the center of $\mathfrak{n}$. We consider a new vector space $\mathfrak{a} \oplus \mathfrak{n}$ as the vector space direct sum of $\mathfrak{a}$ and $\mathfrak{n}$.

On $\mathfrak{a} \oplus \mathfrak{n}$ we define $\langle \cdot, \cdot \rangle$ as the extension of the inner product of $\mathfrak{n}$ that makes $\mathfrak{a} \oplus \mathfrak{n}$ be orthogonal decomposition and the vector B be of unit length. We consider the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{a} \oplus \mathfrak{n}$ that extends the Lie bracket of $\mathfrak{n}$ and satisfies

$$[B, U + Z] = Z + \frac{1}{2}U.$$

Thus, $\mathfrak{a} \oplus \mathfrak{n}$ becomes a solvable Lie algebra with an inner product. The corresponding simply connected Lie group AN, equipped with the induced left-invariant Riemannian metric, is called a *Damek-Ricci space*.

The Levi-Civita connection $\bar{\nabla}$ of a Damek-Ricci space is given by

$$\bar{\nabla}_{sB+V+Y}(rB + U + X) = -\frac{1}{2}J_X V - \frac{1}{2}J_Y U - \frac{1}{2}rV - \frac{1}{2}[U, V] - rY + \frac{1}{2}\langle U, V\rangle B + \langle X, Y\rangle B,$$

where s, $r \in \mathbb{R}$, U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$.

A Damek-Ricci space AN is a symmetric space if and only if AN is isometric to a rank one symmetric space. In this case, using the notation of §6.1, we have that $\mathfrak{v} = \mathfrak{g}_\alpha$ and $\mathfrak{z} = \mathfrak{g}_{2\alpha}$. Indeed, the motivation for the general construction of Damek-Ricci spaces comes from the solvable model AN of a noncompact symmetric space (of rank one) induced by the corresponding Iwasawa decomposition. In this case, AN is either isometric to a complex hyperbolic space $\mathbb{C}H^n$ with constant holomorphic sectional curvature -1 (in this case, $\dim \mathfrak{z} = 1$), or to a quaternionic hyperbolic space $\mathbb{H}H^n$ with constant quaternionic sectional curvature -1 (here $\dim \mathfrak{z} = 3$), or to the Cayley hyperbolic plane $\mathbb{O}H^2$ with minimal sectional curvature -1 ($\dim \mathfrak{z} = 7$). As a limit case, one would obtain the real hyperbolic space $\mathbb{R}H^n$ if one puts $\mathfrak{z} = 0$. Note, for example, that the expression given in Example 6.2 for the Levi-Civita connection of the solvable model of $\mathbb{C}H^n$ coincides (up to rescaling of the curvature) with the general formula for the Levi-Civita connection of a Damek-Ricci space.

6.3. **Isoparametric hypersurfaces in Damek-Ricci spaces.** Now we are in conditions to present the announced construction of the isoparametric families of hypersurfaces in Damek-Ricci spaces. Although the general construction method was developed in [30], some of the ideas behind it trace back to the papers [5] and [29].

Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$. Now let $\mathfrak{w}$ be any proper linear subspace of $\mathfrak{v}$, and define $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$. Using the expression for the Lie bracket of a Damek-Ricci space, one can easily check that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Consider the Lie subgroup S of AN with Lie algebra $\mathfrak{s}$. Using the expression for the Levi-Civita connection of a Damek-Ricci space, it is easy to calculate the shape operator

of S, regarded as a submanifold of AN. It turns out that S is a minimal submanifold of AN, which is homogeneous by construction. Indeed, S is austere: its nonzero principal curvatures occur in pairs with opossite signs. The main theorem in [30] is the following.

Theorem 6.5. *The tubes around S are isoparametric hypersurfaces. Moreover, each of these tubes have constant principal curvatures if and only if the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$, $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$, has constant generalized Kähler angle.*

We soon explain what generalized Kähler angle means. But first note that the these result guarantees that the family of submanifolds of the Damek-Ricci space AN formed by S and all distance tubes around it constitutes an isoparametric family of hypersurfaces. Observe also that this construction admits a lot of freedom, in the sense that $\mathfrak{w}$ can be any vector subspace of $\mathfrak{v}$ different from $\mathfrak{v}$.

The notion of generalized Kähler angle has been introduced in [30] as a generalization of the notion of Kähler angle and quaternionic Kähler angle (see [5]). This concept is crucial, not only to state the characterization above of the examples having constant principal curvatures, but also for the proof of the fact that the tubes are isoparametric. Here we will not explain the details of the proof. But the idea is simple: one has to determine the extrinsic geometry of tubes around a submanifold with known extrinsic geometry. This can be done using Jacobi field theory. The difficult point is to write down and solve the Jacobi equation for a Damek-Ricci space. To sort out this problem, one uses on the one hand a slight modification of the standard Jacobi field theory, which makes use of left-invariant vector fields instead of parallel vector fields. On the other hand, one uses the notion of generalized Kähler angle to simplify the calculations which, otherwise, would be completely unmanageable.

In order to define the generalized Kähler angle of a subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ with respect to some nonzero vector $\xi \in \mathfrak{w}^\perp$, consider the following quadratic form:

$$\begin{array}{rccl} Q_\xi\colon & \mathfrak{z} & \to & \mathbb{R} \\ & Z & \mapsto & \langle (J_Z\xi)^\perp, (J_Z\xi)^\perp \rangle, \end{array}$$

where $(\cdot)^\perp$ denotes orthogonal projection onto $\mathfrak{w}^\perp$. It is clear that the eigenvalues of the quadratic form Q_ξ, for any unit $\xi \in \mathfrak{w}^\perp$, are real numbers in the interval $[0,1]$, and thus can be written in the form $\cos^2(\varphi_i)$, for some angles $\varphi_i \in [0, \pi/2]$, $i = 1, \ldots, m = \dim \mathfrak{z}$. With this notation we define the *generalized Kähler angle* of $\mathfrak{w}^\perp$ with respect to ξ as the m-tuple $(\varphi_1, \ldots, \varphi_m)$. The subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ is said to have *constant generalized Kähler angle* if the generalized Kähler angle of $\mathfrak{w}^\perp$ with respect to any unit $\xi \in \mathfrak{w}^\perp$ is independent of ξ. This is precisely the condition that allows to characterize the examples with constant principal curvatures in Theorem 6.5.

Example 6.6. Consider the complex hyperbolic space $\mathbb{C}H^n$ seen as a Damek-Ricci space. In this case, $m = \dim \mathfrak{z} = 1$ and we regard $\mathfrak{v}$ as a complex Euclidean space $\mathbb{C}^{n-1}$ with complex structure $J = J_Z$, for some unit $Z \in \mathfrak{z}$. Then, given $\xi \in \mathfrak{w}^\perp$ we simply speak about the *Kähler angle* of $\mathfrak{w}^\perp$ with respect to ξ, which is the angle $\varphi \in [0, \pi/2]$ between $J\xi$ and $\mathfrak{w}^\perp$. Note that totally real subspaces $\mathfrak{w}^\perp$ of $\mathfrak{v}$ (i.e. those for which $\langle J\mathfrak{w}^\perp, \mathfrak{w}^\perp \rangle = 0$) are

precisely the subspaces with constant Kähler angle $\pi/2$, whereas complex subspaces (i.e. $J\mathfrak{w}^\perp = \mathfrak{w}^\perp$) are the subspaces with constant Kähler angle 0. If $n = 2$ and hence $\dim \mathfrak{v} = 2$, then any subspace of $\mathfrak{v}$ has constant Kähler angle 0 or $\pi/2$. However, if $n \geq 2$, there always exist subspaces with any constant Kähler angle $\varphi \in [0, \pi/2]$; see [5] for the classification of subspaces with constant Kähler angle of a complex Euclidean space. If $\mathfrak{w}^\perp$ is taken with constant Kähler angle, then the corresponding isoparametric families are homogeneous, as shown in [5]. Moreover, if $n \geq 2$, there always exist subspaces with nonconstant Kähler angle (indeed, infinitely many of them modulo unitary transformation, whenever $n \geq 3$); see [34, §5.4.1]. A simple example is given by taking $\mathfrak{w}$ a one dimensional subspace of $\mathfrak{v}$ and, thus, $\mathfrak{w}^\perp$ is a $(2n-3)$-dimensional subspace of $\mathfrak{v}$. Thus, Theorem 6.5 guarantees the existence of infinitely many noncongruent isoparametric families of hypersurfaces in complex hyperbolic spaces whose principal curvatures are nonconstant and, hence, are not homogeneous.

Example 6.7. The case of the Cayley hyperbolic plane $\mathbb{O}H^2$ is quite interesting. In this setting, $\dim \mathfrak{z} = 7$ and $\mathfrak{v}$ can be identified with the division algebra $\mathbb{O}$ of the octonions, which is an 8-dimensional real vector space. As shown in [5], if we take any subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ with dimension 1, 2, 3, 4, 6, 7 or 8, the tubes around the corresponding minimal submanifold S are always homogeneous hypersurfaces. However, if $\dim \mathfrak{w}^\perp = 5$, none of these tubes is homogeneous. But, in this case, it is easy to see that $\mathfrak{w}^\perp$ has constant generalized Kähler angle (see [30]), which means that the tubes around S have constant principal curvatures by virtue of Theorem 6.5. Thus, we obtain an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in $\mathbb{O}H^2$. This is the only known example with these properties apart from the FKM-families in spheres.

7. Isoparametric submanifolds and isoparametric foliations

So far, we have dealt with the notion of isoparametricity applied to hypersurfaces, that is, to submanifolds with codimension one, in different ambient spaces. However, one can generalize this notion to submanifolds of arbitrary codimension, and the corresponding theory is really rich and appealing. The purpose of this section is to provide an introduction to this generalization, and present the main results known so far, with focus on the classification in complex projective spaces. There are several excellent references to deepen into this and other related topics. Let us mention the articles [93], [94], [95], [49] and [3], as well as the books [77] and [6].

We will start by talking about singular Riemannian foliations in §7.1, since these objects provide the natural framework for the study of isoparametric foliations. Then we will speak about isoparametric submanifolds in spaces of constant curvature in §7.2, where, under some irreducibility assumptions, the classification is known for codimension higher than one. Finally, in §7.3 we present the general definition of isoparametric submanifold in any ambient space, and comment on some known results in spaces of nonconstant curvature.

7.1. Singular Riemannian foliations.

As we have seen in the codimension one case, and will be commented in the next subsection for the arbitrary case, isoparametric submanifolds

define locally a foliation by equidistant submanifolds. Sometimes, this foliation can be extended to fill the whole ambient space, but allowing leaves with different codimensions. An example of this is given by the family of orbits of a cohomogeneity one action. It is important to have a precise notion for this kind of decompositions of a Riemannian manifold into equidistant leaves of different dimensions. The appropriate notion is that of singular Riemannian foliation. These objects were introduced by Molino [69] in his study of Riemannian foliations and constitute nowadays an active field of research. See the articles [3], [62] and [95] for more information. Here we will simply give the definition and some important classes of singular Riemannian foliations.

Definition 7.1. Let $\mathcal{F}$ be a decomposition of a Riemannian manifold $\bar{M}$ into connected injectively immersed submanifolds, called *leaves*, which may have different dimensions. We say that $\mathcal{F}$ is a *singular Riemannian foliation* if the following conditions are satisfied:

(i) $\mathcal{F}$ is a *transnormal system*, that is, every geodesic orthogonal to one leaf remains orthogonal to all the leaves that it intersects, and
(ii) $\mathcal{F}$ is a *singular foliation*, that is, $T_pL = \{X_p : X \in \mathcal{X}_{\mathcal{F}}\}$ for every leaf L in $\mathcal{F}$ and every $p \in L$, where $\mathcal{X}_{\mathcal{F}}$ is the module of smooth vector fields on the ambient manifold that are everywhere tangent to the leaves of $\mathcal{F}$.

The leaves of maximal dimension are called *regular* and the other ones are *singular*. The points of $\bar{M}$ are said to be regular or singular according to the leaves through them. A singular Riemannian foliation is called *regular* if all leaves are regular, that is, if it is a Riemannian foliation. The *dimension* of $\mathcal{F}$ is the maximal dimension of the leaves and its *codimension* is $\dim \bar{M} - \dim \mathcal{F}$.

In these notes, for the sake of brevity, we will usually refer to singular Riemannian foliations simply as *foliations* and we will use the term *regular foliation* to mean (regular) Riemannian foliation.

The first set of examples is given by isometric actions on Riemannian manifolds (see Appendix B for the main concepts concerning isometric actions). Let G be a Lie group that acts on a Riemannian manifold $\bar{M}$ by isometries. Then, the set $\mathcal{F}$ of orbits is called the *orbit foliation* of the action; $\mathcal{F}$ is then a *homogeneous foliation* and its orbits are called *(extrinsically) homogeneous submanifolds*. It is clear that $\mathcal{F}$ is a singular foliation since the set of values of the Killing fields induced by the action at a point $p \in \bar{M}$ coincides with the tangent space $T_p(G \cdot p)$ at p of the orbit $G \cdot p$. The transnormality of $\mathcal{F}$ follows from the fact that $\bar{\nabla} X$ is a skew-symmetric tensor field on $\bar{M}$ for every Killing field X. Hence $\mathcal{F}$ is a singular Riemannian foliation.

Another important example is that of *polar foliations*, also called *singular Riemannian foliations with sections* in the terminology of Alexandrino [2]. Let $\mathcal{F}$ be a foliation on $\bar{M}$. Then $\mathcal{F}$ is said to be polar if, for each point $p \in \bar{M}$, there is an immersed submanifold Σ_p, called *section*, that passes through p and that meets all the leaves and always perpendicularly. It follows that Σ_p is totally geodesic and that the dimension of Σ_p is equal to the codimension of $\mathcal{F}$. When the sections of a polar foliation are flat submanifolds, the foliation is called *hyperpolar*.

If the ambient manifold $\bar{M}$ is complete, the condition of polarity turns out to be equivalent to saying that the distribution made up of the normal spaces to the regular leaves is integrable. In this case, the sections are complete. Moreover, the leaves of a polar foliation on a complete, simply connected Riemannian manifold are always closed submanifolds with globally flat normal bundle (see [63, Theorem 1.2]). Note that, in a complete ambient manifold, codimension one foliations are always polar.

One important question in the study of polar foliations is to decide when polar foliations are orbit foliations of isometric actions. In this case, such homogeneous polar foliations are precisely the orbit foliations of the so-called polar actions (see Appendix B for the definition).

7.2. **Isoparametric submanifolds in real space forms.** Isoparametric submanifolds of arbitrary codimension on real space forms were first studied by Harle [47], Carter and West [17] and, more crucially, by Terng [89]. We give here Terng's definition. It is important to emphasize that this definition is reasonable only for spaces of constant curvature (think of the codimension one case):

Definition 7.2. A submanifold of a space form is called *isoparametric* if its normal bundle is flat and if it has constant principal curvatures in the direction of any parallel normal field.

Recall that we say that a submanifold has flat normal bundle if any normal vector can be extended locally to a parallel (with respect to the normal connection) normal vector field; equivalently, the normal curvature $R^{\perp}$ vanishes identically. The second condition in the definition means that the eigenvalues of the shape operator $\mathcal{S}_{\xi}$ are independent of the point, for any parallel normal vector field ξ.

There is also a notion of isoparametric map that can be used to characterize isoparametric submanifolds in real space forms; see [89]. Here we will content ourselves with stating the main structure results known for simply connected space forms. Thus, as follows from works of Terng [89], [90], and Wu [101], we have:

Theorem 7.3. [89], [90], [101] *Any isoparametric submanifold in $\mathbb{R}^n$, S^n or $\mathbb{R}H^n$ is an open part of a complete isoparametric submanifold.*

Theorem 7.4. [89] *Any complete isoparametric submanifold in the Euclidean space $\mathbb{R}^n$ is the product of an isoparametric submanifold in a sphere times an affine subspace of $\mathbb{R}^n$.*

Theorem 7.5. [101] *Any complete isoparametric submanifold in a real hyperbolic space $\mathbb{R}H^n$ is either an isoparametric submanifold of a totally umbilical hypersurface of $\mathbb{R}H^n$ or the standard product of a hyperbolic totally umbilical submanifold of $\mathbb{R}H^n$ and an isoparametric submanifold of a spherical totally umbilical submanifold of $\mathbb{R}H^n$.*

Thus, the classification problem of isoparametric submanifolds in $\mathbb{R}^n$ and in $\mathbb{R}H^n$ is reduced to the problem in spheres. In all three cases, an important property is the following:

Theorem 7.6. *Each complete isoparametric submanifold in a simply connected real space form is a regular leaf of a singular Riemannian foliation that fills the whole ambient space. The other regular leaves of this foliation are also isoparametric submanifolds.*

This result follows from the general theory developed by Terng. Indeed, these foliations, which are then called *isoparametric foliations*, can be seen as the level sets of the above mentioned isoparametric maps (see [89]).

It turns out that a foliation on a simply connected space form of nonnegative curvature is isoparametric if and only if it is polar. The fact that isoparametric foliations on space forms are polar follows from the general theory of isoparametric submanifolds developed by Terng, whereas the converse is a consequence of the constant curvature of the ambient space (see [91, p. 669] and [2, Theorem 2.7]). However, polar foliations on $\mathbb{R}H^n$ are not so rigid and are not isoparametric in general (see [98, p. 89, Remark 1]).

We have explained in Section 4 that, for isoparametric foliations of codimension one on spheres, there are many inhomogeneous examples and the classification problem is still open. However, the situation for higher codimension is very different. Using theory of Tits buildings, Thorbergsson [92] showed that all such examples are homogeneous. More precisely:

Theorem 7.7. [92] *Every irreducible isoparametric foliation of codimension higher than one on a sphere is the orbit foliation of an s-representation.*

An s-representation is the isotropy representation of a semisimple symmetric space; see Appendix C for more details. Moreover, in the statement, irreducible means the following. Given an isoparametric foliation $\mathcal{F}$ on the unit sphere S^n of a Euclidean space $\mathbb{R}^{n+1}$, one can construct an isoparametric foliation $\hat{\mathcal{F}}$ on $\mathbb{R}^{n+1}$ via homotheties, $\hat{\mathcal{F}} = \{rL : L \in \mathcal{F}, r \in \mathbb{R}\}$. Then we say that $\mathcal{F}$ is irreducible if $\hat{\mathcal{F}}$ is irreducible, that is, $\hat{\mathcal{F}}$ is not the product of two foliations on linear subspaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k+1}$ of $\mathbb{R}^{n+1}$, with $k \in \{1, \ldots, n\}$.

Hence, Thorbergsson's theorem guarantees that the only irreducible isoparametric foliations of codimension at least two on spheres are the restrictions to the unit sphere of T_oM of the orbit foliations of the isotropy representations of irreducible symmetric spaces M of compact type (equivalently, of noncompact type). Finally, the well-known classification of symmetric spaces allows to obtain the explicit classification of irreducible isoparametric foliations of codimension at least two on spheres.

7.3. **Isoparametric submanifold in general Riemannian manifolds.** The attempts to generalize isoparametric submanifolds to ambient spaces of nonconstant curvature have led to several different but related concepts. Our aim is to focus on the notion of isoparametric submanifold proposed by Heintze, Liu and Olmos in [49], see Definition 7.8 below. However, let us briefly comment on other related concepts.

We have already mentioned the notion of polar foliation introduced by Alexandrino [2]. Another important concept, which was introduced by Terng and Thorbergsson [91], is that of *equifocal submanifold* of a compact symmetric space. A closed submanifold of a compact symmetric space is equifocal if it has globally flat and abelian normal bundle and its focal directions and distances are invariant under parallel translation in the normal bundle. This notion of equifocality has been modified by eliminating the requirement of having abelian normal bundle (or, equivalently, having flat sections). Thus, Alexandrino [2] defines an

immersed submanifold of a complete Riemannian manifold to be equifocal if it has globally flat normal bundle, its focal directions and distances are invariant under parallel translation in the normal bundle, and it admits sections. Here, admitting sections means that, for every point p in the submanifold M, there exists a complete, immersed, totally geodesic submanifold Σ_p such that $\nu_p M = T_p\Sigma_p$. It turns out that, with this definition of equifocal submanifold, the regular leaves of polar foliations are equifocal. The converse, the fact that the partition of a Riemannian manifold into the parallel submanifolds determined by an equifocal submanifold is a polar foliation, is also true under some mild assumption; see [3, §4] for more details. Thus, the original notion of equifocality with flat sections turns out to be equivalent to that of hyperpolar foliation, i.e. polar foliation with flat sections.

Terng and Thorbergsson [91] developed a powerful method to study equifocal submanifolds, based on the use of a Riemannian submersion $\mathcal{H} \to G/K$ from a Hilbert space $\mathcal{H}$ of paths to a compact symmetric space G/K, which allows to lift equifocal submanifolds (equivalently, hyperpolar foliations) from G/K to $\mathcal{H}$. This technique was employed by Christ [22] to show the homogeneity of every irreducible hyperpolar foliation of codimension at least two on a simply connected compact symmetric space. Christ's theorem makes use of a homogeneity result for isoparametric submanifolds of a Hilbert space $\mathcal{H}$ (with codimension greater than one if $\mathcal{H}$ is infinite dimensional, or with codimension greater than two if $\mathcal{H}$ is finite dimensional). This result is due to Heintze and Liu [48] and provides a different proof of Thorbergsson's theorem when applied to a finite dimensional Hilbert space.

As announced above, the definition of isoparametric submanifold that we will consider is the one due to Heintze, Liu and Olmos [49], which we present now.

Definition 7.8. An immersed submanifold M of a Riemannian manifold $\bar{M}$ is an *isoparametric submanifold* if the following properties are satisfied:

(i) The normal bundle νM is flat.
(ii) Locally, the parallel submanifolds of M have constant mean curvature in radial directions (see below for explanation).
(iii) M admits sections, i.e. for each $p \in M$ there exists a totally geodesic submanifold Σ_p that meets M at p orthogonally and whose dimension is the codimension of M.

Let us explain the meaning of condition (ii). Since νM is flat, every point $p \in M$ admits an open neighbourhood U where every normal vector can be extended to a parallel normal field. By restricting U further if necessary, we can assume that there is an $s > 0$ such that for all $r < s$ and for every parallel normal field ξ on U, the set $U^{r,\xi} = \{\exp(r\xi_p) : p \in U\}$ is an embedded parallel submanifold of $U \subset M$. The radial vector field $\partial/\partial r = \nabla r$ is normal to every such $U^{r,\xi}$. Then we say that locally the parallel submanifolds of M have constant mean curvature in radial directions if the mean curvature of each $U^{r,\xi}$ is constant with respect to the normal field $\partial/\partial r$.

It was proved in [49, Theorem 2.4] that condition (ii) above may be replaced by the following condition (ii') without changing the notion of isoparametric submanifold:

(ii') Locally, the parallel submanifolds $M^{r,\xi}$ of M have constant mean curvature with respect to any parallel normal field of $M^{r,\xi}$.

This implies that the locally defined parallel submanifolds of an isoparametric submanifold are isoparametric as well [49, Corollary 2.5], and thus define locally a regular foliation where all leaves are isoparametric. When this local foliation can be extended to a global singular Riemannian foliation where the regular leaves are isoparametric, we obtain an isoparametric foliation.

Definition 7.9. A singular Riemannian foliation on a Riemannian manifold is called an *isoparametric foliation* if its regular leaves are isoparametric submanifolds.

Note that, for the codimension one case, the definition of isoparametric submanifold above simplifies and one recovers the definition of isoparametric hypersurface given in Definition 1.2. This definition extends, not only the notion of isoparametric hypersurface in any Riemannian manifold, but also of isoparametric submanifold of a real space form; see [49].

In complete ambient manifolds, isoparametric foliations are always polar, since the distribution of the normal spaces to the regular leaves is integrable, because of condition (iii) in the definition of isoparametric submanifold; cf. §7.1. The converse is not true in general even for codimension one; see [98, p. 89, Remark 1] for counterexamples in the real hyperbolic space. However, homogeneous polar foliations, i.e. orbit foliations of polar actions, are isoparametric.

Theorem 7.10. *The principal orbits of a polar proper action are isoparametric submanifolds. The family of orbits of a polar action on a simply connected complete Riemannian manifold is an isoparametric foliation.*

Proof. Let G be a closed subgroup of the isometry group of the ambient Riemannian manifold $\bar{M}$ that acts polarly on $\bar{M}$. By a general result about polar actions (see [6, Corollary 3.2.5]) we know that each G-equivariant normal vector field (i.e. $g_*\xi_p = \xi_{g(p)}$, for all $g \in G$) along a principal orbit $G \cdot p$ is parallel with respect to the normal connection of $H \cdot p$. This implies that $G \cdot p$ has globally flat normal bundle.

If ξ is a G-equivariant normal vector field along $G \cdot p$, then the shape operators $\mathcal{S}_{\xi_p}$ and $\mathcal{S}_{\xi_q}$ at any two points of $G \cdot p$ are conjugate by means of g_*, where g is any element of G mapping p to q. Thus, $G \cdot p$ has constant principal curvatures with respect to any parallel normal vector field. In particular, it has parallel mean curvature. This is valid for any principal orbit of the action of G. For proper actions, the nearby parallel submanifolds to a principal orbit are principal orbits of the action as well (see [6, §3.1h]). Thus, all principal orbits of a polar action have parallel mean curvature.

By definition of polar action, each principal orbit admits sections through every point. This shows that principal orbits of polar actions are isoparametric.

Finally note that if the ambient space is complete and simply connected, any polar action only has principal or singular orbits (see [63, Corollary 1.3]). Thus, all regular leaves of the corresponding orbit foliation are principal orbits and, as we have shown, are isoparametric. Thus, the orbit foliation is an isoparametric foliation. □

This result is important, since it provides us with a lot of examples of isoparametric foliations, analogously as cohomogeneity one actions provide examples of isoparametric

families of hypersurfaces. However, all such examples are obviously homogeneous. Explicit classifications of polar actions are known in irreducible symmetric spaces of compact type; see for example [56] and [57]. In symmetric spaces of noncompact type, the classification problem of polar actions is much more involved, and we only know complete classifications for real hyperbolic spaces [101] and complex hyperbolic spaces [31]. Beyond symmetric spaces, Fang, Grove and Thorbergsson [39] have recently studied polar actions on simply connected, compact, positively curved manifolds of cohomogeneity greater than one. Their main result shows that these actions are equivariantly diffeomorphic to polar actions on compact rank one symmetric spaces.

But, what do we know about explicit classifications of not necessarily homogeneous isoparametric submanifolds of higher codimensions in specific spaces of nonconstant curvature?

Recently, Lytchak [64] proposed a new approach for the study of polar foliations on symmetric spaces of compact type. Together with Christ's result [22], Lytchak's work implies the homogeneity (and hence the classification, thanks to [55]) of every irreducible isoparametric foliation of codimension at least three on a simply connected irreducible symmetric space of compact type and rank at least two.

In [36], Domínguez-Vázquez obtained a complete classification of irreducible isoparametric foliations of codimension at least two on complex projective spaces $\mathbb{C}P^n$. The main implication of this classification is the existence of inhomogeneous examples, which contrasts with the situation in spheres and Thorbergsson's theorem.

The results in [64] and in [36] are, as far as we know, the first ones involving classifications of isoparametric foliations of higher codimension on spaces of nonconstant curvature. However, an important difference between both contexts is the question of the homogeneity: while for symmetric spaces of rank greater than one the classified examples are homogeneous, rank one symmetric spaces of nonconstant curvature allow a greater diversity of examples, including inhomogeneous foliations of large codimension. The aim of the rest of these notes is to explain the main ideas of the results in [36].

8. Isoparametric foliations on complex projective spaces

In this section we will present the main ideas of the almost complete classification of isoparametric foliations on complex projective spaces obtained in [36]. The exposition here is basically extracted from [35].

The starting point for the study of isoparametric submanifolds in complex projective spaces is their good behaviour with respect to the Hopf map $\pi\colon S^{2n+1} \to \mathbb{C}P^n$. The idea of using the Hopf fibration for the study of geometric objects in $\mathbb{C}P^n$ is not new; it has been used, for example, by Takagi [85] for the classification of homogeneous hypersurfaces in $\mathbb{C}P^n$, by Xiao [102] in his study of isoparametric hypersurfaces in $\mathbb{C}P^n$, or by Podestà and Thorbergsson [79] for the classification of polar actions on $\mathbb{C}P^n$. Using results in [49], one can show the following:

Proposition 8.1. [36] *Let M be a submanifold of $\mathbb{C}P^n$ of positive dimension. Then M is isoparametric if and only if its lift $\pi^{-1}M$ to the sphere S^{2n+1} is isoparametric.*

Using this result, together with the fact that isoparametric submanifolds in spheres can be extended to globally defined analytic isoparametric foliations [89], one can also show that:

Proposition 8.2. [36] *Each isoparametric submanifold of $\mathbb{C}P^n$ is an open part of a unique complete isoparametric submanifold, and this one is a regular leaf of a unique isoparametric foliation on $\mathbb{C}P^n$.*

In view of the previous two propositions, we conclude that, in order to study isoparametric submanifolds of $\mathbb{C}P^n$, it is enough to analyse the projections via the Hopf map π of the isoparametric foliations existing on odd-dimensional spheres S^{2n+1}. We will focus on irreducible isoparametric foliations on $\mathbb{C}P^n$, that is, those foliations for which there is no totally geodesic $\mathbb{C}P^k$, $k \in \{0, \ldots, n-1\}$, being a union of leaves of the foliation. Note that an isoparametric foliation on $\mathbb{C}P^n$ is irreducible if and only if its lift $\pi^{-1}M$ is irreducible as an isoparametric foliation on S^{2n+1}.

By Thorbergsson's result (Theorem 7.7) we know all irreducible isoparametric foliations of codimension higher than one on spheres. Moreover, the classification results of isoparametric hypersurfaces in spheres tell us that, with maybe some exceptions, all examples are either homogeneous, or of FKM-type. Then, it seems natural to propose the following approach in order to obtain an almost complete classification of irreducible isoparametric foliations on complex projective spaces. For each isoparametric foliation $\mathcal{F}$ on S^{2n+1}, decide if it "can be projected" to an isoparametric foliation on $\mathbb{C}P^n$; more precisely, determine if $\mathcal{F}$ is the pullback under the Hopf map of an isoparametric foliation on $\mathbb{C}P^n$ or, equivalently, if the leaves of $\mathcal{F}$ are foliated by the Hopf S^1-fibers. If we do this for all known (homogeneous and inhomogeneous) irreducible isoparametric foliations on S^{2n+1}, we would obtain an almost complete classification of irreducible isoparametric foliations on $\mathbb{C}P^n$.

However, it turns out that it is not trivial to carry out this procedure. The reason is that, somewhat surprisingly, there is not a one-to-one correspondence between congruence classes of isoparametric foliations on S^{2n+1} that can be projected and congruence classes of isoparametric foliations on $\mathbb{C}P^n$. If two foliations on $\mathbb{C}P^n$ are congruent, then their pullbacks are congruent as well, but it can happen that two noncongruent foliations on $\mathbb{C}P^n$ pullback to congruent foliations on S^{2n+1}. This phenomenon (whose analogue does not occur for homogeneous hypersurfaces or polar actions) was discovered by Xiao [102] for the orbit foliations of codimension one induced by the real Grassmannians $SO(n+3)/S(O(2) \times O(n+1))$ with odd n. In [36] we show that this behaviour happens for foliations of higher codimension as well.

But, as shown in [36], there is another phenomenon which is even more interesting. There are homogeneous isoparametric foliations on S^{2n+1} that can be projected to *inhomogenous* isoparametric foliations on $\mathbb{C}P^n$. This happens in codimension one (as noticed by Xiao [102]), but also in higher codimension, which gives rise to the inhomogeneous examples announced above.

In order to analyse the possible projections of a fixed isoparametric foliation $\mathcal{F}$ on an odd-dimensional sphere S^{2n+1}, we can follow two equivalent procedures. The first one

would consist in determining all possible orthogonal transformations $A \in O(2n+2)$ such that the foliation $A(\mathcal{F})$ is the pullback of a foliation on $\mathbb{C}P^n$, then study the congruence in $\mathbb{C}P^n$ of the projections $\pi(A(\mathcal{F}))$ for all possible A, and finally decide the homogeneity of these projections.

However, we will follow a second procedure, which we formalize now. We fix a congruence class of isoparametric foliations on S^{2n+1}, and take any fixed representative of this class, say $\mathcal{F}$. The first step is to find the set $\mathcal{J}_{\mathcal{F}}$ of complex structures on $\mathbb{R}^{2n+2}$ that preserve $\mathcal{F}$. Here and henceforth, by *complex structure* we mean an orthogonal linear transformation $J \in O(2n+2)$ such that $J^2 = -\mathrm{Id}$. We will say that a complex structure *preserves* the foliation $\mathcal{F}$ if $\mathcal{F}$ is the pullback of some foliation on $\mathbb{C}P^n$ under the Hopf map determined by J or, equivalently, if for every $x \in S^{2n+1}$ the Hopf circle $\{\cos(t)x + \sin(t)Jx \in \mathbb{R}^{2n+2} : t \in \mathbb{R}\}$ through x determined by J is contained in the leaf of $\mathcal{F}$ through x. Secondly, we have to determine the quotient set $\mathcal{J}_{\mathcal{F}}/\sim$, where $\sim$ is the equivalence relation "yield congruent foliations on $\mathbb{C}P^n$". Note that $\mathcal{J}_{\mathcal{F}}/\sim$ is isomorphic to the set of congruence classes of isoparametric foliations on $\mathbb{C}P^n$ that pullback under a fixed Hopf map to a foliation congruent to $\mathcal{F}$. Finally, we have to decide which congruence classes correspond to homogeneous foliations on $\mathbb{C}P^n$.

This procedure has been applied to analyse the possible projections of:

- the orbit foliations of the isotropy representations of semisimple symmetric spaces, and
- the isoparametric foliations of codimension one on spheres S^{2n+1} constructed by Ferus, Karcher and Münzner [40], except when $n = 15$.

As a result, and thanks to the classification theorems of isoparametric foliations on spheres, we obtained a classification of irreducible isoparametric foliations on $\mathbb{C}P^n$ of arbitrary codimension q, except for the case $(n, q) = (15, 1)$.

Here, in order to simplify the exposition, we will focus on the first type of foliations, and we refer to [36] for the second type of foliations, and also for details and proofs. It will be convenient to introduce some notation. Let G/K be an irreducible simply connected compact symmetric space of dimension $2n+2$ and rank at least two. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. It is well-known that we can identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$ and regard the isotropy representation of G/K at o as the adjoint representation $\mathrm{Ad}\colon K \to O(\mathfrak{p})$, where $O(\mathfrak{p})$ is the orthogonal group of the Euclidean vector space $\mathfrak{p}$ (endowed with the negative of the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{p}$). Finally, we denote by $\mathcal{F}_{G/K}$ the orbit foliation of the adjoint action $\mathrm{Ad}\colon K \to O(\mathfrak{p})$ restricted to the unit sphere S^{2n+1} of $\mathfrak{p}$. This is an isoparametric foliation whose codimension on S^{2n+1} agrees with the rank of G/K minus 1.

Now we explain the main ideas of how to carry out the three steps in the procedure described above, namely: determination of the complex structures, congruence of the projected foliations, and homogeneity.

8.1. **Complex structures preserving $\mathcal{F}_{G/K}$.** The first step consists in finding all complex structures on $\mathfrak{p}$ that preserve a given foliation $\mathcal{F}_{G/K}$. The key observation is the fact

that the group $\mathrm{Ad}(K)|_{\mathfrak{p}}$ is the largest connected subgroup of $O(\mathfrak{p})$ acting on $\mathfrak{p}$ with the same orbits as the isotropy representation of G/K. This maximality property was proved by Eschenburg and Heintze [38]. Now, let J be a complex structure on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$. Then $\mathcal{T}^1 = \{\cos(t)\mathrm{Id} + \sin(t)J : t \in \mathbb{R}\}$ is a 1-dimensional group leaving invariant the leaves of $\mathcal{F}_{G/K}$. Let K' be the subgroup of $O(\mathfrak{p})$ generated by $\mathrm{Ad}(K)$ and $\mathcal{T}^1$, which is connected and leaves each leaf of $\mathcal{F}_{G/K}$ invariant. Using the maximality property we get that $K' \subset \mathrm{Ad}(K)|_{\mathfrak{p}}$, so $\mathcal{T}^1$ is a subgroup of $\mathrm{Ad}(K)|_{\mathfrak{p}}$. If we differentiate, we get that $J \in \mathrm{ad}(\mathfrak{k})|_{\mathfrak{p}}$, where ad is the adjoint action at the Lie algebra level. Thus, a complex structure J preserves $\mathcal{F}_{G/K}$ if and only if it can be written in the form $J = \mathrm{ad}(X)|_{\mathfrak{p}}$ for some $X \in \mathfrak{k}$. In other words, the set of complex structures on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ can be parametrized by the set

$$\mathcal{J}_{\mathcal{F}_{G/K}} = \{X \in \mathfrak{k} : \mathrm{ad}(X)|_{\mathfrak{p}}^2 = -\mathrm{Id}\},$$

since every transformation $\mathrm{ad}(X)|_{\mathfrak{p}}$, with $X \in \mathfrak{k}$, is skew-symmetric.

Now let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. A general fact about compact groups guarantees that if $X \in \mathfrak{k}$, then there is a $k \in K$ such that $\mathrm{Ad}(k)X \in \mathfrak{t}$. In this situation, one can then show that $X \in \mathcal{J}_{\mathcal{F}_{G/K}}$ if and only if $\mathrm{Ad}(k)X \in \mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$. This means that we will know $\mathcal{J}_{\mathcal{F}_{G/K}}$ once we determine the subset $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$. This restriction to a maximal abelian subalgebra of $\mathfrak{k}$ suggests the utilization of the theory of roots of the compact Lie algebra $\mathfrak{g}$ and, more specifically, the Borel-de Siebenthal theory. Here we give only some basic terminology needed to state the main consequences of the use of this theory in our problem; we refer to our article [36] for more details, and to [54, §VI.8-10] for an introduction to the Borel-de Siebenthal theory.

We say that the symmetric space G/K is inner if the rank of $\mathfrak{g}$ equals the rank of $\mathfrak{k}$. This means that a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is also a maximal abelian subalgebra of $\mathfrak{g}$. Let $\Delta_{\mathfrak{g}}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$ be the corresponding root space decomposition of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. One can show that, for each $\alpha \in \Delta_{\mathfrak{g}}$, the root space $\mathfrak{g}_\alpha$ is either contained in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$. In the first case we say that the root α is *compact*, whereas we call it *noncompact* in the second case.

We can now state an algebraic method that can be used to completely determine the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$ for each symmetric space G/K.

Theorem 8.3. [36] *There exists a complex structure on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ if and only if G/K is an inner symmetric space.*

In this situation, let $T \in \mathfrak{t}$. Then $\mathrm{ad}(T)|_{\mathfrak{p}}$ is a complex structure on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ if and only if $\alpha(T) \in \{-1, 1\}$ for all noncompact roots α.

8.2. **Congruence of the projected foliations.** We are now interested in classifying the complex structures parametrized by the set $\mathcal{J}_{\mathcal{F}_{G/K}}$ in terms of the congruence of the projected isoparametric foliations. More concretely, we have to study the equivalence relation $\sim$ on $\mathcal{J}_{\mathcal{F}_{G/K}}$ defined as follows: given X_1, $X_2 \in \mathcal{J}_{\mathcal{F}_{G/K}}$, we say that $X_1 \sim X_2$ if $\pi_1(\mathcal{J}_{\mathcal{F}_{G/K}})$ and $\pi_2(\mathcal{J}_{\mathcal{F}_{G/K}})$ are congruent foliations on $\mathbb{C}P^n$, where π_1, $\pi_2 \colon S^{2n+1} \to \mathbb{C}P^n$ are the Hopf fibrations determined by the complex structures $\mathrm{ad}(X_1)|_{\mathfrak{p}}$ and $\mathrm{ad}(X_2)|_{\mathfrak{p}}$, respectively.

The main object of study in this subproblem turns out to be the group $\mathrm{Aut}(\mathcal{F}_{G/K})$ of orthogonal transformations of $\mathfrak{p}$ that map leaves of $\mathcal{F}_{G/K}$ to leaves of $\mathcal{F}_{G/K}$. Roughly, the reason for this is that, if J_1 and J_2 are two complex structures preserving $\mathcal{F}_{G/K}$ and with corresponding Hopf maps π_1 and π_2, then $\pi_1(\mathcal{F}_{G/K})$ is congruent to $\pi_2(\mathcal{F}_{G/K})$ if and only if there exists $A \in \mathrm{Aut}(\mathcal{F}_{G/K})$ such that $AJ_1A^{-1} = \pm J_2$.

The determination of the group $\mathrm{Aut}(\mathcal{F}_{G/K})$ can be a very difficult task for an arbitrary singular Riemannian foliation on a sphere. Nonetheless, it can be done for the homogeneous isoparametric foliations $\mathcal{F}_{G/K}$, and also for most of the inhomogeneous foliations of codimension one constructed by Ferus, Karcher and Münzner (although in this case the task is more difficult and involves working with Clifford modules). For the case we are here interested in, it happens that $\mathrm{Aut}(\mathcal{F}_{G/K})$ is canonically isomorphic to the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ of automorphisms of the Lie algebra $\mathfrak{g}$ that restrict to automorphisms of $\mathfrak{k}$; the isomorphism is simply given by the restriction to $\mathfrak{p}$ of the elements of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$. In particular, the adjoint transformations in $\mathrm{Ad}(K)|_{\mathfrak{p}}$ belong to $\mathrm{Aut}(\mathcal{F}_{G/K})$. This readily implies that every $\sim$-equivalence class interesects any maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Thus, we can restrict the study of $\sim$ to the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$, which turns out to be much more manageable in view of the following result.

Theorem 8.4. [36] *Let $\Pi_{\mathfrak{k}}$ be a set of simple roots for $\mathfrak{k}$, and $\bar{C}$ the closed Weyl chamber in $\mathfrak{t}$ defined by the inequalities $\alpha \geq 0$, for all $\alpha \in \Pi_{\mathfrak{k}}$. Let T_1, $T_2 \in \mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$.*

Then $T_1 \sim T_2$ if and only if there is an autormophism φ of the extended Vogan diagram of the pair $(\mathfrak{g}, \mathfrak{k})$ such that $\varphi(T_1) = T_2$.

This result requires some explanations. We recall first that the *extended Dynkin diagram* of $\mathfrak{g}$ consists in the Dynkin diagram of $\mathfrak{g}$ together with an extra node corresponding to the lowest root of the root system of $\mathfrak{g}$; this node is linked to the other nodes according to the usual rules of Dynkin diagrams. Now, we define the *extended Vogan diagram* of the pair $(\mathfrak{g}, \mathfrak{k})$ as the extended Dynkin diagram of $\mathfrak{g}$, where the nodes corresponding to noncompact roots are painted while the ones corresponding to compact roots remain unpainted. Due to the Borel-de Siebenthal theorem, one can always assume that there are at most two painted nodes: one simple root of $\mathfrak{g}$ and, maybe, the lowest root. An automorphism of the extended Vogan diagram is a permutation of its nodes preserving the types of edges between nodes and the colours of the nodes. One can show that every automorphism of the extended Vogan diagram determines (in a unique way) an autormorphism $\varphi\colon \mathfrak{t} \to \mathfrak{t}$ of the root system of $\mathfrak{g}$ that restricts to an automorphism of the root system of $\mathfrak{k}$. For the sake of brevity and since it should not lead to confusion, we did not distinguish between both automorphisms in the statement of Theorem 8.4.

Since for each inner irreducible symmetric space G/K the corresponding extended Vogan diagram is known (see Table 3 for the pictures), one can completely understand the equivalence relation $\sim$ on the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$ and, thus, on $\mathcal{J}_{\mathcal{F}_{G/K}}$. Going through all possible cases of inner symmetric spaces, one can obtain the desired classification of complex structures preserving the irreducible homogeneous isoparametric foliations on S^{2n+1}. As we mentioned earlier, it follows from this classification that the cardinality of $\mathcal{J}_{\mathcal{F}_{G/K}}/\sim$ is at least one for every inner symmetric space G/K, but it is strictly higher than one in

several cases. For instance, its value is $1+[\nu/2]+[p+1-\nu]$ for the complex Grassmannian $SU(p+1)/S(U(\nu)U(p+1-\nu))$ (where $[\cdot]$ denotes the integer part of a real number), whereas such cardinality is 2 for the symmetric space $E_6/SU(6)\cdot SU(2)$.

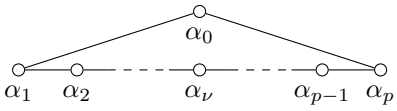
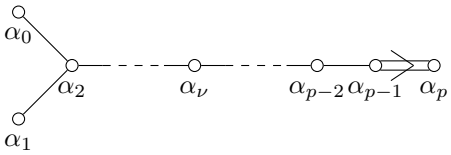
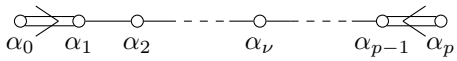
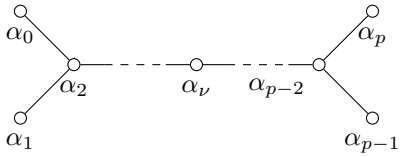
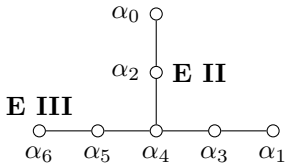
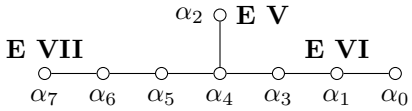
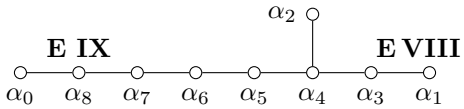
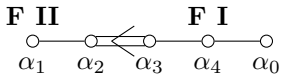
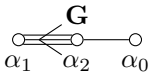
8.3. **Homogeneity.** Although the arguments above complete the classification of irreducible isoparametric foliations of codimension at least two on complex projective spaces, it is important to know which of the examples in the classification are homogeneous. The first observation is that, if $\mathcal{G}$ is a homogeneous isoparametric foliation on $\mathbb{C}P^n$, then its pullback $\pi^{-1}\mathcal{G}\subset S^{2n+1}$ under the Hopf map is homogeneous as well. This implies that $\pi^{-1}\mathcal{G}$ is induced by some s-representation. In other words, $\pi^{-1}\mathcal{G}=\mathcal{F}_{G/K}$ for some semisimple symmetric space G/K, which must necessarily be inner because of Theorem 8.3. However, one can prove even more:

Theorem 8.5. [36] *Let G/K be an irreducible inner symmetric space of rank at least two, $J=\mathrm{ad}(X)|_{\mathfrak{p}}$ a complex structure preserving $\mathcal{F}_{G/K}$, with $X\in\mathfrak{k}$, and π the Hopf map determined by J. Then $\pi(\mathcal{F}_{G/K})$ is homogeneous if and only if G/K is a Hermitian symmetric space and X belongs to the one-dimensional center of $\mathfrak{k}$.*

Since there are inner non-Hermitian symmetric spaces of rank higher than two, Theorems 8.3 and 8.5 readily imply the existence of inhomogeneous irreducible isoparametric foliations of codimension higher than one on complex projective spaces. However, not for all dimensions does the complex projective space $\mathbb{C}P^n$ admit inhomogeneous irreducible isoparametric foliations. We conclude this article with a result that answers this question in a surprising way.

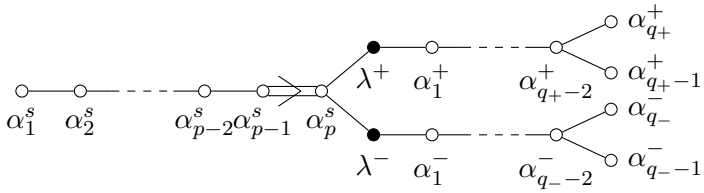
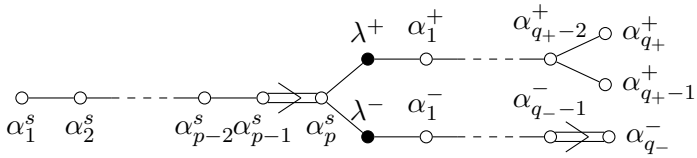
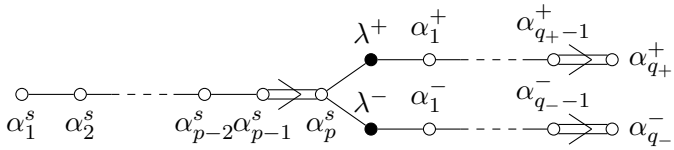
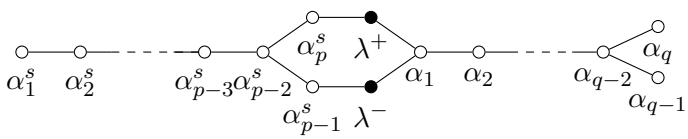
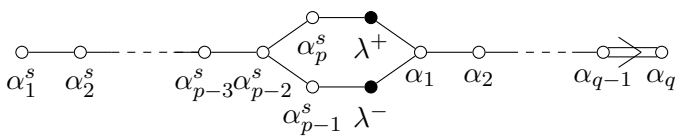
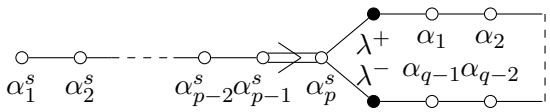
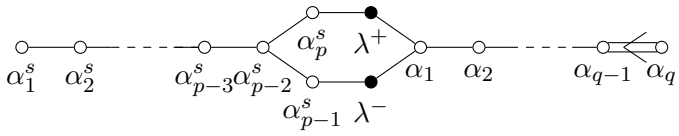
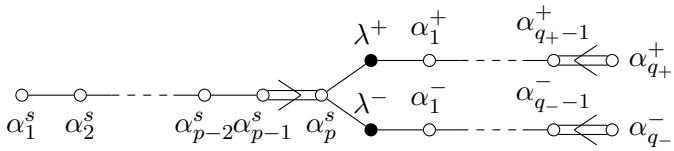
Theorem 8.6. [36] *Every irreducible isoparametric foliation on $\mathbb{C}P^n$ is homogeneous if and only if $n+1$ is a prime number.*

TABLE 3. Extended Vogan diagrams of irreducible compact inner symmetric spaces

	Extended Dynkin diagram	μ	G/K	λ	$N(\mathcal{F}_{G/K})$
A_p		$(1\dots1)$	**A III**	$(1\dots1)$	$1+\left[\frac{\nu}{2}\right]+\left[\frac{p-\nu+1}{2}\right]$ (if $2\nu\neq p+1$) $1+\left[\frac{\nu}{2}\right]$ (if $2\nu=p+1$)
B_p		$(122\dots2)$	**B I**	$(11\dots\overset{\nu}{1}2\dots2)$	1
C_p		$(2\dots221)$	**C I** ($\nu=p$)	$(2\dots221)$	1
			C II ($\nu<p$)	$(1\dots\overset{\nu}{1}2\dots21)$	2 (if $2\nu\neq p$) 1 (if $2\nu=p$)
D_p		$(12\dots211)$	**D I** ($\nu\leq p-2$)	$(1\dots\overset{\nu}{1}2\dots211)$	2 (if $2\nu\neq p$) 1 (if $2\nu=p$)
			D III ($\nu\geq p-1$)	$(12\dots211)$	2
E_6		(122321)	**E II**	(112321)	2
			E III	(122321)	2
E_7		(2234321)	**E V**	(1123321)	1
			E VI	(1234321)	2
			E VII	(2234321)	1
E_8		(23465432)	**E VIII**	(13354321)	1
			E IX	(23465431)	1
F_4		(2432)	**F I**	(2431)	1
			F II	(1321)	$-$ (rank one)
G_2		(32)	**G**	(31)	1

For each extended Dynkin diagram, we provide the maximal root μ and the associated symmetric spaces G/K using Cartan's notation. For every such G/K, we show the corresponding maximal noncompact root λ and the number $N(\mathcal{F}_{G/K})$ of noncongruent isoparametric foliations on the complex projective space induced by $\mathcal{F}_{G/K}$. Roots are specified in coordinates with respect to $\Pi_{\mathfrak{g}}$.

TABLE 4. Lowest weight diagrams of FKM-foliations with $m_1 \leq m_2$

m	Lowest weight diagram	$\mathfrak{h}$	$N(\mathcal{F}_{\mathcal{P}})$
0		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_\pm = 2q_\pm$	2
		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_+ = 2q_+,$ $k_- = 2q_- + 1$	1
		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_\pm = 2q_\pm + 1$	1
1, 7		$\mathfrak{so}(k)$ $k = 2q$	2
		$\mathfrak{so}(k)$ $k = 2q + 1$	1
2, 6		$\mathfrak{u}(k)$ $k = q$	$2 + \left[\frac{k}{2}\right]$
3, 5		$\mathfrak{sp}(k)$ $k = q$	2
4		$\mathfrak{sp}(k_+) \oplus \mathfrak{sp}(k_-)$ $k_\pm = q_\pm$	2

The following data are provided for each value of $m \pmod 8$: the corresponding lowest weight diagrams, the Lie algebra $\mathfrak{h}$ such that $\mathfrak{so}(m+1) \oplus \mathfrak{h}$ is the Lie algebra of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$, and the value of $N(\mathcal{F}_{\mathcal{P}})$.

Appendix A. Jacobi field theory

A rather useful method in submanifold theory is based on employing Jacobi vector fields for the study of the geometric behaviour of a submanifold when this is moved along normal directions. In this appendix (extracted from [34]) we will briefly present the main features of this technique in the Riemannian setting. A more thorough discussion of this method can be found in [6, Chapter 8]. In §A.1 we present the basic definitions and ideas. However, the reader interested only in studying the normal displacement of hypersurfaces can consult directly §A.2 below.

A.1. **General theory and definitions.** Let $\bar{M}$ be a Riemannian manifold of dimension n and $M \subset \bar{M}$ a Riemannian submanifold of $\bar{M}$. For fixed $r > 0$, we define the set

$$M^r = \{\exp(r\xi) : \xi \in \nu M,\ \|\xi\| = 1\}.$$

In general M^r is not a submanifold of $\bar{M}$. But if M^r is a hypersurface then we say that M^r is the *tube* of radius r around M. Locally, if r is sufficiently small, such a set is always a tube. If M^r is a submanifold of $\bar{M}$ but with codimension greater than one, we call it *focal submanifold* of M. In particular, if M^r is a tube around M and M has codimension greater than one, then M is a focal submanifold of M^r.

Let p be any point of M and $\gamma\colon [0,1] \to \bar{M}$ a unit speed geodesic with $\gamma(0) = p$ and $\dot\gamma(0) \in \nu M$. Here and henceforth, $\dot\gamma$ denotes the tangent vector field to the curve γ. Let $F(s,t) = \gamma_s(t)$ be a geodesic variation of $\gamma = \gamma_0$ such that $c(s) = F(s,0) = \gamma_s(0) \in M$ and $\xi(s) = \dot\gamma_s(0) \in \nu M$ for all s. Let ζ be the variational vector field of F. Then ζ is a solution to the initial value problem

$$\zeta'' + \bar{R}(\zeta, \dot\gamma)\dot\gamma = 0, \qquad \zeta(0) = \dot c(0) \in T_pM, \qquad \zeta'(0) = -\mathcal{S}_{\xi(0)}\zeta(0) + \nabla^{\perp}_{\zeta(0)}\xi,$$

where $\mathcal{S}$ is the shape operator of M and the prime $'$ denotes covariant derivative of a vector field along a curve. A Jacobi vector field ζ along γ satisfying $\zeta(0) \in T_pM$ and $\zeta'(0) + \mathcal{S}_{\gamma'(0)}\zeta(0) \in \nu_pM$ is called an *M-Jacobi vector field*.

We say that $\gamma(r)$ is a *focal point* of M along γ if there exists an M-Jacobi vector field ζ along γ such that $\zeta(r) = 0$. A focal point arising from a Jacobi vector field ζ such that $\zeta(0) = 0$, $\zeta'(0) \in \nu M$ and $\zeta(r) = 0$ is a conjugate point of p in $\bar{M}$ along γ.

Assume now that M^r is a submanifold of $\bar{M}$. Let ξ be a smooth curve in νM with $\xi(0) = \dot\gamma(0)$ such that $\|\xi(s)\| = 1$ for all s. Then $F(s,t) = \exp\big(t\,\xi(s)\big)$ is a smooth geodesic variation of γ consisting of geodesics intersecting M perpendicularly. Let ζ be the corresponding M-Jacobi vector field which is the variational vector field of F. Then ζ is determined by the initial values $\zeta(0) = \dot c(0)$ and $\zeta'(0) = \xi'(0)$, where $c(s) = F(s,0)$. For any r, the curve $c_r(s) = F(r,s) = \exp(r\,\xi(s))$ is a smooth curve in M^r. Then,

$$T_{\gamma(r)}M^r = \{\zeta(r) : \zeta \text{ is an } M\text{-Jacobi vector field along } \gamma\}.$$

Let us denote by $\mathcal{S}^r$ the shape operator of M^r. Then it follows that

$$\mathcal{S}^r_{\dot\gamma(r)}\zeta(r) = -\zeta'(r)^{\top}.$$

If M^r is a tube, that is, if M^r is a hypersurface, its shape operator can be described in an efficient way, as we now explain. Let $X \in T_p\bar{M} \ominus \mathbb{R}\dot\gamma(0)$, where $\ominus$ denotes the

orthogonal complement. We introduce the following notation. By B_X we denote the parallel translation of X along the geodesic γ. Let ζ_X be the M-Jacobi vector field along γ given by the following initial conditions

$$\zeta_X(0) = X, \qquad \zeta_X'(0) = -\mathcal{S}_{\dot\gamma(0)}X, \qquad \text{if } X \in T_pM,$$

$$\zeta_X(0) = 0, \qquad \zeta_X'(0) = X, \qquad \text{if } X \in \nu_pM \ominus \mathbb{R}\dot\gamma.$$

We define $D(r)$ by $D(r)B_X(r) = \zeta_X(r)$ for all $X \in T_p\bar M \ominus \mathbb{R}\dot\gamma(0)$. In other words, D is the $\mathrm{End}(\dot\gamma^\perp)$-valued tensor field along γ determined by the following initial value problem

$$D'' + \bar R_{\dot\gamma} \circ D = 0, \qquad D(0) = \begin{pmatrix} \mathrm{Id}_{T_pM} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D'(0) = \begin{pmatrix} -\mathcal{S}_{\dot\gamma(0)} & 0 \\ 0 & \mathrm{Id}_{\nu_pM\ominus\mathbb{R}\dot\gamma(0)} \end{pmatrix},$$

where $\bar R_{\dot\gamma}(v) = \bar R(v,\dot\gamma)\dot\gamma$ for $v \in \dot\gamma^\perp$. The endomorphism $D(r)$ is singular if and only if $\gamma(r)$ is a focal point of M along γ. If this is not the case, M^r is a tube and its shape operator in the direction of $\dot\gamma(r)$ is given by

$$\mathcal{S}^r_{\dot\gamma(r)} = -D'(r)\, D(r)^{-1}.$$

A.2. **Normal displacement of hypersurfaces.** Of special interest is the case when M is a hypersurface. Let us now have a closer look at this case.

Let $M \subset \bar M$ be a hypersurface and ξ a unit normal vector field on an open set of M. Our objective is the study of local geometric properties of the displacement of M in the direction given by ξ at a certain distance r. We can hence assume that ξ is globally defined on M. For $r > 0$ we define the map

$$\begin{array}{rcl} \Phi^r : M & \longrightarrow & \bar M \\ p & \mapsto & \Phi^r(p) = \exp(r\xi_p). \end{array}$$

We denote by η the vector field along Φ^r such that $\eta^r(p) = \dot\gamma_p(r)$ for each $p \in M$, where γ_p is the geodesic of $\bar M$ determined by the initial conditions $\gamma_p(0) = p$ and $\dot\gamma_p(0) = \xi_p$. The map Φ^r is smooth and parametrizes the tube M^r of radius r around M. Clearly, M^r is an immersed submanifold of $\bar M$ if and only if Φ^r is an immersion. It may happen, however, that M^r is a focal submanifold. The fact that M^r has higher codimension depends on the rank of Φ^r.

Let ζ_X be an M-Jacobi vector field. We have $X = \zeta_X(0) \in TM$ and $\zeta_X'(0) = -\mathcal{S}X$ because ξ has unit length and the normal bundle of M has rank one. Then it follows that

$$\Phi^r_* X = \zeta_X(r), \qquad \bar\nabla_X \eta^r = \zeta_X'(r).$$

Thus, Φ^r is not an immersion at $p \in M$ if and only if $\Phi^r(p)$ is a focal point of M along the geodesic γ_p. In this case, the dimension of the kernel of Φ^r_{*p} is called the *multiplicity* of the focal point. If there exists a positive integer k such that $\Phi^r(q)$ is a focal point of M along γ_q with multiplicity k for all q in some open neighbourhood U of p, then, if U is sufficiently small, $\Phi^r|_U$ parametrizes and embedded $(n-1-k)$–dimensional submanifold of $\bar M$, which is a focal submanifold of M. If $\Phi^r(q)$ is not a focal point of M along γ_q for

any q in a sufficiently small neighbourhood U of p, then $\Phi^r|_U$ parametrizes an embedded hypersurface of $\bar{M}$, which is called an *equidistant hypersurface* to M in $\bar{M}$.

If M^r is a hypersurface, its shape operator can be calculated using the endomorphism-valued tensor field D defined above. In this case the initial conditions simplify slightly and D is determined by the initial value problem

$$D'' + \bar{R}_{\dot{\gamma}} \circ D = 0, \qquad D(0) = \mathrm{Id}_{T_pM}, \qquad D'(0) = -\mathcal{S}_{\xi_p}.$$

Finally, let us mention that the notion of equidistant hypersurface can be generalized to arbitrary codimension in the following way. Let M be a submanifold of $\bar{M}$. Assume that M has globally flat normal bundle. For each parallel normal vector field ξ and each sufficiently small $r > 0$, we can consider the set $M^{r,\xi} = \{\exp(r\xi_p) : p \in M\}$. If such a set is a submanifold, then we call it a *parallel submanifold* of M determined by the vector field ξ. Locally and for r sufficiently small, $M^{r,\xi}$ is always a parallel submanifold. Note as well that if M^r is a tube, then it is foliated by parallel submanifolds $M^{r,\xi}$ of M.

Appendix B. Isometric actions

Our purpose here is to review the basic terminology and concepts that arise in the study of isometric actions on Riemannian manifolds. A more detailed reference is [6, Chapter 3].

Let $\bar{M}$ be a Riemannian manifold and G a Lie group acting smoothly on $\bar{M}$ by isometries. This means that we have an *isometric action*, that is, a smooth map

$$\varphi\colon G \times \bar{M} \to \bar{M}, \qquad (g,p) \mapsto gp$$

satisfying $(gg')p = g(g'p)$ for all g, $g' \in G$ and $p \in \bar{M}$, and such that the map

$$\varphi_g\colon \bar{M} \to \bar{M}, \qquad p \mapsto gp$$

is an isometry of $\bar{M}$ for every $g \in G$. If we denote by $I(\bar{M})$ the isometry group of $\bar{M}$, which is known to be a Lie group [72], then we have a Lie group homomorphism $\rho\colon G \to I(\bar{M})$ given by $\rho(g) = \varphi_g$.

For each point $p \in \bar{M}$, the *orbit* of the action of G through p is

$$G \cdot p = \{gp : g \in G\}$$

and the *isotropy group* or *stabilizer* at p is

$$G_p = \{g \in G : gp = p\}.$$

If $G \cdot p = \bar{M}$ for some $p \in \bar{M}$, and hence for each $p \in \bar{M}$, the G-action is said to be *transitive* and $\bar{M}$ is a *homogeneous G-space*. If all leaves are points, the action is said to be *trivial*. An action is called *effective* if the associated map ρ above is injective, which means that G is isomorphic to a subgroup of $I(\bar{M})$. When for every $p \in \bar{M}$ and every g, $h \in G$, the equality $gp = hp$ implies $g = h$, then the action is *free*. If a G-action on $\bar{M}$ is free and transitive we say that G acts *simply transitively* on $\bar{M}$.

Consider two isometric actions $G \times \bar{M} \to \bar{M}$ and $G \times \bar{M}' \to \bar{M}'$. They are said to be *conjugate* or *equivalent* if there is a Lie group isomorphism $\psi\colon G \to G'$ and an isometry $f\colon \bar{M} \to \bar{M}'$ such that $f(gp) = \psi(g)f(p)$ for all $p \in \bar{M}$ and $g \in G$. We say that both isometric actions are *orbit equivalent* if there is an isometry $f\colon \bar{M} \to \bar{M}'$ that maps the

orbits of the G-action on $\bar{M}$ to the orbits of the G'-action on $\bar{M}'$. Clearly, two conjugate actions are orbit equivalent.

An *(extrinsically) homogeneous submanifold* of $\bar{M}$ is an orbit of an isometric action on $\bar{M}$. In general, these orbits will only be immersed submanifolds of $\bar{M}$. With respect to the induced metric, each orbit $G \cdot p$ is a Riemannian homogeneous space $G \cdot p = G/G_p$, on which G acts transitively by isometries.

Each isometric action induces certain orthogonal representations in a natural way. Recall that a representation of a Lie group G on a vector space V is a Lie group homomorphism $\rho\colon G \to GL(V)$ or, equivalently, an action $G \times V \to V$ given by automorphisms of V; when V is a Euclidean space and the automorphisms $\rho(g)$, $g \in G$, are orthogonal transformations of V, we have an orthogonal representation $\rho\colon G \to O(V)$. Let $\varphi\colon G \times \bar{M} \to \bar{M}$ be an isometric action on a Riemannian manifold $\bar{M}$, and let $p \in \bar{M}$. Since the isotropy group G_p fixes p and G_p leaves the orbit $G \cdot p$ invariant, the differential of each isometry $\varphi_g\colon \bar{M} \to \bar{M}$, $p \mapsto gp$, for $g \in G_p$, leaves the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ invariant. Thus, the action

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \qquad (g, X) \mapsto (\varphi_g)_{*p} X,$$

is called the *isotropy representation* of the action φ at p, while

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \qquad (g, \xi) \mapsto (\varphi_g)_{*p} \xi,$$

is called the *slice representation* of the action φ at p.

Let $\bar{M}/G$ be the set of orbits of the action of G on $\bar{M}$, and equip $\bar{M}/G$ with the quotient topology relative to the canonical projection $\bar{M} \to \bar{M}/G$, $p \mapsto G \cdot p$. In general, $\bar{M}/G$ is not a Hausdorff space. In order to avoid this behaviour, the particular type of proper isometric actions was introduced. Thus, the action of G on $\bar{M}$ is *proper* if, for any two points p, $q \in \bar{M}$, there exist open neighbourhoods U_p and U_q of p and q in $\bar{M}$, respectively, such that $\{g \in G : gU_p \cap U_q \neq \emptyset\}$ is relatively compact in G. Equivalently, the map

$$G \times \bar{M} \to \bar{M} \times \bar{M}, \qquad (g, p) \mapsto (p, gp)$$

is a proper map, i.e. the inverse image of each compact set in $\bar{M} \times \bar{M}$ is also compact in $G \times \bar{M}$. Every compact Lie group action is proper. If G is a subgroup of $I(\bar{M})$, then the G-action is proper if and only if G is closed in $I(\bar{M})$. Moreover, if G acts properly on $\bar{M}$, then $\bar{M}/G$ is a Hausdorff space, each isotropy group G_p is compact, and each orbit $G \cdot p$ is closed in $\bar{M}$ and hence an embedded submanifold. In fact, the orbits of an isometric action are closed if and only if the action is orbit equivalent to a proper isometric action, see [27].

We can distinguish three different kinds of orbits of a proper isometric action: principal, exceptional and singular orbits. An orbit $G \cdot p$ is called a *principal orbit* if for each $q \in \bar{M}$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q. The union of all principal orbits is a dense and open subset of $\bar{M}$ and any orbit $G \cdot p$ of a proper action is principal if and only if the slice representation at p is trivial. Each principal orbit is an orbit of maximal dimension. The codimension of any principal orbit is the *cohomogeneity* of the action. A non-principal orbit of maximal dimension is called an *exceptional orbit*. Finally,

a *singular orbit* is an orbit whose dimension is less than the dimension of a principal orbit or, equivalently, an orbit whose codimension is greater than the cohomogeneity.

Another important kind of isometric actions are polar actions. An isometric action of a group G on a Riemannian manifold $\bar{M}$ is called *polar* if its orbit foliation is polar, i.e. if there exists an immersed submanifold Σ of M that intersects all the orbits of the G-action, and for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(G \cdot p)$, are orthogonal. In such a case, the submanifold Σ is totally geodesic and is called a *section* of the G-action. If, in addition, the section Σ is flat in its induced Riemannian metric, the action is called *hyperpolar*. Any polar action admits sections through any given point.

Polar actions are much more rigid than arbitrary isometric actions. For complete, simply connected ambient manifolds $\bar{M}$, the orbits of polar actions are always closed submanifolds, none of them is exceptional, and the image of the group G on the isometry group $I(\bar{M})$ is closed (see [63, Corollary 1.3]). This, in particular, implies that polar actions on complete, simply connected manifolds are orbit equivalent to proper actions. Furthermore, if φ is a polar action of a connected group G on $\bar{M}$, $p \in \bar{M}$ and Σ is a section through p, then the slice representation of such action at p is polar with section $T_p\Sigma$.

Appendix C. Symmetric spaces

Symmetric spaces constitute a particularly nice class of homogeneous spaces. They share, moreover, many connections with the theory of polar actions. Here we provide a quick review on some basic facts about these spaces. Standard references for this topic are [50], [60, 61] and [105].

Firstly, let us fix some notation concerning Lie groups and Lie algebras. As customary, the Lie algebra of a Lie group G will be written with the corresponding gothic letter, in this case, $\mathfrak{g}$. The Lie exponential map will be denoted by Exp. Given $g \in G$, we have the conjugation map $I_g \colon G \to G$, $h \mapsto ghg^{-1}$. Its differential at the identity element $e \in G$ allows to define the Lie group adjoint map $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$, $g \to (I_g)_*$, where $\mathrm{Aut}(\mathfrak{g})$ is the group of automorphisms of the Lie algebra $\mathfrak{g}$, i.e. those linear transformations $\varphi \colon \mathfrak{g} \to \mathfrak{g}$ such that $\varphi[X, Y] = [\varphi X, \varphi Y]$ for all X, $Y \in \mathfrak{g}$. The differential of Ad at e yields the Lie algebra adjoint map $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $X \mapsto \mathrm{ad}(X) = [X, \cdot\,]$. The Killing form of a real Lie algebra $\mathfrak{g}$ is the bilinear form $\mathcal{B} = \mathcal{B}_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$.

Let now M be a Riemannian manifold. Let $o \in M$. Take $r > 0$ sufficiently small so that normal coordinates are defined on the open ball $B_r(o)$. We define the local geodesic symmetry at o as the map $s_o \colon B_r(o) \to B_r(o)$ given by $s_o(\exp_o(tv)) = \exp_o(-tv)$ for $t \in \mathbb{R}$ and $v \in T_oM$. In general, this map is defined only locally. A Riemannian manifold M is said to be *locally symmetric* if at each point there is a ball such that the corresponding local geodesic symmetry is a local isometry. A locally symmetric space is characterized by the fact that $\nabla R = 0$. A connected Riemannian manifold M is called a *(Riemannian) symmetric space* if each local geodesic symmetry s_o can be extended to a global isometry $s_o \colon M \to M$. Since isometries are characterized by their differential at a point, this is equivalent to saying that for each point $o \in M$ there is an involutive isometry of M such

that o is an isolated fixed point of that isometry; this involutive isometry turns out to be s_o.

If M is a connected, complete, locally symmetric Riemannian manifold, then its universal covering is a symmetric space. In particular, every locally symmetric space is locally isometric to a symmetric space. Moreover, every symmetric space is complete and homogeneous.

Now we give a more algebraic description of symmetric spaces. Denote by $G = I(M)^0$ the connected component of the identity of the isometry group $I(M)$ and by $\mathfrak{g}$ the Lie algebra of G. Let $o \in M$ and s_o the geodesic symmetry at o. Define K as the isotropy group of G at o, that is, $K = G_o$, which is compact. The coset space G/K is diffeomorphic to M by means of the map $\Phi\colon G/K \to M$, $gK \mapsto g(o)$. If $\langle\cdot,\cdot\rangle$ denotes the metric obtained by pulling back the metric of M, then Φ becomes an isometry and the metric $\langle\cdot,\cdot\rangle$ is G-invariant, that is, the map $gK \to hgK$ is an isometry for each $h \in G$. The *isotropy representation* of the symmetric space $M \cong G/K$ at o is the orthogonal representation defined by $K \times T_oM \to T_oM$, $(k, v) \mapsto k_*v$.

The map $\sigma\colon G \to G$, $g \mapsto s_o g s_o$, is an involutive automorphism of G, and $G_\sigma^0 \subset K \subset G_\sigma$, where $G_\sigma = \{g \in G : \sigma(g) = g\}$, and G_σ^0 is the connected component of the identity of G_σ . Let θ be the differential of σ at the identity. The Lie algebra of K is given by $\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$, and we define $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$. The space $\mathfrak{p}$ may be identified with T_oM by using the map Φ and taking into account that $\mathfrak{p}$ is a complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$. Thus, $\mathfrak{p}$ inherits an inner product from T_oM which turns out to be $\mathrm{Ad}(K)$-invariant. In fact, the isotropy representation of G/K is equivalent to the adjoint representation of K on $\mathfrak{p}$, $K \times \mathfrak{p} \to \mathfrak{p}$, $(k, X) \mapsto \mathrm{Ad}(k)X$. Moreover, we have the Lie bracket relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to the involution θ (or the point $o \in M$), and θ is called the *Cartan involution*.

The pair (G, K) defined above is an effective (Riemannian) symmetric pair. In general, if G is a connected Lie group and K a compact subgroup, the pair (G, K) is called a *(Riemannian) symmetric pair* if there exists an involutive automorphism σ of G such that $G_\sigma^0 \subset K \subset G_\sigma$, and (G, K) is *effective* if the action of G on $M \cong G/K$ is effective. The isotropy representation of an effective symmetric pair is effective, since isometries are determined by their derivatives. The infinitesimal counterpart of a symmetric pair is the notion of orthogonal symmetric pair. Given a real Lie algebra $\mathfrak{g}$ and a compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we will say that $(\mathfrak{g}, \mathfrak{k})$ is an *orthogonal symmetric pair* if $\mathfrak{k}$ is the fixed point set of an involutive automorphism θ of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{k})$ is said to be effective if $\mathfrak{k} \cap Z(\mathfrak{g}) = 0$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Any effective symmetric pair (G, K) determines an effective orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

Let $M = G/K$ be a symmetric space. The long homotopy sequence of $K \to G \to G/K$ implies that K is connected if M is simply connected and G is connected. Conversely, if G is simply connected and K connected, then M is simply connected.

Let M be a symmetric space and $\widetilde{M}$ its universal covering. Then the De Rham theorem guarantees that $\widetilde{M}$ can be decomposed as $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$. Here $\widetilde{M}_0$ is the

Euclidean factor, that is, $\widetilde{M}_0$ is locally isometric to a Euclidean space, and each $\widetilde{M}_i$, $i = 1, \ldots, k$, is a simply connected, irreducible symmetric space. A symmetric space $M = G/K$ is *irreducible* if its isotropy representation restricted to the identity connected component K^0 of K is an irreducible representation, and is *reducible* otherwise. M is irreducible if and only if its universal covering $\widetilde{M}$ is irreducible.

A *semisimple* symmetric space (or symmetric pair) is one for which the Euclidean factor of its universal covering space has dimension zero. In this case, the Lie algebra of the isometry group of $\widetilde{M}$ is semisimple. A semisimple symmetric space (or symmetric pair) is said to be of *compact type* if all the De Rham factors of its universal covering are compact. It is said to be of *noncompact type* if all the De Rham factors of its universal covering are non-Euclidean, irreducible and noncompact. Again, the Lie algebra $\mathfrak{g}$ of the isometry group of a symmetric space of compact (resp. noncompact) type is compact (resp. noncompact). By definition, an irreducible symmetric space must be one of these three: of Euclidean type (i.e. flat), of compact type, or of noncompact type. If $\mathcal{B}$ is the Killing form of $\mathfrak{g}$, then G/K is of compact type if and only if $\mathcal{B}|_{\mathfrak{p}}$ is negative definite, is of noncompact type if and only if $\mathcal{B}|_{\mathfrak{p}}$ is positive definite, and is of Euclidean type if and only if $\mathcal{B}|_{\mathfrak{p}} = 0$. Moreover, if (G, K) is an effective irreducible symmetric pair of non-Euclidean type, then either G is a simple Lie group, or $(G, K) = (K \times K, \Delta K)$ and G/K is isometric to a compact simple Lie group with bi-invariant metric; here ΔK stands for the diagonal of $K \times K$. If (G, K) is an effective symmetric pair with no Euclidean factor, then $G = I(M)^0$.

There is a duality between symmetric spaces of compact and noncompact type which we explain now. Assume (G, K) is an effective symmetric pair with no Euclidean factor and such that $M = G/K$ is simply connected. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as defined above. We consider the real Lie subalgebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ of the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$, where i is the imaginary unit. Let G^* be the simply connected real Lie group with Lie algebra $\mathfrak{g}^*$. Then we have that G^*/K is a simply connected symmetric space, which we call the *dual* symmetric space of G/K. If G/K is of compact type, then G^*/K is of noncompact type, and if G/K is of noncompact type, then G^*/K is of compact type. Dual symmetric spaces have the same isotropy representation. Duality establishes a one-to-one correspondence between simply connected symmetric spaces of compact and noncompact type, which respects the irreducibility.

Riemannian symmetric spaces have been classified by Cartan. One can find a list of irreducible simply connected symmetric spaces in [50, p. 515–520].

An important subclass of symmetric spaces is that of the Hermitian ones, which we review below. But, first, let us recall some definitions concerning complex, Hermitian and Kähler manifolds. See [103] for more details and proofs.

To start with, let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. In these notes, by *complex structure* on the vector space V we will always understand an orthogonal transformation J of V such that $J^2 = -\operatorname{Id}$. Thus, any endomorphism J of V is a complex structure if and only if any two of the following properties are satisfied: (i) $\langle Jv, Jw \rangle = \langle v, w \rangle$ for all v, $w \in V$ (that is, $J \in O(V)$); (ii) $J^2 = -\operatorname{Id}$; and (iii) $\langle Jv, w \rangle = -\langle v, Jw \rangle$ for all v, $w \in V$ (that is, $J \in \mathfrak{so}(V)$).

A *complex manifold* is a manifold that admits charts with image onto open subsets of $\mathbb{C}^n$ such that the coordinate transitions are holomorphic. This induces an almost complex structure J on M, i.e. an endomorphism of the tangent bundle of M such that $J^2 = -\operatorname{Id}$. If M is Riemannian and complex, and the complex structure J is orthogonal (equivalently, J restricts to a complex structure of each tangent space T_pM, $p \in M$), then M is called a *Hermitian manifold*. A *Kähler manifold* is a Hermitian manifold M satisfying $\nabla J = 0$, where ∇ is the Levi-Civita connection of M. The endomorphism J is known as the *Kähler structure* or the *complex structure* of M.

Thus, a symmetric space M is *Hermitian* if it is a Hermitian manifold and the geodesic symmetries s_p, $p \in M$, are holomorphic transformations. It occurs that every Hermitian symmetric space is Kähler. A symmetric space M is Hermitian if and only if its dual is Hermitian, and every Hermitian symmetric space is simply connected. If, in addition, M is irreducible, then the complex structure J is unique up to sign. An irreducible symmetric pair (G, K) is Hermitian if and only if K is not semisimple. Given an effective irreducible Hermitian symmetric pair (G, K) of non-Euclidean type, then the center of K is isomorphic to $U(1)$ and the induced complex structure J on $\mathfrak{p} \equiv T_o(G/K)$ is given by the $\operatorname{Ad}(K)$-invariant transformation $J = \operatorname{Ad}(i)$, where i stands for the imaginary unit in $U(1)$. Furthermore, every isometry in $I(M)^0$ is holomorphic, and $M = G/K$ is an *inner* or *equal-rank symmetric space*, which means that $\operatorname{rank} G = \operatorname{rank} K$.

It is also possible to define the *rank* of a symmetric space M. This is by definition the dimension of a maximal flat, totally geodesic submanifold of M, or equivalently, the dimension of a maximal abelian subspace of $\mathfrak{p}$. The isotropy representation of a semisimple symmetric space is said to be an *s-representation*. It turns out that the isotropy representation of a semisimple symmetric space M is a polar action on the Euclidean space $T_oM \cong \mathfrak{p}$, and its cohomogeneity is precisely the rank of M. In fact, any maximal abelian subspace of $\mathfrak{p}$ is a section of this representation. This action also induces a polar action on the unit sphere of $T_oM \cong \mathfrak{p}$, which in this case has cohomogeneity equal to the rank of M minus one. A remarkably result by Dadok [24] says that the only homogeneous polar foliations on spheres are the orbit foliations of s-representations.

Of particular interest are the *rank one symmetric spaces*. Together with Euclidean spaces $\mathbb{R}^n$ (which have rank n), rank one symmetric spaces are precisely those manifolds M which are *homogeneous and isotropic*. This means that for any two points p, $q \in M$ and any two tangent vectors $v \in T_pM$, $w \in T_qM$, there is an isometry f of M such that $f(p) = q$ and $f_*(v) = w$. Equivalently, these are the so-called *two-point homogeneous spaces*, i.e. for any four points p_1, p_2, q_1, $q_2 \in M$, there is an isometry f of M such that $f(p_i) = q_i$ for $i = 1$, 2.

Simply connected rank one symmetric spaces of non-Euclidean type are shown in Table 5. Duality of symmetric spaces allows to classify these manifolds into two groups. Those spaces of compact type are spheres and the projective spaces over the algebras of the complex numbers $\mathbb{C}$, of the quaternions $\mathbb{H}$ and of the octonions $\mathbb{O}$. Those of noncompact type are the hyperbolic spaces over the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

Compact	Noncompact	G^{compact}	$G^{\text{noncomp.}}$	K
Spheres S^n	Real hyperbolic spaces $\mathbb{R}H^n$	$SO(n+1)$	$SO(1,n)$	$SO(n)$
Complex projective spaces $\mathbb{C}P^n$	Complex hyperbolic spaces $\mathbb{C}H^n$	$SU(n+1)$	$SU(1,n)$	$\mathrm{S}(U(1)U(n))$
Quaternionic projective spaces $\mathbb{H}P^n$	Quaternionic hyperbolic spaces $\mathbb{H}H^n$	$Sp(n+1)$	$Sp(1,n)$	$Sp(1)Sp(n)$
Cayley projective plane $\mathbb{O}P^2$	Cayley hyperbolic plane $\mathbb{O}H^2$	F_4	F_4^{-20}	$Spin(9)$

Table 5. Duality in rank one symmetric spaces

The spaces in the first row of Table 5 (i.e. spheres S^n and real hyperbolic spaces $\mathbb{R}H^n$) together with Euclidean spaces, are precisely the real space forms, which are the Riemannian manifolds with the simplest curvature tensor.

The second row of Table 5 contains the nonflat *complex space forms*: complex projective spaces $\mathbb{C}P^n$ and complex hyperbolic spaces $\mathbb{C}H^n$. These are Kähler manifolds, so they become Hermitian symmetric spaces.

Quite analogously to the complex case, the third row of Table 5 shows the so-called *quaternionic space forms* $\mathbb{H}P^n$ and $\mathbb{H}H^n$, which can be seen as the most basic examples of quaternionic-Kähler manifolds.

The last row of Table 5 is constituted by two somehow exceptional 16-dimensional manifolds: the Cayley projective and hyperbolic planes, $\mathbb{O}P^2$ and $\mathbb{O}H^2$.

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