New directions in Nonlinear diffusion. Fractional operators

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Outline

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These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

$$\boxed{u_t = \Delta u}$$

a linear example of immense influence in Science.

- The Heat Equation has produced a huge number of concepts, techniques and connections for pure and applied science, for analysts, probabilists, computational people and geometers, for physicists and engineers, and lately in finance and the social sciences.

- Today educated people talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, positive operators in Banach spaces, entropy dissipation, ...

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Nonlinear equations

- The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.
- However, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very not-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x,u,Du) + \mathcal{B}(x,t,u,Du)}$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x,t,u,p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)

- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.
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- Many specific examples, now considered the “classical nonlinear diffusion models”, have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
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Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit the Brownian motion by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- Basic evolution equation

$$\boxed{u_t + (-\Delta)^s u = 0}$$

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Recent Direction. The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy$$

  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s} u$. This one has kernel $C_1 |x-y|^{n-2s}$, which is not integrable.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where P_{ik} denotes the transition function which has a . tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator A as the infinitesimal generator of a Levy process: if X_t is the isotropic α-stable Lévy process we have

$$Au(x) = \lim_{h \to 0} \mathbb{E}(u(x) - u(x + X_h))$$

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The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha} \nabla v) = 0 \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^n.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y \to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre.

Remark. In $\mathbb{R}^n$ all these versions are equivalent. In a bounded domain we have to re-examine all of them. Three main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals.

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Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- Let us see why we have a problem. Take $m = 2$ and differentiate

$$u_t = 2u\Delta u + 2|\nabla u|^2$$

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Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*
To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

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Barenblatt profiles

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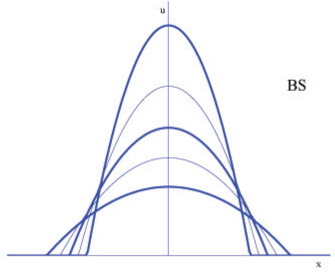
  They are source solutions. *Source means that $u(x,t) \to M\,\delta(x)$ as* $t \to 0$. us
- Explicit formulas (1950, 52):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



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$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

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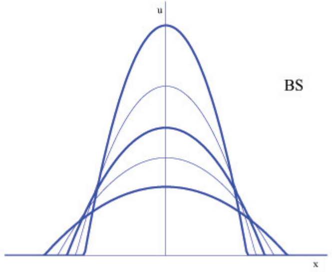

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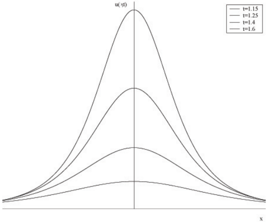

The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for small m, $\quad m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

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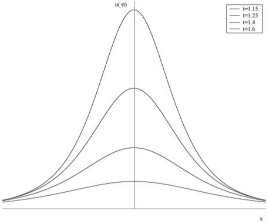

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Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for small m, $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

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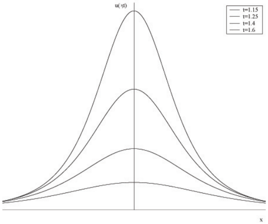

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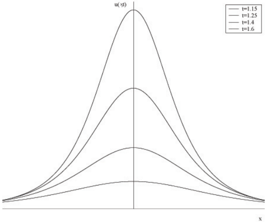

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Outline

Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \ge 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.
- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

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Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel

$$k(x,y) = c|x-y|^{-(n-2s)}$$

  (i.e., a Riesz operator). We have $(-\Delta)^s p = u$.

- The diffusion model with nonlocal effects is thus given by the system

$$\boxed{u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).} \tag{1}$$

  where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u)$

- The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

  where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

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Nonlocal diffusion model

- The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as a large number works on elliptic equations, mainly of the linear or semilinear type (Caffarelli school; Bass, Kassmann, and others)
- There are many works on the subject. Here is a good reference to fractional elliptic work by a young Spanish author

  Xavier Ros-Otón. *Nonlocal elliptic equations in bounded domains: a survey*, Preprint in arXiv:1504.04099 [math.AP].

Nonlocal diffusion Model I. Applications

- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
  This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.
- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L}u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
  $$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

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Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the s interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{3}$$

In one dimension this leads to $u_t = u_x p_x - u^2$, $p_{xx} = -u$. In terms of $v = -p_x = \int u\,dx$ we have

$$v_t = u p_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + v v_x = 0$ which generates shocks in finite time but only if we allow for u to have two signs.

- HYDRODYNAMIC LIMIT. The case $s = 1$ in several dimensions is more interesting because it does not reduce to a simple Burgers equation.

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- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Last reference is proving that the selfsimilar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.
  Uses entropy analysis. Problem is open (and quite interesting in higher dimenions).

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Main estimates for this model

We recall that the equation of M1 is $\partial_t u = \nabla \cdot (u\,\nabla K(u))$, posed in the whole space $\mathbb{R}^n$.

We consider $K = (-\Delta)^{-s}$ for some $0 < s < 1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator. We let $H = K^{1/2} = (-\Delta)^{-s/2}$.

We do next formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions. This is to be justified later by approximation.

- Conservation of mass

$$\frac{d}{dt}\int u(x,t)\,dx = 0. \tag{4}$$

- First energy estimate:

$$\frac{d}{dt}\int u(x,t)\log u(x,t)\,dx = -\int |\nabla Hu|^2\,dx. \tag{5}$$

- Second energy estimate

$$\frac{d}{dt}\int |Hu(x,t)|^2\,dx = -2\int u|\nabla Ku|^2\,dx. \tag{6}$$

Main estimates

- Conservation of positivity: $u_0 \geq 0$ implies that $u(t) \geq 0$ for all times.
- L^∞ estimate. We prove that the L^∞ norm does not increase in time.

  Proof. At a point of maximum of u at time $t = t_0$, say $x = 0$, we have

  $$u_t = \nabla u \cdot \nabla P + u\,\Delta K(u).$$

  The first term is zero, and for the second we have $-\Delta K = L$ where $L = (-\Delta)_q$ with $q = 1 - s$ so that

  $$\Delta K u(0) = -Lu(0) = -\int \frac{u(0) - u(y)}{|y|^{n+2(1-s)}}\,dy \leq 0.$$

  This concludes the proof.
- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved for Model 1. Good comparion works for Model 2 to be presented below, actually, it helps produce a very nice theory.
- Finite propagation is true for model M1. Infinite propagation is true for model M2.

Positivity. Instantaneous Boundedness

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{7}$$

with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.
This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

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Energy and bilinear forms

- **Energy solutions:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. We will review the formulas with attention to the constants that appear since this is not done in [CSV]. The general energy property is as follows: for any F smooth and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\int F(u(t_2))\,dx - \int F(u(t_1))\,dx &= -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt = \\ &\quad -\int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt\end{aligned}$$

where h is a function satisfying $h'(u) = uf'(u)$. We can write the last integral as a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u),u)$$

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s}\iint \nabla v(x)\frac{1}{|x-y|^{n-2s}}\nabla w(y)\,dx\,dy\,. \qquad (8)$$

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where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x)-v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x)-w(y))\, dx\, dy \tag{9}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant K_n depending only on n.

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Additional and Recent work, open problems

- The asymptotic behaviour as $t \to \infty$ is a very interesting topic developed in a paper with Luis Caffarelli. Rates of convergence are found for in dimension $n = 1$ but they are not available for $n > 1$, they are tied to some functional inequalities that are not known,
- The equation is generalized into $u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$ with $m > 1$. Recent work with D. Stan and F. del Teso shows that finite propagation is true for $m \geq 2$ and propagation is infinite is $m < 2$. This is quite different from the standard porous medium case $s = 0$, where $m = 1$ is the dividing value.
- The questions of uniqueness and comparison are solved in dimension $n = 1$ thanks to the trick of integration in space. New tools are needed to make progress in several dimensions.
- The problem in a bounded domain with Dirichlet or Neumann data has not been studied.
- In the standard PME theory, the Wasserstein metrics W_p have proved to be a very interesting tool leading to contractive evolutions. The study in this setting for the present fractional model is only partial, there is work by Carrillo et al. in $n = 1$.

Outline

Part II

Introduction to a second model for comparison

FPME: Second model for fractional Porous Medium Flows

- An alternative natural equation is the equation that we will call FPME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{10}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.
  Understanding the physical situation looks difficult to me , but the modelling on linear an non linear fractional heat equations is done by Stefano Olla, Milton Jara and collaborators, see for instance *M. D. Jara, T. Komorowski, S. Olla*, Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300. *M. Jara, C. Landim, S. Sethuraman*, Probab. Theory Relat. Fields **145** (2009), 565–590.

- Another derivation comes from boundary control problems and it appears in
  Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*, *Adv. Math.* **224** (2010), no. 1, 293–315, where they prove C^α regularity of the solutions.

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Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^{s/2}(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 2$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

  In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

  C^α regularity (if m is not near 0),

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Outline

Outline of work done for model M2

- Existence of self-similar solutions, work by JLV, JEMS 2014. Asymptotic behaviour follows.

  Comparison of models M1 and M2 is quite interesting

- A priori upper and lower estimates of intrinsic, local type. Work with Matteo Bonforte reports on problems posed in $\mathbb{R}^n$ (appeared in ARMA, 2015) and on bounded domains (this is more recent and much less known).

  - *Quantitative positivity and Harnack Inequalities follow.* Against some prejudice due to the nonlocal character of the diffusion, we are able to obtain them here for fractional PME/FDE using the *technique of weighted integrals.*

- Existence of classical solutions and higher regularity for the FPME and the more general model

  $$\partial_t u + (-\Delta)^s \Phi(u) = 0$$

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- Symmetrization (Schwarz and Steiner). Collaboration with Bruno Volzone, two papers at JMPA. *Applying usual symmetrization techniques is not easy and we have many open problems.*
- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
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- A detailed account on such progress is obtained in the papers (cf arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885..

- Operators and Equations in Bounded Domains
  Work that will be presented in the next lecture. It is long time collaboration with Matteo Bonforte and in one instance with Yannick Sire.
  We develop a new programme for Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains, after examining very carefully the concept of FLO in a bounded domain

Future Directions

- Existence of self-similar solutions. See recent papers by JLV in JEMS and arXiv.
  Question : ∃ explicit formulae for the self-similar solutions ? (the influence of Algebra).
- Other nonlocal linear operators (hot topic)
- *p*-Laplacian type fractional flows (JLV paper posted in June 2015 in arXiv).
- Very degenerate nonlinearities, like the Mesa Problem (cf. JLV, arXiv)
- Fast diffusion and extinction. Very singular fast diffusion.
  Non-existence due to instantaneous extinction (paper with Bonforte and Segatti in arXiv, 2015)
- Elliptic theory (main topic, many authors)
- Geostrophic flows (this is more related to Fluid Mechanics)
- Reaction-diffusion and blowup
- Geometrical flows, fractional Yamabe (MMar Gonzalez in Barcelona)
- Chemotaxis systems, ...

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The younger ones

The End

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