Evolution of sharp fronts for the surface quasi-geostrophic equation

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Outline of the lecture

- ☞ Motivation: Vortex Lines for 3D Euler
- ☞ 3D Euler and connections with SQG
- ☞ Vortex Lines (Euler) - Sharp fronts (SQG)
- ☞ Evolution equation for a sharp front
- ☞ Almost Sharp Fronts
- ☞ Limit of Almost Sharp Fronts (Stability)
- ☞ The Spine of an Almost Sharp Front
- ☞ Construction of almost sharp fronts

Motivation: Vortex Lines for 3D Euler

- 3D Euler in vorticity form ($\omega = \mathrm{curl} u$)

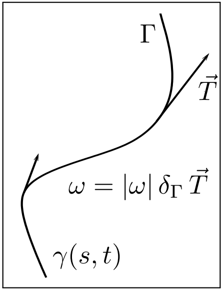
$$\omega_t + u \cdot \nabla \omega = (\nabla u)\omega$$

$$u(x) = \frac{1}{4\pi} \int \frac{x-y}{\|x-y\|^3} \times \omega(y) \mathrm{d}y$$

- Solution of 3D Euler with ω supported on a curve γ

$$\omega = |\omega|\, \delta_\Gamma\, \vec{T}$$

- Vortex lines are advected by the fluid.



Motivation: Vortex Lines for 3D Euler

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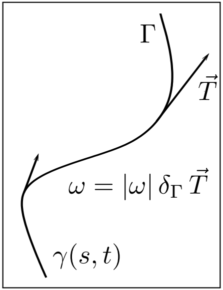

- Vortex lines are advected by the fluid.
- Velocity expression is singular if $\omega = |\omega| \, \delta_\Gamma \, \vec{T}$

$$u(x) \approx \frac{1}{\text{distance to the curve}} \Longrightarrow u \notin L^2$$

- Known approximation - da Rios equation:

$$\boxed{\partial_t \gamma = \kappa \vec{B}}$$

κ - Curvature of γ
$\vec{B}$ - Binormal

The Surface Quasi-Geostrophic Equation

- Active Scalar equation (2-dimensional model).
- Incompressible model for weather prediction.
- Later realized to be a **model for 3D Euler** (Constantin-Majda-Tabak).

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} := \frac{\partial\theta}{\partial t} + u \cdot \nabla\theta = 0,$$

$$u = (-\triangle)^{-\frac{1}{2}} \nabla^{\perp}\theta$$

- Fractional Laplacian defined via the Fourier Transform

$$\widehat{(-\triangle)^{\alpha}} f := |\xi|^{2\alpha} \hat{f} \qquad\qquad (-\triangle)^{-\frac{1}{2}} f = \int_{\mathbb{R}^2} f(y) \frac{1}{|x-y|} \mathrm{d}y$$

- Expression for u is a singular integral (orthogonal Riesz transform)

$$u(x) = \int_{\mathbb{R}^2} (-\theta_{y_2}(y), \theta_{y_1}(y)) \frac{1}{|x-y|} \mathrm{d}y = \int_{\mathbb{R}^2} \theta(y) \Big(-\frac{x_2-y_2}{|x-y|^3}, \frac{x_1-y_1}{|x-y|^3} \Big) \mathrm{d}y$$

3D Euler and SQG

$$\frac{D\theta}{Dt} := \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0$$

3D EULER | **SQG**

$$\omega \sim \nabla^{\perp}\theta$$

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = (\nabla u)\omega \qquad\qquad \frac{\mathrm{D}(\nabla^{\perp}\theta)}{\mathrm{D}t} = (\nabla u)\nabla^{\perp}\theta$$

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- The velocity is recovered via the integrals (K_d homogeneous of degree $1-d$)

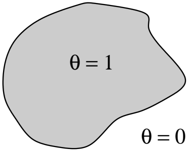
$$u(x)=\int_{\mathbb{R}^3} K_3(y)\omega(x-y)\mathrm{d}y, \qquad\qquad u(x)=\int_{\mathbb{R}^2} K_2(y)\nabla^{\perp}\theta(x-y)\mathrm{d}y.$$

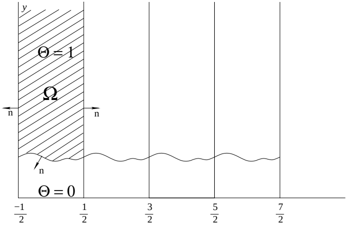
- The strain matrix can be recovered via SIO. $S = \frac{\nabla u + \nabla^{\perp} u}{2}$.
- $|\omega|$ and $|\nabla^{\perp}\theta|$ evolve according to the same type of equation.
- Both systems have conserved energy.
- Beale-Kato-Majda condition for a break down (at time $T_\star$).

$$\int_0^T ||\omega||_{L_x^\infty}(s)\mathrm{d}s \underset{T\to T_\star}{\longrightarrow} \infty, \qquad\qquad \int_0^T ||\nabla^{\perp}\theta||_{L_x^\infty}(s)\mathrm{d}s \underset{T\to T_\star}{\longrightarrow} \infty.$$

- Integral curves of ω (vortex lines) and of $\nabla^{\perp}\theta$ move with the fluid.

Evolution of Sharp Fronts





We want to study the evolution of

$$\begin{cases} \theta(x,y,t) = 1 & y \geq \varphi_0(x) \\ \theta(x,y,t) = 0 & y < \varphi_0(x), \end{cases}$$

where $\varphi_0(x)$ is a smooth periodic function.

MAIN GOAL

Find an evolution equation for a periodic sharp front for SQG using tools available in 3D Euler.

Formal derivation of the equation

$$(-\triangle)^{-\frac{1}{2}}F = \left[\frac{\chi(x,y)}{(x^2+y^2)^{\frac{1}{2}}}\right] * F$$

Since $u = (-\triangle)^{-\frac{1}{2}}\nabla^{\perp}\theta$ and $\nabla^{\perp}\theta(x,y,t) = (-1, -\varphi'(x,t))\delta(y-\varphi(x,t))$

we obtain **(for** (x,y) **not on the front)**

$$u(x,y,t) = -\int_{\mathbb{R}/\mathbb{Z}} \Big(1, \varphi'(\tilde{x},t)\Big) \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} \mathrm{d}\tilde{x}.$$

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$$u(x,y,t) = -\int_{\mathbb{R}/\mathbb{Z}}\Big(1,\varphi'(\tilde{x},t)\Big)\frac{\chi(x-\tilde{x},y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2+(y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde{x}.$$

The integral is divergent **as we approach the front**. We loose one dimension and the kernel is not integrable in one dimension. Since θ is advected by the fluid

$$(\partial_t + u\cdot\nabla)\theta = 0 \Longrightarrow (\partial_t + [u + h\nabla^{\perp}\theta]\cdot\nabla)\theta = 0.$$

We modify u by adding the following terms

$$\Big(1,\varphi'(x,t)\Big)\left[\int_{\mathbb{R}/\mathbb{Z}}\frac{\chi(x-\tilde{x},y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2+(y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde{x}\right].$$

Formal derivation of the equation II

We redefine u to be

$$u(x,y,t) = -\int_{\mathbb{R}/\mathbb{Z}} \Big(0, \varphi'(\tilde{x},t) - \varphi'(x,t)\Big) \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} \mathrm{d}\tilde{x}.$$

Since u is purely vertical and the front is advected by the fluid we have

$$\frac{\partial \varphi}{\partial t}(x,t) = -\int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'(\tilde{x},t) - \varphi'(x,t)}{[(x-\tilde{x})^2 + (\varphi(x,t) - \varphi(\tilde{x},t))^2]^{\frac{1}{2}}} \chi(x-\tilde{x}, \varphi(x,t) - \varphi(\tilde{x},t)) \mathrm{d}\tilde{x},$$

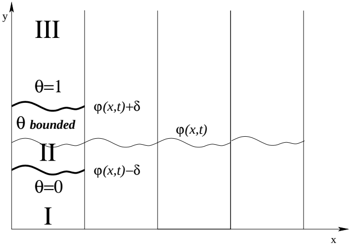
with initial data $\varphi(x,0) = \varphi_0(x)$.

The equation is a nonlinear version of

$$\varphi_t = \mathrm{i}\, D \log |D| \varphi.$$

Almost-Sharp Fronts

Weak solutions of the equation with large gradient ($\sim \dfrac{1}{\delta}$, where 2δ is the thickness of the transition layer for θ).



We consider θ of the following form

$$\begin{aligned} \theta = 1 \text{ if } \quad & y \geq \varphi(x,t) + \delta \\ \theta \ \underline{\textbf{bounded}} \text{ if } \quad & |\varphi(x,t) - y| \leq \delta \\ \theta = 0 \text{ if } \quad & y \leq \varphi(x,t) - \delta \end{aligned}$$

where φ is a smooth periodic function and $0 < \delta < \frac{1}{2}$.

Remark

$\varphi(x,t)$ can be any curve, not necessarily the solution of the sharp front equation.

Derivation

Theorem (D. Córdoba - C. Fefferman - J. R.)

If θ is an almost-sharp front and is a weak solution of SQG, then any curve ϕ in the almost-sharp front satisfies the equation (in the weak formulation)

$$\frac{\partial \phi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\phi'(x,t) - \phi'(\tilde{x},t)}{[(x-\tilde{x})^2 + (\phi(x,t) - \phi(\tilde{x},t))^2]^{\frac{1}{2}}} \chi(x-\tilde{x}, \phi(x,t) - \phi(\tilde{x},t)) \mathrm{d}\tilde{x} +$$

$$+ \boxed{\mathcal{E}rror}$$

with $|\mathcal{E}rror| \leq C\, \delta |log\, \delta|$ where C depends only on $\|\theta\|_{L^\infty}$ and $\|\nabla \phi\|_{L^\infty}$.

Remark

Note that an almost sharp front specifies the function ϕ up to an error of order δ. The above theorem provides an evolution equation for the function ϕ up to an error of order $\delta |log\, \delta|$.

A Spine for Almost Sharp Fronts

Observation

- Given an almost sharp front that changes from 0 to 1 in a strip of thickness δ there exists a "unique" curve that improves the previous result.
- For every time slice we choose it using an intrinsic measure-theoretical approach. (No evolution equation!)

Lemma (Choosing the spine)

Given $\theta \in C^2$, *satisfying* $|\partial^\alpha \theta| \lesssim \delta^{-|\alpha|}, |\alpha| \leq 2$, *and*

$$\theta(x_1, x_2) = \begin{cases} 1 & x_2 > g(x_1) + \delta \\ 0 & x_2 < g(x_1) - \delta \end{cases}$$

there exists a "unique" curve $x_2 = \phi(x_1)$ *such that the difference between the following two vector-valued measures is* $O(\delta^2)$

$$\nabla^\perp \theta \, \mathrm{d}x_1 \mathrm{d}x_2 \qquad - (1, \phi'(x_1)) \mathrm{d}s_\phi \qquad \textit{arc length on curve } \phi$$

- θ does not have to solve any equation.

About the spine

φ is the "centre of mass" of θ

$$\int_{-\infty}^{\infty} \theta_{x_2}(x_1, x_2)(x_2 - \phi(x_1))\mathrm{d}x_2 = 0 \qquad \text{for every } x_1.$$

We are choosing ϕ such that

$$h(x_1) := \int_{-\phi(x_1)-\delta}^{\phi(x_1)+\delta} \theta(x_1, x_2)\mathrm{d}x_2 = -\int_{-\infty}^{\infty} \theta_{x_2}(x_1, x_2)(x_2 - \phi(x_1))\mathrm{d}x_2 = 0.$$

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For every $\Gamma(x_1, x_2)$ with $|\nabla^2\Gamma| \le M$ we have

$$\iint_{\mathbb{R}/_{\mathbb{Z}}\times\mathbb{R}} \Gamma(x_1, x_2, t)\nabla^{\perp}\theta(x_1, x_2, t)\mathrm{d}x_1\mathrm{d}x_2 = -\int_{x_2=\phi(x_1,t)} \Gamma(x_1, x_2)(1, \phi'(x_1, t))\mathrm{d}x_1 + O(M\delta^2).$$

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In the applications Γ arises from the velocity field (among other things), which **unfortunately** does not satisfy $|\nabla^2\Gamma| \leq M$.

Evolution of the spine

Theorem (C. Fefferman - K. Luli - J.R.)

Let θ is an almost sharp front of thickness δ. Define $x_2 = \varphi(x_1, t)$ to be the spine for every time t. Then φ satisfies the equation (in the weak formulation)

$$\frac{\partial \varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'(x,t) - \varphi'(\tilde{x},t)}{[(x-\tilde{x})^2 + (\varphi(x,t) - \varphi(\tilde{x},t))^2]^{\frac{1}{2}}} \chi(x - \tilde{x}, \varphi(x,t) - \varphi(\tilde{x},t)) \mathrm{d}\tilde{x} +$$

$$+ \boxed{\mathcal{E}rror},$$

with $|\mathcal{E}rror| \leq C\,\delta^2 |log\,\delta|$ where C depends only on estimates for $\partial^2\theta$.

Remark

Note that an almost sharp front specifies the function φ up to an error of order δ. The above theorem provides an evolution equation for the function φ up to an error of order $\delta^2 |log\,\delta|$.

Technical issues

- In our case Γ arises from u. We need the weaker version: if $|\nabla^2\Gamma| \leq M|\log\delta|$ or $\Gamma(x_1,x_2) = \Gamma(x_1,\phi(x_1)) + \Gamma_{x_2}(x_1,\phi(x_1))(x_2-\phi(x_1)) + O(M\delta^2|\log\delta|)$ whenever $|x_2-\phi(x_1)| \leq C\delta$ we have

$$\iint_{\mathbb{R}/_{\mathbb{Z}}\times\mathbb{R}} \Gamma(x_1,x_2)\nabla^{\perp}\theta(x_1,x_2,t)\mathrm{d}x_1\mathrm{d}x_2 = -\int_{x_2=\varphi(x_1,t)} \Gamma(x_1,x_2)(1,\varphi'(x_1,t))\mathrm{d}x_1 + O(M\delta^2|\log\delta|).$$

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$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \Gamma(x_1,x_2)\nabla^{\perp}\theta(x_1,x_2,t)\mathrm{d}x_1\mathrm{d}x_2 = -\int_{x_2=\varphi(x_1,t)} \Gamma(x_1,x_2)(1,\varphi'(x_1,t))\mathrm{d}x_1 + O(M\delta^2|\log\delta|).$$

- We only need to use test functions $\gamma(x_1,x_2,t)$ which are functions of x_1 for $|x_2| < C$. We start by considering a weak solution of SQG

$$\int \theta(x_1,x_2,t)\,\partial_t\gamma\,(x_1,x_2,t)dx_2dx_1dt + \int \theta\,(x_1,x_2,t)u(x_1,x_2,t)\cdot\nabla\gamma\,(x_1,x_2,t)\mathrm{d}x_2\mathrm{d}x_1\mathrm{d}t = 0$$

We want to show that the above equation leads to

$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \gamma(x_1,t)\Big\{\text{ Left Hand Side of Sharp Front Equation} = 0\Big\}\mathrm{d}x_1\mathrm{d}t = O(\delta^2|\log\delta|)$$

where $\gamma(x_1,t)$ is a test function (now in one space dimension).

Constructing Almost-Sharp Fronts

- Given a curve and a *transition profile* we want to construct solutions with initial data the rescaled (to have thickness δ) profile, that exist for some positive time independent of δ.
- Want to study the family of solutions on a FIXED DOMAIN. We look for solutions of the form (FIRST SET OF NEW COORDINATES)

$$\theta_\delta(x,y,t) = \Omega(x, \underbrace{\frac{y-\varphi(x,t)}{\delta}}_{\xi}, t),$$

where Ω is smooth and satisfies

$$\begin{cases} \Omega(x,\xi,t) = 1 & \xi > 1, \\ \Omega(x,\xi,t) \text{ smooth} & |\xi| \leq 1, \\ \Omega(x,\xi,t) = 0 & \xi < -1. \end{cases}$$

- **Goal** Prove that for $\delta \leq \delta_0$ the solutions θ_δ exist for at least time T, independent of δ.

No Limit: Second singularity

The velocity is given by (from now own $\chi = 1$)

$$u(x,y,t) = \iint \frac{(-\theta_{\tilde{y}}(\tilde{x},\tilde{y}), \theta_{\tilde{x}}(\tilde{x},\tilde{y}))}{[(x-\tilde{x})^2 + (y-\tilde{y})^2]^{\frac{1}{2}}} \mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

Strategy: write SQG in terms of Ω, x, ξ. Recall $\theta(x,y,t) = \Omega(x,\xi,t)$. We expect

$$\frac{1}{\delta}(\text{Sharp Front Equation for } \varphi) + \text{terms of order 1 in terms of } \Omega, x, \xi$$

$$u \cdot \nabla\theta = \iint \frac{-\theta_{\tilde{y}}(\tilde{x},\tilde{y})\theta_x(x,y) + \theta_{\tilde{x}}(\tilde{x},\tilde{y})\theta_y(x,y)}{[(x-\tilde{x})^2 + (y-\tilde{y})^2]^{\frac{1}{2}}} \mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

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Problem: There is no formal limit!!

Since $\xi = \frac{y - \varphi(x,t)}{\delta}$ we have $y = \varphi(x,t) + \delta\xi$

$$\iint \frac{\Omega_{\tilde{\xi}}(\tilde{x},\tilde{\xi},t)\Omega_x(x,\xi,t) - \Omega_{\tilde{x}}(\tilde{x},\tilde{\xi},t)\Omega_\xi(x,\xi,t)}{[(x-\tilde{x})^2 + (\varphi(x,t) - \varphi(\tilde{x},t) + \delta(\xi-\tilde{\xi}))^2]^{\frac{1}{2}}} \mathrm{d}\tilde{x}\mathrm{d}\tilde{\xi}$$

when $\delta = 0$ the kernel looses the dependence on $\bar{\xi}$ making the integral singular.

It is not possible to compute a formal limit.

Vertical integration regularizes the equation

For this special kind of solution

$$\int \Omega_\xi(x,\xi,t)\mathrm{d}\xi = 1.$$

The singular term (in the limit) becomes

$$\iint \frac{\Omega_{\tilde{\xi}}(\tilde{x},\tilde{\xi},t)\Omega_x(x,\xi,t) - \Omega_{\tilde{x}}(\tilde{x},\tilde{\xi},t)\Omega_\xi(x,\xi,t)}{[(x-\tilde{x})^2 + (\varphi(x,t)-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde{x}\mathrm{d}\tilde{\xi} =$$

$$= \int \frac{\Omega_x(x,\xi,t) - \int \Omega_{\tilde{x}}(\tilde{x},\tilde{\xi},t)\mathrm{d}\tilde{\xi}\ \Omega_\xi(x,\xi,t)}{[(x-\tilde{x})^2 + (\varphi(x,t)-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde{x}.$$

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The singular term (in the limit) becomes

$$\iint \frac{\Omega_{\tilde\xi}(\tilde x,\tilde\xi,t)\Omega_x(x,\xi,t) - \Omega_{\tilde x}(\tilde x,\tilde\xi,t)\Omega_\xi(x,\xi,t)}{[(x-\tilde x)^2 + (\varphi(x,t)-\varphi(\tilde x,t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde x\mathrm{d}\tilde\xi =$$

$$= \int \frac{\Omega_x(x,\xi,t) - \int \Omega_{\tilde x}(\tilde x,\tilde\xi,t)\mathrm{d}\tilde\xi\ \Omega_\xi(x,\xi,t)}{[(x-\tilde x)^2 + (\varphi(x,t)-\varphi(\tilde x,t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde x.$$

Formal integration of the equation with respect to ξ

$$\int \frac{\int \Omega_x(x,\xi,t)\mathrm{d}\xi - \int \Omega_{\tilde x}(\tilde x,\tilde\xi,t)\mathrm{d}\tilde\xi}{[(x-\tilde x)^2 + (\varphi(x,t)-\varphi(\tilde x,t))^2]^{\frac{1}{2}}}\mathrm{d}\tilde x.$$

Now the integral is convergent!!

☞ CLAIM $\int \Omega(x,\xi,t)\mathrm{d}\xi =: h(x,t)$ satisfies a much better equation.

Asymptotics of u. Lack of Limit.

The velocity can be writen in terms of h alone. For $\delta > 0$ it looks like

$$u(x,\xi,t) \approx log\,\delta\ (1,\varphi(x,t)) + \delta\, log\,\delta \text{ term} + \ldots$$

$$u(x,\xi,t) = log\,\delta\alpha(x,t)(1,\varphi(x,t)) - \delta\, log\,\delta\Big\{\alpha_x(x,t)\xi + \alpha_x(x,t)\boxed{h(x,t)} + \alpha(x,t)h_x(x,t)\Big\}(0,1) + \cdots$$

$$\boxed{= \int \Omega(x,\xi,t)d\xi}$$

α fully determined by φ.

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$$u(x,\xi,t) = \log\delta\alpha(x,t)(1,\varphi(x,t))$$
$$-\delta\log\delta\Big\{\alpha_x(x,t)\xi+\alpha_x(x,t)\boxed{h(x,t)}+\alpha(x,t)h_x(x,t)\Big\}(0,1)+\cdots$$

$$\boxed{h(x,t) = \int\Omega(x,\xi,t)d\xi}$$

α fully determined by φ.

And the gradient of θ

$$\nabla\theta = \Omega_x(x,\xi,t)(1,0) + (-\varphi(x,t),1)\frac{1}{\delta}\Omega_\xi(x,\xi,t).$$

And so

$$u\cdot\nabla\theta = \log\delta\Big\{\alpha\Omega_x - \big[\alpha_x\,\xi + \alpha_x h + \alpha h_x\big]\Omega_\xi\Big\} + \cdots$$

The term with the logarithmic divergence disappears

$$\int\alpha\Omega_x - \big[\alpha_x\,\xi + \alpha_x h + \alpha h_x\big]\Omega_\xi \mathrm{d}\xi = 0.$$

Equation for h. Connections with the spine.

Since

$$\int \alpha\Omega_x - \left[\alpha_x\,\xi + \alpha_x h + \alpha h_x\right]\Omega_\xi \mathrm{d}\xi = 0.$$

We find a "simple" equation for $h(x,t) = \int \Omega(x,\xi,t)d\xi$ (IN THE LIMIT)

$$h_t(x,t)+$$

$$+F_1(x,t)\int \frac{h_x(\bar{x},t) - h_{\bar{x}}(\bar{x},t)}{|x-\bar{x}|}\mathrm{d}\bar{x} + F_2(x,t)\int \frac{h(\bar{x},t) - h(\bar{x},t)}{|x-\bar{x}|}\mathrm{d}\bar{x}+$$

$$+F_3(x,t)h_x(x,t) + F_4(x,t)h(x,t)+$$

$$+\int h_{\bar{x}}(\bar{x},t)Q_1(x,\bar{x},t)\mathrm{d}\bar{x} + \int h(\bar{x},t)Q_2(x,\bar{x},t)\mathrm{d}\bar{x} = 0.$$

EQUATION FOR h. CONNECTIONS WITH THE SPINE.

Since

$$\int \alpha\Omega_x - \left[\alpha_x\,\xi + \alpha_x h + \alpha h_x\right]\Omega_\xi \mathrm{d}\xi = 0.$$

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h SATISFIES A “LINEARIZED” SHARP FRONT EQUATION, WHICH IS HOMOGENEOUS!

If $h(x,0) = 0$, that is $\varphi(x,0)$ is the spine, it remains 0, which implies $\varphi(x,t)$ remains as the spine, while solving the sharp front equation (in the limit).

Recast Ω as a transport-like equation.

We use its solution in the original equation for Ω

$$\Omega_t + log\,\delta\Big\{\alpha\Omega_x - \big[\alpha_x\,\xi + \alpha_x h + \alpha h_x\big]\Omega_\xi\Big\} + \cdots$$

which is now a transport-like equation (now h is KNOWN!! - and $= 0$ if φ is the spine).

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2\log\delta \dfrac{1}{(1+(\varphi_x(x,t))^2)^{1/2}} \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = 2\log\delta\left[[\xi + h(x,t)]\dfrac{\varphi_x(x,t)\varphi_{xx}(x,t)}{(1+(\varphi_x(x,t))^2)^{3/2}} - \dfrac{1}{(1+(\varphi_x(x,t))^2)^{1/2}}h_x(x,t)\right] \end{cases}$$

WE NEED TO STUDY Ω AND h IN DIFFERENT COORDINATE SYSTEMS!

The system can be easily integrated if we write the equation using as coordinates arc-length and distance to the curve.

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}\tau} = -\dfrac{2}{L}\log\delta \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}\tau} = \dfrac{2}{L}h_s(s,\tau)\log\delta \end{cases}$$

Model Problem

Model Problem

$$f_t - log\delta f_x = 0$$

Using as new coordinates $(x + log\delta\, t, t)$ the equation becomes

$$\frac{Df}{Dt} = 0$$

To solve the equation, all we need to do is integrate a vector field.
The whole process corresponds to unwinding the logarithmic divergence that appears in $u \cdot \nabla\theta$, before considering the map

$$(x, y, t) \longrightarrow (x, \frac{y - \varphi(x,t)}{\delta}, t)$$

Main Difficulty: Unwinding depends on h.

In the limit h and Ω uncouple, but not for $\delta > 0$.

Unwound Limit equation

Abuse of notation (use x, ξ again for the unwound coordinates)

$$\Omega_t(x,\xi,t)$$

$$+W(x,t)\Omega_\xi(x,\xi,t)\int \Omega_x(x,\bar{\xi},t)\ln\frac{1}{|\xi-\bar{\xi}|}d\bar{\xi}$$

$$+W_1(x,\xi,t)\Omega_x(x,\xi,t)+W_2(x,\xi,t)\Omega_\xi(x,\xi,t)+$$

$$+W_3(x,\xi,t)\Omega_\xi(x,\xi,t)\int \Omega_{\bar{\xi}}(x,\bar{\xi},t)\ln\frac{1}{|\xi-\bar{\xi}|}d\bar{\xi}$$

$$+W_4(x,t)\Omega_\xi(x,\xi,t)\int \Omega_{\bar{\xi}}(x,\bar{\xi},t)\,\bar{\xi}\,\ln\frac{1}{|\xi-\bar{\xi}|}d\bar{\xi}$$

$$+W_5(x,t)\Omega_x(x,\xi,t)\int \Omega_{\bar{\xi}}(x,\bar{\xi},t)\,\bar{\xi}\,\ln\frac{1}{|\xi-\bar{\xi}|}d\bar{\xi}=0$$

Theorem (C. Fefferman & J.R.)

The above equation has a unique solution in the analytic case (in several flavours).

Existence of ASF

Theorem (C. Fefferman & J.R.)

In the analytic case almost-sharp fronts can be constructed for a time independent of the thickness.

- ☞ *Either for functions that convergence asymptotically to the contants 0 and 1 (redoing the whole theory for "almost-constant" patches*
- ☞ *or for analytic functions in the interior of the strip that match the constant values to any fixed "contact order*

Analytic Construction

☞ Study a system with 4 unknowns Ω, h, top boundary, bottom boundary.

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