Nonlocal self-improving properties

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Self-improving properties:

Part 1: The local setting

Meyers estimate

Theorem (Meyers)

Let u be a local weak solution to

$$-\mathrm{div}\,(a(x)Du) = 0 \qquad \textit{in } \mathbb{R}^n,$$

where $a(\cdot)$ is measurable and satisfies

$$\frac{|\xi|^2}{\Lambda} \leq \langle a(x)\xi, \xi\rangle \qquad \textit{and} \qquad |a(x)| \leq \Lambda\,.$$

Then

$$u \in W^{1,2}_{\mathrm{loc}} \Longrightarrow u \in W^{1,2+\delta}_{\mathrm{loc}}$$

for some $\delta > 0$ depending only on n, Λ

The Gehring lemma

This is based on a modification of the seminal result of Gehring:

Theorem (Gehring)

Let $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ *be such that*

$$\left(\fint_B f^p dx\right)^{1/p} \lesssim \left(\fint_B f^q dx\right)^{1/q}$$

for $q < p$ *and for every ball* B. *Then* $f \in L^{p+\delta}_{\text{loc}}(\mathbb{R}^n)$ *for some* $\delta > 0$ *and*

$$\left(\fint_B f^{p+\delta} dx\right)^{1/(p+\delta)} \lesssim \left(\fint_B f^q dx\right)^{1/q}$$

Inhomogeneous case

Theorem (Elcrat-Meyers, Giaquinta-Modica)

Let $u \in W^{1,2}_{\mathrm{loc}}$ *be a weak solution to*

$$-\operatorname{div} a(x, Du) = f \in L^{2+\delta_0},$$

where

$$\frac{|z|^2}{\Lambda} \leq \langle a(x,z), z \rangle \qquad \text{and} \qquad |a(x,z)| \leq \Lambda |z|.$$

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where

$$\frac{|z|^2}{\Lambda} \le \langle a(x,z), z\rangle \qquad \text{and} \qquad |a(x,z)| \le \Lambda |z|.$$

Then

$$u \in W^{1,2} \Longrightarrow u \in W^{1,2+\delta}_{\mathrm{loc}}$$

for some $\delta \in (0, \delta_0]$ *depending only on* n, Λ, δ_0.

Inhomogeneous case

Moreover, the local estimate

$$\left(\fint_{B_R} |Du|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}} \lesssim \left(\fint_{B_{2R}} |Du|^{2}\,dx\right)^{\frac{1}{2}} + \left(\fint_{B_{2R}} |f|^{2+\delta_0}\,dx\right)^{\frac{1}{2+\delta_0}}$$

holds for any ball B_R.

The Gehring lemma with additional terms

Theorem (Gehring-Giaquinta-Modica)

Let $f \in L^p_{\mathrm{loc}}(\Omega)$ *be such that*

$$\left(\fint_{\frac{1}{2}B} f^p dx\right)^{1/p} \lesssim \left(\fint_B f^q dx\right)^{1/q} + \left(\fint_B g^p dx\right)^{1/p}$$

for $q < p$*, then*

$$\left(\fint_{\frac{1}{2}B} f^{p+\delta} dx\right)^{1/(p+\delta)} \lesssim \left(\fint_B f^q dx\right)^{1/q} + \left(\fint_B g^{p+\delta} dx\right)^{1/(p+\delta)}$$

Caccioppoli inequalities imply higher integrability

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{\frac{1}{2}B} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{1,2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

Caccioppoli inequalities imply higher integrability

The proof is very simple: Sobolev-Poincaré yields

$$\left(\fint_{B/2} |Du|^2 dx \right)^{1/2} \lesssim \left(\fint_{B} |Du|^{2n/(n+2)} dx \right)^{(n+2)/2n} ,$$

and the assertion follows from an adaptation of the Gehring lemma.

Gradient oscillations

What about the higher regularity? For solutions to

$$-\mathrm{div}\,(a(x)Du) = 0$$

we have

- $a(\cdot)$ is Dini $\Longrightarrow Du \in C^0$
- $a(\cdot) \in C^{0,\sigma} \Longrightarrow Du \in C^{0,\sigma}$
- $a(\cdot) \in N^{\sigma,q} \Longrightarrow Du \in N^{\sigma,2}$

Recall that $a(\cdot) \in N^{\sigma,q}$ means that

$$\int |a(x+h) - a(x)|^q \, dx \lesssim |h|^{q\sigma}$$

Oscillations of coefficients influence the oscillations of the gradient

Merely measurable coefficients

If the coefficients are merely measurable, there is no gradient oscillation control.

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Indeed, consider $n = 1$ and

$$(a(x)u_x)_x = 0\,, \qquad u(0) = 0\,, \qquad u(1) = 1\,.$$

with

$$0 < \nu \leq a(x) \leq L\,.$$

Then the solution is given by

$$u(x) = M\int_0^x \frac{dt}{a(t)}\,, \quad u_x(x) = \frac{M}{a(x)}\,, \qquad M := \left[\int_0^1 \frac{dt}{a(t)}\right]^{-1}\,.$$

i.e., no gradient differentiability is possible when coefficients are just measurable.

Integrodifferential equations

Part 2: The nonlocal setting

Integrodifferential equations

We consider

$$\mathcal{E}_K(u,\eta) = \int_{\mathbb{R}^n} f\eta\, dx$$

for every test function $\eta \in C_c^\infty(\mathbb{R}^n)$, where

$$\mathcal{E}_K(u,\eta) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [u(x)-u(y)][\eta(x)-\eta(y)]K(x,y)\, dx\, dy\, .$$

The Kernel $K(\cdot,\cdot)$ is assumed to be symmetric and measurable satisfying growth bounds

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2\alpha}}$$

for some $\Lambda \geq 1$.

Integrodifferential equations

Heuristically speaking, the nonlocal equation $\mathcal{E}_K(u,\eta) = \int_{\mathbb{R}^n} f\eta\,dx$ for all $\eta \in C_c^\infty(\mathbb{R}^n)$ is the weak formulation of

$$\mathcal{L}_K u(x) = p.v. \int_{\mathbb{R}^n} (u(x) - u(y))K(x,y)\,dy = \frac{1}{2} f(x)\,.$$

In the case $K(x,y) = c_{n,\alpha}|x-y|^{-n-2\alpha}$ the operator is the fractional Laplacian, and the equation reduces to

$$(-\triangle)^\alpha u = f\,.$$

Furthermore, in the case $K(x,y)=|x-y|^{-n-2\alpha}$ we have

$$\mathcal{E}_K(u,\eta)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{\alpha}}\frac{\eta(x)-\eta(y)}{|x-y|^{\alpha}}\frac{dx\,dy}{|x-y|^{n}}\,,$$

which is the nonlocal analog of

$$\int\langle Du,D\eta\rangle\,dx\,.$$

Integrodifferential equations - some regularity results

- Bass & Kassmann (Comm. PDE, TAMS 05)
- Caffarelli & Silvestre (Comm. PDE 07, lifting and localization)
- Kassmann (Calc. Var. 09, measurable coefficients)
- Caffarelli & Chan & Vasseur (JAMS 07, lifting and localization)
- Bjorland & Caffarelli & Figalli (Adv. Math. 09, p-Laplacean type)
- Caffarelli & Silvestre (Ann. Math. 11, fully nonlinear theory)
- Da Lio & Rivière (Analysis & PDE 11, Adv. Math. 11, systems and half-harmonic maps)
- Di Castro & K. & Palatucci (JFA 14, Poincare 15, p-growth and related theory)

Fractional energies

For $\alpha \in (0,1)$ and $p \in [1,\infty)$, define the seminorm

$$[u]_{\alpha,p}(\mathbb{R}^n) := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha p}} \, dx \, dy \right)^{1/p}$$

Then

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) \, : \, \|u\|_{L^p(\mathbb{R}^n)} + [u]_{\alpha,p}(\mathbb{R}^n) \right\}.$$

In the case $p = 2$ the abbreviation is $H^\alpha(\mathbb{R}^n) = W^{\alpha,2}(\mathbb{R}^n)$.

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The usual gradient can be obtained by letting $\alpha \to 1$, but only after renormalisation by a factor depending on $1 - \alpha$, see Bourgain & Brezis & Mironescu

Integrodifferential equations

Energy solutions are initially considered in

$$u \in H^{\alpha}(\mathbb{R}^n)\,, \quad f \in L^2(\mathbb{R}^n)\,: \qquad \mathcal{E}_K(u,\eta) = \int_{\mathbb{R}^n} f\eta\, dx\,,$$

and the analogue of the Meyers property would be now

$$u \in W^{\alpha,2+\delta}\,, \qquad\qquad \delta > 0$$

upon considering $f \in L^q(\mathbb{R}^n)$ for some $q > 2$

A first result

Theorem (Bass & Ren, JFA 13)

Define the α-gradient

$$\Gamma(x) := \left(\int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^2}{|x - y|^{n+2\alpha}} \, dy \right)^{1/2}$$

for the solution. Then

$$\Gamma \in L^{2+\delta}$$

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This implies, via a delicate yet by-now classical characterisation of Bessel potential spaces due to Strichartz and Stein, that

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This implies, via a delicate yet by-now classical characterisation of Bessel potential spaces due to Strichartz and Stein, that

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for some $\delta > 0$. However, a stronger result actually holds.

Self-improving property

Theorem (K. & Mingione & Sire - Analysis & PDE 2015)

Let $u \in H^{\alpha}$ be a solution to $\mathcal{E}_K(u, \eta) = \int f\eta$ for all $\eta \in C_c^{\infty}$. If $f \in L^{2+\delta_0}$, then

$$u \in W^{\alpha+\delta, 2+\delta} \qquad \text{for some } \delta > 0 \,.$$

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In particular, Sobolev embedding

$$W^{s,q} \hookrightarrow W^{t,p} \qquad \text{for} \quad q > p \quad \text{and} \quad s - \frac{n}{q} = t - \frac{n}{p}$$

gives

$$u \in W^{\alpha+\delta',2+\delta'} \qquad \text{for some } \delta' \in (0,\delta)$$

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gives

$$u \in W^{\alpha+\delta',2+\delta'} \qquad \text{for some } \delta' \in (0,\delta)$$

As we saw, this theorem has no analog in the local case, where the improvement is only in the integrability scale

$$u \in W^{1,2+\delta}_{\text{loc}}$$

The general case

In the local case the most general equation that can be considered is

$$-\text{div}\,(A(x)Du) = -\text{div}\,(B(x)g) + f\,.$$

This corresponds to take in the right hand side all possible orders of differentiation, that in the integer case means taking orders zero and one.

The general case

We therefore consider equations of the type

$$\mathcal{E}_K(u,\eta) = \mathcal{E}_H(g,\eta) + \int_{\mathbb{R}^n} f\eta\, dx \qquad \forall\ \eta \in C_c^\infty(\mathbb{R}^n)\,,$$

where the kernel $H(\cdot)$ satisfies

$$|H(x,y)| \leq \frac{\Lambda}{|x-y|^{n+2\beta}}$$

A model case is obviously given by

$$(-\triangle)^\alpha u = (-\triangle)^\beta g + f\,,$$

where the analysis can be done via Fourier analysis.

Dimension analysis reveals the optimal assumptions

Let us start with

$$(-\triangle)^{\alpha} u = f \in L^{p}\,.$$

C-Z theory gives

$$u \in W^{2\alpha,p}\,.$$

Then we recall the embedding

$$W^{2\alpha,p} \hookrightarrow W^{\alpha,2}$$

provided the following interpolation scale relation holds:

$$2\alpha - \frac{n}{p} = \alpha - \frac{n}{2}\,.$$

This gives

$$f \in L^{\frac{2n}{n+2\alpha}} \longrightarrow f \in L^{\frac{2n}{n+2\alpha}+\delta_0}$$

Dimension analysis reveals the optimal assumptions

Continue with

$$(-\triangle)^{\alpha} u = (-\triangle)^{\beta} g$$

In the case $\alpha = \beta$ we immediately see

$$g \in W^{\alpha,2}$$

In the case $2\beta \geq \alpha$ then we formally invert the operators

$$\partial^{\alpha} u \approx \triangle^{\beta-\alpha/2} g \approx \partial^{2\beta-\alpha} g \in L^2 \,.$$

Therefore we arrive at

$$g \in W^{2\beta-\alpha,2} \longrightarrow g \in W^{2\beta-\alpha+\delta_0,2} \,.$$

Dimension analysis reveals the optimal assumptions

In the case $2\beta < \alpha$ no differentiability is needed on g
Consider $W^{2\beta-\alpha,2}$ as the dual of $W^{\alpha-2\beta,2}$ and eventually observe the embedding

$$W^{\alpha-2\beta,2} \hookrightarrow L^{\frac{2n}{n-2(\alpha-2\beta)}} .$$

But now

$$\left(L^{\frac{2n}{n-2(\alpha-2\beta)}}\right)' = L^{\frac{2n}{n+2(\alpha-2\beta)}};$$

therefore we conclude with

$$g \in L^{\frac{2n}{n+2(\alpha-2\beta)}} \longrightarrow g \in L^{\frac{2n}{n+2(\alpha-2\beta)}+\delta_0} .$$

The Theorem

Theorem (K. & Mingione & Sire)

Under the optimal assumptions

- $f \in L_{\mathrm{loc}}^{\frac{2n}{2+2\alpha}+\delta_0}$
- $g \in W^{2\beta-\alpha+\delta_0,2}$ *if* $2\beta \geq \alpha$
- $g \in L^{\frac{2n}{n+2(\alpha-2\beta)}+\delta_0}$ *if* $2\beta < \alpha$ *,*

any H^α*-solution* u *to the equation*

$$\mathcal{E}_K(u,\eta) = \mathcal{E}_H(g,\eta) + \langle f,\eta\rangle \qquad \forall \eta \in C_c^\infty$$

is such that

$$u \in W_{\mathrm{loc}}^{\alpha+\delta,2+\delta}(\mathbb{R}^n)$$

Part 4: A fractional approach to Gehring lemma

Caccioppoli inequalities imply higher integrability - local case

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ *such that for every ball* $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{B/2} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ *such that*

$$u \in W^{1,2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

Caccioppoli inequalities imply higher integrability - nonlocal case

Theorem (K. & Mingione & Sire)

Let $u \in H^\alpha(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\int_B \fint_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \lesssim \frac{1}{r^{2\alpha}} \fint_B |u(x)-(u)_B|^2\,dx + \int_{\mathbb{R}^n \setminus B} \frac{|u(y)-(u)_B|}{|x_0-y|^{n+2\alpha}}\,dy \fint_B |u(x)-(u)_B|\,dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{\alpha+\delta, 2+\delta}_{\mathrm{loc}}(\mathbb{R}^n)$$

Key observation

- $u \in W^{1,2}$ means that $|Du|^2$ is integrable w.r.t. a **finite** measure (i.e. the Lebesgue measure)

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- $u \in W^{\alpha,2}$ means that

$$\left[\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right]^2$$

is integrable w.r.t. an **infinite** set function, that is

$$E \to \int_E \frac{dx\,dy}{|x - y|^n}\,.$$

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is integrable w.r.t. an **infinite** set function, that is

$$E \to \int_E \frac{dx\,dy}{|x-y|^n}\,.$$

Therefore there are potentially more regularity properties to exploit in the above fractional difference quotient .

Key idea: Dual pairs

To each u and $\varepsilon < (0, 1 - \alpha)$ we associate a function

$$U(x,y) := \frac{|u(x) - u(y)|}{|x-y|^{\alpha+\varepsilon}}$$

and a **finite** and **doubling** measure

$$\mu(E) := \int_E \frac{dx\, dy}{|x-y|^{n-2\varepsilon}}\,.$$

Note that they are in duality in the sense that

$$u \in W^{\alpha,2} \qquad \Longleftrightarrow \qquad U \in L^2(\mu)\,.$$

Strategy: higher integrability for U w.r.t. μ

- We translate the Caccioppoli inequality for u in a reverse Hölder inequality for U w.r.t. μ
- We prove a version of Gehring lemma for dual pairs (μ, U)
- The higher integrability of U turns into the higher differentiability of u
- All estimates heavily degenerate when $\alpha \to 1$ or $\alpha \to 0$

Higher integrability $\Longrightarrow$ higher differentiability

Assume $U \in L^{2+\delta}_{\text{loc}}$, i.e.,

$$\int_{B\times B} U^{2+\delta}\, d\mu = \int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)\alpha+\varepsilon\delta}}\, dx\, dy < \infty\,.$$

Rewrite it as follows:

$$\int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)[\alpha+\varepsilon\delta/(2+\delta)]}}\, dx\, dy < \infty\,.$$

But this means that

$$u \in W^{\alpha+\varepsilon\delta/(2+\delta),2+\delta}_{\text{loc}}(\mathbb{R}^n)\,,$$

i.e., we have gained also differentiability!

Reverse inequality for dual pairs (μ, U)

Proposition (K. & Mingione & Sire)

For every $\sigma \in (0,1)$*, the Caccioppoli inequality implies for the dual pair* (μ, U) *that*

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \leq \frac{c}{\sigma \varepsilon^{1/q-1/2}} \left(\fint_{2\mathcal{B}} U^q d\mu\right)^{1/q} + \frac{\sigma}{\varepsilon^{1/q-1/2}} \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^q \, d\mu\right)^{1/q}$$

holds, where $\mathcal{B} = B \times B$ *and*

$$q \in \left[\frac{2n}{n+2\alpha}, 2\right) .$$

The Gehring lemma for dual pairs (μ, U)

Theorem (K. & Mingione & Sire)

Assume that for every $\sigma \in (0,1)$ *the pair* (μ, U) *satisfies*

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \leq c(\sigma)\left(\fint_{2\mathcal{B}} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q},$$

where $q \in (1,2)$ *and for every choice of* $\mathcal{B} = B \times B$. *Then*

$$U \in L^{2+\delta}_{\text{loc}} \qquad \text{for some} \quad \delta > 0$$

and

$$\left(\fint_{\mathcal{B}} U^{2+\delta}\, d\mu\right)^{1/(2+\delta)} \lesssim \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q}$$

Part 5: Brief sketch of the Gehring lemma for dual pairs

Step 1: Conclusion via a level set inequality

We reduce to prove the Gehring inequality on level sets, that is

$$\int_{\mathcal{B}\cap\{U>\lambda\}} U^2\, d\mu \lesssim \lambda^{2-q} \int_{\mathcal{B}\cap\{U>\lambda\}} U^q\, d\mu$$

holds provided

$$\lambda_0 := \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^2\, d\mu \right)^{1/2} \lesssim \lambda$$

then standard Cavalieri's principle yields the result.

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Warning! There is a need for a rather technical localization argument, which will be omitted here.

Step 1: Conclusion via a level set inequality

Indeed, denoting $U_m = \min(U, m)$, $m \gg \lambda_0$, and $\nu = U^2 d\mu$, we have

$$
\begin{aligned}
\int U_m^\delta \, d\nu &= \delta \int_0^m \lambda^{\delta-1} \nu(\{U > \lambda\}) \, d\lambda \\
&\leq \lambda_0^\delta \int U^2 \, d\mu + \delta \int_{\lambda_0}^m \lambda^{\delta-1} \int_{\{U>\lambda\}} U^2 \, d\mu \, d\lambda \\
&\leq \lambda_0^\delta \int U^2 \, d\mu + c\delta \int_{\lambda_0}^m \lambda^{\delta+1-q} \int_{\{U>\lambda\}} U^q \, d\mu \, d\lambda \\
&\leq \lambda_0^\delta \int U^2 \, d\mu + \frac{c\delta}{\delta + 2 - q} \int U_m^{\delta+2-q} U^q \, d\mu \\
&\leq \lambda_0^\delta \int U^2 \, d\mu + \frac{c\delta}{2 - q} \int U_m^\delta \, d\nu
\end{aligned}
$$

Step 1: Conclusion via a level set inequality

By choosing δ small enough to get

$$\frac{c\delta}{2-q} \leq \frac{1}{2}$$

we obtain, after reabsorption and sending $m \to \infty$,

$$\int U^{2+\delta}\, d\mu \leq 2\lambda_0^{\delta} \int U^2\, d\mu \leq c\lambda_0^{2+\delta}\,.$$

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Thus the goal is to obtain the level set inequality

$$\int_{\mathcal{B}\cap\{U>\lambda\}} U^2\, d\mu \lesssim \lambda^{2-q} \int_{\mathcal{B}\cap\{U>\lambda\}} U^q d\mu\,.$$

Step 2: Global Calderón-Zygmund covering

We will apply C-Z decomposition at level $M\lambda$ in $\mathbb{R}^{\mathbf{2n}}$ for $M \gg 1$ to get a disjoint family of product of dyadic cubes $\{\mathcal{Q}_i\}_i$, where $\mathcal{Q} = Q_1 \times Q_2$ and Q_1 and Q_2 are dyadic cubes in $\mathbb{R}^n$ with same size, such that

$$\left(\fint_{\mathcal{Q}_i} U^2 \, d\mu \right)^{1/2} \approx M\lambda$$

and

$$U \leq M\lambda \qquad \text{outside } \bigcup_i \mathcal{Q}_i$$

We let

$$\mathcal{U}_\lambda := \{\mathcal{Q}_i\}$$

We denote by $k(\mathcal{Q})$ the generation of $\mathcal{Q}$, i.e., $2^{-nk(\mathcal{Q})} = |Q_1|$.

Step 2: Classification of the cubes

We call "off-diagonal" cubes $\mathcal{Q} = Q_1 \times Q_2$ those whose distance from the diagonal is larger than their sidelength:

$$2^{-k(\mathcal{Q})+3} \leq \mathrm{dist}(\mathcal{Q}) := \mathrm{dist}(Q_1, Q_2)\,.$$

The nearly diagonal cubes are then collected to

$$\mathcal{U}_\lambda^d := \left\{ \mathcal{Q} = Q_1 \times Q_2 \in \mathcal{U}_\lambda \ : \ \mathrm{dist}(Q_1, Q_2) < 2^{3-k(\mathcal{Q})} \right\}$$

and we set

$$\mathcal{U}_\lambda^{nd} = \{ \mathcal{Q} \in \mathcal{U}_\lambda \ : \ \mathcal{Q} \notin \mathcal{U}_\lambda^d \}\,.$$

Step 2: Splitting of the analysis

Idea:

- The diagonal cubes in $\mathcal{U}_\lambda^d$ can be treated with the aid of an auxiliary diagonal cover. To treat these we will use the assumed diagonal reverse type Hölder inequality

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \le c(\sigma)\left(\fint_{\mathcal{B}} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q} .$$

Step 2: Splitting of the analysis

Idea:

- The diagonal cubes in $\mathcal{U}_\lambda^d$ can be treated with the aid of an auxiliary diagonal cover. To treat these we will use the assumed diagonal reverse type Hölder inequality

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \leq c(\sigma)\left(\fint_{\mathcal{B}} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q}.$$

- For the non-diagonal cubes in $\mathcal{U}_\lambda^{nd}$ the fractional Poincaré inequality implies *automatically* a reverse type Hölder's inequality. Unfortunately this comes with error terms, whose analysis lead to heavy combinatorial arguments.

Step 3: Diagonal exit time argument and covering

We consider the quantity

$$\Psi(x,\varrho) := \left(\fint_{\mathcal{B}(x,\varrho)} U^2\, d\mu\right)^{1/2}, \quad \mathcal{B}(x,\varrho) \equiv B(x,\varrho)\times B(x,\varrho)\,.$$

Notice that

$$\Psi(x,1) \lesssim \lambda_0 := \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}_1} U^2\, d\mu\right)^{1/2}.$$

Therefore we can find a radius $\varrho(x)$, such that

$$\Psi(x,\varrho(x)) \equiv \left(\fint_{\mathcal{B}(x,\varrho(x))} U^2\, d\mu\right)^{1/2} \approx \lambda$$

whenever $\lambda \geq \lambda_0$ and

$$x \in \{y \in \mathbb{R}^n : \sup_{\varrho>0} \Psi(y,\varrho) > \lambda\}\,.$$

Step 3: Diagonal exit time argument and covering

By Vitali's covering theorem we can now extract an at most countable covering

$$\bigcup_x \mathcal{B}(x, 2\cdot 10^n \varrho(x)) \subset \bigcup_{j\in J_D} \mathcal{B}(x_j, 10^{n+1}\varrho(x_j))$$

and moreover we have

$$\begin{aligned}\sum_j \int_{\mathcal{B}(x_j,10^{n+1}\varrho(x_j))} U^2\, d\mu &\lesssim \lambda^2 \sum_j \mu(\mathcal{B}(x_j, \varrho(x_j))) \\ &= \lambda^2 \sum_j \mu(\mathcal{B}_j)\,.\end{aligned}$$

Step 4: First summation formula

We recall the diagonal reverse Hölder inequality, that is

$$\left(\fint_{\mathcal{B}_j} U^2 d\mu\right)^{1/2} \leq c(\sigma)\left(\fint_{2\mathcal{B}_j} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}_j} U^q\, d\mu\right)^{1/q}.$$

The exit time argument gives

$$\lambda \lesssim \left(\fint_{\mathcal{B}_j} U^2 d\mu\right)^{1/2}$$

so that

$$\lambda \leq c(\sigma)\left(\fint_{2\mathcal{B}_j} U^q d\mu\right)^{1/q} + c\sigma\lambda\,.$$

Step 4: First summation formula

We have, for small enough universal σ, that

$$\lambda \lesssim \left(\fint_{2\mathcal{B}_j} U^q \, d\mu \right)^{1/q}$$

and hence

$$\mu(\mathcal{B}_j) \lesssim \frac{1}{\lambda^q} \int_{2\mathcal{B}_j} U^q \, d\mu \, .$$

Adjusting the constants properly

$$\mu(\mathcal{B}_j) \lesssim \frac{1}{\lambda^q} \int_{2\mathcal{B}_j \cap \{U > \lambda\}} U^q \, d\mu \, ,$$

summation yields

$$\lambda^2 \sum_j \mu(\mathcal{B}_j) \lesssim \lambda^{2-q} \int_{\{U > \lambda\}} U^q \, d\mu \, .$$

Step 4: First summation formula

Therefore, we have

$$\begin{aligned}\int_{\cup_{\mathcal{Q}\in\mathcal{U}_\lambda^d}\mathcal{Q}\cap\{U>M\lambda\}} U^2\,d\mu &\lesssim \sum_j \int_{10^{n+1}\mathcal{B}_j} U^2\,d\mu \\ &\lesssim \lambda^2 \sum_{j\in J_D} \mu(\mathcal{B}_j) \\ &\lesssim \lambda^{2-q} \int_{\{U>\lambda\}} U^q\,d\mu\,,\end{aligned}$$

where we recall that

$$\mathcal{U}_\lambda^d := \left\{\mathcal{Q} = Q_1 \times Q_2 \in \mathcal{U}_\lambda \;:\; \mathrm{dist}(Q_1, Q_2) < 2^{3-k(\mathcal{Q})}\right\}.$$

Step 5: Off-diagonal cubes I

What happens for the off-diagonal cubes?

Step 5: Off-diagonal cubes I

What happens for the off-diagonal cubes? By the fractional Poincaré inequality we get

$$\left(\fint_{\mathcal{Q}} U^2\,d\mu\right)^{1/2} \lesssim \left(\fint_{\mathcal{Q}} U^q\,d\mu\right)^{1/q} + \left(\frac{2^{-k(\mathcal{Q})}}{\operatorname{dist}(\mathcal{Q})}\right)^{\alpha+\varepsilon} \left(\fint_{P_1\mathcal{Q}} U^q\,d\mu\right)^{1/q} + \left(\frac{2^{-k(\mathcal{Q})}}{\operatorname{dist}(\mathcal{Q})}\right)^{\alpha+\varepsilon} \left(\fint_{P_2\mathcal{Q}} U^q\,d\mu\right)^{1/q}$$

for $\frac{2n}{n+2s} \leq q < 2$. We denote here

$$P_1\mathcal{Q} = Q_1 \times Q_1 \qquad P_2\mathcal{Q} = Q_2 \times Q_2$$

for $\mathcal{Q} = Q_1 \times Q_2$.

Step 5: Off-diagonal cubes I

But with a priori nasty diagonal correction terms:

$$\left(\fint_{\mathcal{Q}} U^2 \, d\mu\right)^{1/2} \lesssim \left(\fint_{\mathcal{Q}} U^q \, d\mu\right)^{1/q} + \left(\frac{2^{-k(\mathcal{Q})}}{\operatorname{dist}(\mathcal{Q})}\right)^{\alpha+\varepsilon} \left(\fint_{P_1\mathcal{Q}} U^q \, d\mu\right)^{1/q} + \left(\frac{2^{-k(\mathcal{Q})}}{\operatorname{dist}(\mathcal{Q})}\right)^{\alpha+\varepsilon} \left(\fint_{P_2\mathcal{Q}} U^q \, d\mu\right)^{1/q}.$$

Step 5: Off-diagonal cubes I

It follows that

$$\begin{aligned}\mu(\mathcal{Q}) \lesssim & \frac{1}{\lambda^q} \int_{\mathcal{Q}\cap\{U>\lambda\}} U^q \, d\mu \\ & + \frac{1}{\lambda^q} \frac{\mu(\mathcal{Q})}{\mu(P_1\mathcal{Q})} \left(\frac{2^{-k(\mathcal{Q})}}{\mathrm{dist}(\mathcal{Q})} \right)^{q(\alpha+\varepsilon)} \int_{P_1\mathcal{Q}\cap\{U>\lambda\}} U^q \, d\mu \\ & + \frac{1}{\lambda^q} \frac{\mu(\mathcal{Q})}{\mu(P_2\mathcal{Q})} \left(\frac{2^{-k(\mathcal{Q})}}{\mathrm{dist}(\mathcal{Q})} \right)^{q(\alpha+\varepsilon)} \int_{P_2\mathcal{Q}\cap\{U>\lambda\}} U^q \, d\mu \,.\end{aligned}$$

Step 6: Further classification of off-diagonal cubes

We further classify

$$\mathcal{M}_\lambda^h := \left\{ \mathcal{Q} \in \mathcal{U}_\lambda^{nd} \ : \ \fint_{P_h \mathcal{Q}} U^q \, d\mu \lesssim \lambda^q \right\}, \quad h \in \{1,2\},$$

and

$$\mathcal{N}_\lambda^h := \left\{ \mathcal{Q} \in \mathcal{U}_\lambda^{nd} \ : \ \fint_{P_h \mathcal{Q}} U^q \, d\mu \gtrsim \lambda^q \right\}, \quad h \in \{1,2\},$$

and finally set

$$\mathcal{M}_\lambda := \mathcal{M}_\lambda^1 \cap \mathcal{M}_\lambda^2 \qquad \text{and} \qquad \mathcal{N}_\lambda := \mathcal{N}_\lambda^1 \cup \mathcal{N}_\lambda^2 .$$

Step 7: Soft summation

We then have a first, “soft” summation formula

$$\sum_{\mathcal{Q}\in\mathcal{M}_\lambda} \mu(\mathcal{Q}) \lesssim \frac{1}{\lambda^q} \int_{\{U>\lambda\}} U^q \, d\mu$$

after adjusting parameters suitably.

Step 7: Soft summation

We then have a first, "soft" summation formula

$$\sum_{\mathcal{Q} \in \mathcal{M}_\lambda} \mu(\mathcal{Q}) \lesssim \frac{1}{\lambda^q} \int_{\{U > \lambda\}} U^q \, d\mu$$

after adjusting parameters suitably.

We now need a similar summation formula for $\mathcal{N}_\lambda := \mathcal{N}_\lambda^1 \cup \mathcal{N}_\lambda^2$.

Step 8: Hard summation

We consider those cubes that are not covered by the diagonal covering

$$\mathcal{N}_{\lambda,nd} := \left\{ \mathcal{Q} \in \mathcal{N}_\lambda \ : \ \mathcal{Q} \not\subset \bigcup_{j \in J_D} 10\mathcal{B}_j \right\}$$

and prove the "hard" summation formula obtained with some heavy covering and combinatorics argument

$$\sum_{\mathcal{Q} \in \mathcal{N}_{\lambda,nd}} \mu(\mathcal{Q}) \lesssim \frac{1}{\lambda^q} \int_{\{U > \lambda\}} U^q \, d\mu$$

Step 8: Sketch of the proof for hard summation

Get a disjoint covering of the projections of the bad cubes $\mathcal{N}_\lambda$ and call it

$$\mathcal{PN}_\lambda = \{\mathcal{H}\}$$

and operate the first decomposition

$$\mathcal{N}^h_{\lambda,nd} = \bigcup_{\mathcal{H}\in\mathcal{PN}_\lambda} \mathcal{N}^h_{\lambda,nd}(\mathcal{H}),$$

with

$$\mathcal{N}^h_{\lambda,nd}(\mathcal{H}) := \{\mathcal{Q} \in \mathcal{N}_{\lambda,nd} \ : \ P_h\mathcal{Q} \subset \mathcal{H}\}, \quad h \in \{1,2\}.$$

Then define

$$[\mathcal{N}^h_{\lambda,nd}(\mathcal{H})]_i := \left\{\mathcal{Q} \in \mathcal{N}^h_{\lambda,nd}(\mathcal{H}) \ : \ k(\mathcal{Q}) = i + k(\mathcal{H})\right\}$$

Step 8: Sketch of the proof for hard summation

so that

$$\mathcal{N}^h_{\lambda,nd}(\mathcal{H}) = \bigcup_i [\mathcal{N}^h_{\lambda,nd}(\mathcal{H})]_i$$

Finally, we set

$$[\mathcal{N}^h_{\lambda,nd}(\mathcal{H})]_{i,j} := \left\{ \mathcal{Q} \in [\mathcal{N}^h_{\lambda,nd}(\mathcal{H})]_i \ : \ 2^{j-k(\mathcal{H})} \leq \mathrm{dist}(\mathcal{Q}) < 2^{j+1-k(\mathcal{H})} \right\}$$

to obtain a disjoint decomposition

$$\mathcal{N}^h_{\lambda,nd} = \bigcup_{\mathcal{H} \in P\mathcal{N}_\lambda} \bigcup_{i,j} [\mathcal{N}^h_{\lambda,nd}(\mathcal{H})]_{i,j}$$

Step 8: Sketch of the proof for hard summation

Combinatorics then gives

$$\sum_{\mathcal{Q}\in\mathcal{N}^h_{\lambda,nd}(\mathcal{H})} \frac{\mu(\mathcal{Q})}{\mu(P_h\mathcal{Q})} \left(\frac{2^{-k(\mathcal{Q})}}{\mathrm{dist}(\mathcal{Q})}\right)^{q(\alpha+\varepsilon)} \int_{P_h\mathcal{Q}\cap\{U>\kappa\lambda\}} U^q \, d\mu \lesssim \int_{\mathcal{H}\cap\{U>\kappa\lambda\}} U^q \, d\mu$$

for every $\mathcal{H} \in P\mathcal{N}_\lambda$
Then the hard summation formula follows by summing up on $\mathcal{H} \in P\mathcal{N}_\lambda$, and recalling that $P\mathcal{N}_\lambda$ is a disjoint covering

Step 9: Final summation

Since now

$$\mathcal{U}_\lambda = \mathcal{U}_\lambda^d \cup \mathcal{U}_\lambda^{nd} \qquad \text{and} \qquad \mathcal{U}_\lambda^{nd} = \mathcal{M}_\lambda \cup \mathcal{N}_{\lambda,d} \cup \mathcal{N}_{\lambda,nd} \,,$$

we have actually obtained a summation estimate for all of these collections and hence conclude with the desired inequality:

$$\begin{aligned}
\int_{\{U>\lambda\}} U^2 \, d\mu &\lesssim \lambda^2 \sum_{j \in J_D} \mu(\mathcal{B}_j) + \lambda^2 \sum_{\mathcal{Q} \in \mathcal{M}_\lambda \cup \mathcal{N}_{\lambda,nd}} \mu(\mathcal{Q}) \\
&\lesssim \lambda^2 \sum_{j \in J_D} \mu(\mathcal{B}_j) + \lambda^{2-q} \int_{\{U>\lambda\}} U^q \, d\mu \\
&\lesssim \lambda^{2-q} \int_{\{U>\lambda\}} U^q \, d\mu
\end{aligned}$$

Non-homogeneous equations

Equations of the type

$$\mathcal{E}_K(u,\eta) = \mathcal{E}_H(g,\eta) + \int_{\mathbb{R}^n} f\eta\,dx \qquad \forall\ \eta \in C_c^\infty(\mathbb{R}^n)\,,$$

where

$$|H(x,y)| \leq \frac{\Lambda}{|x-y|^{n+2\beta}}$$

- $f \in L^{2+\delta_0}_{\mathrm{loc}}$
- $g \in W^{2\beta-\alpha+\delta_0,2}$

Non-homogeneous equations - $\beta = \alpha$

We further define

$$G(x,y) := \frac{|g(x) - g(y)|}{|x-y|^{\alpha+\varepsilon}}, \qquad F(x,y) := |f(x)|$$

so that

$$G \in L^{2+\delta_g}_{\mathrm{loc}}(\mathbb{R}^{2n};\mu)\,, \qquad F \in L^{2+\delta_f}_{\mathrm{loc}}(\mathbb{R}^{2n};\mu)$$

Non-homogeneous equations - $\beta = \alpha$

The reverse Hölder inequality provided by the Caccioppoli inequality becomes

$$\begin{aligned}\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \le & \frac{c}{\sigma \varepsilon^{1/q-1/2}} \left(\fint_{2\mathcal{B}} U^q d\mu\right)^{1/q} \\ & + \frac{\sigma}{\varepsilon^{1/q-1/2}} \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^q \, d\mu\right)^{1/q} \\ & + c \left(\fint_{2\mathcal{B}} F^2 \, d\mu\right)^{1/2} \\ & + \frac{c_b [\mu(\mathcal{B})]^{\theta}}{\varepsilon^{1/p-1/2}} \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} G^2 \, d\mu\right)^{1/2} .\end{aligned}$$

Non-homogeneous equations - $\beta = \alpha$

Then a non-homogeneous implementation of the previous argument gives

$$
\begin{aligned}
\left(\fint_{\mathcal{B}} U^{2+\delta}\, d\mu \right)^{1/(2+\delta)} \leq\; & c \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} U^{2}\, d\mu \right)^{1/2} \\
& + c \varrho_0^{\alpha-\varepsilon} \left(\fint_{2\mathcal{B}} F^{2+\delta_0}\, d\mu \right)^{1/(2+\delta_0)} \\
& + c \left(\fint_{2\mathcal{B}} G^{2(1+\delta_1)}\, d\mu \right)^{1/[p(1+\delta_1)]} \\
& + c \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)} \left(\fint_{2^k\mathcal{B}} G^{p}\, d\mu \right)^{1/p}
\end{aligned}
$$

Closing remarks

- The sketch above gives a very simplified overview of the proof, which is actually featuring many more technicalities

Closing remarks

- The sketch above gives a very simplified overview of the proof, which is actually featuring many more technicalities
- Our approach is based only on energy estimates and does not use the linearity of the equation. Therefore it can be easily adapted to the case of nonlinear integrodifferential equations, with possibly degenerate operators of the type

$$\mathcal{E}_K(u,\eta) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))[\eta(x)-\eta(y)]K(x,y)\,dx\,dy$$

with

$$\frac{\Phi(t)t}{|t|^p} \approx 1 \qquad \forall t \in \mathbb{R}\setminus\{0\}\,, \qquad 1 < p < \infty\,.$$

Thank you for your attention!