Remarks on the fast diffusion and the porous medium equation on classes of Riemannian manifolds

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We consider the following nonlinear diffusion equations:

$$u_t = \Delta(|u|^{m-1}u), \quad u(0) = u_0,$$

posed on a Riemannian manifold M, where Δ is the Riemannian Laplacian. Hereafter, all objects appearing are meant in the "Riemannian" sense, e.g. ∇ is the Riemannian gradient, norms are taken w.r.t. the Riemannian volume and so on.

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We shall first deal with the FDE case. Note that the diffusivity coefficient is

$$\frac{m}{u^{1-m}}.$$

If one interprets u as the concentration of some matter that diffuses in M according to the above equation, mass will spread faster when the concentration is lower.

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In this context one has:

- existence of weak solutions, for suitable classes of data, and further basic properties (L^1 contraction, order preservation, energy inequalities);
- smoothing effects, namely

$$\|u(t)\|_\infty \leq C\frac{\|u_0\|_q^\gamma}{t^\alpha}, \quad q \geq q_0 \tag{1}$$

where α, β are explicit positive constants.

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Theorem

Let M be nonparabolic of dimension $N \geq 3$ and of infinite volume. Let $m \in (m_s, 1)$ where $m_s = (N-2)/(N+2)$. Assume moreover that the above smoothing effect holds with $q > d(1-m)/2$, $q \geq 1$. Then the Sobolev inequality holds on M .

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Much more can be said when further assumptions on M are required. In particular it is known that, e.g. when $\sec \leq -k < 0$, than the L^2 spectrum of $-\Delta$ is bounded away from zero, namely there exists $\Lambda > 0$ such that the *gap inequality*

$$\|\nabla u\|_2 \geq \Lambda^{1/2}\|u\|_2 \quad \forall u \in C_c^\infty.$$

Under this condition, *extinction in finite time* can hold. In fact we have:

Theorem

Let M be a Riemannian manifold of infinite volume and with $d \geq 3$. Assume that the Sobolev inequality holds and, moreover, that the gap condition is true. Let $q > 1$, $q \geq d(1-m)/2$. Then if the initial datum u_0 is in $\mathrm{L}^q(M)$ and $t > s \geq 0$ we have:

$$\|u(t)\|_q^{1-m} \leq \|u(s)\|_q^{1-m} - C(t-s).$$

In particular, choosing $s = 0$ and letting t be proportional to $\|u_0\|_q^{1-m}$ shows that $u(t)$ vanishes identically after a finite time $T(u_0)$, with the bound

$$T(u_0) \leq \mathrm{const.}\, \|u_0\|_q^{1-m}. \tag{2}$$

This is a surprising phenomenon, since it holds for all $m < 1$, in sharp contrast with the Euclidean situation when m must be far from 1 for such a result to hold. In a sense, this behaviour reminds more closely of the one typical of the FDE in *bounded Euclidean domains*. We remind briefly about this case.

This is a surprising phenomenon, since it holds for all $m < 1$, in sharp contrast with the Euclidean situation when m must be far from 1 for such a result to hold. In a sense, this behaviour reminds more closely of the one typical of the FDE in *bounded Euclidean domains*. We remind briefly about this case.
Consider a bounded, regular Euclidean domain $\Omega \subset \mathbb{R}^N$ and the equation

$$\begin{cases} u_t = \Delta\left(u^m\right) & \text{in } \Omega \times (0, T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u = u_0 \geq 0 & \text{on } \Omega \times \{0\}\,, \end{cases}$$

where $m \in (0, 1)$ and $T > 0$ is a final time to be discussed later.

It is easy to check that *separable solutions* for the FDE are of the form

$$\left(1-\frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}\,,$$

where $T>0$ is a free parameter (the extinction time) and V is a positive solution of the Emden-Fowler equation

$$-\Delta V=\frac{1}{(1-m)T}\,V^{\frac{1}{m}}\ \text{ in }\Omega\,,\quad V=0\ \text{ on }\partial\Omega\,.$$

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$$m_s = \frac{(N-2)_+}{N+2} .$$

• This can be easily proved by exploiting the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{1+\frac{1}{m}}(\Omega)$ for $m > m_s$.

• Berryman and Holland (ARMA 80) showed that whenever the Emden-Fowler equation admits a *unique* energy solution V , then

$$\lim_{t\to T^-}\left\|\frac{u(t)}{\left(1-\frac{t}{T}\right)^{\frac{1}{1-m}}}-V^{\frac{1}{m}}\right\|_{H_0^1}=0\,.$$

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• DiBenedetto, Kwong and Vespri (Indiana 91) proved a global Harnack principle. That is, for all $\varepsilon > 0$ there holds

$$c_1 \le \frac{u(x,t)}{\left(1-\frac{t}{T}\right)^{\frac{1}{1-m}} \; V(x)^{\frac{1}{m}}} \le c_2 \quad \forall (x,t) \in \Omega \times [\varepsilon, T)$$

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• Finally, Bonforte, G., Vázquez (JMPA 12) proved *uniform convergence in relative error*:

$$\lim_{t\to T^-}\left\|\frac{u(t)}{\left(1-\frac{t}{T}\right)^{\frac{1}{1-m}}V^{\frac{1}{m}}}-1\right\|_\infty=0\,.$$

One can give (exponential) rates of convergence if m is close to one.

Here we shall consider the following analogous problem on the hyperbolic space $\mathbb{H}^N$ ($N \geq 2$):

$$\begin{cases} u_t = \Delta(u^m) & \text{on } \mathbb{H}^N \times (0,T)\,, \\ u = u_0 & \text{on } \mathbb{H}^N \times \{0\}\,, \end{cases}$$

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where $m \in (0, 1)$.

• The metric ds of the hyperbolic space can be written in *spherical coordinates* about a fixed pole o as follows:

$$ds^2 = dr^2 + (\sinh r)^2 d\Theta^2\,,$$

where $r \in [0, \infty)$ (the geodesic distance from the pole) and $d\Theta^2$ is the usual metric on $\mathbb{S}^{N-1}$. This property makes it a *model manifold*.

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- In such spherical coordinates, the FDE on the hyperbolic space reads

$$u_t = \frac{1}{(\sinh r)^{N-1}} \frac{\partial}{\partial r}\left((\sinh r)^{N-1} \frac{\partial(u^m)}{\partial r}\right) + \frac{\Delta_{\mathbb{S}^{N-1}}(u^m)}{(\sinh r)^2}\,.$$

Mancini and Sandeep (Annali Pisa 08) proved existence of positive solutions in $H^1(\mathbb{H}^N)$ (energy solutions) for the Emden-Fowler equation (let $c > 0$)

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when m is supercritical, namely

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Primarily, they proved that all such solutions are *radial* about a pole o and behave like

$$e^{-(N-1)r}$$

as $r \to \infty$. Moreover, the only degree of freedom is the pole itself, i.e. there is uniqueness up to translations.

Theorem (G., Muratori, PLMS 14)

Let u be the solution of the FDE on $\mathbb{H}^N$ (with m supercritical) corresponding to a non-identically zero initial datum $u_0 \geq 0$, which is radial w.r.t. $o \in \mathbb{H}^N$ and belongs to $L^q(\mathbb{H}^N)$ for some $q > \max\{1, N(1-m)/2\}$.

Theorem (G., Muratori, PLMS 14)

Let u *be the solution of the FDE on* $\mathbb{H}^N$ *(with* m *supercritical) corresponding to a non-identically zero initial datum* $u_0 \geq 0$ *, which is radial w.r.t.* $o \in \mathbb{H}^N$ *and belongs to* $L^q(\mathbb{H}^N)$ *for some* $q > \max\{1, N(1-m)/2\}$ *. Let* $T > 0$ *be the extinction time of* u *and* V *the unique positive, energy solution, with pole* o *, of the Emden-Fowler equation*

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$$-\Delta V = \frac{1}{(1-m)T} V^{\frac{1}{m}} \quad \text{on } \mathbb{H}^N.$$

Then

$$\lim_{t \to T^-} \left\| \frac{u(t)}{\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}} - 1 \right\|_\infty = 0 \,.$$

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For all $k \in \mathbb{N}$ there holds

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As a consequence, for any $\varepsilon > 0$ *, there exist* $t_\varepsilon \in (0, T)$ *and* $r_\varepsilon > 0$ *such that*

$$1-\varepsilon \le \frac{\frac{\partial^k u}{\partial r^k}(r,t)}{\left(1-\frac{t}{T}\right)^{\frac{1}{1-m}} \frac{\mathrm{d}^k V^{\frac{1}{m}}}{\mathrm{d} r^k}(r)} \le 1+\varepsilon \quad \forall (r,t) \in [r_\varepsilon, \infty) \times [t_\varepsilon, T)\,.$$

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The case $m > 1$: slow diffusion

Just a few, very recent results are available. The first one we know is by Vázquez, JMPA 15, which proves the following remarkable statements (more details in his talk!!).

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There is a "fundamental solution" U of the PME on $\mathbb{H}^d$, namely a solution which takes a Dirac delta as initial datum, which for small times has the Euclidean behaviour, whereas for t large one has:

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- $tU(r,t)^{m-1} \sim a(\gamma \log t - r + b)_+$ *for some explicit constants;*
- $\|U(t)\|_\infty \sim C\left(\frac{\log t}{t}\right)^{1/(m-1)}$ *as* $t \to +\infty$.

- The results are shown by means of a change of variables which makes the *radial* hyperbolic laplacian become a *weighted* Euclidean laplacian. This turns the hyperbolic evolution into a well-studied weighted Euclidean PME with critical weight (behaving like s^{-2} at infinity)

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- The above upper bound for U then yields a similar upper bound (smoothing effect) for such a set of solutions.
- It is not immediate to extend the latter result to more general manifolds since several *barrier arguments*, rather specific for the case at hand, are used.
- An *absolute* upper bound in terms of $t^{-1/(m-1)}$ is typical of the case of *bounded Euclidean* domains.

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Theorem (G., Muratori, preprint 15)

Let M be a Cartan-Hadamard manifold supporting the gap inequality $\|\nabla u\|_2 \geq \Lambda^{1/2}\|u\|_2 \ \forall u \in C_c^\infty$. Then, for all solutions of the PME on M with L^1 data, the smoothing effect

$$\|u(t)\|_\infty \leq H \left[\frac{\log\left(2 + t\|u_0\|_1^{m-1}\right)}{t} \right]^{\frac{1}{m-1}} \qquad \forall t > 0$$

holds, where H is a suitable positive constant.

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holds, where H is a suitable positive constant.
In particular, the statement is true on any Cartan-Hadamard manifold on which the condition

$$\sec(x) \leq -k < 0 \quad \forall x.$$

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- One can find manifolds, which are in a sense *intermediate* between $\mathbb{R}^N$ and $\mathbb{H}^N$ on which the gap inequality does not hold, but the bound

$$\|u(t)\|_\infty \leq K \left[\log\left(t\,\|u_0\|_1^{m-1} + 2\right)\right]^{\frac{\delta}{(m-1)}} t^{-\frac{1}{m-1}} \quad \forall t > 0\,.$$

holds for a suitable $\delta > 1$, at least for radial solutions.

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  holds for a suitable $\delta > 1$, at least for radial solutions.
- The absolute bound

$$\|u(t)\|_\infty \le B\,t^{-\frac{1}{m-1}} \quad \forall t > 0 \tag{3}$$

  holds in some *weighted* situation as a consequence of a *sub-Poincaré* inequality, even in cases in which the weighted volume is *infinite*.

I shall now briefly address some recent results on the PME with measure data on Cartan-Hadamard manifolds, which in particular imply existence and uniqueness of “fundamental solutions” to the PME. A similar discussion in the Euclidean, fractional case is given in G., Muratori, Punzo, Calc. Var 15.

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Assume that M is a Cartan-Hadamard manifold of dimension $d \geq 3$ and that $\mathrm{Ric}(x) \geq -C(1+\mathrm{dist}(x,o)^2)$.

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Assume that M is a Cartan-Hadamard manifold of dimension $d \geq 3$ and that $\mathrm{Ric}(x) \geq -C(1 + \mathrm{dist}(x, o)^2)$.

Then there exists a weak solution u to the PME, which takes a given finite Radon measure μ as initial datum, satisfies

$$\mu(M) = \int_M u(x,t)\, d\mathcal{V}(x) \quad \textit{for all } t > 0\,,$$

and the smoothing effect

$$\|u(t)\|_\infty \leq K t^{-\alpha} |\mu|(M) \quad \textit{for all } t > 0\,,$$

for suitable explicit α, β .

The next results concern uniqueness for solutions corresponding to *positive* finite measure, and existence of an initial trace.

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Under the above assumptions on M, let μ be a positive, finite measure on M and u_1, u_2 be two nonnegative weak solutions to the PME taking the same initial datum μ. Then $u_1 = u_2$.

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Under the above assumptions on M, *let* μ *be a positive, finite measure on* M *and* u_1, u_2 *be two nonnegative weak solutions to the PME taking the same initial datum* μ. *Then* $u_1 = u_2$.

Theorem

Under the above assumptions on M, *let* u *be a weak solution of the PME on* $(0, T) \times M$. *Suppose in addition that* $u^m \in L^1((0,T); L^1_{\mathrm{loc}}(M))$ *for some* $T > 0$. *Then there exists a finite Radon measure* μ *which is an initial trace in a suitable weak sense. The assumption on* u^m *can be dropped if* $u \geq 0$ *(of course, conditions are however already present in the definition of solution..).*

The proof of the above results is long and technical, and requires several delicate arguments in potential analysis. In particular, we need mean value inequalities for sub- and superharmonic functions, which *need not hold* in general on the manifolds considered in the generality requested. They do indeed hold if, instead of taking mean values w.r.t. the Riemannian measure, one considers objects like

$$\mathfrak{m}_r[u](x) := \int_{\left\{y \in M:\, G(x,y)=\frac{1}{r}\right\}} u(y) \left|\nabla_y G(x,y)\right| dS(y), \qquad (4)$$

given a l.s.c function u , a topic that we had to adapt to the present manifold setting. Then we are able to use potential theoretic methods as introduced by Pierre and used recently by Vázquez (JEMS 14) in the fractional case.

THANK YOU FOR YOUR ATTENTION!