Ancient solutions to Geometric Flows
Lecture No 1

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Ancient and Eternal Solutions

- We will discuss ancient or eternal solutions to parabolic partial differential equations and in particular to geometric flows.
- These are special solutions which exist for all time

$$-\infty < t < T \qquad \text{where} \;\; T \in (-\infty, +\infty].$$

- They appear as blow up limits near a singularity.
- Understanding ancient and eternal solutions often sheds new insight to the singularity analysis

Topics to be discussed

In this series of lectures we will address:

- the classification of ancient solutions to parabolic partial differential equations, with emphasis to geometric flows: Curve shortening flow, Ricci flow and Yamabe flow.
- methods of constructing new ancient solutions from the gluing of two or more solitons (self-similar solutions).
- background, new techniques and future research directions.

Outline of lectures

- Lecture No 1:
  (i) Introduction to fast-diffusion equations on $\mathbb{R}^n$.
  (ii) The Ricci flow on surfaces (logarithmic fast-diffusion).
  (iii) The Yamabe flow on $\mathbb{R}^n$.
- Lecture No 2:
  (i) Liouville's theorem for the heat equation on manifolds.
  (ii) Semilinear diffusion: singularities and eternal solutions.
  (iii) Ancient solutions to the curve shortening flow.
  (iv) Ancient compact solutions to Mean curvature flow.
- Lecture No 3:
  (i) Classification of ancient solutions to the Ricci flow on S^2.
  (ii) Towers of bubbles ancient solutions to the Yamabe flow.
  (iii) Other ancient compact solutions of the Yamabe flow.
  (iv) Open problems.

Fast Diffusion Equations - Introduction

We will discuss non-linear parabolic equations of fast diffusion. Our model is the fast diffusion equation

$$u_t = \Delta u^m = \operatorname{div}(m\, u^{m-1}\, \nabla u), \qquad m < 1.$$

- It appears in physical applications such as diffusion in plasma, thin liquid film dynamics.
- The case $m = 0$ in dimension $n = 2$ corresponds to the Ricci flow on surfaces.
- The case $m = \frac{n-2}{n+2}$ in dimensions $n \geq 3$ corresponds to the Yamabe flow.
- Since, the diffusivity $D(u) = m\, u^{m-1} \uparrow +\infty$, as $u \downarrow 0$ the equation becomes singular at $u = 0$, resulting to the phenomenon of fast-diffusion.

Scaling and the Barenblatt solution

Scaling: If u solves the fast diffusion equation $u_t = \Delta u^m$, then

$$\tilde{u}(x,t) = \gamma^{-1}\, u(\alpha\, x, \beta\, t), \qquad \gamma = \left(\frac{\beta}{\alpha^2}\right)^{\frac{1}{1-m}}$$

also solves the same equation.

Self-Similar solution: There exists a self-similar solution

$$U(x,t) = t^{-\lambda}\left(C + k\,\frac{|x|^2}{t^{2\mu}}\right)^{-\frac{1}{1-m}}$$

with

$$\lambda^{-1} = \frac{2}{n} - (1-m), \quad \mu = \frac{\lambda}{n}, \quad k = \frac{\lambda\,(1-m)}{2mn}.$$

The above is a solution if $\frac{2}{n} - (1-m) > 0$, i.e. $m > \frac{n-2}{n}$.

The exponent $m = \frac{n-2}{n}$ is **critical**.

The Aronson-Bénilan inequality

Every solution u to the fast-diffusion equation

$$u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1$$

satisfies the differential inequality

$$(*) \quad u_t \geq -\frac{\lambda\, u}{t}$$

with $\lambda^{-1} = \frac{2}{n} - (1-m) > 0$ iff $m > \frac{(n-2)_+}{n}$.
The *pressure* $v := \frac{m}{1-m}\, u^{m-1}$ which evolves by the equation

$$v_t = (1-m)\, v\, \Delta v - |\nabla v|^2$$

satisfies the sharp differential inequality

$$(**) \qquad \Delta v \leq \frac{\lambda}{t}.$$

Remark: The Aronson-Bénilan $(*)$ inequality follows from $(**)$.
The differential inequality $(**)$ becomes an equality when v is the Barenblatt solution.

The Aronson-Bénilan Inequality

The inequality

$$(*) \quad u_t \geq -\frac{\lambda\, u}{t}$$

can be re-written as

$$(**) \quad (\log u)_t \geq -\frac{\lambda}{t}.$$

Integrating $(**)$ on a time interval $[t_1, t_2]$ gives:

$$u(x, t_2) \geq u(x, t_1) \left(\frac{t_2}{t_1}\right)^{-\lambda}, \quad \forall x.$$

Hence, if $u(x, t_1) > 0$, we have $u(x, t_2) > 0$, for all $t_2 > t_1$.

We can actually do better than that and compare values of u at different points x and different times t !

The Li-Yau type Harnack Inequality

Combining the Aronson-Bénilan inequality $\Delta v \leq \frac{\lambda}{t}$ with the evolution equation for v gives the following Li-Yau type differential inequality:

$$-v_t + (1-m)\,\lambda\,\frac{v}{t} \geq |\nabla v|^2.$$

Integrating this inequality on optimal paths we obtain:
Harnack Inequality (Auchmuty-Bao and Hamilton)
If $0 < t_1 < t_2$, then

$$v(x_2, t_2) \leq \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_1, t_1) + \frac{\delta}{4}\,\frac{|x_2 - x_1|^2}{t_2^{\delta} - t_1^{\delta}}\, t_1^{\mu} \right].$$

with $\mu = (1-m)\,\lambda > 0$ and $\delta = \frac{2\lambda}{n}$.

Application: Solutions of $u_t = \Delta u^m$ with $\frac{(n-2)_+}{n} < m < 1$ satisfy the lower bound:

$$u(x,t) \geq c(t)\,(1 + |x|^2)^{-\frac{1}{1-m}}.$$

The other Aronson-Bénilan inequality

A simple scaling argument shows that every solution u to the fast-diffusion equation $u_t = \Delta u^m$, $0 \leq m < 1$ satisfies the differential inequality

$$(*) \qquad u_t \leq \frac{u}{(1-m)\,t}.$$

Integrating $(*)$ on a time interval $[t_1, t_2]$ gives:

$$u(x, t_2) \leq u(x, t_1) \left(\frac{t_2}{t_1}\right)^{\frac{1}{1-m}}, \quad \forall x$$

i.e. the L^∞ norm of a solution doesn't blow up.

Remark: In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, solutions u exhibit a regularizing effect from L^1_{loc} to L^∞_{loc}:

$$\sup_{|x| \leq R} u(x, t) \leq F\left(t, R, \int_{B_{2R}} u_0(x)\, dx\right).$$

The Cauchy problem in the super-critical case

Consider the fast-diffusion equation

$$(*) \qquad u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1.$$

In the super-critical case no growth conditions need to be imposed on the initial data for existence. More precisely, it follows from the results of Herrero and Pierre and Dahlberg and Kenig:

- For any nonnegative continuous weak solution u of $(*)$, there exists a unique locally finite Borel measure μ_0 on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n).$$

- The trace μ_0 determines the solution uniquely.
- For any locally finite Borel measure μ_0 on $\mathbb{R}^n$ there exists a continuous weak solution u of $(*)$ in $S_\infty = \mathbb{R}^n \times (0, \infty)$ with initial trace μ_0.

The sub-critical case $m < (n-2)_+/n$

- In the sub-critical case $m < \frac{(n-2)_+}{n}$ the analogues of the above results do not hold true. In particular, there exists no solution with initial data the Dirac mass.
- Chasseigne & Vazquez: introduced a new class of weak solutions.
- Solutions may extinct in finite time. However, if $u(x_0, t) > 0$ for some $x_0 \in \mathbb{R}^n$, then $u(x, t) > 0$ for all $x \in R^n$. As a result locally bounded solutions are C^∞ smooth.
- Bonforte & Vázquez: Harnack type estimates and positivity.
- The Sobolev critical case of exponents $m = \frac{n-2}{n+2}$ is of particular geometric interest as it corresponds to the Ricci flow for $n = 2$ and the Yamabe flow for $n \geq 3$.

The Ricci flow on $\mathbb{R}^2$ - Logarithmic Fast-diffusion

- In 1982 R. Hamilton introduced the Ricci flow, namely the evolution of a Riemannian metric g_{ij} by

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

where R_{ij} denotes the Ricci curvature of the metric g_{ij}.

- If $g_{ij} = u\, g_{euc}$, where g_{euc} denotes the standard Euclidean metric, then in dimension $n = 2$, we have

$$R_{ij} = \frac{1}{2}\, R\, g_{ij}, \qquad R = -\frac{\Delta \log u}{u}.$$

where R denotes the Scalar curvature of g_{ij}.

- Hence, in dim $n = 2$ the evolution of the metric $g_{ij} = u\, g_{euc}$ under the Ricci flow (RF) is equivalent to the equation:

$$\text{(LFD)} \qquad u_t = \Delta \log u.$$

The Yamabe flow on $\mathbb{R}^n$, $n \geq 3$

- In 1987 R. Hamilton introduced the Yamabe flow, namely the evolution of a Riemannian metric $g_{ij} = v^{\frac{4}{n-2}} g_{euc}$ which is conformally equivalent to the standard Euclidean metric g_{euc} by

$$\text{(YF)} \qquad \frac{\partial g_{ij}}{\partial t} = -R\, g_{ij}$$

where R denotes the Scalar curvature of the metric g.

- Since, the scalar curvature R is given in terms of v by $R = -C_n\, v^{-\frac{n+2}{n-2}} \Delta v$, the function v satisfies the equation

$$(v^{\frac{n+2}{n-2}})_t = \Delta v$$

hence $u := v^{\frac{n+2}{n-2}}$ evolves by the fast-diffusion equation

$$u_t = \Delta u^{\frac{n-2}{n+2}}.$$

- The Yamabe flow was used to obtain a different proof of the Yamabe conjecture.

Logaritmic fast-diffusion

Consider the logarithmic fast-diffusion equation

$$(*) \quad u_t = \Delta \log u, \quad \text{in } \mathbb{R}^2 \times [0, T), \;\; T > 0.$$

- Lions and Toscani: $(*)$ arises as a singular limit for finite velocity Boltzmann kinetic models.
- Kurtz: $(*)$ describes the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation.
- In dimension $n = 2$ equation $(*)$ arises as a model for long Va-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected.

Examples of solutions of $u_t = \Delta \log u$ on $\mathbb{R}^2$

- Contracting spheres: $u(x,t) = \frac{8\lambda(T-t)}{(\lambda+|x|^2)^2}, \ \lambda > 0.$
  They are *ancient solutions* which vanish at time $t = T$ and:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u\,dx = \int_{\mathbb{R}^2} R\,u = -4\pi.$$

- Cigar solution: $u(x,t) = \frac{1}{\lambda\,|x|^2 + e^{4\lambda t}}.$
  They are *eternal complete non-compact* solutions which look like cigars and have infinite area, i.e.

$$\int_{R^2} u\,dx = \infty.$$

- Cusp solution: $u(x,t) = \frac{2t}{|x|^2\,\log^2|x|}, \ |x| > 1.$
  They are *complete non-compact* solutions which look like cusps and have finite area.

The Cauchy problem

Consider the Cauchy problem

$$(*) \quad \begin{cases} u_t = \Delta \log u & \text{in} \quad \mathbb{R}^2 \times [0, T) \\ u(\cdot, 0) = f & \text{on} \quad \mathbb{R}^2 \end{cases}$$

with initial data $f \geq 0$. In 1994, jointly with M. del Pino we obtained the following results:

- If $\int_{\mathbb{R}^2} f\, dx < \infty$, then $\forall \mu \geq 0$, $\exists\, u_\mu$ solution of $(*)$ on $\mathbb{R}^2 \times (0, T_\mu)$ with $T_\mu = \frac{1}{2\pi(2+\mu)} \int_{\mathbb{R}^2} f(x)\, dx$ satisfying

$$\frac{d}{dt} \int_{\mathbb{R}^2} u^\mu(x,t)\, dx = -2\pi\,(2+\mu).$$

- If $\int_{\mathbb{R}^2} f\, dx = \infty$, $\exists\, u$ solution of $(*)$ on $\mathbb{R}^2 \times (0, \infty)$.
- If $\int_{\mathbb{R}^2} f\, dx < \infty$, then every solution vanishes at time $T \leq T_{\max}$, with $T_{\max} = \frac{1}{4\pi} \int_{\mathbb{R}^2} f(x)\, dx$.

Remarks on the Cauchy problem

- The maximal solution ($\mu = 0$) defines complete non-compact metrics on $\mathbb{R}^2$ of finite area that behave as cusps at infinity.
- The intermediate solution u_μ with $\mu = 2$ corresponds to smooth metrics on S^2 evolving by the Ricci flow.
- All other solutions u_μ ($\mu \neq 0, 2$) correspond to metrics on orbifolds evolving by the Ricci flow.
- Estéban, Rodriguez and Vázquez : Radial u_μ, $\mu > 0$ are characterized by the outgoing flux at infinity:

$$\lim_{r \to \infty} r \, (\log u_\mu)_r = -(2 + \mu).$$

- Conclusion: A strong non-uniqueness phenomenon takes place.

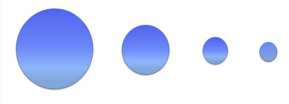
The Ricci flow on S^2 $(\mu = 2)$

- Consider the intermediate solution u of $u_t = \Delta \log u$ with area decay:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,dx = -8\,\pi.$$

- An example is the contracting spheres

$$u_s(x,t) = \frac{8\,(T-t)}{(1+|x|^2)^2}.$$



- It follows that any other such solution satisfies:

$$u(x,t) \sim \frac{C_t}{(1+|x|^2)^2}, \qquad \text{as } |x| \to +\infty.$$

The Ricci flow on S^2

- Any such solution can be lifted on S^2 defining smooth metrics

$$g(\cdot,t) = \bar{u}(\cdot,t)\, g_{S^2}$$

 evolving by the Ricci flow.

- Indeed, if $\Phi : S^2 \setminus \{\psi = \frac{\pi}{2}\} \to \mathbb{R}^2$ denotes the stereographic projection, which maps $\psi = -\pi/2 \in S^2$ to $0 \in \mathbb{R}^2$, then

$$\bar{u}(\cdot,t) := u(\cdot,t) \circ \Phi.$$

- B. Chow & R. Hamilton: Any solution $g(\cdot,t) = \bar{u}\, g_{S}^{2}$ of the Ricci flow on S^2 will converge (after re-scaling) as $t \to T$ to the round sphere solution.

- Equivalently, this describes the vanishing behavior of the solution $u(\cdot,t)$ of $u_t = \Delta \log u$ on $\mathbb{R}^2$.

Vanishing behavior of solutions

If u is a solution of (*) with $\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)dx = -2\pi\,(2+\mu)$, then:

- $\mu = 2$ (Metrics on S^2)
  Y.S Hsu (also B. Chow, Hamilton):

$$u(x,t) \approx \frac{8\lambda\,(T-t)}{(\lambda+|x|^2)^2}, \quad \text{as } t \to T.$$

- $\mu > 2$, $0 < \mu < 2$ (Metrics on Orbifolds).
  Y.S. Hsu: Under radial symmetry, there exist unique constants $\alpha, \beta > 0$, $\alpha + 2\beta = 1$, depending on μ, and a parameter $\gamma > 0$ such that

$$u(x,t) \approx (T-t)^{\alpha}\,\phi_\gamma\Big(\frac{|x|}{(T-t)^{\beta}}\Big), \quad \text{as } t \to T.$$

where ϕ is a solution to the ODE

$$(r\phi'/\phi)'/r + \alpha\phi + \beta r\phi' = 0, \quad \phi_r(0) = 0, \phi(0) = \gamma.$$

The maximal solution

- Consider the maximal solution u of $u_t = \Delta \log u$. Its area decays as:

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t)\, dx = -4\,\pi.$$

  Hence u will vanish at time $T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x)\, dx.$

- Estéban, Rodriguez and Vázquez : If u_0 is compactly supported, then

$$u(x,t) = \frac{2t}{|x|^2 \log^2 |x|}(1 + o(1)), \qquad \forall t < T$$

  however the bound deteriorates as $t \to T$.

- It follows that for all $0 < t < T$, u defines a complete non-compact metric with finite area.

- Problem: Study the singularity formation of the metric $g := u\, g_{euc}$ as t approaches the vanishing time T.

Vanishing behavior of the maximal solution

- D., del Pino & N. Sesum established:

  i. On the outer region: $(T - t) \log |x| > T$, we have

  $$u(x,t) \approx \frac{2T}{|x|^2 \log^2 |x|}, \quad \text{as } t \to T^-.$$

  ii. On the inner region: $(T - t) \log |x| < T$, u has the self-similar profile:

  $$u(x,t) \approx (T-t)^2 \, e^{-\frac{2T}{T-t}} \, \phi(e^{-\frac{T}{T-t}} |x|)$$

  with $\phi(r) = \frac{2T^{-1}}{r^2+b}$ being the cigar metric.

- Our work is based on formal asymptotics previously derived by J.R. King in the rotationally symmetric case.

Our Geometric Estimates

- Our proof of the vanishing behavior of the maximal solution u is based on geometric estimates on the maximum curvature $R_{\max}$ and the width w of the evolving metric $g_{ij} := u\, g_{_{euc}}$ near the vanishing time T of u.
- Definition of the width: Consider families $\mathcal{F}$ of curves Γ homotoping a circle at infinity to a point. Define the width of the metric $ds^2 = u\,(dx^2 + dy^2)$ on the plane

$$w = \inf_{F} \sup_{\Gamma \in \mathcal{F}} L(\Gamma)$$

  where $L(\Gamma) = \int_\Gamma \sqrt{u}\, d\sigma$.
- Note: When $u = u(r)$ is rotationally symmetric then $w = \max_{0 \le r < \infty} 2\pi r \sqrt{u(r)}$.

Our Geometric Estimates

- Theorem [D. & Hamilton] There exist constants $\gamma > 0$ and $C < \infty$ such that

$$\gamma\,(T - t) \le w \le C\,(T - t)$$

and

$$\frac{\gamma}{(T - t)^2} \le R_{\max} \le \frac{C}{(T - t)^2}$$

on $0 < t \le T$.

- It follows that the singularity is of Type II. This is the first type II singularity which was shown to exists in the Ricci flow in any dimension.
- In the rotationally symmetric case $u = u(r, t)$:

$$\gamma\,(T - t) \le \max_{0 \le r < \infty} r\,\sqrt{u}(r, t) \le C\,(T - t)$$

Inner region convergence

- We first set $\bar{u}(x,\tau) = \tau^2\, u(x,t),\ \tau = \frac{1}{T-t}$.
- For $\tau_k \to \infty$ set

$$\bar{u}_k(y,\tau) = \alpha_k\, \bar{u}(\sqrt{\alpha_k}\, y, \tau + \tau_k)$$

where $\alpha_k = [\bar{u}(0,\tau_k)]^{-1}$ so that $\bar{u}_k(0,0) = 1$.

- It follows that $\bar{u}_k$ satisfies the equation

$$\bar{u}_\tau = \Delta \log \bar{u} + \frac{2\bar{u}}{\tau + \tau_k}, \quad -\tau_k + \frac{1}{T} < \tau < \infty.$$

- Set $\bar{R}_k = -\Delta \log \bar{u}_k / \bar{u}_k$. Our maximum curvature a-priori estimates imply that

$$-\frac{C}{(\tau + \tau_k)^2} \le \bar{R}_k(y,\tau) \le C.$$

Inner region convergence

- Theorem 1: (D. & Sesum): Passing to a subsequence, $\tau_k \to \infty$, $\bar{u}_k(y,\tau) = \alpha_k\, \bar{u}(\sqrt{\alpha_k}\, y, \tau + \tau_k)$ converges, uniformly on compact subsets to a complete eternal solution U of

$$(\star) \qquad U_\tau = \Delta \log U, \qquad \text{on } \mathbb{R}^2 \times (-\infty, +\infty)$$

of bounded width and curvature.

- Theorem 1:(Classification of Eternal Solutions) (D. & Sesum): The only eternal solutions of $(\star)$ with $0 < R(\cdot, t) \le C(t)$ are the soliton (self-similar) solutions of the form

$$U(y,t) = \frac{1}{\lambda |y - y_0|^2 + e^{4\tau}}, \quad \lambda > 0.$$

- Conclusion: We obtain the inner region asymptotics

$$u(x, t_k) \approx \frac{(T - t_k)^2}{\lambda |x|^2 + \alpha_k}, \quad |x| \le M\sqrt{\alpha_k}$$

with $\alpha_k = (T - t_k)^{-2}\, u(0, t_k)$.

Vanishing behavior of the maximal solution

- Moreover limits are unique (among sequences $\tau_k \to +\infty$).
- We conclude that on the inner region: $(T-t)\log|x| < T$, u has the self-similar profile:

$$u(x,t) \approx (T-t)^2\, e^{-\frac{2T}{T-t}}\, \phi(e^{-\frac{T}{T-t}}|x|)$$

with $\phi(r) = \frac{2T^{-1}}{r^2+b}$ being the cigar metric.

- You then show that on outer region: $(T-t)\log|x| > T$, we have

$$u(x,t) \approx \frac{2T}{|x|^2 \log^2|x|}, \quad \text{as } t \to T^-.$$

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