Fractional nonlinear degenerate diffusion equations on bounded domains

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References:

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[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Sys. (2015).*

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Outline of the talk

- The setup of the problem
- Existence and uniqueness
- First pointwise estimates
- Upper Estimates
- Weighted Estimates
- Harnack Inequalities
- Asymptotic behaviour of nonnegative solutions

The setup of the problem

- **Introduction**
- **Assumption on the operator $\mathcal{L}$**
- **Assumption on the nonlinearity F**
- **Mild Solutions and Monotonicity Estimates**
- **Assumption on the inverse operator $\mathcal{L}^{-1}$**
- **Examples of operators**
- **The "dual" formulation of the problem**
- **Existence and uniqueness of weak dual solutions**

Introduction

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\,F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e , then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
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Assumption on the nonlinearity F

Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous and non-decreasing function, with $F(0)=0$. Moreover, it satisfies the condition:

(N1) $F\in C^1(\mathbb{R}\setminus\{0\})$ and $F/F'\in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0,\mu_1>0$ s.t.

$$\frac{1}{m_1}=1-\mu_1\le\left(\frac{F}{F'}\right)'\le 1-\mu_0=\frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r)=F'(r)=0$ or $r=0$.

The main example will be

$$F(u)=|u|^{m-1}u,\qquad \text{with } m>1\,,\text{ and}\qquad \mu_0=\mu_1=\frac{m-1}{m}<1,.$$

which corresponds to the nonlocal porous medium equation studied in $[BV1]$.
A simple variant is the combination of two powers:

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Existence of Mild Solutions and Monotonicity Estimates

Theorem (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all nonnegative $u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

$$(1) \qquad t \mapsto t^{\frac{1}{\mu_0}}\, F(u(t,x)) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}\,, \qquad \text{which implies} \qquad \partial_t u \ge -\frac{1-\mu_0}{\mu_0}\,\frac{u}{t}$$

or equivalently, that the function

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Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

(K1)
$$0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

(K2)
$$c_{0,\Omega}\delta_\gamma(x)\,\delta_\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right)$$

where

$$\delta_\gamma(x) := \mathrm{dist}(x, \partial\Omega)^\gamma\,.$$

When the operator $\mathcal{L}$ has a first nonnegative eigenfunction Φ_1, we can rewrite (K2) as

(K3)
$$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1\right)$$

Indeed, (K2) implies that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta_\gamma$.

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Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,
$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{}\,(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$
This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
if $0 < s < 1$, we can use the representation
$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}}\, dz,$$
where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:
$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}.$$

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The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x) \phi_j(x) \, \mathrm{d}x , \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1 .$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0 , \qquad \text{in } (0,\infty) \times \partial\Omega .$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) , \quad \text{with } \gamma = 1$$

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The Restricted Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

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We can also treat more general operators of SFL and RFL type:
Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) ,$$

with bounded measurable coefficients, which are uniformly elliptic. The uniform ellipticity allows to build a self-adjoint operator on $L^2(\Omega)$ with discrete spectrum (λ_k, ϕ_k). Using the spectral theorem, we can construct the spectral power of such operator, defined as follows:

$$\mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x) \qquad \text{where} \qquad \hat{f}_k = \int_\Omega f(x) \phi_k(x) \, dx .$$

Such operators enjoy (K3) estimates with $\gamma = 1$

(K3) $\quad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{K}(x,y) \le \dfrac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\dfrac{\phi_1(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\dfrac{\phi_1(y)}{|x-y|^\gamma} \wedge 1 \right)$

We can treat the class of intrinsically ultra-contractive operators introduced by Davies and Simon.

We can also treat more general operators of SFL and RFL type:

Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form:

$$A = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j)\,,$$

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We can treat the class of intrinsically ultra-contractive operators introduced by Davies and Simon.

Fractional operators with general kernels. Consider integral operators of the following form

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y)\right) \frac{K(x,y)}{|x-y|^{N+2s}}\, dy\,.$$

where K is a measurable symmetric function bounded between two positive constants, satisfying

$$\Big|K(x,y) - K(x,x)\Big|\, \chi_{|x-y|<1} \le c|x-y|^{\sigma}\,, \qquad \text{with } 0 < s < \sigma \le 1\,,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right)\,, \quad \text{with } \gamma = s$$

Censored fractional Laplacians and operators with general kernels.
Introduced by Bogdan et al. in 2003.

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$

where $a(x,y)$ is a measurable symmetric function bounded between two positive constants, satisfying some further assumptions; a sufficient assumption is $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption (K_4):

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This bounds has been proven by Chen, Kim and Song (2010).
Remarks.

- This is a third model of Dirichlet fractional Laplacian (for $a(x,y) = C_{N,s}$). This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, when $s \in (0, 1/2]$ this corresponds to "Neumann" boundary conditions.

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Sums of two fractional operators. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption $(K4)$ with $\gamma = s$. The limit case $s = 1$ and $\sigma \in (0,1)$ satisfies the strongest assumption $(K4)$ with $\gamma = s = 1$.

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Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

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$$A_s f(x) = \text{P.V.}\int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x)\cdot y\chi_{|y|\le 1}\right)\chi_{|y|\le 1}\mathrm{d}\nu(y)\,.$$

where the measure ν on $\mathbb{R}^N\setminus\{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1\vee|x|^2\,\mathrm{d}\nu(y) < \infty\,.$$

More precisely $\mathrm{d}\nu(y) = j(y)\,\mathrm{d}y$ with $j:(0,\infty)\to(0,\infty)$ is given by

$$j(r) := \int_0^\infty \frac{e^{r^2/(4t)}}{(4\pi t)^{N/2}}\mathrm{d}\mu(r) \qquad \text{with} \qquad \int_0^\infty (1\vee t)\mathrm{d}\mu(t) < \infty\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) in the form with $s = 1$ and $\gamma = 1$.

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Schrödinger equations for non-symmetric diffusions. In the case

$$\mathcal{L} = A + \mu \cdot \nabla + \nu \,,$$

where A is uniformly elliptic both in divergence and non-divergence form:

$$A_1 = \frac{1}{2}\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) \qquad \text{or} \qquad A_2 = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial_{ij} \,,$$

We assume C^1 coefficient a_{ij}, uniformly elliptic.
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The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -F(u)\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{K}(x,y)u(t,y)\,\mathrm{d}y\,.$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

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Weak Dual Solutions

A function u is a *weak dual* solution to the Dirichlet Problem for $\partial_t + \mathcal{L}^{-1}F(u) = 0$ in $Q_T = (0,T)\times\Omega$ if:

- $u \in C((0,T) : L^1_{\delta_\gamma}(\Omega))\,,\ F(u) \in L^1\left((0,T) : L^1_{\delta_\gamma}(\Omega)\right)$;
- The following identity holds for every $\psi/\delta_\gamma \in C^1_c((0,T) : L^\infty(\Omega))$:

$$\int_0^T\int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,dx\,dt - \int_0^T\int_\Omega F(u)\,\psi\,dx\,dt = 0.$$

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** is a weak dual solution to Equation $\partial_t + \mathcal{L}^{-1}F(u) = 0$ such that moreover

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We will use a special class of weak dual solutions:

The class $\mathcal{S}_p$ of weak dual solutions

We consider a class $\mathcal{S}_p$ of nonnegative weak dual solutions u to the (HDP) with initial data in $0 \le u_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$, such that:

(i) the map $u_0 \mapsto u(t)$ is "almost" order preserving in $\mathrm{L}^1_{\delta_\gamma}(\Omega)$, namely $\exists\, C > 0$ s.t.

$$\|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \le C\,\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \qquad \text{for all } 0 \le t_0 \le t.$$

(ii) for all $t > 0$ we have $u(t) \in L^p(\Omega)$ for some $p \ge 1$.

We prove that the mild solutions of Crandall and Pierre fall into this class:

Proposition. Semigroup solutions with $u_0 \in \mathrm{L}^p$ are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in \mathrm{L}^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

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Reminder about Mild solutions and their properties

Mild solutions, or semigroup solutions have been obtained by Benilan, Crandall and Pierre via Crandall-Liggett type theorems; the underlying idea is the use of an Implicit Time Discretization (ITD) method: consider the following partition of $[0,T]$

$$t_k = \frac{k}{n}T\,, \quad \text{for any } 0 \le k \le n\,, \quad \text{with } t_0 = 0\,, \, t_n = T\,, \text{ and } \quad h = t_{k+1} - t_k = \frac{T}{n}\,.$$

For any $t \in (0,T)$, the (unique) semigroup solution $u(t,\cdot)$ is obtained as the limit in $L^1(\Omega)$ of the solutions $u_{k+1}(\cdot) = u(t_{k+1},\cdot)$ which solve the following elliptic equation (u_k is the datum, is given by the previous iterative step)

$$h\mathcal{L}F(u_{k+1}) + u_{k+1} = u_k \qquad \text{or equivalently} \qquad \frac{u_{k+1} - u_k}{h} = -\mathcal{L}F(u_{k+1})\,.$$

Usually such solutions are difficult to treat since a priori they are merely very weak solutions. We can prove the following result:

Proposition. Semigroup solutions with $u_0 \in L^p$ are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

The "dual" formulation of the problem

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Existence and uniqueness of weak dual solutions

Theorem. Existence of weak dual solutions (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in L^1_{\delta_\gamma}(\Omega)$ there exists a minimal weak dual solution to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : L^1_{\delta_\gamma}(\Omega))$. Mild solutions are weak dual solutions and the set of such solutions has the properties needed to form a class of type $\mathcal{S}$.

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First Pointwise Estimates

Theorem. (M.B. and J. L. Vázquez)

Let $u \geq 0$ be a solution in the class $\mathcal{S}_p$ of very weak solutions to Problem (CDP) with $p > N/2s$. Then,

$$\int_\Omega u(t_1,x)\mathbb{K}(x,x_0)\,\mathrm{d}x \leq \int_\Omega u(t_0,x)\mathbb{K}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \geq t_0 \geq 0\,.$$

Moreover, for almost every $0 \leq t_0 \leq t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0)\, F(u(t_0,x_0)) \leq \int_\Omega \left[u(t_0,x) - u(t_1,x)\right] \mathbb{K}(x,x_0)\,\mathrm{d}x$$
$$\leq (m_0 - 1)\frac{t_1^{\frac{1}{\mu_0}}}{t_0^{\frac{1-\mu_0}{\mu_0}}}\, F(u(t_1,x_0))\,.$$

Remark. As a consequence of the above inequality and Hölder inequality, we have that $\mathcal{S}_p = \mathcal{S}_\infty$, when $p > N/2s$.

Sketch of the proof of the First Pointwise Estimates

We would like to take as test function

$$\psi(t,x)=\psi_1(t)\psi_2(x)=\chi_{[t_0,t_1]}(t)\,\mathbb{K}(x_0,x)\,,$$

This is not admissible in the Definition of Weak Dual solutions.
Plugging such test function in the definition of weak dual solution gives the formula

$$\int_\Omega u(t_0,x)\mathbb{K}(x_0,x)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{K}(x_0,x)\,\mathrm{d}x=\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\,.$$

This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates, recalling that $\frac{1}{\mu_0}=\frac{m_0}{m_0-1}$

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{is nondecreasing in } t>0 \text{ for a.e. } x\in\Omega\,.$$

to get for all $0\le t_0\le t_1$:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m_0}{m_0-1}}(t_1-t_0)F(u(t_0,x_0))\le\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\le\frac{m_0-1}{t_0^{\frac{1}{m_0-1}}}t_1^{\frac{m_0}{m_0-1}}F(u(t_1,x_0)).\ \square$$

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Upper Estimates

- **Absolute upper bounds**
  - **Absolute bounds**
  - **The power case. Absolute bounds and boundary behaviour**
- **Smoothing Effects**
  - L^1-L^∞ **Smoothing Effects**
  - $L^1_{\delta_\gamma}$-L^∞ **Smoothing Effects**
  - **Backward in time Smoothing effects**

Absolute upper bounds

Theorem. (Absolute upper estimate) (M.B. & J. L. Vázquez)

Let u be a nonnegative weak dual solution corresponding to $u_0 \in L^1_{\delta_\gamma}(\Omega)$. Then, there exists universal constants $K_0, K_1, K_2 > 0$ such that the following estimates hold true for all $t > 0$:

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \le F^*\left(\frac{K_1}{t}\right) .$$

Moreover, there exists a time $\tau_1(u_0)$ with $0 \le \tau_1(u_0) \le K_0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \le 1 \qquad \text{for all } t \ge \tau_1 ,$$

so that

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_2}{t^{\frac{1}{m_i-1}}} \qquad \text{with} \qquad \begin{cases} i = 0 & \text{if } t \le K_0 \\ i = 1 & \text{if } t \ge K_0 \end{cases}$$

The Legendre transform of F is defined as a function $F^* : \mathbb{R} \to \mathbb{R}$ with

$$F^*(z) = \sup_{r\in\mathbb{R}} \left(zr - F(r)\right) = z\,(F')^{-1}(z) - F\left((F')^{-1}(z)\right) = F'(r)\,r + F(r)\,,$$

with the choice $r = (F')^{-1}(z)$.

The power case. Absolute bounds and boundary behaviour

Theorem. (Absolute upper estimate and boundary behaviour) (M.B. & J. L. Vázquez)

Let u be a weak dual solution. Then, there exists universal constants $K_1, K_2 > 0$ such that the following estimates hold true: (K1) assumption implies:

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}, \qquad \text{for all } t > 0.$$

Moreover, (K2) assumption implies:

$$u(t,x) \le K_2 \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and } x \in \Omega.$$

Remark.

- This is a very strong regularization *independent* of the initial datum u_0.
- The boundary estimates are sharp, since we will obtain lower bounds with matching powers.
- This bounds give a sharp time decay for the solution, but only for large times, say $t \ge 1$. For small times we will obtain a better time decay when $0 < t < 1$, in the form of smoothing effects

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The power case. Absolute bounds and boundary behaviour

Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}}(t_1-t_0)\,u^m(t_0,x_0)\le\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,\mathrm{d}x-\int_\Omega u(t_1,x)G_\Omega(x,x_0)\,\mathrm{d}x\,.$$

for any $u\in\mathcal{S}_p$, all $0\le t_0\le t_1$ and all $x_0\in\Omega$. Choose $t_1=2t_0$ to get

$$(*)\qquad \boxed{u^m(t_0,x_0)\le\frac{2^{\frac{m}{m-1}}}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,\mathrm{d}x\,.}$$

Recall that $u\in\mathcal{S}_p$ with $p>N/(2s)$, means $u(t)\in\mathrm{L}^p(\Omega)$ for all $t>0$, so that:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,\mathrm{d}x\le\frac{c_0}{t_0}\|u(t_0)\|_{\mathrm{L}^p(\Omega)}\,\|G_\Omega(\cdot,x_0)\|_{\mathrm{L}^q(\Omega)}<+\infty$$

since $G_\Omega(\cdot,x_0)\in\mathrm{L}^q(\Omega)$ for all $0<q<N/(N-2s)$, so that $u(t_0)\in\mathrm{L}^\infty(\Omega)$ for all $t_0>0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,\mathrm{d}x\le\|u(t_0)\|_{\mathrm{L}^\infty(\Omega)}\frac{c_0}{t_0}\int_\Omega G_\Omega(x,x_0)\,\mathrm{d}x\,.$$

Taking the supremum over $x_0\in\Omega$ of both sides, we get:

$$\boxed{\|u(t_0)\|^{m-1}_{\mathrm{L}^\infty(\Omega)}\le\frac{c_0}{t_0}\sup_{x_0\in\Omega}\int_\Omega G_\Omega(x,x_0)\,\mathrm{d}x\le\frac{K_1^{m-1}}{t_0}}\qquad\square$$

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Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

Theorem. (Weighted $L^1 - L^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \le K_6 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{L^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{L^1(\Omega)}$.
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Corollary.

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

WEIGHTED $L^1 - L^\infty$ SMOOTHING EFFECT FOR SMALL TIMES:

$$\|u(t)\|_{L^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t \le \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}.$$

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$$\|u(t)\|_{L^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(d+\gamma)\vartheta_{1,\gamma}}}, \qquad \text{for all } t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}.$$

Moreover, the condition $t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{L^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

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Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.

Corollary. (Backward weighted $L^1 - L^\infty$ smoothing effects)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

For small times: for all $t, h > 0$ and for all $0 \le t \le \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{L^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{0,\gamma}}{m_0-1}} \frac{\|u(t+h)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}}.$$

For large times: for all $t, h > 0$ and for all $t \ge \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{L^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m_1-1}} \frac{\|u(t+h)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(N+\gamma)\vartheta_{1,\gamma}}}.$$

Moreover, the condition $t \ge \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{L^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{L^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{L^1(\Omega)}$.

Corollary. (Backward weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effects)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

For small times: for all $t, h > 0$ and for all $0 \le t \le \|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{0,\gamma}}{m_0-1}} \frac{\|u(t+h)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}}.$$

For large times: for all $t, h > 0$ and for all $t \ge \|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m_1-1}} \frac{\|u(t+h)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(N+\gamma)\vartheta_{1,\gamma}}}.$$

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Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.

Weighted L^1 estimates

- L^1 **estimates with** Φ_1 **weight**
- L^1 **estimates with** δ_γ **weight**

Weighted $\mathrm{L}^1_{\Phi_1}$ estimates

To simplify the presentation, we first treat the case in which $\mathcal{L}$ has a first nonnegative eigenfunction Φ_1; we recall that $\Phi_1 \asymp \delta_\gamma$ on $\overline{\Omega}$, by hyp. (K2).

Proposition. (Weighted L^1 estimates for ordered solutions)

Let $u \geq v$ be two ordered weak dual solutions to the Problem (CDP) corresponding to the initial data $0 \leq u_0, v_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then for all $t_1 \geq t_0 \geq 0$

$$\int_\Omega \big[u(t_1,x) - v(t_1,x)\big]\Phi_1(x)\,\mathrm{d}x \leq \int_\Omega \big[u(t_0,x) - v(t_0,x)\big]\Phi_1(x)\,\mathrm{d}x\,.$$

Moreover, for all $0 \leq \tau_0 \leq \tau, t < +\infty$
such that either $t,\tau \leq K_0$ or $\tau_0 \geq K_0$, we have

$$\int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\Phi_1(x)\,\mathrm{d}x \leq \int_\Omega \big[u(t,x) - v(t,x)\big]\Phi_1(x)\,\mathrm{d}x$$
$$+ K_8\,\|u(\tau_0)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s(m_i-1)\vartheta_{i,\gamma}}\,|t-\tau|^{2s\vartheta_{i,\gamma}} \int_\Omega \big[u(\tau_0,x) - v(\tau_0,x)\big]\Phi_1\,\mathrm{d}x$$

where $i = 0$ if $t,\tau \leq \|u(\tau_0)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{\frac{2s}{d+\gamma}}$ and $i = 1$ if $t,\tau \geq \|u(\tau_0)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{\frac{2s}{d+\gamma}}$.

Weighted $\mathrm{L}^1_{\delta_\gamma}$ estimates

Taking any nonnegative function $\psi \in \mathrm{L}^\infty(\Omega)$, using assumption (K2) gives

$$\mathcal{L}^{-1}\psi(x) \asymp \delta_\gamma(x) \qquad \text{for a.e. } x \in \Omega\,.$$

This will imply the monotonicity of some L^1-weighted norm.

Proposition. (Weighted L^1 estimates for ordered solutions)

Let $u \geq v$ be two ordered weak dual solutions to the Problem (CDP) corresponding to $0 \leq u_0, v_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$. Then for all $0 \leq \psi \in \mathrm{L}^\infty(\Omega)$ and all $0 \leq \tau \leq t$

$$\int_\Omega \big[u(t,x) - v(t,x)\big]\mathcal{L}^{-1}\psi(x)\,\mathrm{d}x \leq \int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\mathcal{L}^{-1}\psi(x)\,\mathrm{d}x\,.$$

As a consequence, there exists a constant $C_{\Omega,\gamma} > 0$ such that for all $0 \leq \tau \leq t$

$$\int_\Omega \big[u(t,x) - v(t,x)\big]\,\delta_\gamma(x)\,\mathrm{d}x \leq C_{\Omega,\gamma}\int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\,\delta_\gamma(x)\,\mathrm{d}x\,.$$

Moreover...

Weighted $L^1_{\delta_\gamma}$ estimates

Let's put $\boxed{\Psi_1 = \mathcal{L}^{-1}\delta_\gamma}$, which bears some similarity with the formula $\Phi_1 = \lambda_1^{-1}\mathcal{L}^{-1}\Phi_1$.

Proposition. (Weighted L^1 estimates for ordered solutions) Continued

Moreover, for all $0 \le \tau_0 \le \tau, t < +\infty$ such that either $t, \tau \le K_0$ or $\tau_0 \ge K_0$, we have

$$\int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\Psi_1(x)\,dx \le \int_\Omega \big[u(t,x) - v(t,x)\big]\Psi_1(x)\,dx$$

$$+ K_8 \,\|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s(m_i-1)\vartheta_{i,\gamma}} \,|t-\tau|^{2s\vartheta_{i,\gamma}} \int_\Omega \big[u(\tau_0,x) - v(\tau_0,x)\big]\delta_\gamma(x)\,dx$$

where $i = 0$ if $t, \tau \le \|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$ and $i = 1$ if $t, \tau \ge \|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$.

Remark. The above inequality, together with monotonicity, allows to prove that weak dual solutions constructed by approximation from below by mild solutions belong to the space

$$u \in C([0,\infty) : L^1_{\delta_\gamma}(\Omega)).$$

Harnack inequalities

- **Global Harnack Principle**
- **Local Harnack inequalities**

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Global Harnack Principle) (M.B. & J. L. Vázquez)

There exist universal constants $H_0, H_1, L_0 > 0$ such that setting

$$t_* = \frac{L_0}{\left(\int_\Omega u_0 \delta_\gamma \,\mathrm{d}x\right)^{m-1}} ,$$

we have that for all $t \geq t_*$ and all $x \in \Omega$, the following inequality holds:

$$H_0 \, \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq H_1 \, \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}$$

Recall that δ_γ is the first eigenfunction of $\mathcal{L}$.

Remarks.

- This inequality implies local Harnack inequalities of elliptic type
- Useful to study the sharp asymptotic behaviour

Local Harnack inequalities

Theorem. (Local Harnack Inequalities of Elliptic Type) (M.B. & J. L. Vázquez)

There exist constants $H_R, L_0 > 0$ such that setting $t_* = L_0 \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have that for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, the following inequality holds:

$$\sup_{x \in B_R(x_0)} u(t,x) \le H_R \inf_{x \in B_R(x_0)} u(t,x)$$

The constant H_R depends on $\mathrm{dist}(B_R(x_0), \partial\Omega)$.

Corollary. (Local Harnack Inequalities of Backward Type)

Under the runninig assumptions, for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, we have:

$$\sup_{x \in B_R(x_0)} u(t,x) \le 2H_R \inf_{x \in B_R(x_0)} u(t+h,x) \qquad \text{for all } 0 \le h \le t_* .$$

Summary

Asymptotic behaviour of nonnegative solutions

- Convergence to the stationary profile
- Convergence with optimal rate

Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}},$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative weak dual solution to the (CDP), then we have (unless $u \equiv 0$)

$$\lim_{\tau \to \infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0.$$

The previous theorem admits the following corollary.

Theorem. (Elliptic problem) (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

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Convergence with optimal rate

Theorem. (Sharp asymptotic with rates) (M.B. , Y. Sire, J. L. Vázquez)

Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k} \left[\frac{\int_\Omega \Phi_1 \, \mathrm{d}x}{\int_\Omega u_0 \Phi_1 \, \mathrm{d}x} \right]^{m-1}$$

such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{2}{m-1} \frac{t_0}{t_0 + t} \,.$$

The constant $\overline{k} > 0$ only depends on $m, N, s,$ and $|\Omega|$.

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 3
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.

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The End

Muchas Gracias!!!

Thank You!!!

Grazie Mille!!!