Lyapunov functionals for the heat equation and sharp inequalities

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Outline

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Young's inequality

- Beckner, (1975); Brascamp and Lieb (1976)

$$\|f * g\|_r \leq (A_p A_q A_{r'})^n \|f\|_p \|g\|_q.$$

- $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q, r < \infty$ and $1/p + 1/q = 1 + 1/r$. The constant A_m

$$A_m = \left(\frac{m^{1/m}}{m'^{1/m'}} \right)^{1/2}$$

primes always denote dual exponents, $1/m + 1/m' = 1$.

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Method of proof

- The best constants in Young's inequality were found by Beckner using tensorisation arguments and rearrangements of functions.
- Brascamp and Lieb derived them from a more general inequality, which is nowadays known as the Brascamp-Lieb inequality.
- The expression of the best constant, in the case in which both f and g are probability density functions, is obtained by noticing that Young's inequality is saturated by Gaussian densities.

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Reverse Young's inequality

- Brascamp and Lieb noticed that the sharp form of Young inequality also holds in the so-called reverse case

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- where now $0 < p, q, r < 1$ while, as in Young's inequality, $1/p + 1/q = 1 + 1/r$. In this case, however, the dual exponents p', q', r' are negative, and the constant A_m

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- The original proof in Brascamp-Lieb was rather complicated, and used tensorisation, Schwarz symmetrisation, Brunn-Minkowski and some not so intuitive phenomenon for the measure in high dimension.
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Other proofs II

- A different proof in Bennett, Bez (2009).
- Bennett-Bez gave a semigroup proof of the optimal Young inequality identifying a super-solution of the heat equation.
- We will start here from a Lyapunov functional, by proving its monotonicity. Fundamental tool the invariance of the functional with respect to the scaling dilation

$$f(x) \to f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a^n}\right), \quad a > 0.$$

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Connections with other inequalities

- Lieb enlightened connections of the sharp form of Young inequality with other inequalities.
- By letting $p, q, r \to 1$ in the sharp form of Young's inequality reduces to another well-known inequality in information theory, known as Shannon's entropy power inequality

$$e^{2H(f*g)} \geq e^{2H(f)} + e^{2H(g)},$$

- In Shannon's inequality f and g are probability density functions, and

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- The proof is based on an identity which couples Fisher's information with Shannon's entropy functional
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A Lyapunov functional

- The proof of Stam is based on the following argument. Let $f(x,t)$ and $g(x,t)$ be two solutions of the heat equation with different coefficients of diffusion, corresponding to the initial data $f(x)$ (respectively $g(x)$).
- Consider the evolution in time of the functional $\Theta_{f,g}(t)$ defined by

$$\Theta_{f,g}(t) = \frac{e^{2H(f(t))} + e^{2H(g(t))}}{e^{2H(f(t)*g(t))}}.$$

- Evaluating the time derivative of $\Theta_{f,g}(t)$, shows that $\Theta_{f,g}(t)$ is increasing in time, and converges towards the constant value $\Theta_{f,g}(+\infty) = 1$.

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Hölder's inequality

- Hölder's inequality for integrals states that, if $p, q > 1$ are such that $1/p + 1/q = 1$

$$\int_{\mathbb{R}} |f(x)g(x)|\, dx \leq \left(\int_{\mathbb{R}} |f(x)|^p\, dx \right)^{1/p} \left(\int_{\mathbb{R}} |g(x)|^q\, dx \right)^{1/q}.$$

- Moreover, there is equality if and only f and g are such that there exist positive real numbers a and b such that $af^p(x) = bg^q(x)$ almost everywhere.

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Classical proof

- Hölder's inequality can be proven in many ways, for example resorting to Young's inequality for constants, which states that, if $1/p + 1/q = 1$

$$cd \leq \frac{c^p}{p} + \frac{d^q}{q},$$

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A new proof

- A different way to achieve Hölder's inequality is the following. Let $\Phi_{u,v}(t)$ be the functional

$$\Phi_{u,v}(t) = \int_{\mathbb{R}} u(x,t)^{1/p} v(x,t)^{1/q}\, dx,$$

where $1/p + 1/q = 1$.

- $u(x,t)$ and $v(x,t)$, $t > 0$, are solutions to the heat equation corresponding to the initial values $u(x) \in L^1(\mathbb{R})$ (respectively $v(x) \in L^1(\mathbb{R})$).

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- The functional $\phi_{u,v}(t)$ is increasing in time from

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A new proof III

- Let us evaluate the time derivative of $\Phi(t)$

$$\Phi'_{u,v}(t) = \int_{\mathbb{R}} \left[(u(x,t)^{1/p})_t v(x,t)^{1/q} + u(x,t)^{1/p} (v(x,t)^{1/q})_t \right] dx =$$

$$\int_{\mathbb{R}} \left[\frac{1}{p} u^{1/p-1} v^{1/q} u_{xx} + \frac{1}{q} u^{1/p} v^{1/q-1} v_{xx} \right] dx =$$

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- Integrating by parts we get

$$\Phi'_{u,v}(t) = \frac{1}{pq} \int_{\mathbb{R}} u^{1/p}(x,t) v^{1/q}(x,t) \left(\frac{u_x(x,t)}{u(x,t)} - \frac{v_x(x,t)}{v(x,t)} \right)^2 dx \geq 0$$

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Consequences

- The functional $\Phi_{u,v}(t)$ is increasing in time. Note that the time derivative of the functional is equal to zero if and only if, for every $t > 0$

$$\frac{u_x(x,t)}{u(x,t)} - \frac{v_x(x,t)}{v(x,t)} = 0.$$

for all x

- This condition can be rewritten as

$$\frac{d}{dx} \log \frac{u(x,t)}{v(v,t)} = 0.$$

- $\Phi'(t) = 0$ if and only if $u(x,t) = c\, v(x,t)$ for some positive constant c.

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Consequences II

- The functional $\Phi(t)$ is monotone increasing, and it will reach its eventual maximum value as time $t \to \infty$.
- The computation of the limit value uses in a substantial way the scaling invariance of Φ.
- The value of $\Phi_{u,v}(t)$ does not change if we scale $u(x,t)$ and $v(x,t)$ according to

$$u(x,t) \to U(x,t) = \sqrt{1+2t}\, u(x\sqrt{1+2t}, t)$$
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- It is well-known that

$$\lim_{t\to\infty} U(x,t) = M_1(x) \int_{\mathbb{R}} u(x)\,dx \qquad \lim_{t\to\infty} V(x,t) = M_1(x) \int_{\mathbb{R}} v(x)\,dx,$$

- $M_1(x)$ is the Gaussian density in $\mathbb{R}$ of variance equal to 1.
- The monotonicity of the functional $\Phi(t)$ implies the inequality

$$\int_{\mathbb{R}} u(x)^{1/p} v(x)^{1/q}\,dx \le \left(\int_{\mathbb{R}} u(x)\,dx\right)^{1/p} \left(\int_{\mathbb{R}} v(x)\,dx\right)^{1/q},$$

with equality if and only if $u(x) = cv(x)$ for some constant c.

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Comments

- Despite its apparent complexity, this way of proof is based on a solid physical argument, namely the monotonicity in time of a Lyapunov functional of the solution to the heat equation.
- This gives a clear indication that many inequalities reflect the physical principle of the tendency of a system to move towards the state of maximum entropy.
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The Lyapunov functional related to Young's inequality

- The key functional to study is

$$\Psi_{u,v}(t) = \left(\int_{\mathbb{R}} \left(u(x,t)^{1/p} * v(x,t)^{1/q} \right)^r dx \right)^{1/r}.$$

- As in Young's inequality, $1/p + 1/q = 1 + 1/r$
- $u(x,t)$ and $v(x,t)$ are solutions of the heat equation corresponding to the initial data $u(x)$ (respectively $v(x)$). However, these solutions correspond to two different heat equations, with different coefficients of diffusions, say α and β.

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Main result

Theorem

Let $1/p + 1/q = 1 + 1/r$, *and* $u(x,t)$ *and* $v(x,t)$, $t > 0$, *are solutions to the heat equation corresponding to the initial values* $u(x) \in L^1(\mathbb{R})$ *(respectively* $v(x) \in L^1(\mathbb{R})$*). Then, if* $p, q, r > 1$, *and the diffusion coefficients are given by* $\alpha = q'/p$ *(respectively* $\beta = p'/q$*), or by a multiple of them,* $\Psi_{u,v}(t)$ *is increasing in time from*

$$\Psi_{u,v}(t=0) = \left(\int_{\mathbb{R}} \left(u(x)^{1/p} * v(x)^{1/q} \right)^r dx \right)^{1/r},$$

to the limit value

$$\lim_{t\to\infty} \Psi_{u,v}(t) = (A_p A_q A_{r'})^{1/2} \left(\int_{\mathbb{R}} u(x)\, dx \right)^{1/p} \left(\int_{\mathbb{R}} v(x)\, dx \right)^{1/q}.$$

Proof

- Compute the time derivative of the functional $\Psi_{u,v}(t)$. To shorten, let us denote

$$h(x,t) = u(x,t)^{1/p} * v(x,t)^{1/q}.$$

- Computations give

$$\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}} h^r(x,t)\,dx = -(\alpha+\beta)(r-1)\int_{\mathbb{R}} h^{r-2}(x,t)\,(h_x(x,t))^2\,dx +$$

$$\alpha\frac{p}{p'}\int_{\mathbb{R}} h^{r-1}(x,t)A(u^{1/p},v^{1/q})(x,t)\,dx +$$

$$\beta\frac{q}{q'}\int_{\mathbb{R}} h^{r-1}(x,t)B(u^{1/p},v^{1/q})(x,t)\,dx.$$

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Proof

- A and B are defined by

$$A(f,g)(x,t) = \int_{\mathbb{R}} \frac{(f_x(x-y,t))^2}{f(x-y)} g(y,t)\,dy,$$

$$B(f,g)(x,t) = \int_{\mathbb{R}} f(x-y) \frac{(g_y(y,t))^2}{g(y)}\,dy.$$

- Since

$$\frac{d\Psi_{u,v}(t)}{dt} = \Psi_{u,v}(t)^{1-r} \frac{d}{dt} \int_{\mathbb{R}} h^r(x,t)\,dx,$$

the sign of the time derivative of the functional $\Psi_{u,v}(t)$ depends of the sign of the expression on the right-hand side of the previous equality.

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Key result

Lemma

Let $f(x)$ and $g(x)$ be probability density functions such that both $A(f,g)$ and $B(f,g)$ are well defined. Then, for all positive constants a, b and $r > 0$

$$(a^2 + b^2 + 2abr) \int_{\mathbb{R}} (f * g)^{r-2} ((f * g)_x)^2 \, dx \leq$$

$$a^2 \int_{\mathbb{R}} (f * g)^{r-1} A(f,g) \, dx + b^2 \int_{\mathbb{R}} (f * g)^{r-1} B(f,g) \, dx.$$

Moreover, there is equality if and only if, for any positive constant c and constants m_1, m_2, f and g are Gaussian densities, $f(x) = M_{ca}(x - m_1)$ and $g(x) = M_{cb}(x - m_2)$.

Proof II

- If we choose $a^2 = \alpha p/p'$, $b^2 = \beta q/q'$, the coefficient of the term on the left-hand side

$$a^2 + b^2 + 2abr = \alpha \frac{p}{p'} + \beta \frac{q}{q'} + 2\sqrt{\alpha\beta}\sqrt{\frac{pq}{p'q'}}r.$$

- Let us introduce, for any given $r > 1$ the function

$$\Gamma(\alpha, \beta) = (\alpha + \beta)(r - 1) - \left(\alpha \frac{p}{p'} + \beta \frac{q}{q'} + 2\sqrt{\alpha\beta}\sqrt{\frac{pq}{p'q'}}r\right).$$

- The function Γ is jointly convex, and possesses the half-line of extremals

$$\beta = \frac{p}{q'} \cdot \frac{p'}{q}\alpha.$$

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Proof III

- Along the half-line the functional $\Phi_{u,v}(t)$ is increasing with respect to t.
- Scaling $u(x,t)$ and $v(x,t)$, we conclude that the functional will keep its maximum value as time goes to infinity, and

$$\lim_{t\to\infty} \Phi_{u,v}(t) = \left(\int_{\mathbb{R}} u(x)\,dx\right)^{1/p} \left(\int_{\mathbb{R}} v(x)\,dx\right)^{1/q} C(p,q,r)$$

- The constant

$$C(p,q,r) = \left(\int_{\mathbb{R}} \left(M_{q'/p}(x)^{1/p} * M_{p'/q}(x)^{1/q}\right)^r dx.\right)^{1/r} = (A_p A_q A_{r'})^{1/2}.$$

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