Lyapunov functionals for the heat equation and sharp inequalities

Giuseppe Toscani

Department of Mathematics
University of Pavia, Italy

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Outline

1. Inequalities and Lyapunov functionals
   - Introduction
   - Entropy power inequality

2. Young’s inequality
   - A new proof of Hölder inequality
   - Lyapunov functionals and Young’s inequality
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   - Introduction
   - Entropy power inequality

2. Young’s inequality
   - A new proof of Hölder inequality
   - Lyapunov functionals and Young’s inequality
Young’s inequality

- Beckner, (1975); Brascamp and Lieb (1976)

\[ \|f \ast g\|_r \leq (A_p A_q A_{r'})^n \|f\|_p \|g\|_q. \]

- \( f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), 1 < p, q, r < \infty \) and \( 1/p + 1/q = 1 + 1/r \). The constant \( A_m \)

\[ A_m = \left( \frac{m^{1/m}}{m'^{1/m'}} \right)^{1/2} \]

primes always denote dual exponents, \( 1/m + 1/m' = 1 \).
Young’s inequality

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Method of proof

- The best constants in Young’s inequality were found by Beckner using tensorisation arguments and rearrangements of functions.
- Brascamp and Lieb derived them from a more general inequality, which is nowadays known as the Brascamp-Lieb inequality.
- The expression of the best constant, in the case in which both $f$ and $g$ are probability density functions, is obtained by noticing that Young’s inequality is saturated by Gaussian densities.
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Reverse Young’s inequality

Brascamp and Lieb noticed that the sharp form of Young inequality also holds in the so-called reverse case

\[ \| f * g \|_r \geq (A_p A_q A_{r'})^n \| f \|_p \| g \|_q. \]

where now \( 0 < p, q, r < 1 \) while, as in Young’s inequality, \( 1/p + 1/q = 1 + 1/r \). In this case, however, the dual exponents \( p', q', r' \) are negative, and the constant \( A_m \)

\[ A_m = \left( \frac{m^{1/m}}{|m'|^{1/|m'|}} \right)^{1/2}. \]
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\[ A_m = \left( \frac{m^{1/m}}{|m'|^{1/|m'|}} \right)^{1/2}. \]
The proof of this sharp reverse Young inequality was subsequently simplified by Barthe, (1998).

The original proof in Brascamp-Lieb was rather complicated, and used tensorisation, Schwarz symmetrisation, Brunn-Minkowski and some not so intuitive phenomenon for the measure in high dimension.

The proof of the main result of Barthe relies on a parametrization of functions which was suggested by Brunn’s proof of the Brunn-Minkowski inequality.
Other proofs

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A different proof in Bennett, Bez (2009).

Bennett-Bez gave a semigroup proof of the optimal Young inequality identifying a super-solution of the heat equation.

We will start here from a Lyapunov functional, by proving its monotonicity. Fundamental tool the invariance of the functional with respect to the scaling dilation

\[ f(x) \rightarrow f_a(x) = \frac{1}{a^n} f \left( \frac{x}{a^n} \right), \quad a > 0. \]
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Connections with other inequalities

- Lieb enlightened connections of the sharp form of Young inequality with other inequalities.
- By letting $p, q, r \to 1$ in the sharp form of Young’s inequality reduces to another well-known inequality in information theory, known as Shannon’s entropy power inequality

$$e^{2H(f \ast g)} \geq e^{2H(f)} + e^{2H(g)}$$

- In Shannon’s inequality $f$ and $g$ are probability density functions, and

$$H(f) = -\int_{\mathbb{R}} f(x) \log f(x) \, dx$$
Entropic power inequality

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The first rigorous proof of entropy power inequality was given by Stam (1959), and generalized by Blackman (1965) to $n$-dimensional random vectors.

The proof is based on an identity which couples Fisher’s information with Shannon’s entropy functional.

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The proof of Stam is based on the following argument. Let \( f(x, t) \) and \( g(x, t) \) be two solutions of the heat equation with different coefficients of diffusion, corresponding to the initial data \( f(x) \) (respectively \( g(x) \)).

Consider the evolution in time of the functional \( \Theta_{f,g}(t) \) defined by

\[
\Theta_{f,g}(t) = \frac{e^{2H(f(t))} + e^{2H(g(t))}}{e^{2H(f(t)*g(t))}}.
\]

Evaluating the time derivative of \( \Theta_{f,g}(t) \), shows that \( \Theta_{f,g}(t) \) is increasing in time, and converges towards the constant value \( \Theta_{f,g}(+\infty) = 1 \).
A Lyapunov functional

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$$\Theta_{f,g}(t) = \frac{\exp(2H(f(t))) + \exp(2H(g(t)))}{\exp(2H(f(t)\ast g(t)))}.$$ 

- Evaluating the time derivative of $\Theta_{f,g}(t)$, shows that $\Theta_{f,g}(t)$ is increasing in time, and converges towards the constant value $\Theta_{f,g}(+\infty) = 1$. 
Hölder’s inequality

- Hölder’s inequality for integrals states that, if \( p, q > 1 \) are such that \( \frac{1}{p} + \frac{1}{q} = 1 \)

\[
\int \left| f(x)g(x) \right| \, dx \leq \left( \int |f(x)|^p \, dx \right)^{1/p} \left( \int |g(x)|^q \, dx \right)^{1/q}.
\]

- Moreover, there is equality if and only if \( f \) and \( g \) are such that there exist positive real numbers \( a \) and \( b \) such that \( af^p(x) = bg^q(x) \) almost everywhere.
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Hölder’s inequality can be proven in many ways, for example resorting to Young’s inequality for constants, which states that, if $1/p + 1/q = 1$

$$cd \leq \frac{c^p}{p} + \frac{d^q}{q},$$

for all nonnegative $c$ and $d$

Equality is achieved if and only if $c^p = d^q$. 
A new proof of Hölder inequality

Classical proof

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- Equality is achieved if and only if $c^p = d^q$. 
A new proof of Hölder’s inequality is the following. Let $\Phi_{u,v}(t)$ be the functional

$$
\Phi_{u,v}(t) = \int_{\mathbb{R}} u(x, t)^{1/p} v(x, t)^{1/q} \, dx,
$$

where $1/p + 1/q = 1$.

$u(x, t)$ and $v(x, t)$, $t > 0$, are solutions to the heat equation corresponding to the initial values $u(x) \in L^1(\mathbb{R})$ (respectively $v(x) \in L^1(\mathbb{R})$).
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A new proof

- A different way to achieve Hölder’s inequality is the following. Let $\Phi_{u,v}(t)$ be the functional

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where $1/p + 1/q = 1$.

- $u(x,t)$ and $v(x,t)$, $t > 0$, are solutions to the heat equation corresponding to the initial values $u(x) \in L^1(\mathbb{R})$ (respectively $v(x) \in L^1(\mathbb{R})$).
The functional $\phi_{u,v}(t)$ is increasing in time from

$$\Phi_{u,v}(t = 0) = \int_{\mathbb{R}} u(x)^{1/p} v(x)^{1/q} \, dx,$$

to

$$\lim_{t \to \infty} \Phi_{u,v}(t) = \int_{\mathbb{R}} u(x) \, dx \int_{\mathbb{R}} v(x) \, dx.$$
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A new proof II

- The functional \( \phi_{u,v}(t) \) is **increasing in time** from

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A new proof III

Let us evaluate the time derivative of $\Phi(t)$

$$\Phi'_{u,v}(t) = \int_{\mathbb{R}} \left[ (u(x, t)^{1/p})_t v(x, t)^{1/q} + u(x, t)^{1/p} (v(x, t)^{1/q})_t \right] \, dx =$$

$$\int_{\mathbb{R}} \left[ \frac{1}{p} u^{1/p-1} v^{1/q} \, u_{xx} + \frac{1}{q} u^{1/p} v^{1/q-1} \, v_{xx} \right] \, dx =$$

$$\int_{\mathbb{R}} \left[ \frac{1}{p} u^{-1/q} v^{1/q} \, u_{xx} + \frac{1}{q} u^{1/p} v^{-1/p} \, v_{xx} \right] \, dx.$$

Integrating by parts we get

$$\Phi'_{u,v}(t) = \frac{1}{pq} \int_{\mathbb{R}} u^{1/p}(x, t) v^{1/q}(x, t) \left( \frac{u_x(x, t)}{u(x, t)} - \frac{v_x(x, t)}{v(x, t)} \right)^2 \, dx \geq 0$$
A new proof of Hölder inequality

Let us evaluate the time derivative of $\Phi(t)$

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\Phi_{u,v}'(t) = \int_{\mathbb{R}} \left[ (u(x,t)^{1/p})_t v(x,t)^{1/q} + u(x,t)^{1/p} (v(x,t)^{1/q})_t \right] \, dx =
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Consequences

- The functional $\Phi_{u,v}(t)$ is increasing in time. Note that the time derivative of the functional is equal to zero if and only if, for every $t > 0$

$$\frac{u_x(x,t)}{u(x,t)} - \frac{v_x(x,t)}{v(x,t)} = 0.$$  

for all $x$

- This condition can be rewritten as

$$\frac{d}{dx} \log \frac{u(x,t)}{v(x,t)} = 0.$$  

- $\Phi'(t) = 0$ if and only if $u(x, t) = c v(x, t)$ for some positive constant $c$.  

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The functional $\Phi(t)$ is monotone increasing, and it will reach its eventual maximum value as time $t \to \infty$.

The computation of the limit value uses in a substantial way the scaling invariance of $\Phi$.

The value of $\Phi_{u,v}(t)$ does not change if we scale $u(x,t)$ and $v(x,t)$ according to

$$u(x,t) \to U(x,t) = \sqrt{1+2t} u(x \sqrt{1+2t}, t)$$
$$v(x,t) \to V(x,t) = \sqrt{1+2t} v(x \sqrt{1+2t}, t).$$
Consequences II

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Consequences III

- It is well-known that

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\lim_{t \to \infty} U(x, t) = M_1(x) \int_{\mathbb{R}} u(x) \, dx \quad \lim_{t \to \infty} V(x, t) = M_1(x) \int_{\mathbb{R}} v(x) \, dx,
\]

- \(M_1(x)\) is the Gaussian density in \(\mathbb{R}\) of variance equal to 1.
- The monotonicity of the functional \(\Phi(t)\) implies the inequality

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\int_{\mathbb{R}} u(x)^{1/p} v(x)^{1/q} \, dx \leq \left( \int_{\mathbb{R}} u(x) \, dx \right)^{1/p} \left( \int_{\mathbb{R}} v(x) \, dx \right)^{1/q},
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with equality if and only if \(u(x) = cv(x)\) for some constant \(c\).
- Setting \(f = u^{1/p}\) and \(g = v^{1/q}\) proves both Hölder inequality and the equality cases.
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Despite its **apparent** complexity, this way of **proof** is based on a **solid physical argument**, namely the monotonicity in time of a Lyapunov functional of the solution to the heat equation.

This gives a clear indication that **many inequalities** reflect the physical principle of the tendency of a system to move towards the state of maximum entropy.

This idea also applies to prove **Young’s inequality**.
A new proof of Hölder inequality

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- This gives a clear indication that **many inequalities** reflect the **physical principle of the tendency of a system to move towards the state of maximum entropy**.
- This idea also applies to prove **Young’s inequality**.
The key functional to study is

$$\Psi_{u,v}(t) = \left( \int_{\mathbb{R}} \left( u(x, t)^{1/p} \ast v(x, t)^{1/q} \right)^r \, dx \right)^{1/r}.$$ 

As in Young’s inequality, $1/p + 1/q = 1 + 1/r$

$u(x, t)$ and $v(x, t)$ are solutions of the heat equation corresponding to the initial data $u(x)$ (respectively $v(x)$). However, these solutions correspond to two different heat equations, with different coefficients of diffusions, say $\alpha$ and $\beta$. 
The Lyapunov functional related to Young’s inequality

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Main result

Theorem

Let \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \), and \( u(x, t) \) and \( v(x, t) \), \( t > 0 \), are solutions to the heat equation corresponding to the initial values \( u(x) \in L^1(\mathbb{R}) \) (respectively \( v(x) \in L^1(\mathbb{R}) \)). Then, if \( p, q, r > 1 \), and the diffusion coefficients are given by \( \alpha = q'/p \) (respectively \( \beta = p'/q \)), or by a multiple of them, \( \Psi_{u,v}(t) \) is increasing in time from

\[
\Psi_{u,v}(t = 0) = \left( \int_{\mathbb{R}} \left( u(x)^{1/p} \ast v(x)^{1/q} \right)^r \, dx \right)^{1/r},
\]

to the limit value

\[
\lim_{t \to \infty} \Psi_{u,v}(t) = (A_p A_q A_{r'})^{1/2} \left( \int_{\mathbb{R}} u(x) \, dx \right)^{1/p} \left( \int_{\mathbb{R}} v(x) \, dx \right)^{1/q}.
\]
Proof

- Compute the time derivative of the functional $\Psi_{u,v}(t)$. To shorten, let us denote

$$h(x, t) = u(x, t)^{1/p} \ast v(x, t)^{1/q}.$$ 

- Computations give

$$\frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}} h^r(x, t) \, dx = - (\alpha + \beta)(r - 1) \int_{\mathbb{R}} h^{r-2}(x, t) (h_x(x, t))^2 \, dx +$$

$$\alpha \frac{p}{p'} \int_{\mathbb{R}} h^{r-1}(x, t) A(u^{1/p}, v^{1/q})(x, t) \, dx +$$

$$\beta \frac{q}{q'} \int_{\mathbb{R}} h^{r-1}(x, t) B(u^{1/p}, v^{1/q})(x, t) \, dx.$$
Proof

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Proof

• $A$ and $B$ are defined by

\[
A(f, g)(x, t) = \int_{\mathbb{R}} \frac{(f_x(x - y, t))^2}{f(x - y)} g(y, t) \, dy,
\]

\[
B(f, g)(x, t) = \int_{\mathbb{R}} f(x - y) \frac{(g_y(y, t))^2}{g(y)} \, dy.
\]

• Since

\[
\frac{d\Psi_{u,v}(t)}{dt} = \Psi_{u,v}(t)^{1-r} \frac{d}{dt} \int_{\mathbb{R}} h^r(x, t) \, dx,
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the sign of the time derivative of the functional $\Psi_{u,v}(t)$ depends of the sign of the expression on the right-hand side of the previous equality.
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the sign of the time derivative of the functional $\Psi_{u,v}(t)$ depends on the sign of the expression on the right-hand side of the previous equality.
Key result

**Lemma**

Let $f(x)$ and $g(x)$ be probability density functions such that both $A(f, g)$ and $B(f, g)$ are well defined. Then, for all positive constants $a$, $b$ and $r > 0$

\[
(a^2 + b^2 + 2abr) \int_{\mathbb{R}} (f \ast g)^{r-2} ((f \ast g)_x)^2 \, dx \leq
\]

\[
a^2 \int_{\mathbb{R}} (f \ast g)^{r-1} A(f, g) \, dx + b^2 \int_{\mathbb{R}} (f \ast g)^{r-1} B(f, g) \, dx.
\]

Moreover, there is equality if and only if, for any positive constant $c$ and constants $m_1, m_2$, $f$ and $g$ are Gaussian densities, $f(x) = M_{ca}(x - m_1)$ and $g(x) = M_{cb}(x - m_2)$. 
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Proof II

- If we choose \( a^2 = \alpha p/p' \), \( b^2 = \beta q/q' \), the coefficient of the term on the left-hand side

\[
a^2 + b^2 + 2abr = \alpha \frac{p}{p'} + \beta \frac{q}{q'} + 2\sqrt{\alpha \beta} \sqrt{\frac{pq}{p'q'}} r. \]

- Let us introduce, for any given \( r > 1 \) the function

\[
\Gamma(\alpha, \beta) = (\alpha + \beta)(r - 1) - \left( \alpha \frac{p}{p'} + \beta \frac{q}{q'} + 2\sqrt{\alpha \beta} \sqrt{\frac{pq}{p'q'}} r \right). \]

- The function \( \Gamma \) is jointly convex, and possesses the half-line of extremals

\[
\beta = \frac{p}{q'} \cdot \frac{p'}{q} \alpha. \]
Proof II

- If we choose $a^2 = \alpha p/p'$, $b^2 = \beta q/q'$, the coefficient of the term on the left-hand side

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Proof III

- Along the half-line the functional \( \Phi_{u,v}(t) \) is increasing with respect to \( t \).
- Scaling \( u(x, t) \) and \( v(x, t) \), we conclude that the functional will keep its maximum value as time goes to infinity, and

\[
\lim_{t \to \infty} \Phi_{u,v}(t) = \left( \int_{\mathbb{R}} u(x) \, dx \right)^{1/p} \left( \int_{\mathbb{R}} v(x) \, dx \right)^{1/q} C(p, q, r)
\]

- The constant

\[
C(p, q, r) = \left( \int_{\mathbb{R}} \left( M_{q'/p}(x)^{1/p} \ast M_{p'/q}(x)^{1/q} \right)^r \, dx \right)^{1/r} = (A_p A_q A_{r'})^{1/2}.
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- Easy to generalize to dimension \( n > 1 \).
Lyapunov functionals and Young’s inequality

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