Regularity results for fractional filtration equations

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Fractional filtration equation

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(\varphi(u)) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(\cdot, 0) = f \end{cases}$$

- EXPONENTS: $0 < \sigma < 2$
- INITIAL DATA: $f \in L^1(\mathbb{R}^N), \quad f \geq 0$
- NON-LINEARITIES: φ continuous, non-decreasing, $\varphi(0) = 0$
  - Examples: $\varphi(u) = \begin{cases} u^m, & m > 0 \\ \log(1+u), & N = \sigma = 1 \end{cases}$

A non-negative, self-adjoint, linear operator

$$A^\alpha u(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-tA}u(x) - u(x)\right) \frac{dt}{t^{1+\alpha}}$$

- $\lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+\alpha}}, \qquad \lambda > 0$

$$(-\Delta)^{\sigma/2}u(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})}\int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right)\frac{dt}{t^{1+\frac{\sigma}{2}}}$$

- $\mathcal{F}\big((-\Delta)^{\sigma/2}u\big)(\xi) = |\xi|^\sigma\,\mathcal{F}(u)(\xi)$
- $(-\Delta)^{\sigma/2}u(x) = C_{N,\sigma}\,\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+\sigma}}\,dy$

Motivation

- Non-linear generalization of the *fractional heat equation*

$$u_t + (-\Delta)^{\sigma/2} u = 0$$

- Non-local generalization of the *filtration equation*

$$u_t - \Delta(\varphi(u)) = 0$$

(Other possible generalizations [Caffarelli-Vázquez, 2009])

- Hydrodynamic limit of zero range processes [Jara, 2009]

- $\varphi(u) = \log(1+u) \rightsquigarrow$ Nonlocal "critical" viscous transport equation

Mild solution

- THEOREM [Crandall-Pierre, 1982]: A linear operator such that

$$\exists \textit{ mild} \text{ solution (ITD): } \begin{cases} u_t + Au = 0, \\ u(0) = f \in L^1 \end{cases}$$

$$+$$

conditions on φ

$$\Downarrow$$

$$\exists \text{ mild solution: } \begin{cases} u_t + A\varphi(u) = 0, \\ u(0) = f \in L^1 \end{cases}$$

- Abstract construction $\Rightarrow$
  - Not enough information to prove that mild $\Rightarrow$ weak.
  - No estimates $\Rightarrow$ No further properties.

Integration by parts / Functional space

- $$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\varphi\, \psi &= \int_{\mathbb{R}^N} |\xi|^{\sigma} \hat{\varphi}\, \hat{\psi} = \int_{\mathbb{R}^N} |\xi|^{\sigma/2} \hat{\varphi}\, |\xi|^{\sigma/2} \hat{\psi} \\ &= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\varphi\, (-\Delta)^{\sigma/4}\psi\end{aligned}$$

Integration by parts / Functional space

- $$\begin{aligned}\int_{\mathbb{R}^N}(-\Delta)^{\sigma/2}\varphi\,\psi &= \int_{\mathbb{R}^N}|\xi|^{\sigma}\hat{\varphi}\,\hat{\psi}=\int_{\mathbb{R}^N}|\xi|^{\sigma/2}\hat{\varphi}\,|\xi|^{\sigma/2}\hat{\psi}\\ &= \int_{\mathbb{R}^N}(-\Delta)^{\sigma/4}\varphi\,(-\Delta)^{\sigma/4}\psi\end{aligned}$$

- $$\begin{aligned}\|\varphi\|_{\dot{H}^{\sigma/2}} &= \left(\int_{\mathbb{R}^N}|\xi|^{\sigma}|\hat{\varphi}|^2\,d\xi\right)^{1/2}=\|(-\Delta)^{\sigma/4}\varphi\|_2\\ &= \left(\int_{\mathbb{R}^N}\left(C_{N,\sigma}\text{P.V.}\int_{\mathbb{R}^N}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+\frac{\sigma}{2}}}\,dy\right)^2\,dx\right)^{1/2}\end{aligned}$$

Weak/strong solutions

Weak (L^1-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $\varphi(u) \in L^2_{\text{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$

- $\displaystyle \int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \psi}{\partial t}\, dxds - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(\varphi(u))(-\Delta)^{\sigma/4}\psi\, dxds = 0,$

  $\forall \psi \in C^1_{\mathsf{c}}(\mathbb{R}^N \times (0,\infty))$

- $u(\cdot,0) = f$ a.e.

Weak/strong solutions

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  $\forall \psi \in C^1_{\mathsf{c}}(\mathbb{R}^N \times (0,\infty))$
- $u(\cdot, 0) = f$ a.e.

Strong solution (uniqueness)

- Weak (L^1-energy) solution
- $\partial_t u \in L^1_{\mathsf{loc}}(\mathbb{R}^N \times (0,\infty))$ + ...

Difficulties

- $(-\Delta)^{\sigma/4}(\varphi\psi) = ?$
- $(-\Delta)^{\sigma/4}(\varphi \circ \psi) = ?$

- $(-\Delta)^{\sigma/4}\psi$ not compactly supported even when ψ is
    - $(-\Delta)^{\sigma/4}$ *non-local* operator

σ-harmonic extension [Caffarelli-Silvestre, 2007]

$$v = \mathrm{E}(g) : \quad \begin{cases} L_\sigma v \equiv \mathrm{div}(y^{1-\sigma}\nabla v) = 0, & \mathbb{R}^{N+1}_+ = \{x \in \mathbb{R}^N,\ y > 0\} \\ v(x,0) = g(x), & x \in \mathbb{R}^N \end{cases}$$

- $v(x,y) = \displaystyle\int_{\mathbb{R}^N} P(x-\xi,y)g(\xi)\,d\xi, \qquad P(x,y) = d_{N,\sigma}\frac{y^\sigma}{(|x|^2+|y|^2)^{(N+\sigma)/2}}$
- $\displaystyle\lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = \sigma \lim_{y\to 0^+}\frac{v(x,y)-v(x,0)}{y^\sigma}$

$$= \sigma \lim_{y\to 0^+}\int_{\mathbb{R}^N}\frac{P(x-\xi,y)}{y^\sigma}(g(\xi)-g(x))\,d\xi = -\frac{\sigma d_{N,\sigma}}{C_{N,\sigma}}(-\Delta)^{\sigma/2}g$$

$$\frac{\partial v}{\partial y^\sigma} \equiv \mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = -(-\Delta)^{\sigma/2}g$$

An equivalent local problem

- $w = \mathrm{E}(\varphi(u)), \quad u = \varphi^{-1}(\mathrm{Tr}(w))$

$$\begin{cases} L_\sigma w = 0, & (x,y) \in \mathbb{R}^{N+1}_+,\ t > 0, \\ \dfrac{\partial w}{\partial y^\sigma} = \dfrac{\partial \varphi^{-1}(w)}{\partial t}, & x \in \mathbb{R}^N, y = 0,\ t > 0, \\ w = \varphi(f), & x \in \mathbb{R}^N, y = 0,\ t = 0. \end{cases}$$

- Some proofs (e.g. existence) may be easier in this formulation!
- Dynamical boundary conditions (Amann, Escher, Fila, Vitillaro, ...)

Weak solution (local problem)

Weak (L^1-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $w \in L^2_{\rm loc}((0,\infty) : X^\sigma(\mathbb{R}^{N+1}_+))$;
- $\displaystyle\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \psi}{\partial t}\, dxds - \mu_\sigma \int_0^\infty \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla w, \nabla \psi\rangle \, dxdyds = 0,$
  $\forall \psi \in C^1_{\mathsf{c}}\left(\overline{\mathbb{R}^{N+1}_+} \times (0,\infty)\right)$;
- $u(\cdot, 0) = f$ a.e.

► $\displaystyle \|v\|_{X^\sigma} = \left(\mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} |\nabla v|^2 \, dxdy\right)^{1/2}$

Extension / trace operators

- $\mathrm{E} : \dot{H}^{\sigma/2}(\mathbb{R}^N) \to X^\sigma(\mathbb{R}^{N+1}_+)$ isometry [Caffarelli-Silvestre, 2007]

- $$\mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle \nabla \mathrm{E}(\varphi), \nabla \mathrm{E}(\psi)\rangle = \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\varphi\,(-\Delta)^{\sigma/4}\psi$$

- $$\mathrm{Tr}(\Psi_1) = \mathrm{Tr}(\Psi_2)$$
  $$\Downarrow$$
  $$\int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle \nabla \mathrm{E}(\varphi), \nabla \Psi_1\rangle = \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle \nabla \mathrm{E}(\varphi), \nabla \Psi_2\rangle$$

- $\mathrm{Tr} : X^\sigma(\mathbb{R}^{N+1}_+) \to \dot{H}^{\sigma/2}(\mathbb{R}^N)$ surjective and continuous

- TRACE EMBEDDING: $\|\Phi\|_{X^\sigma} \geq \|\mathrm{E}(\mathrm{Tr}(\Phi))\|_{X^\sigma} = \|\mathrm{Tr}(\Phi)\|_{\dot{H}^{\sigma/2}}$

Steps towards regularity

- Mild $\Rightarrow$ weak
  - Energy estimates for associated elliptic problem
- Weak + bounded $\Rightarrow$ strong
  - Steklov averages
- Weak + conditions on the initial data $\Rightarrow$ bounded for $t \geq \tau > 0$
  - Approximation by bounded strong solutions
- Weak + bounded $\Rightarrow$ C^α for some $\alpha \in (0,1)$
  - [Athanasopoulos-Caffarelli, 2009] + positivity
- C^α for some $\alpha \in (0,1) \Rightarrow \underbrace{C^{1,\beta} \text{ for all } \beta \in (0,1)}_{\text{classical}} \Rightarrow$ Further regularity

$C^{\alpha} \Rightarrow C^{1,\beta}$ ($\sigma = 1$, $N = 1$) [dPQRV, preprint]

$$u_0 = u(x_0, t_0), \qquad \varphi'(u_0) > 0$$

$$\begin{aligned}
\partial_t u + \varphi'(u_0)(-\Delta)^{1/2} u &= -(-\Delta)^{1/2} \big(\varphi(u) - \varphi'(u_0)u\big) \\
&= -(-\Delta)^{1/2} \underbrace{\Big(\varphi(u) - \big(\varphi(u_0) + \varphi'(u_0)(u - u_0)\big)\Big)}_{F(u)}
\end{aligned}$$

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$$\begin{aligned}u(x,t) = &\int_{\mathbb{R}} P(x - x_1, \varphi'(u_0)t) f(x_1)\, dx_1 \\ &- \frac{1}{\varphi'(u_0)} \int_0^t \int_{\mathbb{R}} P(x - x_1, \varphi'(u_0)(t - t_1)) (-\Delta)^{1/2} F(u(x_1, t_1))\, dx_1 dt_1\end{aligned}$$

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$$= -(-\Delta)^{1/2}\underbrace{\left(\varphi(u) - \left(\varphi(u_0) + \varphi'(u_0)(u - u_0)\right)\right)}_{F(u)}$$

$$u(x,t) = \int_{\mathbb{R}} P(x - x_1, \varphi'(u_0)t) f(x_1)\, dx_1$$

$$- \frac{1}{\varphi'(u_0)} \underbrace{\int_0^t \int_{\mathbb{R}} (-\Delta)^{1/2} P(x - x_1, \varphi'(u_0)(t - t_1)) F(u(x_1, t_1))\, dx_1 dt_1}_{g(x,t)}$$

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$$P(x,t) = \frac{1}{\pi}\frac{t}{x^2+t^2}, \qquad (-\Delta)^{1/2} P(x,t) = \frac{1}{\pi}\frac{x^2-t^2}{(x^2+t^2)^2} \equiv A(x,t)$$

Doubling Hölder [Caffarelli-Vasseur, 2010]

- Step 1: C^α for $\alpha \in (0, 1/2) \Rightarrow C^{2\alpha}$
  - $F(u_0) = F'(u_0) = 0$
  - $|F''(u)| = |\varphi''(u)| \leq C$ for $0 < \delta \leq u \leq K$ + positivity
  - $|F(u(x,t))| = \frac{|F''(\xi)|}{2} |u(x,t) - u(x_0,t_0)|^2 \leq C|(x,t) - (x_0,t_0)|^{2\alpha}$
- Step 2: C^α for all $\alpha \in (0,1) \Rightarrow C^{1,\beta}$ for all $\beta \in (0,1)$
  - Same ideas for second increments
- Examples: $\varphi(u) = u^m,\ (1+u)^m, \quad m > 0, \qquad \varphi(u) = \log(1+u)$
- Extensions: $N \geq 1$, $0 < \sigma < 2$

Smoothing effect PME

$$\blacktriangleright \begin{cases} u_t + (-\Delta)^{\sigma/2}(u^m) = 0, \\ u(\cdot,0) = f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \end{cases} \qquad m > 0$$

THEOREM: $p > \max\{1, (1-m)N/\sigma\}$ $\qquad (\underline{\geq})$

$$\Downarrow$$

$$\sup_{x\in\mathbb{R}^N} |u(x,t)| \leq C\, t^{-N/(N(m-1)+\sigma p)} \|f\|_p^{\sigma p/(N(m-1)+\sigma p)}$$

- Functional inequalities + Moser iteration

Stroock-Varopoulos inequality, $0 < \sigma < 2$ [S, 1984], [V, 1985]

THM: $\psi' = (\Psi')^2 \quad \Rightarrow \quad \displaystyle\int_{\mathbb{R}^N} \psi(v)(-\Delta)^{\sigma/2} v \geq \int_{\mathbb{R}^N} \left|(-\Delta)^{\sigma/4}\Psi(v)\right|^2$

$$
\begin{aligned}
\int_{\mathbb{R}^N} \psi(v)(-\Delta)^{\sigma/2} v &= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\psi(v)(-\Delta)^{\sigma/4} v \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \mathrm{E}(\psi(v)), \nabla\, \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ \psi' \quad \langle \nabla\, \mathrm{E}(v), \nabla\, \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\, |\nabla \Psi(\mathrm{E}(v))|^2 \\
&\geq \int_{\mathbb{R}^N} \left|(-\Delta)^{\sigma/4}\Psi(v)\right|^2
\end{aligned}
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ \psi' \quad \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v)\rangle \\
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \; \psi' \quad \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v) \rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \left| \nabla \Psi(\mathrm{E}(v)) \right|^2 \\
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \psi(\mathrm{E}(v)), \nabla \mathrm{E}(v) \rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} (\Psi')^2 \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v) \rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \left| \nabla \Psi(\mathrm{E}(v)) \right|^2 \\
&\geq \int_{\mathbb{R}^N} \left| (-\Delta)^{\sigma/4} \Psi(v) \right|^2
\end{aligned}
$$

Hardy-Littlewood-Sobolev's inequality

THEOREM [H-L, 1928], [S, 1938]: $0 < \sigma < \min\{2, N\}$

$$\Downarrow$$

$$\|\phi\|_{\frac{2N}{N-\sigma}} \leq C\|(-\Delta)^{\sigma/4}\phi\|_2, \qquad (\dot{H}^{\sigma/2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-\sigma}}(\mathbb{R}^N))$$

$$1 = N \leq \sigma < 2?$$

Classical Nash inequality [Nash, 1958]

THEOREM: $0 < \sigma \le 2, \quad \phi \in L^1(\mathbb{R}^N) \cap \dot{H}^{\sigma/2}(\mathbb{R}^N)$

$$\Downarrow$$

$$\|\phi\|_2^{1+\frac{\sigma}{N}} \le C\|\phi\|_1^{\sigma/N}\,\|\phi\|_{\dot{H}^{\sigma/2}}$$

- $\displaystyle\int_{|\xi|\ge\rho} |\hat{\phi}(\xi)|^2\,d\xi \le \rho^{-\sigma}\int_{|\xi|\ge\rho}|\xi|^\sigma|\hat{\phi}(\xi)|^2\,d\xi \le \rho^{-\sigma}\|\phi\|^2_{\dot{H}^{\sigma/2}}$

- $\displaystyle\int_{|\xi|\le\rho} |\hat{\phi}(\xi)|^2\,d\xi \le \omega_N\rho^N\|\phi\|_1^2$

- $\|\phi\|_2^2 = \|\hat{\phi}\|_2^2 \le \rho^{-\sigma}\|\phi\|^2_{\dot{H}^{\sigma/2}} + \omega_N\rho^N\|\phi\|_1^2$

A Nash-Gagliardo-Nirenberg inequality [dPQRV, to appear]

THEOREM: $0 < \sigma < 2, \quad \phi \in L^p(\mathbb{R}^N) \cap \dot{H}^{\sigma/2}(\mathbb{R}^N)$

$$\Downarrow$$

$$\|\phi\|_{\frac{N(p+2)}{2N-\sigma}}^{1+\frac{p}{2}} \leq C \|\phi\|_p^{p/2} \, \|\phi\|_{\dot{H}^{\sigma/2}}$$

- Stroock-Varopoulos: $\displaystyle\int_{\mathbb{R}^N} \phi^{\frac{p}{2}} (-\Delta)^{\frac{\sigma}{4}} \phi \geq C \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\sigma}{8}} \phi^{\frac{p+2}{4}} \right|^2$
- Hardy-Littlewood-Sobolev: $\displaystyle\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\sigma}{8}} \phi^{\frac{p+2}{4}} \right|^2 \geq \left(\int_{\mathbb{R}^N} \phi^{\frac{N(p+2)}{2N-\sigma}} \right)^{\frac{2N-\sigma}{2N}}$
- Hölder: $\displaystyle\int_{\mathbb{R}^N} \phi^{\frac{p}{2}} (-\Delta)^{\frac{\sigma}{4}} \phi \leq \|\phi\|_p^{p/2} \, \|\phi\|_{\dot{H}^{\sigma/2}}$

Smoothing effect PME: proof (Moser's iteration)

- Multiply by u^{p_k-1}, integrate on $\mathbb{R}^N \times (t_k, t_{k+1})$, $t_k = (1-2^{-k})t$:

$$\int_{\mathbb{R}^N} u^{p_k}(\cdot, t_k) = p_k \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}(u^m) u^{p_k-1} + \int_{\mathbb{R}^N} u^{p_k}(\cdot, t_{k+1})$$

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$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot, \tau)\|_2^2 \, d\tau \qquad \text{(SV inequality)}$$

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$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot, \tau)\|_2^2 \, d\tau$$

$$\geq \frac{C}{\|u(\cdot, t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot, \tau)\|_{p_k}^{p_k} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot, \tau)\|_2^2 \, d\tau \quad (L^p\text{-decay})$$

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$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2 \, d\tau$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{p_k}^{p_k} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2 \, d\tau$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1} \, d\tau \quad \text{(NGN inequality)}$$

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- Multiply by u^{p_k-1}, integrate on $\mathbb{R}^N \times (t_k, t_{k+1})$, $t_k = (1-2^{-k})t$:

$$\int_{\mathbb{R}^N} u^{p_k}(\cdot, t_k) = p_k \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}(u^m) u^{p_k-1} + \int_{\mathbb{R}^N} u^{p_k}(\cdot, t_{k+1})$$

$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot, \tau)\|_2^2 \, d\tau$$

$$\geq \frac{C}{\|u(\cdot, t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot, \tau)\|_{p_k}^{p_k} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot, \tau)\|_2^2 \, d\tau$$

$$\geq \frac{C}{\|u(\cdot, t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot, \tau)\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1} \, d\tau$$

$$\geq \frac{C 2^{-(k+1)} t}{\|u(\cdot, t_k)\|_{p_k}^{p_k}} \|u(\cdot, t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1} \qquad (L^p\text{-decay})$$

Smoothing effect PME: proof (Moser's iteration)

- $\|u(\cdot,t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}} \le \left(\frac{c}{t}\right)^{\frac{1}{2p_k+m-1}} 2^{\frac{k+1}{2p_k+m-1}} \|u(\cdot,t_k)\|_{p_k}^{\frac{2p_k}{2p_k+m-1}}$

- $p_{k+1} \equiv \dfrac{N(2p_k+m-1)}{2N-\sigma} > p_k$ if $p_0 = p > \dfrac{(1-m)N}{\sigma}$

- Iteration

Smoothing effect LFDE

- $$\begin{cases} u_t + (-\Delta)^{1/2}\log(1+u) = 0, \\ u(\cdot,0) = f \in \mathcal{X} := \left\{ f \geq 0 : \int_{\mathbb{R}} (1+f)\log(1+f) < \infty \right\} \end{cases}$$
- $L^1_+(\mathbb{R}) \cap L^p(\mathbb{R}) \subset \mathcal{X} \subset L^1_+(\mathbb{R}), \quad p > 1$
- $\times \log(1+u), \quad \times \partial_t(\log(1+u))$:
  - $u(\cdot,t) \in \mathcal{X}$ for all $t > 0$
  - $$\underbrace{\int_{\mathbb{R}} |(-\Delta)^{1/4}\log(1+u)(x,t)|^2\,dx}_{2E(t)} \leq \frac{1}{t}\int_{\mathbb{R}} ((1+f)\log(1+f) - f)$$

THEOREM: $f \in \mathcal{X} \quad \Rightarrow \quad u(\cdot,t) \in L^\infty(\mathbb{R})$ for all $t > 0$

A Trudinger type inequality [Strichartz, 1972]

NGN inequality: $\|\phi\|_{\frac{N(p+2)}{2N-\sigma}}^{1+\frac{p}{2}} \leq C\,p\|\phi\|_p^{p/2}\,\|\phi\|_{\dot{H}^{\sigma/2}}$

- $\sigma < N$: $\frac{N(p+2)}{2N-\sigma} > p \Leftrightarrow p < \frac{2N}{N-\sigma}$
- $\sigma \geq N = 1$: $\frac{N}{2N-\sigma} \geq 1 \Rightarrow \frac{N(p+2)}{2N-\sigma} \geq p+2$
  - $\sigma > N$: $\phi \in L^\infty(\mathbb{R})$
  - $\sigma = N$: $\phi \in L^q(\mathbb{R}),\ q \geq p, \quad \phi \notin L^\infty(\mathbb{R})$

THEOREM: $\|\phi\|_{H^{1/2}(\mathbb{R})} \leq 1 \quad \Rightarrow \quad \int_{\mathbb{R}} (e^{\alpha\phi^2(x)} - 1)\,dx \leq 1$

Smoothing effect LFED: proof

Step 1: $L^p \to L^\infty, \quad p > 1$

- Multiply the equation by $\dfrac{u^{p-1}}{p-1} + \dfrac{u^p}{p}$

Step 2: $\mathcal{X} \to L^2$

- $f \in \mathcal{X} \Rightarrow w := \log(1+u)(\cdot, t) \in H^{1/2}(\mathbb{R})$.
  - NGN inequality + energy estimate
- $e^{w^2/c} - 1 \in L^1(\mathbb{R})$
  - Trudinger inequality
- $u = e^w - 1 \in L^2(\mathbb{R})$
  - Calculus inequality: $(e^w - 1)^2 \leq (e^c - 1)(e^{w^2/c} - 1)$

A nonlocal transport equation

- Hilbert transform: $$H(v)(y) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{v(s)}{y-s}\,ds$$
- $N = 1$: $$(-\Delta)^{1/2}v(y) = H(\partial_y v)(y) = \partial_y H(v)(y)$$
- Bäcklund transformation:

$$\left.\begin{array}{l} \bullet\ y = \displaystyle\int_0^x (1+u(s,t))\,ds - c(t), \\ \quad c'(t) = H(\log(1+u(x,t)))(0,t) \\ \\ \bullet\ \tau = t, \end{array}\right\} \quad (y,\tau) = J(x,t)$$

  - $v(y,\tau) = \log(1+u(x,t))$
  - $\partial_x y = 1+u, \quad \partial_t y = -\underbrace{H(\log(1+u))}_{\widetilde{H}(v)}, \quad \widetilde{H}(v) = H(v\circ J)\circ J^{-1}$

$$\partial_\tau v - \widetilde{H}(v)\partial_y v = -\partial_y \widetilde{H}(v)$$

A nonlocal transport equation: global existence

- $v(\cdot,\tau) \in L^1(\mathbb{R}) \;\Leftrightarrow\; u(\cdot,t) \in \mathcal{X}$
  - $\displaystyle\int_{\mathbb{R}} v(y,\tau)\,dy = \int_{\mathbb{R}} (1+u(x,t))\log(1+u(x,t))\,dx$

THEOREM: $v(x,0) = v_0 \in L^1(\mathbb{R})$, $v_0 \geq 0$

$\Downarrow$

$\exists$! classical global in time solution

- positive and bounded for $\tau > 0$
- …

$\widetilde{H} \to H$

$$\partial_\tau v - H(v)\partial_y v = -\partial_y H(v)$$

- 1-D model for 2-D Quasi-Geostrophic and 3-D incompressible Euler
  - [Córdoba-Córdoba-Fontelos, 2005]
- Earlier 1-D models:
  - [Constantin-Lax-Majda, 1985], [Baker-Li-Morlet, 1996], ...
- $v_0 \in H^{1/2}(\mathbb{R})$:
  - No viscosity $\Rightarrow$ Blow-up (in C^1) occurs for some data
  - Critical viscosity $\Rightarrow$ global $\exists$, smoothness [Kiselev, 2010]

Open problems / Work in progress

- $C^{1,\beta}$ for all $\beta \quad \Rightarrow \quad C^\infty$
- $C^{1,\beta}$ regularity for $\sigma \neq 1$
- $\varphi(u) = \log u$
- $m \to 0^+$
- $\varphi(u) = u^m, \quad 1 - \sigma < m < 0 \qquad (N = 1,\ \sigma > 1)$
- $\partial_\tau v - H(v)\partial_y v = -\partial_y H(v)$
- …

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