Regularity theory and Asymptotic behaviors in Integro-differential equations

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1. Introduction to Nonlocal Equations

$$\mathcal{L}[u](x) = (1-\sigma)c_\sigma \int_{\mathbf{R}^n} \left[u(x+y) - u(x) - y \cdot \nabla u(x)\, \chi_{B_1}(y)\right] K(y)dy$$

where

$$K(y) \approx \frac{1}{|y|^{n+2\sigma}}, \quad 0 < \sigma < 1.$$

Let $\delta(u,x,y) = u(x+y) - u(x) - y \cdot \nabla u(x)\, \chi_{B_1}(y)$.

- If u is C^2 in a nbd of x,

$$\frac{\delta(u,x,y)}{|y|^{n+2\sigma}} \approx \frac{|D^2u(x)||y|^2}{|y|^{n+2\sigma}} \approx \frac{C}{|y|^{n+2\sigma-2}}$$

  is integrable in B_1 if $\sigma < 1$.

- If $|\delta(u,x,y)| \leq C$,

$$\left|\frac{\delta(u,x,y)}{|y|^{n+2\sigma}}\right| < \frac{C}{1+|y|^{n+2\sigma}}$$

  for $|y| > 1$, is integrable if $\sigma > 0$.

Symmetry v.s. Nonsymmetry

- If $K(y)$ is symmetric Kernels i.e. $K(-y) = K(y)$,

$$\mathcal{L}[u](x) = (1-\sigma)c_\sigma \int_{\mathbf{R}^n} [u(x+y) + u(x-y) - 2u(x)]\, K(y)dy.$$

  The effect of "$y \cdot \nabla u(x)\, \chi_{B_1}(y)$" vanishes.

$$(-\triangle)^\sigma [u](x) = (1-\sigma)c_\sigma \int_{\mathbf{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\sigma}} dy$$

  is called a fractional Laplacian.
- If K is not symmetric, then "$y \cdot \nabla u(x)\chi_{B_1}(y)$" persists. And $\mathcal{L}[u](x) = (1-\sigma)c_\sigma \int_{\partial\Omega} \delta(u,x,y)K(y)dy$ still has "$y \cdot \nabla u(x)\chi_{B_1}(y)$" even though $K(y)$ is symmetric since $\partial\Omega$ is not symmetric with respect to the origin.

Jump Process

The Laplacian is the infinitesimal generator of the Brownian motion which is continuous process.

$$\mathcal{L}[u](x) = (1-\sigma)c_\sigma \int_{\mathbf{R}^n} \left[u(x+y) - u(x) - y \cdot \nabla u(x)\, \chi_{B_1}(y)\right] K(y)dy$$

describes the infinitesimal generator of a given purely jump processes,. i.e. processes without diffusion or drift part. Nonlinear integro-differential operators come from the stochastic control theory related with

$$\mathcal{I}u(x) = \sup_\alpha \mathcal{L}_\alpha u(x),$$

or game theory associated with

$$\mathcal{I}u(x) = \inf_\beta \sup_\alpha \mathcal{L}_{\alpha\beta} u(x), \tag{1}$$

when the stochastic process is of Lèvy type allowing jumps.

Nonlocal Curvature

The minimizer u_η of

$$\eta \int |\nabla u|^2 dx + \frac{1}{\eta} \int F(u) dx$$

with a double well potential converges to $u_0 = \chi_\Omega - \chi_{\Omega^c}$ and the interface satisfies the mean curvature equation. In phase transition "free energy" , nonlocal energy corresponds to processes with large scale correlation (like solidification) where information far away from the interface has direct influence in it.

$$E_\eta(u) = \eta \int \int [u(x) - u(y)]^2 K(x,y) dxdy + \frac{1}{\eta} \int F(u) dx$$

If the kernel is $K \approx \frac{1}{|x-y|^{n+2\sigma}}$, then $u_0 = \chi_\Omega - \chi_{\Omega^c}$ minimize

$$E(\Omega) = (1-\sigma) \int \int \frac{[u(x) - u(y)]^2}{|x-y|^{n+2\sigma}} dx \quad \text{or} \quad (1-\sigma) \int \int \frac{\chi_\Omega(x) \chi_{\Omega^c}(y)}{|x-y|^{n+2\sigma}} dxdy.$$

(Other examples) Fractional Mean Curvature flows, Fractional Conformal Laplacian and fractional Yamabe problems

2. Point of views

(1) via Fourier transformation

- $\widehat{(-\triangle)}^{\sigma}[u] = |\xi|^{2\sigma}\hat{u}$
- $(-\triangle)^{\sigma_1}(-\triangle)^{\sigma_2}[u] = (-\triangle)^{(\sigma_1+\sigma_2)}[u]$ and $(-\triangle)^0[u] = u$.
- If $\sigma\rangle 1$, then $(-\triangle)^{\sigma}[u] = -\triangle u$.

- - - It works for linear equations,

(2)

$$(-\triangle)^{\sigma}u = f(u)$$

implies

$$u(x) = \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-2\sigma}} f(u)dy.$$

When $f(u) = u^{\frac{n+2\sigma}{n-2\sigma}}$, then u is invariant under Kelvin Transform and it has some scaling invariance.

(3) Dirichlet to Neumann map

- - - Some nonlocal equation can be local equation by extending it to higher dimensional space.
Observation: $\triangle_x u^* + u^*_{yy} = 0$ with $u^*(x,0) = u(x)$ implies

$$\begin{aligned} (-\partial_z)^2 u^* = u^*_{yy} = (-\triangle_x)u^* \\ \partial_y u^* = -(-\triangle_x)^{1/2} u^* \end{aligned} \qquad (2)$$

At $y = 0$, $u \rangle (-\triangle_x)^{1/2} u$ is a map from Dirichlet data to Neumann data.

Now we obtain the same interpretation for any fractional power $(-\triangle)^\sigma$ $(0 < \sigma < 1)$ as the Dirichlet to Neumann operator of an appropriate extension $u^*(x,y)$ satisfying the following extension problem:

$$\begin{cases} -\nabla(y^a \nabla u^*(x,y)) = 0 & \text{in } \mathbf{R}^n \times \mathbf{R}^+ \\ u^*(x,0) = u(x) & \text{on } \mathbf{R}^n \\ -(-\triangle)^\sigma u(x) = \lim_{y\to 0} y^a u^*_y(x,y), \quad (a = 1 - 2\sigma), & \text{on } \mathbf{R}^n. \end{cases} \qquad (3)$$

Remark:

(4) The Balance in the inetegral

- - - It woks for general operators. In some case, the integral can be decomposed to

$$f(x) \geq \int_{B_1} \frac{\Lambda\delta_+}{|y|^{n+2\sigma}} - \int_{B_1} \frac{\lambda\delta_+}{|y|^{n+2\sigma}} - \int_{\mathbf{R}^n\setminus B_1} \frac{\Lambda\delta_+ - \lambda\delta_+}{|y|^{n+2\sigma}}$$

First, $\int_{\mathbf{R}^n\setminus B_1} \frac{\Lambda\delta_+ - \lambda\delta_+}{|y|^{n+2\sigma}}$ is bounded.
And if a concave function touches u at x from above in B_1, then $\delta_- = 0$ on B_1.
Now we can control

$$f(x) = \int_{B_1} \frac{\Lambda\delta_-}{|y|^{n+2\sigma}} = \sum_k \int_{B_{2^{-k}}\setminus B_{2^{-(k+1)}}} \frac{\Lambda\delta_+}{|y|^{n+2\sigma}}.$$

3. Harnack Inequalities and Hölder Continuity
3.1 Elliptic Local Equation

Theorem (Harnack Inequality)

If u is a nonnegative harmonic function in $B_R(0)$, then we have

$$\sup_{B_{R/2}} u \leq C \inf_{B_{R/2}} u$$

for a uniform constant $C > 0$.

Corollary (Oscillation Lemma)

$$\mathrm{osc}_{B_{R/2}} u \leq \gamma\, \mathrm{osc}_{B_R} u \quad \textit{for some } 0 < \gamma < 1.$$

by applying Harnack Inequality on $M(R) - u$ and $u - m(R)$ for $M(R) = \sup_{B_R} u$ and $M(R) = \inf_{B_R} u$

Theorem (Hölder Regularity)

$$||u||_{C^\alpha(B_{R/2})} \leq C||u||_{L^\infty(B_R)} \quad \textit{for some } 0 < \alpha < 1.$$

3.2 Parabolic Local Equation

Theorem (Harnack Inequality)

$\mathrm{Q}_r \fallingdotseq Q_r(0, r^\sigma) = B_r \times (0, r^\sigma]$,
$\mathrm{Q}_r^- \fallingdotseq \mathrm{Q}_r + (0, r^\sigma) = B_r \times (r^\sigma, 2r^\sigma)$ *(Cube at past time)* ,
$\mathrm{Q}_r^+ \fallingdotseq \mathrm{Q}_r^- + (0, 2r^\sigma) = B_r \times (3r^\sigma, 4r^\sigma)$ *(Cube at future time)*, $\mathrm{Q}_r(x_0, t_0) = \mathrm{Q}_r + (x_0, t_0)$.
If u is a nonnegative caloric function in Q_{3R}*, then we have*

$$\sup_{\mathrm{Q}_R^-} u \le C \inf_{\mathrm{Q}_R^+} u$$

for a uniform constant $C > 0$.

There is a time delay to control the lower bound in a small neighborhood of a point by the current value at the point, which is a main difference between elliptic and parabolic equations.
Oscillating Lemma and *Hölder Regularity* come from the similar arguments.

Nonlocal Equations

- Classical Harnack Inequality for fractional Laplacian fails if u is nonnegative only on a bounded set B_R (Kassmann). So we assume $u \geq 0$ in $\mathbb{R}^n$.
- Harnack inequality doesn't imply Hölder regularity directly because of the assumption: $u \geq 0$ in $\mathbb{R}^n$.
- If the kernel is allowed to oscillate between two different σ_1 and σ_2, there could be a discontinuous solution (Barlow, Bass, Chen & Kassmann'09)

Known Results in Nonlocal Equations

- Probabilistic Approach
  (Song& Vondracek '04, Bass & Levin '02, Kassmann, Chen...)
  Yes for Linear Equation with symmetric or nonsymmetric Kernel.
- Analytic Approch
  (Caffarelli& Silvestre '07)
  Yes for Nonlinear Elliptic Equation with symmetric Kernel.
  (Sung Hoon Kim & L. '09)
  Yes for the singular nonlocal parabolic equation $u_t + (-\triangle)^{\sigma} u^m = 0$.
  (Yong-Cheol Kim & L. '09)
  Yes for Nonlinear Elliptic Equation with non- symmetric Kernel for $1 < \sigma < 2$.
  (Yong-Cheol Kim & L. '10)
  Yes for Nonlinear Elliptic Equation in some large class with non- symmetric Kernel for $0 < \sigma < 2$.
  (H.C. Lara and G. Dávila '11)
  Yes for Nonlinear Elliptic Equation in some small class with non- symmetric Kernel for $0 < \sigma < 2$ after dropping off ∇u.
  (Yong-Cheol Kim & L. '12)
  Yes for Nonlinear Parabolic Equation with symmetric Kernel for $0 < \sigma < 2$.

Key Observations

(1.) Ellipric equation with Nonsymmetric Kernel

- $\int_{\mathbf{R}^n} \frac{|y\cdot\nabla u(x)\,\chi_{B_1}(y)|}{|y|^{n+\sigma}} dy$ needs to be controlled.
- The equation is not scaling invariant due to $|\chi_{B_1}(y)|$
- The influence of the gradient term is very large if we want to control the solution in a small region.
- The error term can be regarded as a drift term. $(1 < \sigma \leq 2)$ and $(0 < \sigma \leq 1)$ requires different technique due to the difference of the blow rate as $|y|$ approaches to zero and the decay rate as $|y|$ approaches to infinity.
- When $(1 < \sigma \leq 2)$, better decay rate of kernel allows Hölder regularity in a larger class where there is scaling-invariance: $y \cdot \nabla u(x)\,\chi_{B_R}(y)$ for all $R \geq 1/2$ has been considered.

(2.) Parabolic Equation with symmetric Kernel

The quations nonlocal in space variable , but local in time variable.

Classes of Operators

Set

$$\mathcal{L}[u](x) = (2-\sigma)c_\sigma \int_{\mathbb{R}^n} \Big[u(x+y) - u(x) - y\cdot\nabla u(x)\,\chi_{B_1}(y)\Big] K(y)dy,$$

$$\mathcal{L}_{\alpha\beta}[u](x;\nabla\varphi) = (2-\sigma)c_\sigma \int_{\mathbb{R}^n} \Big[u(x+y) - u(x) - y\cdot\nabla\varphi(x)\,\chi_{B_1}(y)\Big] K_{\alpha\beta}(y)dy,$$

and

$$\mathcal{I}[u](x;\nabla\varphi(x)) = \sup_\alpha \inf_\beta \mathcal{L}_{\alpha\beta}[u](x;\nabla\varphi).$$

Lemma

u is a viscosity supersolution to $\mathcal{I}u = f$ on Ω if and only if $\mathcal{I}u(x;\nabla\varphi(x))$ is well-defined and

$$\mathcal{I}u(x;\nabla\varphi(x)) \le f(x)$$

for any $x \in \Omega$ and any C^2-function φ touching u from below at x.

$$\mu(u, x, y; \nabla\varphi) = (u(x + y) - u(x)) + y \cdot \nabla\varphi(x)\chi_{B_R}(y)$$

μ is the difference between the value of u and its linear approximation.

$$\mu^+(u, x, y; \nabla\varphi) = \max(\mu(u, x, y; \nabla\varphi), 0)$$

$$\mu^-(u, x, y; \nabla\varphi) = \max(-\mu(u, x, y; \nabla\varphi), 0).$$

$$\mathcal{M}^+[u](x; \nabla\varphi(x)) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda\mu^+ - \lambda\mu^-}{|y|^{n+\sigma}} \, dy,$$

$$\mathcal{M}^-[u](x; \nabla\varphi(x)) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\lambda\mu^+ - \Lambda\mu^-}{|y|^{n+\sigma}} \, dy.$$

- We consider a collection of operators such that $\mathcal{M}^-[u](x;\nabla\varphi(x)) \leq \mathcal{I}[u](x;\nabla\varphi(x)) \leq \mathcal{M}^+[u](x;\nabla\varphi(x))$.
- $\mathcal{S}^+(\lambda,\Lambda,|f|)$ is a collection of super-solutions of $\mathcal{M}^-[u] = |f(x)|$ and $\mathcal{S}^-(\lambda,\Lambda,|f|)$ is a collection of sub-solutions of $\mathcal{M}^+[u] = -|f(x)|$.
- Any solution of $\mathcal{I}[u] = f(x)$ belongs to $\mathcal{S}(\lambda,\Lambda,|f|) = \mathcal{S}^+(\lambda,\Lambda,|f|) \cap \mathcal{S}^-(\lambda,\Lambda,|f|)$.
- If $u \in \mathcal{S}(\lambda,\Lambda,|f|)$, then this class is invariant under the scaling: $\frac{u(\eta x+x_0)}{\eta^\sigma} \in \mathcal{S}(\lambda,\Lambda,|f|)$ and $u(\eta x + x_0) \in \mathcal{S}(\lambda,\Lambda,\eta^\sigma|f|)$ for $0 < \eta < 1$.

Subclass for Elliptic Nonlocal Equations with nonsymmetric kernel

Definition

Let $0 < \eta \le 1$ and $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$, where $\mathfrak{L}$ is a class of linear integro-differential operators. Then we say that $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}_{\eta}$ if, for $R \in (0,1]$, there are $\mathcal{B}^{\pm}_{R} : \mathbf{R}^n \to \mathbf{R}$ such that

1. *$\mathcal{B}^{\pm}_{R}(\cdot)$ is homogeneous of degree one, i.e. $\mathcal{B}^{\pm}_{R}(0) = 0$ and $\mathcal{B}^{\pm}_{R}(\nabla u) = \mathcal{B}^{+}_{R}\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u|$ for $|\nabla u| \neq 0$,*
2. $|\{\nu \in S^{n-1} : \mathcal{B}^{\pm}_{R}(\nu) < 0\}| \ge \eta |S^{n-1}| > 0$,
3. *$\mathcal{M}^{-}_{\mathfrak{L},R,\eta}u(x) \le \mathcal{I}u(x) - \mathcal{I}0(x) \le \mathcal{M}^{+}_{\mathfrak{L},R,\eta}u(x)$ whenever $u \in \mathrm{C}^{1,1}[x] \cap \mathrm{B}(\mathbb{R}^n)$ for $x \in B_R$,*

where $\mathcal{M}^{\pm}_{\mathfrak{L},R,\eta}u(x) := \mathcal{M}^{\pm}_{\mathfrak{L},R}u(x) \pm \mathcal{B}^{\pm}_{R}(\nabla u(x)) \pm (2-\sigma)CR^{1-\sigma}|\nabla u(x)|$.

Definition

For $0 < R < 1$, the drift vector $\mathfrak{b}_{\mathcal{L},R}$ of a linear operator $\mathcal{L}$ at R is defined by

$$\mathfrak{b}_{\mathcal{L},R} = (2-\sigma)\int_{B_1 \setminus B_R} y\,K(y)\,dy.$$

Lemma

Let $0 < \sigma < 2$ *and* $0 < R < 1$. *Let* $\mathfrak{L}$ *be a class of linear integro-differential operators. Then we have the following results:*
(1) *If* $\mathcal{L} \in \mathfrak{L}$, *then* $\mathcal{L} \in \mathcal{S}_\eta^{\mathfrak{L}}$ *for* $0 < \eta \le \frac{1}{2}$.
(2) *Assume that there is a unit vector* $\mathfrak{a}$ *such that for any nonzero drift vector* $\mathfrak{b}_{\mathcal{L}_i,R}$, $\left\langle \mathfrak{a}, \frac{\mathfrak{b}_{\mathcal{L}_i,R}}{|\mathfrak{b}_{\mathcal{L}_i,R}|} \right\rangle > 0$. *If* $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$ *satisfies that*

$$\min_{i=1,\cdots,N} \mathcal{L}_i u(x) \le \mathcal{I} u(x) - \mathcal{I} 0(x) \le \max_{i=1,\cdots,N} \mathcal{L}_i u(x)$$

whenever $u \in \mathrm{C}^{1,1}[x] \cap \mathrm{B}(\mathbb{R}^n)$ *for* $x \in B_R$, *then* $\mathcal{I} \in \mathcal{S}_\eta^{\mathfrak{L}}$ *for some* $\eta > 0$.
(3) *Assume that there is a vector* $\mathfrak{a} \in S^{n-1}$ *and* $\eta > 0$ *such that for any nonzero drift vectors* $\mathfrak{b}_{\mathcal{L}_\alpha,R}$, $\left\langle \mathfrak{a}, \frac{\mathfrak{b}_{\mathcal{L}_\alpha,R}}{|\mathfrak{b}_{\mathcal{L}_\alpha,R}|} \right\rangle \ge \eta^{\frac{1}{n-1}}$ *for any* $\alpha \in I$. *If* $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$ *satisfies that*

$$\min_{\alpha \in I} \mathcal{L}_\alpha u(x) \le \mathcal{I} u(x) - \mathcal{I} 0(x) \le \max_{\alpha \in I} \mathcal{L}_\alpha u(x)$$

whenever $u \in \mathrm{C}^{1,1}[x] \cap \mathrm{B}(\mathbb{R}^n)$ *for* $x \in B_R$, *then we have* $\mathcal{I} \in \mathcal{S}_\eta^{\mathfrak{L}}$.

Lemma

(1) *Let* $0 < \sigma < 2$ *and let* $\mathfrak{L}$ *be a class of linear integro-differential operators. If* $\mathcal{L} \in \mathcal{S}^{\mathfrak{L}}$ *with a symmetric kernel* K*, then* $\mathcal{L} \in \mathcal{S}_{\eta}^{\mathfrak{L}}$ *for some* $\eta \in (0, 1]$*. In addition, if* $\mathcal{L}_{\alpha}$ *has symmetric kernel for all* α*, then* $\sup_{\alpha} \mathcal{L}_{\alpha}, \inf_{\alpha} \mathcal{L}_{\alpha} \in S_{\eta}^{\mathfrak{L}}$ *for* $1 \geq \eta > 0$*.*
(2) *If* $1 < \sigma < 2$*, then* $\mathcal{S}^{\mathfrak{L}_0} = \mathcal{S}_{\eta}^{\mathfrak{L}_0}$ *for some* $\eta \in (0, 1]$*.*

Theorems for Elliptic Nonlocal Equations

Theorem

Let $\sigma_0 \in (1,2)$. If u is a bounded function on $\mathbb{R}^n$ such that

$$\mathcal{M}^-_{\mathfrak{L}_0} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}$$

or

$$\mathcal{M}^-_{\mathfrak{L}_0,R,\eta} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0,R,\eta} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \leq 1 \text{ on } B_{2R}$$

in the viscosity sense, then there is some constant $\alpha > 0$ (depending only on λ, Λ, n and σ_0) such that

$$\|u\|_{C^\alpha(B_{R/2})} \leq \frac{C}{R^\alpha}\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0 \Big)$$

where $C > 0$ is some universal constant depending only on α.

Theorem

For a given $\sigma_0 \in (0,2)$, *let* $\sigma_0 < \sigma < 2$. *If* $u \in \mathrm{B}(\mathbb{R}^n)$ *is a positive function such that*

$$\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}$$

or

$$\mathcal{M}^-_{\mathfrak{L}_0,R,\eta} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0,R,\eta} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \le 1 \text{ on } B_{2R}$$

in the viscosity sense, then there is some constant $C > 0$ *depending only on* $\lambda, \Lambda, n, \sigma_0$, *and* $\|u\|_{L^\infty(\mathbb{R}^n)}$ *such that*

$$\sup_{B_{R/2}} u \le C\Big(\inf_{B_{R/2}} u + C_0\Big).$$

Steps of Proof

- Nonlocal A-B-P estimate:
  It gives the lower bound of the contact set between the function and the concave envelope , $\Gamma(u)$, in terms of it's supremum, $\sup u^+$ when u is nonpositive outside a given domain.
- Construction of Barrier $\Psi(x)$: It will be used to transform a general solution $u(x)$ to a function $u(x) + \Psi(x)$ which satisfies the configuration at the condition of A-B-P estimate.

- Decay estimate for the upper level sets

$$\Big|\{u > t\} \cap B_r(x_0)\Big| \leq Cr^n(u(x_0) + C_0 r^\sigma)t^{-\eta_0}$$

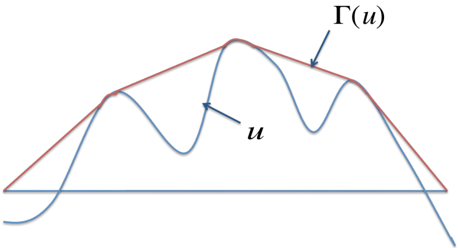
. C-Z decomposition is used at this step.

- Reducing the oscillation of solution with uniform factor $0 < \gamma < 1$.
Main Induction: there are $0 < m_k < M_k < \infty$ and $0 < \alpha < 1$ such that $|M_k - m_k| < \left(\frac{1}{2^k}\right)^\alpha$ and $m_k \leq u(x) \leq M_k$ for all $x \in B_{1/2^k}$.
Idea: apply the decay estimate on

$$w = \max\left(\frac{u(x/2^k) - m_k}{(M_k + m_k)/2}, 0\right) \geq 0 \quad \text{on } \mathbb{R}^n$$

by showing that w satisfies a nice equation in $B_{3/4}$ since $w > 0$ in B_1,

Gradient Map



$\Gamma(u)$ is the concave envelope of u in B_R. $\nabla\Gamma$ is the gradient map. Set $\mathcal{C}(u,\Gamma,B_R)$ $=\{y\in B_R : u(y)=\Gamma(y)\}$ be the contact set between u and $\Gamma(u)$.
$\mathcal{C}^+(u,\Gamma,B_R;b)=\mathcal{C}(u,\Gamma,B_R)\cap\{y\in B_R : b\cdot\nabla\Gamma(y)\geq 0\}$. $B_d\subset\nabla\Gamma(B_R)=\nabla\Gamma(\mathcal{C}(u,\Gamma,B_R))$ for $d=\frac{\sup_{B_R}u}{R}$ and

$$\left(\frac{\sup_{B_R}u}{R}\right)^n\leq C|\nabla\Gamma|(B_R). \tag{4}$$

And $B_d^+(b)=\{y\in B_d : y\cdot b>0\}\subset\nabla\Gamma(\mathcal{C}^+(u,\Gamma,B_R;b))$ is enough to have (4).

$1 < \sigma < 2$

Lemma

$\mathcal{M}^+_{\mathfrak{L}} u \geq -f$ on B_R, there exists some constant $C > 0$ depending only on n, λ and Λ (but not on σ) such that for any $x \in \mathcal{C}(u, \Gamma, B_R)$ and any $M > 0$ there is some $k \in \mathbb{N} \cup \{0\}$ such that

$$\left|\overline{R}_k(x)\right| \leq C \frac{R^{\sigma-2}(f(x) + J_\sigma(R)|\nabla\Gamma(x)|)}{M} |R_k(x)| \quad (5)$$

where $\overline{R}_k(x) = \{y \in R_k(x) : \mu^-(u, x, y; \nabla\Gamma) \geq M r_k^2\}$ (: a bad set)
and $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$ for $r_k = \varrho_0 2^{-\frac{1}{2-\sigma} - k} R$, $\varrho_0 = 1/(16\sqrt{n})$(: a ring)
and $J_\sigma(R)$ is $\frac{1}{1-\sigma}(1 - R^{1-\sigma})$ for $\sigma \in (0,1) \cup (1,2)$ and $-\log(R)$ for $\sigma = 1$. (: a blow-up rate)

$\left|\overline{R}_k(x)\right|$ measures the size of a bad set where the solution has been bended too much. A-B-P estimate tells us that the the size of bad set is controlled by f and $|\nabla\Gamma(x)|$ with blow-up rate. Such rate will be controlled when we prove the decay estimate of the upper level set of u for $1 < \sigma < 2$, but it blows up for $0 < \sigma \leq 1$.

Idea of Proof

Γ touches u from above and then we have $\mu(u,x,y;\nabla\Gamma)\le 0$ for $y\in B_R$.

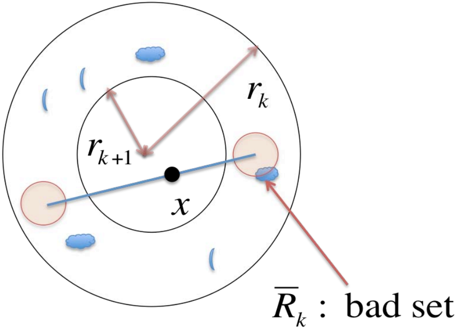
$$
\begin{aligned}
-f(x) &\le \mathcal{M}^+_{\mathfrak{L}} u(x;\nabla\Gamma)\\
&= (2-\sigma)\left(\int_{\mathbb{R}^n}\frac{-\lambda\mu^-(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy+\int_{\mathbb{R}^n}\frac{\Lambda\mu^+(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy\right)\\
&\le (2-\sigma)\int_{B_{r_0}(x)}\frac{-\lambda\mu^-(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy+(2-\sigma)\omega_n\Lambda\, J_\sigma(R)\,|\nabla\Gamma(x)|
\end{aligned}
$$

$$
\begin{aligned}
\frac{f(x_0)}{2-\sigma} &\ge \lambda\sum_{k=0}^{\infty}\int_{R_k(x_0)}\frac{-\mu(u,x_0,y;\nabla\Gamma(x))}{|y|^{n+\sigma}}\,dy-\omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)|\\
&\ge c\sum_{k=0}^{\infty}M_0\,\frac{r_k^2}{r_k^\sigma}\,CR^{\sigma-2}\,\frac{f(x_0)+J_\sigma(R)|\nabla\Gamma(x_0)|}{M_0}-\omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)|.
\end{aligned}
\tag{6}
$$

Thus this implies that

$$
\begin{aligned}
f(x_0)+(2-\sigma)\omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)| &\ge \frac{c\rho_0^2}{1-2^{-(2-\sigma)}}C(f(x_0)+J_\sigma(R)|\nabla\Gamma(x_0)|)\\
&\ge C(f(x_0)+(2-\sigma)J_\sigma(R)|\nabla\Gamma(x_0)|)
\end{aligned}
$$

, which implies the uniform bound of $C>0$.



1. The bad set $\overline{R}_k(x)$ of Γ has small measure in a ring $B_{r_k}(x) \setminus B_{r_{k+1}}(x)$. Since Γ is a concave function, there is no bad set in smaller ball $B_{r_{k+1}}(x)$.
2. If we can control oscillation of the concave function Γ, then the gradient image $\nabla\Gamma$ will be controlled in smaller ball.
3. Then for each x in the contact set $\mathcal{C}(u, \Gamma, B_R)$, we have a rectangle where the gradient image $\nabla\Gamma$ is controllable and such rectangles will make a covering of the contact set.

Nonlocal A-B-P estimate

Theorem

Let u and Γ be functions as in Lemma 3. Set $g_\eta(z) = (|z|^{n/(n-1)} + \eta^{n/(n-1)})^{1-n}$. Then there exist a finite family $\{Q_j\}_{j=1}^m$ of open cubes Q_j with diameters d_j such that

(a) *Any two cubes Q_i and Q_j do not intersect,* (b) $\mathcal{C}(u,\Gamma,B_R) \subset \bigcup_{j=1}^m \overline{Q}_j$,

(c) $\mathcal{C}(u,\Gamma,B_R) \cap \overline{Q}_j \neq \phi$ *for any Q_j,* (d) $d_j \leq \varrho_0 2^{-\frac{1}{2-\sigma}} R$ *where* $\varrho_0 = 1/(16\sqrt{n})$,

(e) $\int_{\overline{Q}_j} g_\eta(\nabla\Gamma(y))\,\mathrm{det}(D^2\Gamma(y))^- \,dy \leq CR^{n(\sigma-2)}(\sup_{\overline{Q}_j}(J_\sigma(R)^n + \eta^{-n}|f|^n)|Q_j|$,

(f) $|\{y \in 4\sqrt{n}\,Q_j : u(y) \geq \Gamma(y) - CR^{(\sigma-2)}(\sup_{\overline{Q}_j}(f + J_\sigma(R)|\nabla\Gamma|)d_j^2\}| \geq \eta_0|Q_j|$,

where the constants $C > 0$ and $\eta_0 > 0$ depend on n, Λ and λ (but not on σ).

We may consider the nonlocal A-B-P Estimate as *a kind of Riemann sum* of the Local A-B-P Estimate. The sum is given with a covering of the contact set since the gradient map has contribution only on the contact set between the solution and the concave envelope. But the size of rectangle is order of $2^{-\frac{1}{2-\sigma}}$, which goes to zero as $\sigma \to 2$ and then we recover the local A-B-P Estimate.

Key Lemma

Lemma

$u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity supersolution to $\mathcal{M}^-_{\mathfrak{L}_0} u \le \varepsilon_0/R^\sigma$ with $\sigma \ne 1$ or $\mathcal{L}u \le \varepsilon_0/R^\sigma$ with $\sigma \in (\sigma_0, 2)$ on $B_{2\sqrt{n}R}$ such that $u \ge 0$ on $\mathbb{R}^n$ and $\inf_{Q_R} u \le 1$, then $|\{u \le M\} \cap Q_R| \ge \nu |Q_R|$

If we apply this lemma inductively, we have the geometric decay of level sets.

Lemma

$$\Big|\{u > t\} \cap Q_R\Big| \le C\, t^{-\varepsilon_*} |Q_R|, \ \forall\, t > 0.$$

Idea of proof of Harnack Inequality Set $M = \sup_{Q_R} u$ with $u(0) = 1$ and $u \ge 0$. We apply the geometric decay estimate on u and $(M - u)_+$. Then $|\{u \ge M/2\}|$ and $|\{M - u \ge M/2\}|$ will be very small if M is very large, but it is a contradiction since $\{u \ge M/2\} \cup \{M - u \ge M/2\} = Q_R$

First notice $\mathcal{C}(u,\Gamma,B_R)\subset\{u\leq M\}\cap Q_R$. Apply A-B-P estimate on $u+\Psi$

$$\ln\left(\frac{[(\sup_{B_{2\sqrt{n}R}} v)/R]^n}{\eta^n}+1\right)\leq C\int_{\mathcal{C}(u,\Gamma,B_R)} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^- \,dy$$

$$\leq C\left(\sum_j \sup_{\overline{Q}_j}\left((R^{\sigma-2}J_\sigma(R))^n+\eta^{-n}(R^{(\sigma-2)}(\psi+\varepsilon_0)/R^\sigma\right)^n|Q_j|\right) \qquad (7)$$

$$\leq C\left((R^{\sigma-1}J_\sigma(R))^n+\eta^{-n}\sum_j\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\right).$$

Here we note that $K_R := Exp((R^{\sigma-1}J_\sigma(R))^n)\leq C<\infty$ for $1<\sigma<2$ and $R<1$. If we set $\eta=\left(\sum_j\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\right)^{1/n}$ in (10), then we have that

$$1\leq\sup_{B_{2\sqrt{n}R}} v\leq C\varepsilon_0+CR\left(\sum_j\left((\sup_{\overline{Q}_j}\psi)/R^2\right)^n|Q_j|\right)^{1/n}. \qquad (8)$$

$\frac{1}{2^{1/n}}R\leq C\left(\sum_j\left(\sup_{Q_j}\psi\right)^n|Q_j|\right)^{1/n}$ implies $\frac{1}{2}|Q_R|\leq C\left(\sum_{Q_j\cap B_{R/4}\neq\phi}|Q_j|\right)\leq|\{u\leq M\}\cap Q_R|$.

$0 < \sigma \leq 1$

Lemma

For any $x \in C^+(u, \Gamma, B_R; b)$ and any $M > 0$ there is some $k \in \mathbb{N} \cup \{0\}$ such that

$$\left|\widetilde{R}_k(x)\right| \leq C \frac{(R^{\sigma-2} f(x) + R^{-1}|\nabla\Gamma(x)|)}{M} |R_k(x)| \tag{9}$$

where $\widetilde{R}_k(x) = \{y \in R_k(x) : \mu^-(u, x, y; \nabla\Gamma) \geq M_0 r_k^2\}$.

Take any $x \in C_\eta^+(u, \Gamma, B_R)$. From the definition of $\mathcal{S}_\eta^{\mathfrak{L}_0}$, we have that

$$\begin{aligned}\mathcal{I}u(x; \nabla\Gamma) &\leq \mathcal{M}^+_{\mathfrak{L}_0,R} u(x; \nabla\Gamma) + \mathcal{B}^+_R(\nabla\Gamma(x)) + (2-\sigma)CR^{1-\sigma}|\nabla\Gamma(x)| \\ &\leq \mathcal{M}^+_{\mathfrak{L}_0,R} u(x; \nabla\Gamma) + (2-\sigma)CR^{1-\sigma}|\nabla\Gamma(x)|\end{aligned}$$

because $\mathcal{B}^+_R(\nabla\Gamma(x)) \leq 0$ from the assumption and $\mu^+_R(u, x, \cdot; \nabla\Gamma) = 0$ on $\mathbb{R}^n$. The conclusion comes from similar arguments

Main Step at Key Lemma

$$\begin{aligned}\ln\left(\frac{[(\sup_{B_{2\sqrt{n}R}} v)/R]^n}{\eta^n}+1\right) &\leq C\int_{C(u,\Gamma,B_R)} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^- \, dy \\ &\leq C\left(\sum_j \sup_{\overline{Q}_j}\left((R^{-1}J_\sigma(R))^n+\eta^{-n}(R^{(\sigma-2)}(\psi+\varepsilon_0)/R^\sigma\right)^n|Q_j|\right) \\ &\leq C\left((J_\sigma(R))^n+\eta^{-n}\sum_j \sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\right).\end{aligned} \tag{10}$$

where $J_\sigma(R)=\frac{1}{1-\sigma}(1-R^{1-\sigma})<C<\infty$ for some $C>0$ since $0<\sigma<1$.

Definition

Let $u : \mathbb{R}^n \times I \to \mathbb{R}$ be a function which is not positive on $\partial_p^* Q_{r/2}$ and is upper semicontinuous on $\overline{Q}_r$.
(i) $u(x,t)$ is called concave in $\mathbb{R}^n \times I$ if $u(x,t)$ is concave in x and nondecreasing in t.
(ii) The concave envelope $\Gamma(y,s)$ of u in Q_{2r} is defined as

$$\Gamma(y,s) = \begin{cases} \inf\{v(y,s) : v \in \Pi,\ v > u^+ \text{ in } Q_{2r}\} & \text{in } Q_{2r} \\ 0 & \text{in } \partial_p^* Q_{2r}, \end{cases}$$

where Π is the family of all concave functions v in Q_{2r} such that $v \leq 0$ on $\partial_p Q_{2r}$.
(iii) The *normal map* $\mathfrak{N}_u$ of $u : Q \to \mathbb{R}$ is given by

$$\mathfrak{N}_u(x,t) = \{(p,h) \in \mathbb{R}^n \times (\tau_1, t] : u(y,s) \leq u(x,t) + p \cdot (y-x),\ \forall (y,s) \in Q, \text{ and } u(x,t) - p \cdot x = h\} \tag{11}$$

Lemma

If $v \in C^{2,1}(\overline{Q}_r(x_0, t_0))$ *is concave in* x *and increasing in* t *and if* $v = 0$ *on* $\partial_p Q_r(x_0, t_0)$*, then we have that*

$$(a) \quad \int_{B_r(x_0)} v(x, t_0)\det(D^2 v(x, t_0))dx = (n+1)\int_{Q_r(x_0,t_0)} \partial_t v(x, t)\det(D^2 v(x, t))dxdt,$$

(b)

$$\mathfrak{N}_v(Q_r) = \int_{Q_r} \partial_t v(x, t)\det(D^2 v(x, t))^- dxdt,$$

(c)

$$\max_{x \in B_r} v(x, t_0) \le Cr\,\left(\mathfrak{N}_v(Q_r(x_0, t_0))\right)^{\frac{1}{n+1}}.$$

5.2 Main Steps at Parabolic A-B-P Estimate

1. Choose any point in the contact set $\mathcal{C}(u, \Gamma, Q_1) := \{(x,t) : u(x,t)\Gamma(x,t)\}$. Note date $\det(D^2 u(x,t)) = 0$ in $Q_1 \backslash \mathcal{C}(u, \Gamma, Q_1)$.
2. For a fixed time, carry out elliptic argument to control bad set in a cube of space variable .
3. Control the growth rate concave envelope at a cube of space-time variable using the property of the concave envelope.
   For any $(x,t) \in Q_1 \cap \mathcal{C}$, we have that

$$0 \leq \sup_{B_r(x)} \Gamma_t(y,t) \leq \left((1 - cr) \sup_{B_r(x,t) \cap \mathcal{C}} \Gamma_t(y,t) + \frac{1}{c} r \sup_{Q_1} \Gamma_t(y,s) \right) \text{ on } \overline{Q}_1$$

   for some uniform constant $c > 0$.

Parabolic A-B-P Estimate

Theorem

There exist a finite family $\{K_k\}$ of pairwise disjoint $(n+1)$-dimensional cubes with sidelength d_k such that

(a) $\mathcal{C}(u, \Gamma, Q_1) \subset \bigcup_k K_k$,

(b) $\mathcal{C}(u, \Gamma, Q_1) \cap K_k \neq \phi$ for any k,

(c) $r_k \leq 2\rho_0 2^{-\frac{1}{2-\sigma}}$ for any k,

(d) $|\mathfrak{R}_\Gamma(K_k)| \leq C\left(\sup_{2\overline{K}_k} f + d_k \sup_{Q_1} f\right)^{n+1} \times |\mathcal{C}(u, \Gamma, Q_1) \cap K_k|$

(e) $\left(\sup_{Q_1} u^+\right)^{n+1} \leq C\, |\mathfrak{R}_\Gamma(Q_1)|$

where the constants $C > 0$ depends on n, Λ and λ (but not on σ) and $2\overline{K}_k$ is cube with $2d_k$ containing K_k.

6. Degenerate nonlocal Equation of Porous Medium Type (Joint work with Sunghoon Kim)

In this chapter, $0 < \sigma < 1$. $\sigma = 1$ corresponds to the standard Laplacian. We consider initial value problem with fractional fast diffusion:

$$\begin{cases} (-\triangle)^{\sigma} u^m + u_t = 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbf{R}^n \backslash \Omega \\ u(x,0) = u_0(x) & \text{non-negative and } \dot{H}^{\sigma}_0\text{-bounded} \end{cases} \tag{12}$$

in the range of exponents $\frac{n-2\sigma}{n+2\sigma} < m < 1$, with $0 < \sigma < 1$.

$$\begin{cases} (-\triangle)^{\sigma} v + \left(v^{\frac{1}{m}}\right)_t = 0 & \text{in } \Omega \\ v = 0 & \text{on } \mathbf{R}^n \backslash \Omega \\ v(x,0) = v_0(x) = u_0^m(x) & \text{in } \Omega, \end{cases} \tag{M.P}$$

We consider the extension problem.

$$\begin{cases} \nabla(y^a \nabla v^*) = 0 & \text{in } y > 0 \\ \lim_{y\to 0} y^a v_y^*(x,y,t) = (v^{\frac{1}{m}})_t(x,0,t) & x \in \Omega \\ v^*(x,0,t) = 0 & \text{on } \mathbf{R}^n \backslash \Omega \end{cases} \tag{13}$$

for $a = 1 - 2\sigma$. Since the diffusion coefficients $D(v) = |v|^{1-\frac{1}{m}}$ goes to infinity as $v \to 0$, we need to control the oscillation of v from below. Hence, we consider the new function w^* derived from v^* such that $w^*(x,y,t) = M - v^*(x,y,t+t_0)$ with $M = M(t_0) = \sup_{t \geq t_0 > 0} v^*$. After showing the bound of L^∞-norm of v, we know that the solution satisfies

$$v^*(\cdot, t) \leq M(t_0) < \infty \qquad (t \geq t_0).$$

From this, we get to a familiar situation:

$$\begin{cases} \nabla(y^a \nabla w^*) = 0 & \text{in } y > 0 \\ -\lim_{y\to 0^+} y^a \nabla_y w^*(x,y) = \left[(M - w^*)^{\frac{1}{m}}\right]_t (x,0) & x \in \Omega \\ w^*(x,0,t) = M & \text{on } \mathbf{R}^n \backslash \Omega. \end{cases} \tag{14}$$

Theorem

(From $L^{\frac{n-2\sigma}{n+2\sigma}}$ to L^{∞}) Let $v(x,t)$ be a function in $L^{\infty}(0,T;L^{\frac{2n}{n-2\sigma}}(\Omega)) \cap L^2(0,T;\dot{H}_0^{\sigma}(\mathbf{R}^n))$, then

$$\sup_{x\in\Omega}|v(x,T)| \leq C^* \frac{||v_0||_{L^{\frac{2n}{n-2\sigma}}(\Omega)}}{T^{\frac{mn}{2mn-(n-2\sigma)(1+m)}}}$$

for some constant $C^* > 0$.

For the second theorem, we need better control of v.

Theorem (Hölder regularity of fractional FDE)

For $x_0 = (x_0^1, \cdots, x_0^n)$, we define $Q_r(x_0,t_0) = [x_0^i - r, x_0^i + r]^n \times [t_0 - r^{2\sigma}, t_0]$, for $t_0 > r^{2\sigma} > 0$. Assume now that $[x_0^i - r, x_0^i + r]^n \subset \Omega$ and $v(x,t)$ is bounded in $\mathbf{R}^n \times [t_0 - r^{2\sigma}, t_0]$, then there exist constants γ and β in $(0,1)$ that can be determined a priori only in terms of the data, such that v is C^{β} in $Q_{\gamma r}(x_0,t_0)$.

Properties

1. (Scaling Invariance)
$$u'(x',t') = L^{\frac{2\sigma}{m-1}} T^{-\frac{1}{m-1}} u\left(\frac{x'}{L}, \frac{t'}{T}\right)$$
2. (L^1-contraction)
$$\int_\Omega \left[u(x,t) - \tilde{u}(x,t)\right]_+ dx \leq \int_\Omega \left[u(x,\tau) - \tilde{u}(x,\tau)\right]_+ dx \tag{15}$$
As a consequence,
$$\|u(t) - \tilde{u}(t)\|_1 \leq \|u_0 - \tilde{u}_0\|_1. \tag{16}$$
3. Comparison Principle
4. Mass conservation in Cauchy problem. $\int_{\mathbf{R}^n} u_0(x)dx = \int_{\mathbf{R}^n} u(x,t)dx$ for $t > 0$.
5. (Extinction in Finite Time in the bounded domain with zero data) There exists $T^* > 0$ such that $u(\cdot,t) = 0$ for all $t \geq T^*$, i.e.,
$$\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = 0$$
6. (No waiting time) Even for $u_0(x)$ having a compact support, $u(x,t) > 0$ for $t > 0$.

Estimates on Finite Extinction Time

Lemma

When $\frac{n-2\sigma}{n+2\sigma} < m < 1$, *there exists a positive constant* C *such that the solution* $v = u^m$ *of* (M.P) *satisfies*

$$T^* - t \leq C\left(\int_{\mathbf{R}^n} v^{\frac{m+1}{m}}(x,t)\,dx\right)^{\frac{1-m}{1+m}}.$$

Lemma

When $\frac{n-2\sigma}{n+2\sigma} < m < 1$, *the solution* v *satisfies*

$$\int_{\mathbf{R}^n} v^{\frac{m+1}{m}}(x,t)\,dx \leq \left(1 - \frac{t}{T^*}\right)^{\frac{1+m}{1-m}} \int_{\mathbf{R}^n} v^{\frac{m+1}{m}}(x,0)\,dx. \qquad (17)$$

Proof of Theorem for L^∞-norm

Multiplying the equation by the function $v_k = (v - C_k)_+$ and integrating in space, $\mathbf{R}^n$, we have

$$\frac{1}{m}\int_{\mathbf{R}^n} \frac{d}{dt}\left[\int_0^{v_k} (\xi + C_k)^{\frac{1}{m}-1}\,\xi\, d\xi\right] dx + \int_{\mathbf{R}^n} v_k\,[(-\triangle)^\sigma v_k]\; dx \le 0 \tag{18}$$

since

$$(C_k + v_k)^{\frac{1}{m}} v_k = (C_k + \xi)^{\frac{1}{m}} \xi\Big|_{\xi=0}^{\xi=v_k} = \int_0^{v_k} \frac{d}{d\xi}\left[(C_k + \xi)^{\frac{1}{m}} \xi\right] d\xi.$$

After some computation, we have $U_k \le 2C(m+1)\left(\int_\Omega v_k^{\frac{1+m}{m}}(s)\,dx\right)^{\frac{2m}{1+m}}$ for

$$U_k = \sup_{t \ge T_k} \int_\Omega v_k^{\frac{1}{m}+1}\,dx + (m+1)\int_{T_k}^{\infty} \|v_k\|^2_{\dot{H}^\sigma}\,dt.$$

By applying time average and Hölder regularity,

$$U_k \le C' \frac{\left(2^{\frac{4m}{1+m}\left(1+\frac{(1+m)\sigma}{mn}\right)-\frac{2}{1+m}}\right)^k}{t_0^{1-\frac{1-m}{1+m}} N^{\frac{4m}{1+m}\left(1+\frac{(1+m)\sigma}{mn}\right)-2}} U_{k-1}^{\frac{2m}{1+m}\left(1+\frac{(1+m)\sigma}{mn}\right)}$$

And $U_0 \le \|v(\cdot,0)\|^2_{L^{\frac{2n}{n-2\sigma}}}$. If $\|v(\cdot,0)\|^2_{L^{\frac{2n}{n-2\sigma}}}$ is smaller than a uniform number, $U_k \rangle 0$ as $k \rangle \infty$.

Proof of Theorem for C^α-norm

1. For the technical reason, we consider the equation for the $w = \text{maximum} - v$.
2. Wr consider local Energy estimate for the extension problem since the values on the local region has large influence from the whole domain. Technically the multiple of the cut-off function η and w is hard to applied the nonlocal operator. And the extension problem is easy to handle since it is a local equation. But the cut-off function is only depends on x, not z. Still it can be played as an cut-off function in (x, z) variable for a little bit larger level set. And we need to use intrinsic scale, for example the scaled parabolic cylinder is

$$Q_k(\theta_0) = B_{R_k} \times \left(-\theta_0^{-\alpha} R_k^{2\sigma}, 0\right)$$

3. If local L^2-norm is uniformly small for u such that $|u| < 1$ in a parabolic cylinder, $|u| < 1/2$ in a half cylinder as the control for L^∞-norm.
4. For general case, there is a time t^*, when the solution is close to the supremum with nontrivial measure. Through DiBenedetto's trick, we can show the solution is close to the supremum with nontrivial measure for $t^* < t < 0$.
5. Use De Giorgi's isoperimetric inequality to show that the upper level set is uniformly small after uniform step. Then the local L^∞-estimate says the supremum decreases with a uniform amount.

Theorem

Under the above assumptions on u_0 and m, we have the following property near the extinction time of a solution $u(x,t)$: for any sequence $\{u(x,t_n)\}$, we have a subsequence $t_{n_k} \to T^*$ and a $\varphi(x)$ such that

$$\lim_{k\to\infty}(T^* - t_{n_k})^{-1/(1-m)}\left|u(x,t_{n_k}) - U(x,t_{n_k};T^*)\right| \to 0$$

uniformly in compact subset of Ω for $U(x,t;T^*) = (T^* - t)^{1/(1-m)}\varphi^{1/m}(x)$ where φ is a eigen-function of fully nonlinear equation

$$\begin{cases} (-\triangle)^\sigma\varphi = \dfrac{1}{1-m}\varphi^{\frac{1}{m}} & \text{in } \Omega \\ \varphi = 0 & \text{on } \mathbf{R}^n\backslash\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

7.Geometry of the Ground State (Joint work with Sunghoon Kim)

$$\begin{cases} (-\triangle)^{\frac{1}{2}}\varphi = \lambda\varphi^p & \text{in } \Omega \\ \varphi > 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \mathbf{R}^n \backslash \Omega. \end{cases} \tag{19}$$

The main question we address is motivated by the following conjecture:

Conjecture

Let φ_σ be the ground state eigenfunction for the symmetric stable processes of index $0 < \sigma < 1$ killed upon leaving the interval $I = (-1, 1)$. Then φ_σ is concave on I.

Rodrigo Bañuelos, Tadeusz Kulczycki, Pedro J. Méndez-Hernández. *On the shape of the ground state eigenfunction for stable processes.* Potential Anal. 24(2006), no. 3

The solution u of

$$\begin{cases} (-\triangle)^{\sigma} u + u_t = 0 & (x,t) \in \mathbf{R}^n \times [0,\infty) \\ u(x,t) > 0 & x \in \mathbf{R}^n \\ u(x,t) = 0 & x \in \mathbf{R}^n \backslash \Omega \\ u(x,0) = g(x) & x \in \mathbf{R}^n \end{cases}$$

is the trace, $u^*(x,0,t)$, of u^* satisfying the following extension problem:

$$\begin{cases} \triangle_x u^* + u^*_{,zz} = 0 & \text{in } \mathbf{R}^n \times \mathbf{R}^+ \times [0,\infty) \\ u^*_z(x,0,t) = u^*_t(x,0,t) & x \in \Omega \\ u^*(x,0,t) = 0 & x \in \mathbf{R}^n \backslash \Omega \\ u^*(x,0,0) = g(x) & x \in \mathbf{R}^n. \end{cases}$$

This relation between u and u^* is very useful to discuss the geometric properties.

Theorem

Let Ω be a convex bounded domain, and let $u_0 \geq 0$ be a continuous and bounded initial function. If $(u_0)^{-\frac{2}{n+1}}$ is strictly convex, then the solution u^* is power-convex in the space variable x for all $t, z > 0$, i.e., $D_x^2(u^*)^{-\frac{2}{n+1}} \geq 0$.

Corollary

Under the same condition, the stationary profile $\varphi(x)$ of $u(x,t)$ is power-convex, i.e., $D_x^2\left(\varphi(x)\right)^{-\frac{2}{n+1}} \geq 0$.

Lemma

If Ω is smooth and strictly convex, then $\varphi(x)$ is strictly power-convex: there exists a constant $c_1 > 0$ such that

$$D_x^2\left(\varphi(x)\right)^{-\frac{2}{n+1}} \geq c_1 \mathbf{I}.$$

The constant c_1 depends on φ and Ω.

Parabolic Approximation and Nontriviality of the Limit

Lemma (Approximation Lemma)

For every $u_0 \in \dot{H}_0^{\frac{1}{2}}(\Omega)$*, we have*

$$|e^{\lambda_1 t}u(x,t) - a_1\varphi(x)| \le Ce^{-(\lambda_2-\lambda_1)t} \tag{20}$$

and

$$\|e^{\lambda_1 t}u(x,t) - a_1\varphi(x)\|_{C_x^k(\Omega)} \le CKe^{-(\lambda_2-\lambda_1)t} \tag{21}$$

for $k = 1, 2, \cdots$.

Put

$$e^{\lambda_1 t}u(x,t) - a_1\varphi(x) = e^{-(\lambda_2-\lambda_1)t}\eta(x,t).$$

Then, $\eta(x,t)$ satisfies the equation $\eta_t + (-\triangle)^{\frac{1}{2}}\eta = \lambda_2\eta$. The regularity on η gives the conclusion.

Lemma

If u^* is a solution of the problem, then the function u^* is bounded and

$$\|u^*\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^+)} \leq \sup_{x\in\mathbf{R}^n} u_0(x).$$

$$\|u^*_t\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^+)} \leq C\|u_0\|_{H^{1/2}(\mathbf{R}^n)}$$

$$\|u^*_z\|_{L^\infty(\Omega)} \leq C \qquad at\ z=0$$

Lemma

If u^* is a solution of the problem , and if $|u^*| \leq 1$ on the cylinder $B_1^+ \times [0,1]$, Then

$$|u^*_{x_i}| \leq C, \quad |u^*_{x_i x_i}| \leq C$$

$$|u^*_{zz}| \leq C$$

$(i = 1, \cdots, n)$ *on the cylinder* $B_{1/2}^+ \times \left[\frac{1}{2}, 1\right]$ *for some constant* $C > 0$.

(i) (Bernstein Computation) Find a sub-solution to apply maximum principle:

$$X = A{u^*}^2 + \eta^2 (u^*_{x_i})^2$$

where A is a large number and η is a function satisfying

$$\eta(x,z,t) = \left(e^{\frac{z^2 t}{2}} + \frac{t}{2}\left(z + \frac{t^2}{4}\right) - 1\right)(1 - |x|^2 - z^2).$$

Observe $(u^*_{x_i})^2 \leq X$ on $B_{1/2}^+ \times \left[\frac{1}{2}, 1\right]$.
(ii) Apply the same method for

$$X = A{u^*_{x_i}}^2 + \eta^2 (u^*_{x_i,x_i})^2$$

(ii) $u^*_{zz} = -\sum_i u^*_{x_i,x_i}$.

Boundary Regularity

Lemma (Boundary Regularity for u)

Let Ω be a convex subset of $\mathbf{R}^n$ with $n > 1$ and u is the solution. Then,

$$\left|u^2(x,t) - c_0 \textbf{dist}(x,\partial\Omega)\right| \le C_0 \left(\textbf{dist}(x,\partial\Omega)\right)^{1+\gamma}, \qquad t > 0$$

near the boundary $\partial\Omega$ for some uniform constant C_0 and $0 < \gamma < 1$.

First we have

$$\overline{c}_1 \left(\text{dist}\left(x,\partial\Omega\right)\right)^{\frac{1}{2}} \le u(x,t) \le \overline{c}_2 \left(\text{dist}\left(x,\partial\Omega\right)\right)^{\frac{1}{2}}, \qquad \text{near the boundary } \partial\Omega. \tag{22}$$

u stays away from the upper bound or lower bound at an interior point with a fixed amount δ. Harnack estimate in the interior gives a ball where u stays away from the upper or lower bound. Such improvement in a ball with uniform measure will make improvement all the way to the boundary by comparing super- or sub-solution having the improvement. Then we have improvement.

$$\overline{c}_3 \left(\text{dist}\left(x,\partial\Omega_{\frac{1}{4}}\right)\right)^{\frac{1}{2}} \le u_{\frac{1}{4}}(x,t) \le \overline{c}_4 \left(\text{dist}\left(x,\partial\Omega_{\frac{1}{4}}\right)\right)^{\frac{1}{2}}, \qquad \left(\frac{3t_1}{4} \le t \le t_1\right)$$

for some constants $\overline{c}_3 < \overline{c}_4$ such that

$$\overline{c}_1 \le \overline{c}_3 \le \overline{c}_4 \le \tilde{c}_2 \qquad \text{and} \qquad \overline{c}_4 - \overline{c}_3 < \tilde{c}_2 - \overline{c}_1.$$

Hence, we get, for $v = u^2$,

$$\left(u^{-\frac{2}{n+1}}\right)_{\nu\nu} = \left(v^{-\frac{1}{n+1}}\right)_{\nu\nu} = -\frac{1}{n+1}\left[\frac{vv_{\nu\nu} - \left(\frac{n+2}{n+1}\right)v_\nu^2}{v^{\frac{2n+3}{n+1}}}\right]$$
$$\approx \frac{(n+2)v_\nu^2}{(n+1)^2 v^{\frac{2n+3}{n+1}}} \to \infty \qquad \text{as } x \in \Omega \to x_0 \in \partial\Omega. \tag{23}$$

Lemma (Estimate for u^* near the infinity)

Assume that $u(x,t)$ is a bounded and integrable function with a compact support. Let $\hat{u}(x,z,t)$ be the convolution of u and Poisson kernel p of the Harmonic extension:

$$u^*(x,z,t) = \int_{\mathbf{R}^n} p(x,\xi,z)u(\xi,t)\,d\xi$$

where

$$p(x,\xi,z) = \frac{c_n z}{(|x-\xi|^2 + z^2)^{\frac{n+1}{2}}}, \qquad z > 0,\ x,\xi \in \mathbf{R}^n,\ c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

Then, there exist some constant c_1 and R_1 such that $\hat{u}$ is power-convex in the space variable x for all $t > 0$ on $\{(x,z) : |x| > c_1 R_1, 0 < z < R_1\}$, i.e.,

$$D_x^2\left[(u^*)^{-\frac{2}{n+1}}\right] \geq 0 \qquad \forall |x| > c_1 R_1,\ 0 < z < R_1. \tag{24}$$

Approximation

For any $0 < \delta < \delta'$, let us consider the solution $\tilde{u}_\delta$ of

$$
\begin{cases}
\triangle_x \tilde{u}_\delta + \tilde{u}_{\delta,zz} = 0 & (x,t) \in B_{\frac{1}{\delta}} \times \mathbf{R}^+, \quad 0 < z < \frac{1}{\delta} \\
\tilde{u}_{\delta,z}(x,0,t) = \tilde{u}_{\delta,t}(x,0,t) & (x,t) \in \Omega \times \mathbf{R}^+ \\
\tilde{u}_\delta(x,0,t) = \delta\xi(x,t) & x \in B_{\frac{1}{\delta}} \backslash \Omega \\
\tilde{u}_\delta(x,z,t) = \delta & (x,z) \in \partial B_{\frac{1}{\delta}} \times [0,1/\delta] \\
\tilde{u}_\delta\,(x,1/\delta,t) = \delta\zeta(x) & (x,t) \in B_{\frac{1}{\delta}} \times \mathbf{R}^+ \\
\tilde{u}_\delta(x,z,0) = \overline{u}(x,z) & x \in B_{\frac{1}{\delta}},\ 0 \le z \le \frac{1}{\delta}
\end{cases}
\tag{25}
$$

Let's define the positive functions $\zeta(x) \in C_0^\infty(B_{\frac{1}{\delta}})$ satisfying

$$(\zeta)^{-\frac{2}{n+1}} \quad : \text{bounded and strictly convex}$$

and $\xi(x,t) \in C^\infty(\{B_{\frac{1}{\delta}} \backslash \Omega\} \times \mathbf{R}^+)$ such that

$$
\begin{cases}
\xi(x,t) = 1 & \text{on } x \in \partial\Omega, \\
\xi(x,t) = \delta & \text{on } x \in \partial B_{\frac{1}{\delta}}, \\
\left([u(x,t) + \delta]\,\chi_{\{\Omega\}} + \delta\xi\chi_{\{B_{\frac{1}{\delta}} \backslash \Omega\}} \right)^{-\frac{2}{n+1}} & : \text{strictly convex.}
\end{cases}
$$

Lemma

Let Ω be a convex bounded domain and let $\tilde{u}_0 \geq 0$ be a continuous and bounded initial function. If $(u_0)^{-\frac{2}{n+1}}$ is strictly convex, then the solution $\tilde{u}_\delta$ is power-convex in the space variable x for all $t > 0$, i.e., $D_x^2\left[(\tilde{u}_\delta)^{-\frac{2}{n+1}}\right] \geq 0$.

Let $w = (\tilde{u}_\delta)^{-\frac{2}{n+1}}$. Then w satisfies

$$0 = w^2 \triangle_{(x,z)} w - \frac{(n+3)}{2} w|\nabla_{(x,z)} w|^2 \qquad \text{in } B_{\frac{1}{\delta}} \times [0, 1/\delta] \times \mathbf{R}^+ \tag{26}$$

and

$$w_z(x,0,t) - w_t(x,0,t) = 0 \qquad x \in B_{\frac{1}{\delta}} \tag{27}$$

To estimate the minimum of the second derivatives, we look at the quantity

$$w_{\overline{\alpha}\overline{\alpha}}(x_0, z_0, t) + \varepsilon\psi(t) = \inf_{(x,z)\in B_{\frac{1}{\delta}}\times\left[0,\frac{1}{\delta}\right], s\in[0,t]} \ \inf_{e_\beta\in\mathbf{R}_x^n, |e_\beta|=1} \left[w_{\beta\beta}(x,z,s) + \varepsilon\psi(s)\right]$$

$(1 \leq \beta \leq n)$, which is taken along a direction $e_{\overline{\alpha}} \in \mathbf{R}_x^n$, $(|e_{\overline{\alpha}}| = 1)$, in which the minimum of the second directional derivative is achieved.

Case 1. $x_0 \in \partial B_{\frac{1}{\delta}}$ and $z_0 > 0$. Nondegneracy of the gradient on the boundary implies a contradiction.
Case 2. We consider the case : $z_0 = 0$ and $x_0 \in \partial\Omega$. Boundary estimate for nonlocal equations implies $w_{\overline{\alpha\alpha}} \rangle \infty$ and a contradiction.
Case 3. We next deal with the case $x_0 \in \Omega$ and $z_0 = 0$.

$$Z_w(x,z,t) = \inf_{e_\beta \in \mathbf{R}^n} w_{\beta\beta}(x,z,t) + \varepsilon\psi(t)$$

$$\begin{aligned} 0 \geq (Z_w)_t &= w_{\alpha\alpha,t}(x_1,0,t_1) + \varepsilon\psi_t(t_1) = (Z_w)_z + \varepsilon\psi_t(t_1), \\ \text{and} \qquad 0 &\leq (Z_w)_z = w_{\alpha\alpha,z}(x_1,0,t_1). \end{aligned}$$

Hence, by choosing the function $\psi(t)$ with $\psi_t(t) > 0$, we get a contradiction.
Case 4. Finally, let us consider the case that $x_0 \in B_{\frac{1}{\delta}}$ and $0 < z_0 < 1/\delta$.

$$Z = w_{,\tilde{\alpha}\tilde{\beta}}\eta^{\tilde{\alpha}}\eta^{\tilde{\beta}} + \varepsilon\psi(t), \qquad\qquad e_{\tilde{\alpha}},\ e_{\tilde{\beta}} \in \mathbf{R}^{n+1}_{(x,z)}$$

with

$$\eta^{\tilde{\beta}}(x,z) = \delta_{\overline{\alpha}\tilde{\beta}} + c_{\overline{\alpha}}y^{\tilde{\beta}} + \frac{1}{2}c_{\overline{\alpha}}c_{\tilde{\gamma}}y^{\tilde{\gamma}}y^{\tilde{\beta}}.$$

Select $c_{\overline{\alpha}}$ and $\psi(t)$ so that Z has a contradiction at the maximum point.
Finally select $\psi(t)$ which satisfies the condition in the cases to have a contradiction.

8. The other works and On going projects

1. Regularity theory for Fully Nonlinear Integro-differential equations with nonsymmetric kernels (2010, 2011 with Youngcheol Kim)
2. Regularity theory for Parabolic Fully Nonlinear Integro-differential equations (2012 with Youngcheol Kim)
3. The properties of global solutions with super-linear forcing term, u^p. (Jinmyoung Seok)
4. Nonlocal Curvature Flows (with Hyeseon Kim)
5. Geostropic equations (with Jihoon Kim)
6. Homogenization with perforated boundaries (with Minha Kim, Martin)

Thank You!